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Free Probability Setting

16.1 A Few Notions about Algebras and Tracial States

Definition 16.1. *A C^*-algebra $(\mathcal{A}, *)$ is a complex algebra equipped with an involution $*$ and a norm $||.||_{\mathcal{A}}$ such that $\mathcal{A}$ is complete for the norm $\|\cdot\|_{\mathcal{A}}$ and, for any $X, Y \in \mathcal{A}$,*

$$\|XY\|_{\mathcal{A}} \leq \|X\|_{\mathcal{A}} \, \|Y\|_{\mathcal{A}}, \quad \|X^*\|_{\mathcal{A}} = \|X\|_{\mathcal{A}}, \quad \|XX^*\|_{\mathcal{A}} = \|X\|_{\mathcal{A}}^2.$$

$X \in \mathcal{A}$ is self-adjoint iff $X^* = X$. $\mathcal{A}_{sa}$ denote the set of self-adjoint elements of $\mathcal{A}$. A C^*-algebra $(\mathcal{A}, *)$ is unital if it contains a neutral element I.

$\mathcal{A}$ can always be realized as a sub-C^*-algebra of the space $B(H)$ of bounded linear operators on a Hilbert space H. For instance, if $\mathcal{A}$ is a unital C^*-algebra furnished with a positive linear form τ, one can always construct such a Hilbert space H by completing and separating $L^2(\tau)$ (this is the Gelfand–Neumark–Segal construction, see [186, Theorem 2.2.1]). We shall restrict ourselves to this case in the sequel and denote by H a Hilbert space equipped with a scalar product $\langle ., . \rangle_H$ such that $\mathcal{A} \subset B(H)$.

Definition 16.2. *If $\mathcal{A}$ is a sub-C^*-algebra of $B(H)$, $\mathcal{A}$ is a von Neumann algebra iff it is closed for the weak topology, generated by the semi-norms $\{p_{\xi,\eta}(X) = \langle X\xi, \eta \rangle_H, \xi, \eta \in H\}$.*

Let us notice that by definition, a von Neumann algebra contains only bounded operators. The theory nevertheless allows us to consider unbounded operators thanks to the notion of affiliated operators. A densely defined self-adjoint operator X on H is said to be affiliated to $\mathcal{A}$ iff for any Borel function f on the spectrum of X, $f(X) \in \mathcal{A}$ (see [167, p.164]). Here, $f(X)$ is well defined for any operator X as the operator with the same eigenvectors as X and eigenvalues given by the image of those of X by the map f. Murray and von Neumann have proved that if X and Y are affiliated with $\mathcal{A}$, $aX + bY$ is also affiliated with $\mathcal{A}$ for any $a, b \in \mathbb{C}$.

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A state τ on a unital von Neumann algebra $(\mathcal{A}, *)$ is a linear form on $\mathcal{A}$ such that $\tau(\mathcal{A}_{sa}) \subset \mathbb{R}$ and:
1. **Positivity** $\tau(\mathbf{A}\mathbf{A}^*) \geq 0$, for any $\mathbf{A} \in \mathcal{A}$.
2. **Total mass** $\tau(I) = 1$.

A tracial state satisfies the additional hypothesis:
3. **Traciality** $\tau(\mathbf{AB}) = \tau(\mathbf{BA})$ for any $\mathbf{A}, \mathbf{B} \in \mathcal{A}$.

The couple $(\mathcal{A}, \tau)$ of a von Neumann algebra equipped with a state τ is called a W^*- probability space.

Exercise 16.3. *1. Let $n \in \mathbb{N}$, and consider $\mathcal{A} = M_n(\mathbb{C})$ as the set of bounded linear operators on $\mathbb{C}^n$. For any $v \in \mathbb{C}^n$, $\langle v, v\rangle_{\mathbb{C}^n} = \sum_{i=1}^n |v_i|^2 = \|v\|_{\mathbb{C}^n}^2 = 1$,*

$$\tau_v(M) = \langle v, Mv\rangle_{\mathbb{C}^n}$$

is a state. There is a unique tracial state on $M_n(\mathbb{C})$ that is the normalized trace

$$\frac{1}{n}\mathrm{Tr}(M) = \frac{1}{n}\sum_{i=1}^n M_{ii}.$$

2. Let $(X, \Sigma, d\mu)$ be a classical probability space. Then $\mathcal{A} = L^\infty(X, \Sigma, d\mu)$ equipped with the expectation $\tau(f) = \int f d\mu$ is a (non-)commutative probability space. Here, $L^\infty(X, \Sigma, d\mu)$ is identified with the set of bounded linear operators on the Hilbert space H obtained by separating $L^2(X, \Sigma, d\mu)$ (by the equivalence relation $f \simeq g$ iff $\mu((f-g)^2) = 0$). The identification follows from the multiplication operator $M(f)g = fg$. Observe that $\mathcal{A}$ is weakly closed for the semi-norms $(\langle f, .g\rangle_H, f, g \in L^2(\mu))$ as $L^\infty(X, \Sigma, d\mu)$ is the dual of $L^1(X, \Sigma, d\mu)$.

3. Let G be a discrete group, and $(e_h)_{h\in G}$ be a basis of $\ell^2(G)$. Let $\lambda(h)e_g = e_{hg}$. Then, we take $\mathcal{A}$ to be the von Neumann algebra generated by the linear span of $\lambda(G)$. The (tracial) state is the linear form such that $\tau(\lambda(g)) = 1_{g=e}$ (e = neutral element).

We refer to [200] for further examples and details.

The notion of law $\tau_{X_1,\ldots,X_m}$ of m operators $(X_1, \ldots, X_m)$ in a W^*-probability space $(\mathcal{A}, \tau)$ is simply given by the restriction of the trace τ to the algebra generated by $(X_1, \ldots, X_m)$, that is by the values

$$\tau_{X_1,\ldots,X_m}(P) := \tau(P(X_1, \ldots, X_m)), \quad P \in \mathbb{C}\langle X_1, \ldots X_m\rangle$$

where $\mathbb{C}\langle X_1, \ldots X_m\rangle$ denotes the set of polynomial functions of m non-commutative variables.

16.2 Space of Laws of m Non-commutative Self-adjoint Variables

Following the above description, laws of m non-commutative self-adjoint variables can be seen as elements of the set $\mathcal{M}^{(m)}$ of linear forms τ on the set of polynomial functions of m non-commutative variables $\mathbb{C}\langle X_1, \ldots X_m\rangle$ furnished with the involution

$$(X_{i_1}X_{i_2}\cdots X_{i_n})^* = X_{i_n}X_{i_{n-1}}\cdots X_{i_1}$$

and such that:

1. **Positivity** $\tau(PP^*) \geq 0$, for any $P \in \mathbb{C}\langle X_1, \ldots X_m\rangle$,
2. **Traciality** $\tau(PQ) = \tau(QP)$ for any $P, Q \in \mathbb{C}\langle X_1, \ldots X_m\rangle$,
3. **Total mass** $\tau(I) = 1$.

This point of view is identical to the previous one. Indeed, being given $\mu \in \mathcal{M}^{(m)}$ such that

$$|\mu(X_{i_1}\cdots X_{i_k})| \leq R^k$$

for any choices of $i_1, \cdots, i_k \in \{1, \cdots, m\}$ and k and some finite constant R, we can construct a W^*-probability space $(\mathcal{A}, \tau)$ and operators $(X_1, \ldots, X_m)$ such that

$$\mu = \tau_{X_1,\ldots,X_m}. \tag{16.1}$$

We refer to [6, 167, 200] for such a construction.

The topology under consideration is usually in free probability the $\mathbb{C}\langle X_1, \ldots X_m\rangle$-* topology that is $\{\tau_{X_1^n,\ldots,X_m^n}\}_{n\in\mathbb{N}}$ converges to $\tau_{X_1,\ldots,X_m}$ iff for every $P \in \mathbb{C}\langle X_1, \ldots X_m\rangle$,

$$\lim_{n\to\infty} \tau_{X_1^n,\ldots,X_m^n}(P) = \tau_{X_1,\ldots,X_m}(P).$$

If $(X_1^n, \ldots, X_m^n)_{n\in\mathbb{N}}$ are non-commutative variables whose law $\tau_{X_1^n,\ldots,X_m^n}$ converges to $\tau_{X_1,\ldots,X_m}$, then we shall also say that $(X_1^n, \ldots, X_m^n)_{n\in\mathbb{N}}$ converges in law (or in distribution) to $(X_1, \ldots, X_m)$.

Such a topology is reasonable when one deals with uniformly bounded non-commutative variables. In fact, if we consider for $R \in \mathbb{R}^+$,

$$\mathcal{M}_R^{(m)} := \{\mu \in \mathcal{M}^{(m)} : \mu(X_i^{2p}) \leq R^p, \ \forall p \in \mathbb{N}, \ 1 \leq i \leq m\},$$

then $\mathcal{M}_R^{(m)}$, equipped with this $\mathbb{C}\langle X_1, \ldots X_m\rangle$-* topology, is a Polish space (i.e a complete metric space). In fact, $\mathcal{M}_R^{(m)}$ is compact by the Banach–Alaoglu theorem. A distance is for instance given by

$$d(\mu, \nu) = \sum_{n\geq 0} \frac{1}{2^n}|\mu(P_n) - \nu(P_n)|$$

where $\{P_n\}_{n\in\mathbb{N}}$ is a dense sequence of polynomials with operator norm bounded by one when evaluated at any set of self-adjoint operators with operator norms bounded by R.

This notion is the generalization of laws of m real-valued variables bounded by a given finite constant R, in which case the $\mathbb{C}\langle X_1, \ldots X_m\rangle$-* topology driven by polynomial functions is the same as the standard weak topology. Actually, it is not hard to check that $\mathcal{M}_R^{(1)} = \mathcal{P}([-R,R])$. However, it may be useful to consider more general topologies, compatible with the existence of unbounded operators, as might be encountered for instance when considering the deviations of large random matrices. One way to do that is to change the set of test functions (as one does in the case $m = 1$ where bounded continuous test functions are often chosen to define the standard weak topology). In [55], the set of test functions was chosen to be the complex vector space $\mathcal{CC}_{st}^m(\mathbb{C})$ generated by the Stieltjes functionals

$$ST^m(\mathbb{C}) = \left\{ \prod_{1\le i\le n}^{\rightarrow} \left(z_i - \sum_{k=1}^{m} \alpha_i^k X_k \right)^{-1} ; \quad z_i \in \mathbb{C}\backslash\mathbb{R}, \alpha_i^k \in \mathbb{Q}, n \in \mathbb{N} \right\} \tag{16.2}$$

where $\prod^{\rightarrow}$ denotes the non-commutative product. It can be checked easily that, with such type of test functions, $\mathcal{M}^{(m)}$ is again a Polish space.

A particular important example of non-commutative laws is given by the empirical distribution of matrices.

Definition 16.4. *Let $N \in \mathbb{N}$ and consider m Hermitian matrices $A_1^N, \ldots,$ $A_m^N \in \mathcal{H}_N^m$. Then, the empirical distribution of the matrices $(A_1^N, \ldots, A_m^N)$ is given by*

$$\mathbf{L}_{A_1^N,\ldots,A_m^N}(P) := \frac{1}{N}\mathrm{Tr}\left(P(A_1^N,\ldots,A_m^N)\right), \quad \forall P \in \mathbb{C}\langle X_1, \cdots X_m\rangle.$$

Exercise 16.5. *Show that if the spectral radius of $(A_1^N, \ldots, A_m^N)$ is uniformly bounded by R, $\mathbf{L}_{A_1^N,\ldots,A_m^N}$ is an element of the set $\mathcal{M}_R^{(m)}$ of non-commutative laws. Moreover, if $(A_1^N, \ldots, A_m^N)_{N\in\mathbb{N}}$ is a sequence such that*

$$\lim_{N\to\infty} \mathbf{L}_{A_1^N,\ldots,A_m^N}(P) = \tau(P), \quad \forall P \in \mathbb{C}\langle X_1, \cdots X_m\rangle,$$

show that $\tau \in \mathcal{M}_R^{(m)}$.

It is actually a long-standing question posed by A. Connes to know whether all $\tau \in \mathcal{M}^{(m)}$ can be approximated in such a way. In the case $m = 1$, the question amounts to asking if for all $\mu \in \mathcal{P}([-R,R])$, there exists a sequence $(\lambda_1^N, \ldots, \lambda_N^N)_{N\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^N} = \mu.$$

This is well known to be true by the Birkhoff's theorem (which is based on the Krein–Milman theorem), but still an open question when $m \geq 2$.

Bibliographical Notes. Introductory notes to free probability can be found in [118, 200, 202, 203]. Basics on operator algebra theory are taken from [83, 167].