5

Global Determination of the Basin of Attraction

Up to now we have focussed on the construction of a Lyapunov function, i.e. a function with negative orbital derivative. For such a function in a neighborhood of an exponentially asymptotically stable equilibrium one can always find a Lyapunov basin as described above. Since the basin of attraction is an open set and the Lyapunov basins are compact sets, they are always proper subsets of the basin of attraction. Hence, the best we can expect is that, given a compact subset of the basin of attraction, we find a Lyapunov basin larger than this compact set with our method.

For the results of this section we assume that f is bounded in $A(x_0)$ which can be achieved by studying the dynamically equivalent system $\dot{x} = g(x)$ where $g(x) = \frac{f(x)}{1+\|f(x)\|^2}$, cf. Remark 2.5. In particular, the basin of attraction of x_0 is the same for the two systems $\dot{x} = f(x)$ and $\dot{x} = g(x)$. We can show that given a compact set $K_0 \subset A(x_0)$ one obtains a Lyapunov basin larger than K_0 by approximating the function V. The approximation can either be direct or using the Taylor polynomial of V. This result uses an estimate of $|[V(x) - V(x_0)] - [v(x) - v(x_0)]|$ near x_0. Note, that this estimate is possible although the approximation v only uses the values of the orbital derivative $V'(x)$ and not of $V(x)$. The reason is that V is a smooth function in x_0. Thus, the result does not hold for approximations of the function T.

The result requires a sufficiently dense grid. Even if the set $v'(x) < 0$ is already quite large, the largest sublevel set of v probably only provides a small Lyapunov basin. In order to enlarge the Lyapunov basin one has to use a denser grid – not only where $v'(x) > 0$, not only near the boundary of the former Lyapunov basin, but in the whole expected basin of attraction.

We consider again the example (2.11). A series of Lyapunov basins with denser grids is shown in Figures 5.1 to 5.3. Note that here the sets $v'(x) < 0$ do not change significantly, but the values of v and hence the Lyapunov basins do. However, the enlargement of the Lyapunov basins is not monotonous, cf. Figures 5.1 and 5.2. This indicates that in practical applications the error is in fact smaller than predicted by the corresponding theorem. Figure 5.3

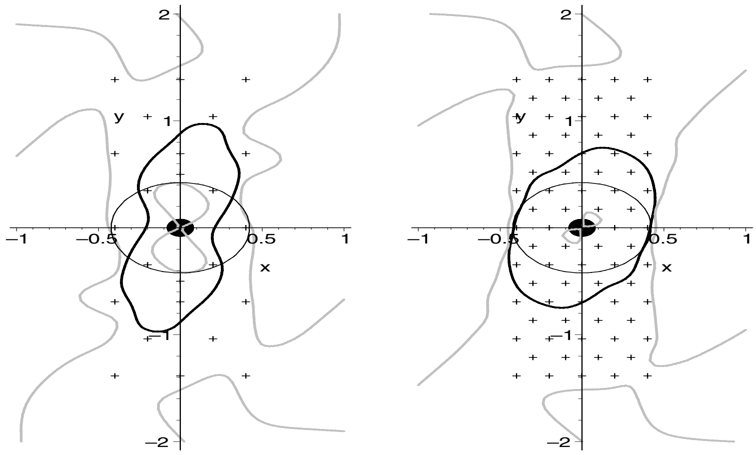

Fig. 5.1. The function v is the approximation of the Lyapunov function V satisfying $V'(x, y) = -(x^2 + y^2)$ for the example (2.11). The figures show the set $v'(x, y) = 0$ (grey), a level set of $v(x, y)$ (black) and a local Lyapunov basin (thin black). Left: grid density $\alpha = 0.4$. Right: grid density $\alpha = 0.2$ (cf. also Figures 5.2 and 5.3). Compare the Lyapunov basins (black) in both figures: although we used more grid points (black +) in the right-hand figure, the left-hand Lyapunov basin is no subset of the right-hand one.

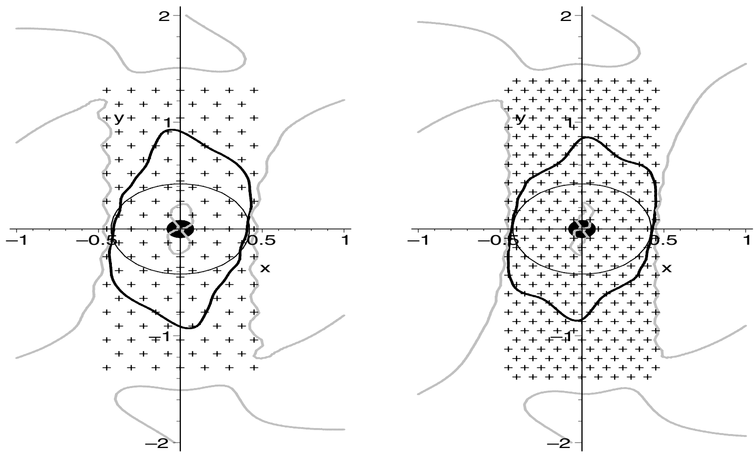

Fig. 5.2. The set $v'(x, y) = 0$ (grey), a level set of $v(x, y)$ (black), the grid points (black +) and a local Lyapunov basin (thin black). Left: grid density $\alpha = 0.15$. Right: grid density $\alpha = 0.1$ (cf. also Figure 5.3). The example considered is (2.11).

compares the best result (484 grid points) with the numerically calculated boundary of the basin of attraction, an unstable periodic orbit. For the data of the grids and the calculations of all figures, cf. Appendix B.2.

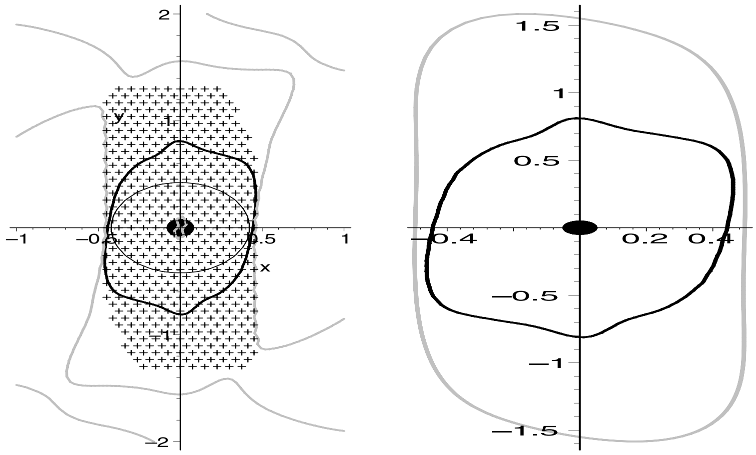

Fig. 5.3. The set $v'(x, y) = 0$ (grey), a level set of $v(x, y)$ (black), the grid points (black +) and a local Lyapunov basin (thin black). Left: grid density $\alpha = 0.075$. Right: the same Lyapunov basin (black) is shown together with the numerically calculated periodic orbit which is the boundary of the basin of attraction (grey). The example considered is (2.11).

Given a Lyapunov function q and a Lyapunov basin K, Theorem 2.24 implies $K \subset A(x_0)$. Thus, on the one hand, we search for a function q, the orbital derivative of which is negative in $K \setminus \{x_0\}$, i.e. a Lyapunov function. On the other hand, K is supposed to be a sublevel set of q, i.e. K is a Lyapunov basin. We have discussed the construction of a Lyapunov function in the preceding chapter, but can we thus find a Lyapunov basin? The appropriate question concerning a *global* Lyapunov function is, whether we can cover any compact set $K_0 \subset A(x_0)$ with our approach, supposed that the grid is dense enough.

The precise result which we will obtain reads: Let K_0 be a compact set with $x_0 \in \mathring{K}_0 \subset K_0 \subset A(x_0)$. Then there is an open set B with $\overline{B} \subset A(x_0)$, a compact set $B \supset K \supset K_0$ and a function q obtained by our method, such that

1. $K = \{x \in B \mid q(x) \leq (R^*)^2\}$ for an $R^* \geq 0$,
2. $q'(x) < 0$ holds for all $x \in K \setminus \{x_0\}$.

In other words, q is a Lyapunov function with Lyapunov basin K and thus they fulfill the conditions of Theorem 2.24. This can be achieved for the approximant v_W of V via W. For the direct approximation of V by v one has to use the extension v^* due to the local behavior of v. For the approximation t of T these results do not hold since T is not defined and smooth in x_0 which is necessary for the proof. For T, we consider a mixed approximation in Section 5.2.

We show in Section 5.1.1 that we can cover any compact subset of $A(x_0)$ when approximating the function V with $V' = -p(x)$. Because of possible numerical problems near x_0 – we have to choose a very dense grid here and this leads to a high condition number of the interpolation matrix – we also discuss a mixed approximation of V^* in Section 5.2 where, additionally to the orbital derivative, the values of the approximated function are given on a non-characteristic hypersurface. In case of the approximation of T this mixed approximation is the only possibility to cover an arbitrary compact subset of $A(x_0)$. The mixed approximation is particularly appropriate to approximate the basin of attraction step by step.

5.1 Approximation via a Single Operator

In this section we consider the approximation of a single operator. In Section 5.1.1 we approximate the function V satisfying $V'(x) = -p(x)$ via the operator $Dq(x) = q'(x)$ of the orbital derivative, where we follow [24]. In Section 5.1.2 we approximate the function $W(x) = \frac{V(x)}{\mathfrak{n}(x)}$ via the operator $D_m W(x) = W'(x) + m(x)W(x)$.

5.1.1 Approximation via Orbital Derivatives

Approximating the function $V'(x) = -p(x)$, cf. Theorem 2.46, we obtain an error estimate for the values of v and show in Theorem 5.1 a converse theorem to Theorem 2.24. However, by this theorem we will need grid points near the equilibrium which may lead to difficulties in the numerical calculation.

Theorem 5.1. *Let x_0 be an equilibrium of $\dot{x} = f(x)$ where $f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ such that the real parts of all eigenvalues of $Df(x_0)$ are negative. Moreover, assume that $\sup_{x \in A(x_0)} \|f(x)\| < \infty$ or, more generally, $\sup_{x \in \mathbb{R}^n} \|f(x)\| < \infty$ holds; this can be achieved using (2.2).*

We consider the radial basis function $\Psi(x) = \psi_{l,k}(c\|x\|)$ with $c > 0$, where $\psi_{l,k}$ denotes the Wendland function with $k \in \mathbb{N}$ and $l := \left\lfloor \frac{n}{2} \right\rfloor + k + 1$. Let $\mathbb{N} \ni \sigma \geq \sigma^* := \frac{n+1}{2} + k$. Let $\mathfrak{q}$ be a (local) Lyapunov function with Lyapunov basin $\tilde{K} := \tilde{K}_r^{\mathfrak{q}}(x_0)$. Let $K_0 \subset A(x_0)$ be a compact set with $\tilde{K} \subset \overset{\circ}{K}_0$.

Then there is an open set B with $\overline{B} \subset A(x_0)$ and an $h^* > 0$, such that for all reconstructions $v \in C^{2k-1}(\mathbb{R}^n, \mathbb{R})$ of the Lyapunov function V of Theorem 2.46 where $V'(x) = -p(x)$ with respect to the grid $X_N \subset \overline{B} \setminus \{x_0\}$ with fill distance $h \leq h^*$ there is an extension v^* as in Theorem 4.8 and a compact set $K \supset K_0$ such that

- *$(v^*)'(x) < 0$ holds for all $x \in K \setminus \{x_0\}$,*
- *$K = \{x \in B \mid v^*(x) \leq (R^*)^2\}$ for an $R^* \in \mathbb{R}^+$.*

In other words, v^* is a Lyapunov function with Lyapunov basin K.

Proof: Let $V \in C^\sigma(A(x_0), \mathbb{R})$ be the function of Theorem 2.46 which satisfies $V'(x) = -p(x)$ for all $x \in A(x_0)$ and $V(x_0) = 0$. Set $R := \sqrt{\max_{x \in K_0} V(x)} > 0$ and

$$\begin{aligned} K_1 &= \{x \in A(x_0) \mid V(x) \leq R^2\}, \\ K_2 &= \{x \in A(x_0) \mid V(x) \leq R^2 + 2\}, \\ B &= \{x \in A(x_0) \mid V(x) < R^2 + 3\}. \end{aligned}$$

Then obviously $K_0 \subset K_1 \subset K_2 \subset B \subset \overline{B} \subset A(x_0)$ and B is open, cf. Theorem 2.46; note that $\sup_{x \in A(x_0)} \|f(x)\| < \infty$. All these sets are positively invariant. Let $\tilde{B}$ be an open set with $\overline{B} \subset \tilde{B} \subset \overline{\tilde{B}} \subset A(x_0)$, e.g. $\tilde{B} = \{x \in A(x_0) \mid V(x) < R^2 + 4\}$. Let $\chi \in C_0^\infty(\mathbb{R}^n, [0,1])$ be a function with $\chi(x) = 1$ for $x \in \overline{B}$ and $\chi(x) = 0$ for $\mathbb{R}^n \setminus \tilde{B}$. Thus, $\chi \in C_0^\infty(\mathbb{R}^n) \subset \mathcal{F}$. Set $\tilde{a} := \|\chi\|_{\mathcal{F}}$ and $V_0 = V \cdot \chi$; then $V_0 \in C_0^\sigma(\mathbb{R}^n, \mathbb{R})$ and $V_0(x) = V(x)$ holds for all $x \in \overline{B}$. Lemma 3.13 implies $V_0 \in \mathcal{F}$. Choose $r_0' > 0$ so small that $\overline{B_{r_0'}(x_0)} = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r_0'\} \subset \tilde{K}$ and

$$2r_0' \max_{\tilde{r} \in [0, r_0']} \left| \frac{d}{dr} \psi(\tilde{r}) \right| \leq \frac{1}{4\|V_0\|_{\mathcal{F}}^2} \tag{5.1}$$

hold where $\psi(r) := \psi_{l,k}(cr)$. This is possible since $\frac{d}{dr}\psi(r) = O(r)$ for $r \to 0$, cf. Proposition 3.11. Choose $r_0 > 0$ such that

$$\Omega := \{x \in A(x_0) \mid V_0(x) = r_0^2\} \subset \overline{B_{r_0'}(x_0)}$$

holds. Ω is a non-characteristic hypersurface by Lemma 2.37 and, hence, by Theorem 2.38 there exists a function $\theta \in C^\sigma(A(x_0) \setminus \{x_0\}, \mathbb{R})$ defined implicitly by $S_{\theta(x)} x \in \Omega$.

Set $\theta_0 := \max_{x \in \overline{B}} \theta(x) > 0$. Define $\min_{x \in \overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)} p(x) =: M_0 > 0$. Define $h^* > 0$ such that $h^* < \left(\frac{1}{C^*} \min \left(\frac{1}{2\theta_0}, M_0 \right) \right)^{\frac{1}{\kappa}}$ holds with C^* and κ as in Theorem 4.2 where $K_0 = \overline{B}$.

Let $X_N \subset \overline{B} \setminus \{x_0\}$ be a grid with fill distance $h \leq h^*$. For the approximant v of $V_0 (= V \text{ in } \overline{B})$ we set $\tilde{b} := -v(x_0)$. For the function $\tilde{v} := v + \tilde{b} \cdot \chi$ we have $\tilde{v}(x_0) = v(x_0) + \tilde{b} = 0$. For $x^* \in \overline{B_{r_0'}(x_0)}$ we have with $\delta_{x^*}, \delta_{x_0} \in \mathcal{F}^*$, cf. Lemma 3.22,

$$\begin{aligned} |V_0(x^*) - \tilde{v}(x^*)| &= |(\delta_{x^*} - \delta_{x_0})(V_0 - v - \tilde{b}\chi)| \\ &= |(\delta_{x^*} - \delta_{x_0})(V_0 - v)| \\ &\leq \|\delta_{x^*} - \delta_{x_0}\|_{\mathcal{F}^*} \cdot \|V_0 - v\|_{\mathcal{F}} \\ &\leq \|\delta_{x^*} - \delta_{x_0}\|_{\mathcal{F}^*} \cdot \|V_0\|_{\mathcal{F}} \text{ by Proposition 3.34.} \end{aligned} \tag{5.2}$$

Moreover, the Taylor expansion yields the existence of an $\tilde{r} \in [0, \rho]$, where $\rho := \|x^* - x_0\| \leq r_0'$ such that

$$\begin{aligned}\|\delta_{x^*}-\delta_{x_0}\|^2_{\mathcal{F}^*} &= (\delta_{x^*}-\delta_{x_0})^x(\delta_{x^*}-\delta_{x_0})^y\Psi(x-y)\\ &= (\delta_{x^*}-\delta_{x_0})^x\left[\psi(\|x-x^*\|)-\psi(\|x-x_0\|)\right]\\ &= 2\left[\psi(0)-\psi(\|x^*-x_0\|)\right]\\ &= -2\frac{d}{dr}\psi(\tilde{r})\rho\\ &\leq \frac{1}{4\|V_0\|^2_{\mathcal{F}}},\text{ cf. (5.1).}\end{aligned}$$

Hence, we have with (5.2)

$$|V_0(x^*)-\tilde{v}(x^*)| \leq \frac{1}{2} \qquad \text{for all } x^* \in \overline{B_{r_0'}(x_0)}.$$

For $x \in \Omega$, i.e. $V(x) = V_0(x) = r_0^2$, we have $x \in \overline{B_{r_0'}(x_0)}$ and hence $\tilde{v}(x) \leq V_0(x) + \frac{1}{2} = r_0^2 + \frac{1}{2}$ and $\tilde{v}(x) \geq V_0(x) - \frac{1}{2} = r_0^2 - \frac{1}{2}$. For $v(x) = \tilde{v}(x) - \tilde{b}\chi(x)$ we thus have

$$v(x) \in \left[r_0^2 - \tilde{b} - \frac{1}{2}, r_0^2 - \tilde{b} + \frac{1}{2}\right] \qquad \text{for all } x \in \Omega. \tag{5.3}$$

For the orbital derivatives we have, using Theorem 4.2

$$|v'(x) - V_0'(x)| \;=\; |v'(x) + p(x)| \leq C^* h^\kappa =: \iota \text{ for all } x \in \overline{B}. \tag{5.4}$$

Since $C^*h^\kappa = \iota < M_0$ by definition of h^*, we have $v'(x) < -p(x) + M_0 \leq 0$ for all $x \in \overline{B} \setminus \tilde{B}^{\mathfrak{q}}_r(x_0)$. Hence, we can apply Theorem 4.8 and obtain an extension v^* of v, such that $v^*(x) = av(x) + b$ holds for all $x \in \overline{B} \setminus \tilde{B}^{\mathfrak{q}}_r(x_0)$ and $(v^*)'(x) < 0$ holds for all $x \in \overline{B} \setminus \{x_0\}$. Now set

$$\begin{aligned}K &= \{x \in B \mid v^*(x) \leq a(R^2 + 1 - \tilde{b}) + b =: (R^*)^2\}\\ &= \{x \in B \setminus \tilde{K} \mid v(x) \leq R^2 + 1 - \tilde{b}\} \cup \tilde{K}.\end{aligned}$$

The equation follows from the fact that $\tilde{K}$ is a subset of both sets, for the proof see below.

We will show that $K_1 \subset K \subset K_2$ holds. Then $K_0 \subset K$, K is a compact set and $(v^*)'(x) < 0$ holds for all $x \in K \setminus \{x_0\}$.

We show $K_1 \subset K$. For $x \in K_1 \setminus \tilde{B}^{\mathfrak{q}}_r(x_0)$ we have in particular $\theta_0 \geq \theta(x) \geq 0$ and with $\iota < \frac{1}{2\theta_0}$ we obtain

$$
\begin{aligned}
v(x) &= v(S_{\theta(x)}x) - \int_0^{\theta(x)} v'(S_\tau x)\,d\tau \\
&\leq r_0^2 - \tilde{b} + \frac{1}{2} + \int_0^{\theta(x)} (-V_0'(S_\tau x) + \iota)\,d\tau \text{ by (5.3) and (5.4)} \\
&\leq \underbrace{V_0(S_{\theta(x)}x) - \int_0^{\theta(x)} V_0'(S_\tau x)\,d\tau}_{=V_0(x)} + \frac{1}{2} + \theta(x)\iota - \tilde{b} \\
&\leq V_0(x) + \frac{1}{2} + \theta_0\iota - \tilde{b} \\
&\leq R^2 + 1 - \tilde{b}, \text{ i.e. } x \in K.
\end{aligned}
$$

Since in particular for $x \in \partial\tilde{K}$ the inequality $v(x) \leq R^2+1-\tilde{b}$ and thus $v^*(x) \leq a(R^2+1-\tilde{b})+b$ holds true and, moreover, v^* decreases along solutions, $\tilde{K} \subset K$ follows. In particular we have $a(R^2+1-\tilde{b})+b > 0$ since $v^*(x_0) = \mathfrak{q}(x_0) \geq 0$. Altogether, we have $K_1 \subset K$.

For the inclusion $K \subset K_2$ we show that for $x \in B \setminus K_2$ the inequality $v(x) > R^2 + 1 - \tilde{b}$ and thus $v^*(x) > a(R^2 + 1 - \tilde{b}) + b$ holds true. If $x \in B \setminus K_2 \subset A(x_0)$, then we have $\theta(x) \leq \theta_0$ and

$$
\begin{aligned}
v(x) &= v(S_{\theta(x)}x) - \int_0^{\theta(x)} v'(S_\tau x)\,d\tau \\
&\geq r_0^2 - \tilde{b} - \frac{1}{2} + \int_0^{\theta(x)} (-V_0'(S_\tau x) - \iota)\,d\tau \text{ by (5.3) and (5.4)} \\
&\geq \underbrace{V_0(S_{\theta(x)}x) - \int_0^{\theta(x)} V_0'(S_\tau x)\,d\tau}_{=V_0(x)} - \frac{1}{2} - \theta(x)\iota - \tilde{b} \\
&\geq V_0(x) - \frac{1}{2} - \theta_0\iota - \tilde{b} \\
&> R^2 + 2 - 1 - \tilde{b}, \text{ i.e. } x \notin K
\end{aligned}
$$

since $\iota < \frac{1}{2\theta_0}$. This proves the theorem. □

5.1.2 Taylor Polynomial

In the following theorem we consider the function V where $V'(x) = -\|x - x_0\|^2$. We do not approximate V by this equation for the orbital derivative, but we approximate the function $W(x) = \frac{V(x)}{\mathfrak{n}(x)}$ as in Section 4.2.3 which satisfies $D_m W(x) = -\frac{\|x-x_0\|^2}{\mathfrak{n}(x)}$. The proof is similar to the one of Theorem 5.1.

Theorem 5.2. *Let x_0 be an equilibrium of $\dot{x} = f(x)$, where $f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ such that the real parts of all eigenvalues of $Df(x_0)$ are negative. Moreover,* *assume that* $\sup_{x\in A(x_0)}\|f(x)\|<\infty$ *or, more generally,* $\sup_{x\in\mathbb{R}^n}\|f(x)\|<\infty$ *holds; this can be achieved using (2.2).*

We consider the radial basis function $\Psi(x)=\psi_{l,k}(c\|x\|)$ *with* $c>0$*, where* $\psi_{l,k}$ *denotes the Wendland function with* $k\in\mathbb{N}$ *and* $l:=\lfloor\frac{n}{2}\rfloor+k+1$*. Let* V *be the Lyapunov function of Theorem 2.46 with* $V'(x)=-\|x-x_0\|^2$ *and* $V(x_0)=0$*, and* $\mathfrak{n}(x)=\sum_{2\le|\alpha|\le P}c_\alpha(x-x_0)^\alpha+M\|x-x_0\|^{2H}$ *as in Definition 2.56, and let* $W(x)=\frac{V(x)}{\mathfrak{n}(x)}\in C^{P-2}(A(x_0),\mathbb{R})$ *with* $W(x_0)=1$*, cf. Proposition 2.58. Let* $\sigma\ge P\ge 2+\sigma^*$*, where* $\sigma^*:=\frac{n+1}{2}+k$*. Let* $K_0\subset A(x_0)$ *be a compact set with* $x_0\in\mathring{K}_0$*.*

Then there is an open set B *with* $\overline{B}\subset A(x_0)$*, such that for all reconstructions* $w\in C^{2k-1}(\mathbb{R}^n,\mathbb{R})$ *of* W *with respect to a grid* $X_N\subset\overline{B}\setminus\{x_0\}$ *which is dense enough in the sense of Theorem 4.10, there is a compact set* $K\supset K_0$ *such that with* $v_W(x):=\mathfrak{n}(x)w(x)$

- $v_W'(x)<0$ *holds for all* $x\in K\setminus\{x_0\}$*,*
- $K=\{x\in B\mid v_W(x)\le(R^*)^2\}$ *for an* $R^*\in\mathbb{R}^+$*.*

In other words, v_W *is a Lyapunov function with Lyapunov basin* K*.*

Proof: Let $V\in C^\sigma(A(x_0),\mathbb{R})$ be the function of Theorem 2.46 which satisfies $V'(x)=-\|x-x_0\|^2$ for all $x\in A(x_0)$ and $V(x_0)=0$. Then $W\in C^{P-2}(A(x_0),\mathbb{R})$ satisfies $W'(x)+\frac{\mathfrak{n}'(x)}{\mathfrak{n}(x)}W(x)=-\frac{\|x-x_0\|^2}{\mathfrak{n}(x)}$ for all $x\in A(x_0)\setminus\{x_0\}$, cf. (4.21). Note that $P-2\ge\sigma^*$. Set $R:=\sqrt{\max_{x\in K_0}V(x)}>0$ and

$$\begin{aligned}K_1&=\{x\in A(x_0)\mid V(x)\le R^2\},\\K_2&=\{x\in A(x_0)\mid V(x)\le R^2+2\},\\B&=\{x\in A(x_0)\mid V(x)<R^2+3\}.\end{aligned}$$

Then obviously $K_0\subset K_1\subset K_2\subset B\subset\overline{B}\subset A(x_0)$ and B is open, cf. Theorem 2.46; note that $\sup_{x\in A(x_0)}\|f(x)\|<\infty$. All these sets are positively invariant. Let $\tilde{B}$ be an open set with $\overline{B}\subset\tilde{B}\subset\overline{\tilde{B}}\subset A(x_0)$, e.g. $\tilde{B}=\{x\in A(x_0)\mid V(x)<R^2+4\}$. Let $\chi\in C_0^\infty(\mathbb{R}^n,[0,1])$ be a function with $\chi(x)=1$ for $x\in\overline{B}$ and $\chi(x)=0$ for $\mathbb{R}^n\setminus\tilde{B}$. Thus, $\chi\in C_0^\infty(\mathbb{R}^n)\subset\mathcal{F}$. Set $\tilde{a}:=\|\chi\|_{\mathcal{F}}$ and $W_0=W\cdot\chi$; then $W_0\in C_0^{P-2}(\mathbb{R}^n,\mathbb{R})$ and $W_0(x)=W(x)$ holds for all $x\in\overline{B}$. Lemma 3.13 implies $W_0\in\mathcal{F}$. Choose $r_0'>0$ so small that $\overline{B_{r_0'}(x_0)}\subset K_0$,

$$r_0'\le\frac{1}{\left(4C\sqrt{\Psi(0)}\|W_0\|_{\mathcal{F}}\right)^{\frac{1}{2}}}\tag{5.5}$$

$$\text{and }2(r_0')^5\max_{\tilde{r}\in[0,r_0']}\left|\frac{d}{dr}\psi(\tilde{r})\right|\le\frac{1}{(4C\|W_0\|_{\mathcal{F}})^2}\tag{5.6}$$

hold, where $\mathfrak{n}(x)\le C\|x-x_0\|^2$ for all $x\in\overline{B}$, cf. Proposition 2.58, 2., and $\psi(r):=\psi_{l,k}(cr)$. This is possible since $\frac{d}{dr}\psi(r)=O(r)$ for $r\to 0$, cf. Proposition 3.11. Choose $r_0>0$ such that

$$\Omega := \{x \in A(x_0) \mid V(x) = r_0^2\} \subset \overline{B_{r_0'}(x_0)}$$

holds. Ω is a non-characteristic hypersurface by Lemma 2.37 and hence, by Theorem 2.38 there exists a function $\theta \in C^\sigma(A(x_0) \backslash \{x_0\}, \mathbb{R})$ defined implicitly by $S_{\theta(x)} x \in \Omega$. Set $\theta_0 := \max_{x \in \overline{B}} \theta(x) > 0$. Let

$$\tilde{c} < \min \left(\frac{1}{2\theta_0 C M_0}, \frac{1}{C} \right), \tag{5.7}$$

where $M_0 := \max_{x \in \overline{B}} \|x - x_0\|^2$ and choose a grid $X_N \subset \overline{B} \setminus \{x_0\}$ according to Theorem 4.10.

For the approximant w of $W_0 (= W$ in $\overline{B})$ we set $\tilde{b} := 1 - w(x_0)$. With $\delta_{x_0} \in \mathcal{F}^*$, cf. Lemma 3.25, and $W_0(x_0) = 1$, cf. Proposition 2.58, 3., we have

$$\begin{aligned} |\tilde{b}| &= |\delta_{x_0}(W_0 - w)| \\ &\leq \|\delta_{x_0}\|_{\mathcal{F}^*} \cdot \|W_0 - w\|_{\mathcal{F}} \\ &\leq \sqrt{\Psi(0)} \cdot \|W_0\|_{\mathcal{F}} \end{aligned} \tag{5.8}$$

by Proposition 3.37 since $\|\delta_{x_0}\|_{\mathcal{F}^*}^2 = \delta_{x_0}^x \delta_{x_0}^y \Psi(x - y) = \Psi(0)$. For the function $\tilde{w} := w + \tilde{b} \cdot \chi$ we have $\tilde{w}(x_0) = w(x_0) + \tilde{b} = 1$. For $x^* \in \overline{B_{r_0'}(x_0)}$ we have thus

$$\begin{aligned} |W_0(x^*) - \tilde{w}(x^*)| &= |(\delta_{x^*} - \delta_{x_0})(W_0 - w - \tilde{b} \cdot \chi)| \\ &= |(\delta_{x^*} - \delta_{x_0})(W_0 - w)| \\ &\leq \|\delta_{x^*} - \delta_{x_0}\|_{\mathcal{F}^*} \cdot \|W_0 - w\|_{\mathcal{F}} \\ &\leq \|\delta_{x^*} - \delta_{x_0}\|_{\mathcal{F}^*} \cdot \|W_0\|_{\mathcal{F}} \text{ by Proposition 3.37.} \end{aligned}$$

Moreover, the Taylor expansion yields the existence of an $\tilde{r} \in [0, \rho]$ where $\rho := \|x^* - x_0\| \leq r_0'$ such that

$$\begin{aligned} \|\delta_{x^*} - \delta_{x_0}\|_{\mathcal{F}^*}^2 &= (\delta_{x^*} - \delta_{x_0})^x (\delta_{x^*} - \delta_{x_0})^y \Psi(x - y) \\ &= (\delta_{x^*} - \delta_{x_0})^x \left[\psi(\|x - x^*\|) - \psi(\|x - x_0\|)\right] \\ &= 2\left[\psi(0) - \psi(\|x^* - x_0\|)\right] \\ &= -2\psi'(\tilde{r})\rho \\ &\leq \frac{1}{(4C \cdot (r_0')^2 \|W_0\|_{\mathcal{F}})^2}, \text{ cf. (5.6).} \end{aligned}$$

Hence, for all $x^* \in \overline{B_{r_0'}(x_0)}$ we have

$$|W_0(x^*) - \tilde{w}(x^*)| \leq \frac{1}{4C \cdot (r_0')^2}. \tag{5.9}$$

For $v_W(x) = \mathfrak{n}(x) w(x) = \mathfrak{n}(x)[\tilde{w}(x) - \tilde{b}\chi(x)]$ we have for all $x^* \in \overline{B_{r_0'}(x_0)}$

$$\begin{aligned}
|V(x^*) - v_W(x^*)| &= \mathfrak{n}(x^*)[W_0(x^*) - \tilde{w}(x^*) + \tilde{b}\chi(x^*)] \\
&\leq \underbrace{\max_{x \in B_{r_0'}(x_0)} \mathfrak{n}(x)}_{\leq C\cdot(r_0')^2} \left[|W_0(x^*) - \tilde{w}(x^*)| + |\tilde{b}|\right] \\
&\leq \frac{1}{4} + \frac{1}{4} \qquad \text{by (5.5), (5.8) and (5.9).}
\end{aligned}$$

Thus,

$$v_W(x) \in \left[r_0^2 - \frac{1}{2}, r_0^2 + \frac{1}{2}\right] \qquad \text{for all } x \in \Omega. \tag{5.10}$$

For the orbital derivatives we have, using Theorem 4.10, (4.18) and (4.19)

$$v_W'(x) < 0 \qquad \text{for all } x \in \overline{B} \setminus \{x_0\} \text{ and} \tag{5.11}$$

$$|v_W'(x) + \|x - x_0\|^2| \leq \tilde{c}C\|x - x_0\|^2 \ \leq \ \tilde{c}CM_0 \ \leq \ \frac{1}{2\theta_0} \tag{5.12}$$

by (5.7) for all $x \in \overline{B}$.

Now set

$$K = \{x \in B \mid v_W(x) \leq R^2 + 1 =: (R^*)^2\}.$$

We will show that $K_1 \subset K \subset K_2$ holds. Then $K_0 \subset K$, K is a compact set and $v_W'(x) < 0$ holds for all $x \in K \setminus \{x_0\}$, cf. (5.11).

We show $K_1 \subset K$. Let $x \in K_1$. We distinguish between the cases $\theta(x) < 0$ and $\theta(x) \geq 0$. If $\theta(x) < 0$, then

$$\begin{aligned}
v_W(x) &= v_W(S_{\theta(x)}x) - \int_0^{\theta(x)} v_W'(S_\tau x)\, d\tau \\
&\leq v_W(S_{\theta(x)}x) \\
&\leq r_0^2 + \frac{1}{2} \text{ by (5.10)} \\
&\leq R^2 + 1,
\end{aligned}$$

since $R^2 = \max_{x \in K_0} V(x) \geq \max_{x \in \Omega} V(x) = r_0^2$.

Now assume $\theta_0 \geq \theta(x) \geq 0$. We have

$$\begin{aligned}
v_W(x) &= v_W(S_{\theta(x)}x) - \int_0^{\theta(x)} v_W'(S_\tau x)\, d\tau \\
&\leq r_0^2 + \frac{1}{2} + \int_0^{\theta(x)} \left(\|S_\tau x - x_0\|^2 + \frac{1}{2\theta_0}\right) d\tau \text{ by (5.10) and (5.12)} \\
&\leq \underbrace{V(S_{\theta(x)}x) - \int_0^{\theta(x)} V'(S_\tau x)\, d\tau}_{=V(x)} + \frac{1}{2} + \frac{\theta(x)}{2\theta_0} \\
&\leq V(x) + 1 \\
&\leq R^2 + 1,
\end{aligned}$$

i.e. $x \in K$. Hence, $K_1 \subset K$.

For the inclusion $K \subset K_2$ we show that for $x \in B \setminus K_2$ the inequality $v_W(x) > R^2+1$ holds true. If $x \in B \setminus K_2 \subset A(x_0)$, then we have $0 \leq \theta(x) \leq \theta_0$ and

$$\begin{aligned} v_W(x) &= v_W(S_{\theta(x)}x) - \int_0^{\theta(x)} v_W'(S_\tau x)\,d\tau \\ &\geq r_0^2 - \frac{1}{2} + \int_0^{\theta(x)} \left(\|S_\tau x - x_0\|^2 - \frac{1}{2\theta_0} \right) d\tau \text{ by (5.10) and (5.12)} \\ &\geq \underbrace{V(S_{\theta(x)}x) - \int_0^{\theta(x)} V'(S_\tau x)\,d\tau}_{=V(x)} - \frac{1}{2} - \frac{\theta(x)}{2\theta_0} \\ &\geq V(x) - \frac{1}{2} - \frac{1}{2} \\ &> R^2 + 2 - 1, \end{aligned}$$

i.e. $x \notin K$. This proves the theorem. □

5.2 Mixed Approximation

For T (and also for V) one can use a mixed approximation. Here, additionally to the orbital derivative Q', the values of Q are given on an $(n-1)$-dimensional manifold, a *non-characteristic hypersurface*. Such a non-characteristic hypersurface can be given by the level set of a (local) Lyapunov function within its Lyapunov basin. With this method one can also cover each compact subset of the basin of attraction by a Lyapunov basin when approximating the function T or V. In the case of T, where $T'(x) = -1$, the level sets of the function T and thus also of t up to a certain error have a special meaning: a solution needs the time $T_2 - T_1$ from the level set $T = T_2$ to the level set $T = T_1$.

Moreover, one can exhaust the basin of attraction by compact sets: starting with a local Lyapunov function and a corresponding local Lyapunov basin K_0, one obtains a larger Lyapunov basin K_1 through mixed approximation using the boundary ∂K_0 as a non-characteristic hypersurface. The boundary ∂K_1 is again a non-characteristic hypersurface and hence one obtains a sequence of compact sets $K_0 \subset K_1 \subset \ldots$ which exhaust $A(x_0)$. In Figure 5.4 we show the first step of this method with the function V^* for example (2.11): starting with a local Lyapunov basin K_0 (magenta), we obtain a larger Lyapunov basin K_1 using mixed approximation. In [23] the same example with a different grid X_N is considered, and one more step is calculated.

In this section we approximate the function $Q = T$ or $Q = V^*$ via its orbital derivatives and its function values. The orbital derivatives are given on

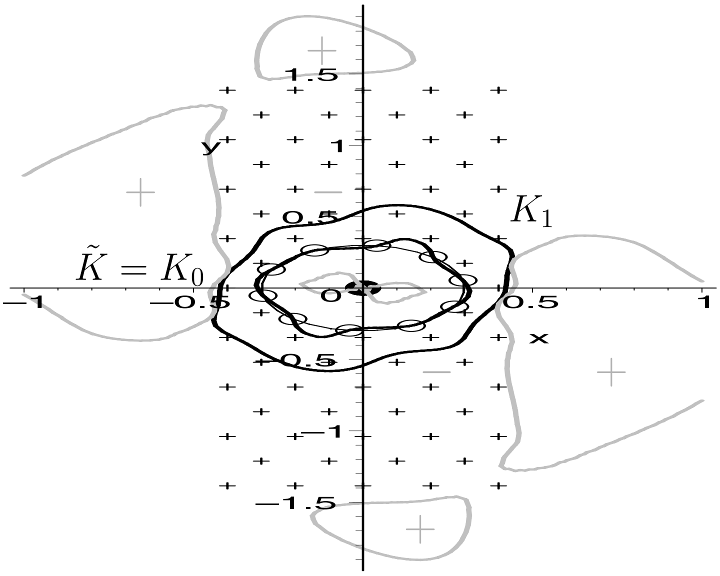

Fig. 5.4. Mixed approximation of V^* where $(V^*)'(x, y) = -(x^2 + y^2)$ with given value $V^*(x, y) = 1$ on the boundary of the local Lyapunov basin $\tilde{K} = K_0$ (thin black). We used a grid of $M = 10$ points (black circles) on $\partial\tilde{K}$ for the values of the approximation v and a second grid of $N = 70$ points (black +) with $\alpha = 0.2$ for the orbital derivative v'. The sign of $v'(x, y)$ (grey), and the level sets $v(x, y) = 1$ and $v(x, y) = 1.1$ (black) are shown. The level set $v(x, y) = 1.1$ is the boundary of K_1 (black) which is a subset of the basin of attraction for (2.11).

a grid X_N, whereas the function values are given on a different grid $X_M^0 \subset \Omega$ where Ω is a non-characteristic hypersurface. In most cases, Ω is given by a level set of a Lyapunov function q, e.g. a local Lyapunov function. This mixed interpolation problem was discussed in Section 3.1.3, cf. Definition 3.7. Any compact subset $K_0 \subset A(x_0)$ can be covered by a Lyapunov basin K obtained by a mixed approximation via radial basis functions as we prove in Section 5.2.1.

Moreover, we can approach the basin of attraction stepwise by a sequence of Lyapunov functions q_i, $i = 1, 2, \ldots$ with Lyapunov basins $K_i \supset K_{i-1}$. The advantage of this approach is that one can use a grid X_N outside K_{i-1} in each step, cf. Section 5.2.2.

5.2.1 Approximation via Orbital Derivatives and Function Values

We approximate the Lyapunov function T satisfying $T' = -\bar{c}$. Note that we fix the values of T on the boundary of a Lyapunov basin, i.e. on a non-characteristic hypersurface.

Theorem 5.3. *Consider the function* $\Psi(x) = \psi_{l,k}(c\|x\|)$ *with* $c > 0$, *where* $\psi_{l,k}$ *denotes the Wendland function with* $k \in \mathbb{N}$ *and* $l := \lfloor \frac{n}{2} \rfloor + k + 1$. *Let* $f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$, *where* $\mathbb{N} \ni \sigma \geq \sigma^* := \frac{n+1}{2} + k$. *Moreover, assume that* $\sup_{x \in A(x_0)} \|f(x)\| < \infty$ *or, more generally,* $\sup_{x \in \mathbb{R}^n} \|f(x)\| < \infty$ *holds. Let* $\mathfrak{q}$ *be a Lyapunov function with Lyapunov basin* $\tilde{K} := \tilde{K}_r^{\mathfrak{q}}(x_0)$ *according to Definition 2.23. Define* $\Omega := \partial\tilde{K}$. *Let* K_0 *be a compact set with* $\tilde{K} \subset \mathring{K}_0 \subset K_0 \subset A(x_0)$ *and let* $H \in C^\sigma(\Omega, \mathbb{R}_0^+)$.

Then there is an open set B *with* $\overline{B} \subset A(x_0)$ *and constants* $h_0^*, h^* > 0$ *such that for every reconstruction* t *of* T *in the sense of Definition 3.7, where* T *is defined in Theorem 2.38 with* $T(x) = H(x)$ *for* $x \in \Omega$, *using grids* $X_N \subset \overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$ *with fill distance* $h \leq h^*$ *and* $X_M^0 \subset \Omega$ *with fill distance* $h_0 \leq h_0^*$, *there is an extension* $t^* \in C^{2k-1}(\mathbb{R}^n, \mathbb{R})$ *of* t *as in the Extension Theorem 4.8, such that:*

There is a compact set $B \supset K \supset K_0$ *with*

1. $K = \{x \in B \mid t^*(x) \leq (R^*)^2\}$ *for an* $R^* \in \mathbb{R}^+$,
2. $(t^*)'(x) < 0$ *for all* $x \in K \setminus \{x_0\}$.

In other words, t^* *is a Lyapunov function with Lyapunov basin* K.

Proof: We assume without loss of generality that $\mathfrak{q}(x_0) = 0$. Ω is a non-characteristic hypersurface and we define the function θ for all $x \in A(x_0) \setminus \{x_0\}$ implicitly by

$$S_{\theta(x)}x \in \Omega, \text{ i.e. } \mathfrak{q}(S_{\theta(x)}x) = r^2,$$

cf. Theorem 2.38, and set

$$\theta_0 := \max_{x \in K_0 \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)} \theta(x) \geq 0,$$
$$K_1 := \left\{x \in A(x_0) \setminus \tilde{B}_r^{\mathfrak{q}}(x_0) \mid \theta(x) \leq \theta_0\right\} \cup \tilde{B}_r^{\mathfrak{q}}(x_0).$$

Then obviously $K_0 \subset K_1$, and K_1 is positively invariant.

Define $T \in C^\sigma(A(x_0) \setminus \{x_0\}, \mathbb{R})$ as in Theorem 2.38, i.e. $T'(x) = -\bar{c}$ for $x \in A(x_0) \setminus \{x_0\}$ and $T(x) = H(x)$ for $x \in \Omega$. We set $c_M := \max_{x \in \Omega} H(x) = \max_{x \in \Omega} T(x)$ and $c_m := \min_{x \in \Omega} H(x) = \min_{x \in \Omega} T(x)$. With $\theta^* := \frac{c_M - c_m + \frac{3}{2}\bar{c}\,\theta_0 + 2}{\frac{1}{2}\bar{c}} > \theta_0$ we define the following sets

$$K_2 := \left\{x \in A(x_0) \setminus \tilde{B}_r^{\mathfrak{q}}(x_0) \mid \theta(x) \leq \theta^*\right\} \cup \tilde{B}_r^{\mathfrak{q}}(x_0),$$
$$B := \left\{x \in A(x_0) \setminus \tilde{B}_r^{\mathfrak{q}}(x_0) \mid \theta(x) < \theta^* + 1\right\} \cup \tilde{B}_r^{\mathfrak{q}}(x_0).$$

Then obviously $K_1 \subset K_2 \subset B$, both K_2 and B are positively invariant and B is open, cf. Proposition 2.44; note that $\sup_{x \in A(x_0)} \|f(x)\| < \infty$.

We modify T in $\tilde{B}_r^{\mathfrak{q}}(x_0)$ and outside $\overline{B}$ such that $T \in C_0^\sigma(\mathbb{R}^n, \mathbb{R}) \subset \mathcal{F}$ and T remains unchanged in $\overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$, cf. the proof of Theorem 4.1.

We apply Theorem 3.41 to $K = \overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$ and to grids $X_N \subset \overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$ with fill distance $h \leq \left(\frac{\bar{c}}{2C^*}\right)^{\frac{1}{\kappa}} =: h^*$ and $X_M^0 \subset \Omega$ with fill distance $h_0 \leq \frac{1}{C_0^*} =: h_0^*$, where C^* and C_0^* are as in Theorem 3.41. We obtain an approximation $t \in C^{2k-1}(\mathbb{R}^n, \mathbb{R})$ for which the following inequality holds for all $x \in \overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$, cf. (3.49),

$$-\frac{3}{2}\bar{c} \leq t'(x) \leq -\frac{1}{2}\bar{c} < 0. \tag{5.13}$$

The Extension Theorem 4.8, applied to $K = \overline{B}$, $\mathfrak{q}$ and $q = t$ implies

$$(t^*)'(x) < 0 \qquad \text{for all } x \in \overline{B} \setminus \{x_0\}. \tag{5.14}$$

Note that for the function t^* we have $t^*(x) = at(x) + b$ for all $x \notin \tilde{B}_r^{\mathfrak{q}}(x_0)$. We set $R := \sqrt{c_M + 1 + \frac{3}{2}\bar{c}\,\theta_0}$ and define

$$\begin{aligned} K &:= \{x \in B \mid t^*(x) \leq aR^2 + b =: (R^*)^2\} \\ &= \{x \in B \setminus \tilde{B}_r^{\mathfrak{q}}(x_0) \mid t(x) \leq R^2\} \cup \tilde{B}_r^{\mathfrak{q}}(x_0). \end{aligned}$$

The equation follows from the fact that $\tilde{B}_r^{\mathfrak{q}}(x_0)$ is a subset of both sets, for the proof see below. Note that by (3.50) of Theorem 3.41 we have the following result for all $x \in B \setminus \{x_0\}$:

$$t(S_{\theta(x)}x) \in [c_m - 1, c_M + 1] \tag{5.15}$$

since $S_{\theta(x)}x \in \Omega$.

We will show that $K_1 \subset K \subset K_2$ holds. Note that $K_1 \subset K$ implies $K_0 \subset K$. $K \subset K_2 \subset B$, on the other hand, shows that K is a compact set; 2. then follows from (5.14).

We show $K_1 \subset K$: For $x \in K_1 \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$, we have with $0 \leq \theta(x) \leq \theta_0$, the positive invariance of K_1 and (5.15)

$$\begin{aligned} c_M + 1 &\geq t(S_{\theta(x)}x) \\ &= t(x) + \int_0^{\theta(x)} t'(S_\tau x)\, d\tau \\ &\geq t(x) - \frac{3}{2}\bar{c}\,\theta(x) \text{ by (5.13)} \\ t(x) &\leq c_M + 1 + \frac{3}{2}\bar{c}\,\theta_0 = R^2 \end{aligned}$$

and hence $x \in K$. For $x \in \tilde{B}_r^{\mathfrak{q}}(x_0)$, we have by the Extension Theorem 4.8 and (5.15) $t^*(x) \leq \max_{\xi \in \Omega} t^*(\xi) \leq a(c_M + 1) + b \leq (R^*)^2$. Thus, $x \in K$.

We show $K \subset K_2$: Assume in contradiction that there is an $x \in B \setminus K_2$ with $t(x) \leq R^2$ and $\theta(x) > \theta^*$ – note that $\tilde{B}_r^{\mathfrak{q}}(x_0) \subset K_2$ by construction. By (5.15) we have

$$\begin{aligned} c_m - 1 &\leq t(S_{\theta(x)}x) \\ &= t(x) + \int_0^{\theta(x)} t'(S_\tau x)\, d\tau \\ &\leq t(x) - \frac{1}{2}\bar{c}\,\theta(x) \text{ by (5.13)} \\ t(x) &> c_m - 1 + \frac{1}{2}\bar{c}\,\theta^* = R^2 \end{aligned}$$

by definition of θ^*. This is a contradiction and thus $K \subset K_2$, which proves the theorem. □

The following corollary shows that the difference of the values of t corresponds to the time which a solution needs from one level set to another up to the error $\max_\xi |t'(\xi) + \bar{c}|$.

Corollary 5.4 *Let the assumptions of Theorem 5.3 hold. For $x \in \mathbb{R}^n$ and $\tilde{t} > 0$ let $S_\tau x \in K \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$ hold for all $\tau \in [0, \tilde{t}]$. Denote $\rho_1 := t(x)$ and $\rho_0 := t(S_{\tilde{t}}x)$. Moreover, let $\max_{\xi \in K \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)} |t'(\xi) + \bar{c}| =: \iota < \bar{c}$ hold (by the assumptions of Theorem 5.3, in particular (5.13), $\iota = C^* h^\kappa \leq \frac{\bar{c}}{2}$ is an upper bound).*

Then the time $\tilde{t}$ fulfills

$$\frac{\rho_1 - \rho_0}{\bar{c} + \iota} \leq \tilde{t} \leq \frac{\rho_1 - \rho_0}{\bar{c} - \iota}.$$

PROOF: We have $\rho_0 - \rho_1 = \int_0^{\tilde{t}} t'(S_\tau x)\, d\tau$. Since $|t'(S_\tau x) + \bar{c}| \leq \iota$ holds for all $\tau \in [0, \tilde{t}]$, we have $(-\bar{c} - \iota)\,\tilde{t} \leq \rho_0 - \rho_1 \leq (-\bar{c} + \iota)\,\tilde{t}$, which proves the corollary. □

Now we consider the Lyapunov function V satisfying $V'(x) = -p(x)$. Fixing the values on a non-characteristic hypersurface Ω, we have to consider the function V^*, cf. Proposition 2.51, which satisfies $(V^*)'(x) = -p(x)$ for $x \in A(x_0) \setminus \{x_0\}$ and $V^*(x) = H(x)$ for $x \in \Omega$, where H is a given function.

Theorem 5.5. *Consider the function $\Psi(x) = \psi_{l,k}(c\|x\|)$ with $c > 0$, where $\psi_{l,k}$ denotes the Wendland function with $k \in \mathbb{N}$ and $l := \lfloor \frac{n}{2} \rfloor + k + 1$. Let $f \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$, where $\mathbb{N} \ni \sigma \geq \sigma^* := \frac{n+1}{2} + k$. Moreover, assume that $\sup_{x \in A(x_0)} \|f(x)\| < \infty$ or, more generally, $\sup_{x \in \mathbb{R}^n} \|f(x)\| < \infty$ holds. Let $\mathfrak{q}$ be a Lyapunov function with Lyapunov basin $\tilde{K} := \tilde{K}_r^{\mathfrak{q}}(x_0)$ according to Definition 2.23. Define $\Omega := \partial\tilde{K}$. Let K_0 be a compact set with $\tilde{K} \subset \mathring{K}_0 \subset K_0 \subset A(x_0)$ and let $H \in C^\sigma(\Omega, \mathbb{R}_0^+)$.*

Then there is an open set B with $\overline{B} \subset A(x_0)$ and constants $h_0^*, h^* > 0$ such that for every reconstruction v of V^* in the sense of Definition 3.7, where V^* is defined in Proposition 2.51 with $V^*(x) = H(x)$ for $x \in \Omega$, using grids $X_N \subset \overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$ with fill distance $h \leq h^*$ and $X_M^0 \subset \Omega$ with fill distance $h_0 \leq h_0^*$, there is an extension $v^* \in C^{2k-1}(\mathbb{R}^n, \mathbb{R})$ of v as in the Extension Theorem 4.8, such that:

There is a compact set $B \supset K \supset K_0$ with

1. $K = \{x \in B \mid v^*(x) \leq (R^*)^2$ *for an* $R^* \in \mathbb{R}^+$,
2. $(v^*)'(x) < 0$ *for all* $x \in K \setminus \{x_0\}$.

In other words, v^* *is a Lyapunov function with Lyapunov basin* K.

PROOF: We assume without loss of generality that $\mathfrak{q}(x_0) = 0$. Ω is a non-characteristic hypersurface and we define the function θ for all $x \in A(x_0)\setminus\{x_0\}$ implicitly by

$$S_{\theta(x)}x \in \Omega, \text{ i.e. } \mathfrak{q}(S_{\theta(x)}x) = r^2,$$

cf. Theorem 2.38, and set

$$\theta_0 := \max_{x \in K_0 \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)} \theta(x) \geq 0,$$
$$K_1 := \left\{x \in A(x_0) \setminus \tilde{B}_r^{\mathfrak{q}}(x_0) \mid \theta(x) \leq \theta_0\right\} \cup \tilde{B}_r^{\mathfrak{q}}(x_0).$$

Then $K_0 \subset K_1$, and K_1 is positively invariant. We set $\epsilon := \frac{1}{2} \inf_{x \notin \tilde{B}_r^{\mathfrak{q}}(x_0)} p(x) > 0$, $p_M := \max_{x \in K_1} p(x)$, $c_M := \max_{x\in\Omega} H(x) = \max_{x\in\Omega} V^*(x)$ and $c_m := \min_{x\in\Omega} H(x) = \min_{x\in\Omega} V^*(x)$. With $\theta^* := \frac{c_M - c_m + (p_M+\epsilon)\theta_0 + 2}{\epsilon} > \theta_0$ we define the following sets

$$K_2 := \left\{x \in A(x_0) \setminus \tilde{B}_r^{\mathfrak{q}}(x_0) \mid \theta(x) \leq \theta^*\right\} \cup \tilde{B}_r^{\mathfrak{q}}(x_0),$$
$$B := \left\{x \in A(x_0) \setminus \tilde{B}_r^{\mathfrak{q}}(x_0) \mid \theta(x) < \theta^* + 1\right\} \cup \tilde{B}_r^{\mathfrak{q}}(x_0).$$

Then obviously $K_1 \subset K_2 \subset B$, both K_2 and B are positively invariant and B is open, cf. Proposition 2.51; note that $\sup_{x \in A(x_0)} \|f(x)\| < \infty$.

We modify V^* in $\tilde{B}_r^{\mathfrak{q}}(x_0)$ and outside $\overline{B}$ such that $V^* \in C_0^\sigma(\mathbb{R}^n, \mathbb{R}) \subset \mathcal{F}$ and V^* remains unchanged in $\overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$, cf. the proof of Theorem 4.1.

We apply Theorem 3.41 to $K = \overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$ and to grids $X_N \subset \overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$ with fill distance $h \leq \left(\frac{\epsilon}{C^*}\right)^{\frac{1}{\kappa}} =: h^*$ and $X_M^0 \subset \Omega$ with fill distance $h_0 \leq \frac{1}{C_0^*} =: h_0^*$, where C^* and C_0^* are as in Theorem 3.41. We obtain a function $v \in C^{2k-1}(\mathbb{R}^n, \mathbb{R})$ for which the following inequality holds for all $x \in \overline{B} \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$, cf. (3.49),

$$-p(x) - \epsilon \ \leq v'(x) \leq \ -p(x) + \epsilon < 0. \tag{5.16}$$

The Extension Theorem 4.8, applied to $K = \overline{B}$, $\mathfrak{q}$ and $q = v$ implies

$$(v^*)'(x) < 0 \qquad \text{for all } x \in \overline{B} \setminus \{x_0\}. \tag{5.17}$$

Note that for the function v^* we have then $v^*(x) = av(x) + b$ for all $x \notin \tilde{B}_r^{\mathfrak{q}}(x_0)$. We set $R := \sqrt{c_M + 1 + (p_M + \epsilon)\theta_0}$ and define

$$K := \{x \in B \mid v^*(x) \leq aR^2 + b =: (R^*)^2\}$$
$$= \{x \in B \setminus \tilde{B}_r^{\mathfrak{q}}(x_0) \mid v(x) \leq R^2\} \cup \tilde{B}_r^{\mathfrak{q}}(x_0).$$

The equation follows from the fact that $\tilde{B}_r^{\mathfrak{q}}(x_0)$ is a subset of both sets, for the proof see below. Note that by (3.50) of Theorem 3.41 we have the following result for all $x \in B \setminus \{x_0\}$:

$$v(S_{\theta(x)}x) \in [c_m - 1, c_M + 1] \tag{5.18}$$

since $S_{\theta(x)}x \in \Omega$.

We will show that $K_1 \subset K \subset K_2$ holds. Note that $K_1 \subset K$ implies $K_0 \subset K$. $K \subset K_2 \subset B$, on the other hand, shows that K is a compact set; 2. then follows from (5.17).

We show $K_1 \subset K$: For $x \in K_1 \setminus \tilde{B}_r^{\mathfrak{q}}(x_0)$, we have with $0 \leq \theta(x) \leq \theta_0$, the positive invariance of K_1 and (5.18)

$$\begin{aligned} c_M + 1 &\geq v(S_{\theta(x)}x) \\ &= v(x) + \int_0^{\theta(x)} \underbrace{v'(S_t x)}_{\geq -p(S_t x) - \epsilon} \, dt \quad \text{by (5.16)} \\ &\geq v(x) - (p_M + \epsilon)\theta(x) \\ v(x) &\leq c_M + 1 + (p_M + \epsilon)\theta_0 = R^2, \end{aligned}$$

and hence $x \in K$. For $x \in \tilde{B}_r^{\mathfrak{q}}(x_0)$, we have by the Extension Theorem 4.8 and (5.18) $v^*(x) \leq \max_{\xi \in \Omega} v^*(\xi) \leq a(c_M + 1) + b \leq (R^*)^2$. Thus, $x \in K$.

We show $K \subset K_2$: Assume in contradiction that there is an $x \in B \setminus K_2$ with $v(x) \leq R^2$ and $\theta(x) > \theta^*$ – note that $\tilde{B}_r^{\mathfrak{q}}(x_0) \subset K_2$ by construction. By (5.18) we have

$$\begin{aligned} c_m - 1 &\leq v(S_{\theta(x)}x) \\ &= v(x) + \int_0^{\theta(x)} \underbrace{v'(S_t x)}_{\leq -p(S_t x) + \epsilon} \, dt \quad \text{by (5.16)} \\ &\leq v(x) + (-2\epsilon + \epsilon)\theta(x) \\ v(x) &> c_m - 1 + \epsilon\theta^* = R^2 \end{aligned}$$

by definition of θ^*. This is a contradiction and thus $K \subset K_2$, which proves the theorem. □

5.2.2 Stepwise Exhaustion of the Basin of Attraction

Using Theorem 5.3 or Theorem 5.5 we can stepwise exhaust the basin of attraction, cf. also Section 6.3. We assume that $\sup_{x \in A(x_0)} \|f(x)\| < \infty$.

Calculate a local Lyapunov function $\mathfrak{q}$ and a corresponding local Lyapunov basin $\tilde{K} = \tilde{K}_r^{\mathfrak{q}}(x_0)$. Denote $q_0 := \mathfrak{q}$, $K_0 := \tilde{K}$ and $r_0 := r$, and set $B_0 = \mathbb{R}^n$. This is the departing point for a sequence of compact Lyapunov basins K_i, $i = 1, 2, \ldots$, with $K_{i+1} \supset K_i$ and $\bigcup_{i \in \mathbb{N}} K_i = A(x_0)$.

Now assume that a Lyapunov function q_i with Lyapunov basin $K_i = \tilde{K}^{q_i}_{r_i}(x_0)$ and neighborhood B_i is given. The only information we need of the Lyapunov function and the Lyapunov basin is the boundary $\partial K_i =: \Omega_{i+1}$. Hence, if q_i is the extension of a function $\tilde{q}_i$, it suffices to know the set ∂K_i either given by $\partial \tilde{K}^{\tilde{q}_i}_{\tilde{r}_i}(x_0) = \{x \in B_i \mid \tilde{q}_i(x) = \tilde{r}_i^2\}$ or by $\partial \tilde{K}^{q_i}_{r_i}(x_0) = \{x \in B_i \mid q_i(x) = r_i^2\}$.

Let $B_{i+1} \supset K_i$ be an open set which will be specified below. Choose grids $X_N \subset \overline{B_{i+1}} \setminus K_i$ – in practical applications we let the grid be slightly larger, also including points in K_i near ∂K_i – and $X_M^0 \subset \Omega_{i+1} := \partial K_i$. Now approximate either $Q = T$ or $Q = V^*$ by a mixed approximation with respect to the grids X_N and X_M^0 and the values $Q(\xi_j) = H(\xi_j) = 1$ for $\xi_j \in X_M^0$. Make the grids dense enough so that for the reconstruction q there is a set $S_{i+1} := \{x \in B_{i+1} \mid 1 - \epsilon_{i+1} \leq q(x) \leq \tilde{r}_{i+1}^2\}$ with $\epsilon_{i+1} > 0$ such that $q'(x) < 0$ holds for all $x \in S_{i+1}$ and $\partial K_i \subset \overset{\circ}{S}_{i+1}$. Then there is an extension q^* of q with $\mathfrak{q} = q_i$ such that $q_{i+1}(x) := q^*(x)$ is a Lyapunov function with Lyapunov basin $K_{i+1} := S_{i+1} \cup K_i = \{x \in B_{i+1} \mid q^*(x) \leq a\tilde{r}_{i+1}^2 + b =: r_{i+1}^2\}$.

We show that with this method $\bigcup_{i \in \mathbb{N}} K_i = A(x_0)$ holds and we can thus stepwise exhaust the basin of attraction, if we choose B_{i+1} properly. To show this, we reprove the induction step from K_i to K_{i+1} again. Let K_i^* be a sequence of compact sets with $K_{i+1}^* \supset K_i^*$ and $\bigcup_{i \in \mathbb{N}} K_i^* = A(x_0)$, e.g. $K_i^* = S_{-i}\tilde{K}$ where S_{-i} denotes the flow and $\tilde{K}$ is the local Lyapunov basin defined above. By Proposition 2.44, the sets K_i^* are compact and since for all $z \in A(x_0)$ there is a finite time T^* with $S_{T^*} z \in \tilde{K}$, we obtain $\bigcup_{i \in \mathbb{N}} K_i^* = A(x_0)$.

Now we reprove the induction step from K_i to K_{i+1}. For given i choose $l_{i+1} > l_i$ so large that $K_i \subset \overset{\circ}{K^*_{l_{i+1}}} \subset K^*_{l_{i+1}} \subset A(x_0)$ holds. Such an l_{i+1} exists due to the compactness of K_i. Then Theorem 5.3 or 5.5 with $\mathfrak{q} = q_i$, $\tilde{K} = K_i$ and $K_0 = K^*_{l_{i+1}}$ implies that there is an open set $B =: B_{i+1}$ and a Lyapunov function q^* with Lyapunov basin $K =: K_{i+1}$ with $K_{i+1} \supset K^*_{l_{i+1}}$. This shows $\bigcup_{i \in \mathbb{N}} K_i = A(x_0)$.

For examples of the stepwise exhaustion, cf. Section 6.3 and Figure 5.4 as well as [23].