6

Bohr's type real part estimates

6.1 Introduction

This chapter is connected with two classical assertions of the analytic functions theory, namely, with Hadamard-Borel-Carathéodory inequality

$$|f(z) - f(0)| \le \frac{2r}{R-r} \sup_{|\zeta|<R} \Re\big\{f(\zeta) - f(0)\big\}, \tag{6.1.1}$$

and with Bohr's inequality

$$\sum_{n=0}^{\infty} |c_n z^n| \le \sup_{|\zeta|<R} |f(\zeta)| \tag{6.1.2}$$

for the majorant of the Taylor's series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{6.1.3}$$

where $|z| \le R/3$ in (6.1.2) and the value $R/3$ cannot be improved.

In the chapter we deal, similarly to Aizenberg, Grossman and Korobeinik [6], Bénéteau, Dahlner and Khavinson [13], Djakov and Ramanujan [35], with the value of l_q-norm (quasi-norm, for $0 < q < 1$) of the remainder of the Taylor series (6.1.3). The particular case $q = \infty$ in all subsequent inequalities of the chapter can be obtained by passage to the limit as $q \to \infty$.

In Section 2 we prove the inequality

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \le \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}} \, ||\Re f||_1 \tag{6.1.4}$$

with the sharp constant, where $r = |z| < R$, $m \ge 1$, $0 < q \le \infty$.

Section 3 contains corollaries of (6.1.4) for analytic functions f in D_R with bounded $\Re f$, with $\Re f$ bounded from above, with $\Re f > 0$, as well as for bounded analytic functions. In particular, we obtain the estimate

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\},$$

with the best possible constant. This estimate, taken with $q = 1$, $m = 1$, is a refinement of (6.1.1). Other inequalities, which follow from (6.1.4), contain the supremum of $|\Re f(\zeta)| - |\Re f(0)|$ or $|f(\zeta)| - |f(0)|$ in D_R, as well as $\Re f(0)$ in the case $\Re f > 0$ on D_R. Each of these estimates specified for $q = 1$ and $m = 1$ refines a certain Hadamard-Borel-Carathéodory type inequality with a sharp constant.

Note that a sharp estimate of the full majorant series by the supremum modulus of f was obtained by Bombieri [19] for $r \in [R/3, R/\sqrt{2}]$.

In Section 4 we give modifications of Bohr's theorem as consequences of our inequalities with sharp constants derived in Section 3. For example, if the function (6.1.3) is analytic on D_R, then for any $q \in (0, \infty]$, integer $m \geq 1$ and $|z| \leq R_{m,q}$ the inequality

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \sup_{|\zeta|<R} \Re\{e^{-i \arg f(0)} f(\zeta)\} - |f(0)| \tag{6.1.5}$$

holds, where $R_{m,q} = r_{m,q} R$, and $r_{m,q}$ is the root of the equation

$$2^q r^{mq} + r^q - 1 = 0$$

in the interval $(0, 1)$ if $0 < q < \infty$, and $r_{m,\infty} = 2^{-1/m}$. In particular,

$$r_{1,q} = (1 + 2^q)^{-1/q} \quad \text{and} \quad r_{2,q} = 2^{1/q}\left(1 + \sqrt{1 + 2^{q+2}}\,\right)^{-1/q}. \tag{6.1.6}$$

Note that $R_{m,q}$ is the radius of the largest disk centered at $z = 0$ in which (6.1.5) takes place.

Some of the inequalities presented in Section 4 contain known analogues of Bohr's theorem with $\Re f$ in the right-hand side (see Aizenberg, Aytuna and Djakov [3], Paulsen, Popescu and Singh [73], Sidon [85], Tomić [88]).

In Section 5 we give a generalization of assertions in Sections 3 and 4 for the case $q = m = 1$. In particular, we prove the so called Bohr's theorem for non-concentric circles stated below.

Let the function f, analytic and bounded in D_R be given in the neighbourhood of $a \in D_R$ by the Taylor series

$$\sum_{n=0}^{\infty} c_n(a)(z - a)^n,$$

and let $d_a = \text{dist}\,(a, \partial D_R)$. Then for any z in the disk

$$|z - a| \leq \frac{d_a(2R - d_a)}{4R - d_a}$$

the inequality

$$\sum_{n=0}^{\infty} |c_n(a)(z-a)^n| \leq \sup_{|\zeta|<R} |f(\zeta)| \tag{6.1.7}$$

holds. Moreover,

$$\frac{d_a(2R - d_a)}{4R - d_a}$$

is the radius of the largest disk centered at a in which (6.1.7) takes place for all f.

Note that passage to the limit as $R \to \infty$ in the above assertion implies an analogue of Bohr's theorem for a half-plane $\mathbb{H} \subset \mathbb{C}$. The radius of the largest disk centered at $a \in \mathbb{H}$ with a placed at the distance d_a from $\partial\mathbb{H}$ in which the inequality

$$\sum_{n=0}^{\infty} |c_n(a)(z-a)^n| \leq \sup_{\zeta\in\mathbb{H}} |f(\zeta)|$$

holds for all bounded analytic functions in $\mathbb{H}$ is equal to $d_a/2$.

Note that Aizenberg [10] recently proved a theorem containing as particular cases the theorems due to Bohr [18], Sidon [85], Tomić [88], Aizenberg, Aytuna and Djakov [3], as well as assertions given in Section 4 of the present chapter for $q = 1, m = 1$. The following notions are essentially used in [10]. Let $G \subset \mathbb{C}$ be any domain, and let $\tilde{G}$ be the convex hull of G. A point $p \in \partial G$ is called a point of convexity if $p \in \partial\tilde{G}$. A point of convexity p is called regular if there exists a disk $D \subset G$ such that $p \in \partial D$.

Aizenberg's theorem claims that if the function (6.1.3) is analytic on D_1 and $f(D_1) \subset G$ with $\tilde{G} \neq \mathbb{C}$, then the inequality

$$\sum_{n=1}^{\infty} |c_n z^n| < \text{dist}(c_0, \partial\tilde{G})$$

holds for $|z| < 1/3$. The constant $1/3$ cannot be improved if ∂G contains at least one regular point of convexity. A multidimensional analog of this theorem is given in [10] as well.

6.2 Estimate for the l_q-norm of the Taylor series remainder by $||\Re f||_1$

In the sequel, we use the notation $r = |z|$ and $D_\varrho = \{z \in \mathbb{C} : |z| < \varrho\}$.

We start with a sharp inequality for an analytic function f. The right-hand side of the inequality contains the norm in the space $L_1(\partial D_R)$.

Proposition 6.1. *Let the function* (6.1.3) *be analytic on* D_R *with* $\Re f \in h_1(D_R)$, *and let* $q > 0$, $m \geq R$, $|z| = r < R$. *Then the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}} \, ||\Re f||_1 \tag{6.2.1}$$

holds with the sharp constant.

Proof. 1. Proof of inequality (6.2.1). Let a function f, analytic in D_R with $\Re f \in h_1(D_R)$ be given by (6.1.3). By Corollary 5.1

$$|c_n| \leq \frac{1}{\pi R^{n+1}} \, ||\Re f||_1 \tag{6.2.2}$$

for any $n \geq 1$.

Using (6.2.2), we find

$$\begin{aligned} \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} &\leq \frac{1}{\pi R} \left\{ \sum_{n=m}^{\infty} \left(\frac{r}{R} \right)^{nq} \right\}^{1/q} ||\Re f||_1 \\ &= \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}} \, ||\Re f||_1 \end{aligned}$$

for any z with $|z| = r < R$.

2. Sharpness of the constant in (6.2.1). By (6.2.1), the sharp constant $C(r)$ in

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq C(r) \, ||\Re f||_1 \tag{6.2.3}$$

satisfies

$$C(r) \leq \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}}. \tag{6.2.4}$$

We show that the converse inequality for $C(r)$ holds as well.

Let $\rho > R$. Consider the families of analytic functions in $\overline{D}_R$

$$f_\rho(z) = \frac{z}{z - \rho}, \qquad w_\rho(z) = f_\rho(z) - \beta_\rho, \tag{6.2.5}$$

depending on the parameter ρ, with the real constant β_ρ defined by

$$||\Re f_\rho - \beta_\rho||_1 = \min_{c \in \mathbb{R}} ||\Re f_\rho - c||_1.$$

Then, for any real constant c

$$||\Re w_\rho - c||_1 \geq ||\Re w_\rho||_1.$$

Setting here

$$c = A_\rho = \max_{|\zeta|=R} \Re w_\rho(\zeta)$$

and taking into account

$$||\Re w_\rho - A_\rho||_1 = \int_{|\zeta|=R} \big[A_\rho - \Re w_\rho(\zeta)\big]|d\zeta|$$

$$= 2\pi R\{A_\rho - \Re w_\rho(0)\} = 2\pi R \max_{|\zeta|=R} \Re\{w_\rho(\zeta) - w_\rho(0)\},$$

we arrive at

$$2\pi R \max_{|\zeta|=R} \Re\{w_\rho(\zeta) - w_\rho(0)\} \geq ||\Re w_\rho||_1. \tag{6.2.6}$$

In view of

$$c_n(\rho) = \frac{w_\rho^{(n)}(0)}{n!} = -\frac{1}{\rho^n} \quad \text{for} \quad n \geq 1,$$

we find

$$\sum_{n=m}^{\infty} |c_n(\rho) z^n|^q = \sum_{n=m}^{\infty} \left(\frac{r}{\rho}\right)^{nq} = \frac{r^{mq}}{\rho^{(m-1)q}(\rho^q - r^q)}. \tag{6.2.7}$$

By (6.2.5), (1.4.6) and (1.4.7) we have

$$\max_{|\zeta|=R} \Re\big\{w_\rho(\zeta) - w_\rho(0)\big\} = \max_{|\zeta|=R} \Re\big\{f_\rho(\zeta) - f_\rho(0)\big\} = \frac{R}{\rho + R}. \tag{6.2.8}$$

It follows from (6.2.3), (6.2.6), (6.2.7) and (6.2.8) that

$$C(r) \geq \frac{(\rho + R)r^m}{2\pi R^2 \rho^{m-1}(\rho^q - r^q)^{1/q}}. \tag{6.2.9}$$

On passing to the limit as $\rho \downarrow R$ this becomes

$$C(r) \geq \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}}, \tag{6.2.10}$$

which together with (6.2.4) proves the sharpness of the constant in (6.2.1). □

6.3 Other estimates for the l_q-norm of the Taylor series remainder

In this section we obtain estimates with sharp constants for the l_q-norm (quasi-norm for $0 < q < 1$) of the Taylor series remainder for bounded analytic functions and analytic functions whose real part is bounded or one-side bounded.

We start with a theorem concerning analytic functions with real part bounded from above which refines Hadamard-Borel-Carathéodory inequality (6.1.1).

Theorem 6.1. *Let the function* (6.1.3) *be analytic on* D_R *with* $\Re f$ *bounded from above, and let* $q > 0$, $m \geq 1$, $|z| = r < R$. *Then the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\} \tag{6.3.1}$$

holds with the sharp constant.

Proof. We write (6.2.1) for the disk D_ϱ, $\varrho \in (r, R)$, with f replaced by $f - \omega$, where ω is an arbitrary real constant. Then

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{r^m}{\pi \varrho^m (\varrho^q - r^q)^{1/q}} \, ||\Re f - \omega||_{L_1(\partial D_\varrho)}. \tag{6.3.2}$$

Putting here

$$\omega = \mathcal{A}_f(R) = \sup_{|\zeta|<R} \Re f(\zeta)$$

and taking into account that

$$||\Re f - \mathcal{A}_f(R)||_{L_1(\partial D_\varrho)} = 2\pi\rho\{\mathcal{A}_f(R) - \Re f(0)\} = 2\pi\rho \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\},$$

we find

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\},$$

which implies (6.3.1) after the passage to the limit as $\varrho \uparrow R$.

Hence, the sharp constant $C(r)$ in

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq C(r) \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\} \tag{6.3.3}$$

obeys

$$C(r) \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}}. \tag{6.3.4}$$

To get the lower estimate for $C(r)$, we shall use functions f_ρ given by (6.2.5). Taking into account the equality

$$f_\rho^{(n)}(0) = w_\rho^{(n)}(0),$$

as well as (6.3.3), (6.2.7) and (6.2.8), we arrive at

$$C(r) \geq \frac{(\rho + R) r^m}{R\rho^{m-1}(\rho^q - r^q)^{1/q}}. \tag{6.3.5}$$

Passing to the limit as $\rho \downarrow R$ in the last inequality, we obtain

$$C(r) \geq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}}, \qquad (6.3.6)$$

which together with (6.3.4) proves the sharpness of the constant in (6.3.1). □

Remark 6.2. Inequality (6.3.1) for $q = m = 1$ is well known (see, e.g. Polya and Szegö [76], III, Ch. 5, § 2). Adding $|c_0|$ and $|f(0)|$ to the left- and right-hand sides of (6.3.1) with $q = m = 1$, respectively, and replacing $-\Re f(0)$ by $|f(0)|$ in the resulting relation, we arrive at

$$\sum_{n=0}^{\infty} |c_n z|^n \leq \frac{R+r}{R-r}|f(0)| + \frac{2r}{R-r} \sup_{|\zeta|<R} \Re f(\zeta),$$

which is a refinement of the Hadamard-Borel-Carathéodory inequality

$$|f(z)| \leq \frac{R+r}{R-r}|f(0)| + \frac{2r}{R-r} \sup_{|\zeta|<R} \Re f(\zeta)$$

(see, e.g., Burckel [23], Ch. 6 and references there, Titchmarsh [87], Ch. 5).

The next assertion contains a sharp estimate for analytic functions on D_R with bounded real part. It is a refinement of the inequality

$$|f(z) - f(0)| \leq \frac{2r}{R-r} \sup_{|\zeta|<R} \{|\Re f(\zeta)| - |\Re f(0)|\}$$

which follows from (6.1.1).

Theorem 6.2. *Let the function* (6.1.3) *be analytic on D_R with bounded real part, and let $q > 0$, $m \geq 1$, $|z| = r < R$. Then the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \{|\Re f(\zeta)| - |\Re f(0)|\} \qquad (6.3.7)$$

holds with the sharp constant.

Proof. Setting

$$\omega = \mathcal{R}_f(R) = \sup_{|\zeta|<R} |\Re f(\zeta)|$$

in (6.3.2) and making use of the equalities

$$||\Re f - \mathcal{R}_f(R)||_{L_1(\partial D_\varrho)} = 2\pi\rho\{\mathcal{R}_f(R) - \Re f(0)\} = 2\pi\rho \sup_{|\zeta|<R} \{|\Re f(\zeta)| - \Re f(0)\},$$

we arrive at

$$\left\{\sum_{n=m}^{\infty}|c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<R}\{|\Re f(\zeta)| - \Re f(0)\}.$$

This estimate leads to

$$\left\{\sum_{n=m}^{\infty}|c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R}\{|\Re f(\zeta)| - \Re f(0)\} \qquad (6.3.8)$$

after the passage to the limit as $\varrho \uparrow R$. Replacing f by $-f$ in the last inequality, we obtain

$$\left\{\sum_{n=m}^{\infty}|c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R}\{|\Re f(\zeta)| + \Re f(0)\},$$

which together with (6.3.8) results at (6.3.7).

Let us show that the constant in (6.3.7) is sharp. By $C(r)$ we denote the best constant in

$$\left\{\sum_{n=m}^{\infty}|c_n z^n|^q\right\}^{1/q} \leq C(r) \sup_{|\zeta|<R}\{|\Re f(\zeta)| - |\Re f(0)|\}. \qquad (6.3.9)$$

As shown above, $C(r)$ obeys (6.3.4).

We introduce the family of analytic functions in $\overline{D}_R$

$$g_\rho(z) = \frac{\rho}{z-\rho} + \frac{\rho^2}{\rho^2 - R^2}, \qquad (6.3.10)$$

depending on a parameter $\rho > R$. By (5.4.22) and (5.4.26) we have

$$\sup_{|\zeta|<R}\{|\Re g_\rho(\zeta)| - |\Re g_\rho(0)|\} = \frac{R}{\rho + R}. \qquad (6.3.11)$$

Taking into account that the functions (6.2.5) and (6.3.10) differ by a constant, and using (6.3.9), (6.2.7) and (6.3.11), we arrive at (6.3.5). Passing there to the limit as $\rho \downarrow R$, we conclude that (6.3.6) holds, which together with (6.3.4) proves the sharpness of the constant in (6.3.7). $\square$

The following assertion contains an estimate with the sharp constant for bounded analytic functions in D_R. It gives a refinement of the estimate

$$|f(z) - f(0)| \leq \frac{2r}{R-r} \sup_{|\zeta|<R}\{|f(\zeta)| - |f(0)|\}$$

which follows from (6.1.1).

Theorem 6.3. *Let the function* (6.1.3) *be analytic and bounded on D_R, and let $q > 0$, $m \geq 1$, $|z| = r < R$. Then the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \{|f(\zeta)| - |f(0)|\} \qquad (6.3.12)$$

holds with the sharp constant.

Proof. Setting

$$\omega = \mathcal{M}_f(R) = \sup_{|\zeta|<R} |f(\zeta)|$$

in (6.3.2) and using the equalities

$$||\Re f - \mathcal{M}_f(R)||_{L_1(\partial D_\varrho)} = 2\pi\rho\{\mathcal{M}_f(R) - \Re f(0)\} = 2\pi\rho \sup_{|\zeta|<R} \{|f(\zeta)| - \Re f(0)\},$$

we obtain

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<1} \{|f(\zeta)| - \Re f(0)\}.$$

Passage to the limit as $\varrho \uparrow R$ gives

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \{|f(\zeta)| - \Re f(0)\}.$$

Replacing f by $fe^{i\alpha}$, we arrive at

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \{|f(\zeta)| - \Re\big(f(0)e^{i\alpha}\big)\},$$

which implies (6.3.12) by the arbitrariness of α.

Let us show that the constant in (6.3.12) is sharp. By $C(r)$ we denote the best constant in

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq C(r) \sup_{|\zeta|<R} \{|f(\zeta)| - |f(0)|\}. \qquad (6.3.13)$$

As shown above, $C(r)$ obeys (6.3.4).

We consider the family h_ρ of analytic functions in $\overline{D}$, defined by (6.3.10). By (5.4.34) we have

$$\sup_{|\zeta|<R} \{|g_\rho(\zeta)| - |g_\rho(0)|\} = \frac{R}{\rho + R}. \qquad (6.3.14)$$

Taking into account that the functions (6.2.5) and (6.3.10) differ by a constant, and using (6.3.13), (6.2.7) and (6.3.14), we arrive at (6.2.9). Passing there to the limit as $\rho \downarrow R$, we obtain (6.2.10), which together with (6.2.4) proves the sharpness of the constant in (6.3.12). $\square$

Remark 6.3. We note that a consequence of (5.1.6) is the inequality

$$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq \frac{r^m}{R^{m-1}(R^q - r^q)^{1/q}} \, \frac{[\mathcal{M}_f(R)]^2 - |f(0)|^2}{\mathcal{M}_f(R)} \tag{6.3.15}$$

with the constant factor in the right-hand side twice as small as in (6.3.12) and sharp, which can be checked using the sequence of functions given by (6.3.10) and the limit passage as $\rho \downarrow R$. Inequality (6.3.15) for $q = 1, m = 1$ with $\mathcal{M}_f(R) \leq 1$ was derived by Paulsen, Popescu and Singh [73].

The next assertion refines the inequality

$$|f(z) - f(0)| \leq \frac{2r}{R - r} \, \Re f(0)$$

resulting from (6.1.1) for analytic functions in D_R with $\Re f > 0$.

Theorem 6.4. *Let the function* (6.1.3) *be analytic with positive* $\Re f$ *on* D_R*, and let* $q > 0$*,* $m \geq 1$*,* $|z| = r < R$*. Then the inequality*

$$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \, \Re f(0) \tag{6.3.16}$$

holds with the sharp constant.

Proof. Setting $\omega = 0$ in (6.3.2), with f such that $\Re f > 0$ in D_R, we obtain

$$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \, \Re f(0),$$

which leads to (6.3.16) as $\varrho \uparrow R$.

Thus, the sharp constant $C(r)$ in

$$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq C(r) \, \Re f(0) \tag{6.3.17}$$

obeys the estimate (6.3.4).

To show the sharpness of the constant in (6.3.16), consider the family of analytic functions in $\overline{D}_R$

$$h_\rho(z) = \frac{\rho}{\rho - z} - \frac{\rho}{\rho + R}, \tag{6.3.18}$$

depending on the parameter $\rho > R$. By (5.4.40), the real part of h_ρ is positive in D_R. Taking into account that the functions (6.2.5) and (6.3.18) differ by a constant and using (6.3.17), (6.2.7) and $\Re h_\rho(0) = R(\rho+1)^{-1}$, we arrive at (6.3.5). Passing there to the limit as $\rho \downarrow R$, we obtain (6.3.6), which together with (6.3.4) proves the sharpness of the constant in (6.3.16). $\square$

6.4 Bohr's type theorems

In this section we collect some corollaries of the theorems in Sect. 3.

Corollary 6.1. *Let the function* (6.1.3) *be analytic on* D_R, *and let*

$$\sup_{|\zeta|<R} \Re\{e^{-i \arg f(0)} f(\zeta)\} < \infty, \tag{6.4.1}$$

where $\arg f(0)$ *is replaced by a real number if* $f(0) = c_0 = 0$.

Then for any $q \in (0, \infty]$, *integer* $m \geq 1$, *and* $|z| \leq R_{m,q}$ *the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \sup_{|\zeta|<R} \Re\{e^{-i \arg f(0)} f(\zeta)\} - |f(0)| \tag{6.4.2}$$

holds, where $R_{m,q} = r_{m,q} R$, *and* $r_{m,q}$ *is the root of the equation* $2^q r^{mq} + r^q - 1 = 0$ *in the interval* $(0,1)$ *if* $0 < q < \infty$, *and* $r_{m,\infty} = 2^{-1/m}$. *Moreover,* $R_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (6.4.2) *takes place for all* f. *In particular,* (6.1.6) *holds.*

Proof. Obviously, the condition

$$\frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \leq 1$$

for the sharp constant in (6.3.1) holds if $|z| \leq R_{m,q}$. Therefore, the disk of radius $R_{m,q}$ centered at $z = 0$ is the largest disk, where the inequality

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \sup_{|\zeta|<R} \Re f(\zeta) - \Re f(0) \tag{6.4.3}$$

holds for all f.

Suppose first that $f(0) \neq 0$. Setting $e^{-i \arg f(0)} f$ in place of f in (6.4.3) and noting that the coefficients $|c_n|$ in the left-hand side of (6.4.3) do not change, when $\Re f(0)$ is replaced by $|f(0)| = |c_0|$, we arrive at (6.4.2). In the case $f(0) = c_0$ we chose the value α of $\arg f(0)$ in such a way that ((6.4.1) holds, then replace f by $f e^{-i\alpha}$ in (6.4.3) and hence arrive at (6.4.2). $\square$

Inequality (6.4.2) with $q = 1, m = 1$ becomes

$$\sum_{n=1}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<R} \Re\{e^{-i \arg f(0)} f(\zeta)\} - |f(0)| \tag{6.4.4}$$

with $|z| \leq R/3$, where $R/3$ is the radius of the largest disk centered at $z = 0$ in which (6.4.4) takes place. Note that (6.4.4) is equivalent to a sharp inequality obtained by Sidon [85] in his proof of Bohr's theorem and to the inequality derived by Paulsen, Popescu and Singh [73].

For $q = 1, m = 2$ inequality (6.4.2) is

$$\sum_{n=2}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<R} \Re\{e^{-i \arg f(0)} f(\zeta)\} - |f(0)|, \tag{6.4.5}$$

where $|z| \leq R/2$ and $R/2$ is the radius of the largest disk about $z = 0$ in which (6.4.5) takes place.

The next assertion follows from Theorem 6.3. For $q = 1, m = 1$ it contains Bohr's inequality (6.1.2).

Corollary 6.2. *Let the function* (6.1.3) *be analytic and bounded on* D_R. *Then for any* $q \in (0, \infty]$, *integer* $m \geq 1$, *and* $|z| \leq R_{m,q}$ *the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \sup_{|\zeta|<R} |f(\zeta)| - |f(0)| \tag{6.4.6}$$

holds, where $R_{m,q}$ *is defined in Corollary* 6.1. *Moreover,* $R_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (6.4.6) *takes place for all* f. *In particular,* (6.1.6) *holds.*

For $q = 1, m = 2$ inequality (6.4.6) takes the form

$$|c_0| + \sum_{n=2}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<R} |f(\zeta)|, \tag{6.4.7}$$

where $|z| \leq R/2$. The value $R/2$ of the radius of the disk where (6.4.7) holds cannot be improved. Note that the inequality

$$|c_0|^2 + \sum_{n=1}^{\infty} |c_n z^n| \leq 1, \tag{6.4.8}$$

was obtained by Paulsen, Popescu and Singh [73] for functions (6.1.3) satisfying the condition $|f(\zeta)| \leq 1$ in D_R and is valid for $|z| \leq R/2$. The value $R/2$ of the radius of the disk where (6.4.8) holds is sharp. Comparison of (6.4.7) and (6.4.8) shows that none of these inequalities is a consequence of the other one.

We conclude this section by an assertion which follows from Theorem 6.4.

Corollary 6.3. *Let the function* (6.1.3) *be analytic, and* $\Re\{e^{-i\arg f(0)}f\} > 0$ *on* D_R. *Then for any* $q \in (0, \infty]$, *integer* $m \geq 1$, *and* $|z| \leq R_{m,q}$ *the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq |f(0)| \tag{6.4.9}$$

holds, where $R_{m,q}$ *is the same as in Corollary* 6.1. *Moreover,* $R_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (6.4.9) *takes place for all* f. *In particular,* (6.1.6) *holds.*

Note that the inequality (6.4.9) for $q = 1, m = 1$ with $|z| \leq R/3$ was obtained by Aizenberg, Aytuna and Djakov [3] (see also Aizenberg, Grossman and Korobeinik [6]).

6.5 Variants and extensions

In this section we generalize Theorems 6.1-6.4 and Corollaries 6.1-6.3 restricting ourselves to the case $q = m = 1$. We consider analytic functions f in D_R with the Taylor expansion

$$\sum_{n=0}^{\infty} c_n(a)(z-a)^n \tag{6.5.1}$$

in a neighbourhood of an arbitrary point $a \in D_R$ instead of the expension around $z = 0$.

The estimate for the rest of the Taylor series around $a \in D_R$ follows from inequality (5.7.2). Its sharpness is demonstrated with the help of the same families of test functions as in the proof of Theorem 5.1. We use the notation $d_a = \operatorname{dist}(a, \partial D_R)$ as before.

Theorem 6.5. *Let* f *be analytic on* D_R, *and let* (6.5.1) *be its Taylor expansion in a neighbourhood of* $a \in D_R$. *Then for any* $z, |z-a| = r < d_a$ *the inequality*

$$\sum_{n=1}^{\infty} |c_n(a)(z-a)^n| \leq \frac{2Rr}{(2R-d_a)(d_a-r)}\, \mathcal{Q}_a(f)\,, \tag{6.5.2}$$

holds with the best possible constant, where $\mathcal{Q}_a(f)$ *is each of the following expressions:*

(i) $\sup\limits_{|\zeta|<R} \Re f(\zeta) - \Re f(a)$,

(ii) $\sup\limits_{|\zeta|<R} |\Re f(\zeta)| - |\Re f(a)|$,

(iii) $\sup\limits_{|\zeta|<R} |f(\zeta)| - |f(a)|$,

(iv) $\Re f(a)$, *if* $\Re f > 0$ *on* D_R.

Proof. 1. Proof of inequality (6.5.2). Let a function f, analytic on D_R be given by (6.5.1) in a neighbourhood of $a \in D_R$. By 5.7.2,

$$\sum_{n=1}^{\infty} |c_n(a)(z-a)^n| = \sum_{n=1}^{\infty} \frac{|f^{(n)}(a)|}{n!} r^n \le \frac{2R}{R+r_a} \left\{ \sum_{n=1}^{\infty} \left(\frac{r}{R-r_a} \right)^n \right\} \mathcal{Q}_a(f)$$

$$= \frac{2rR}{(R+r_a)(R-r_a-r)} \mathcal{Q}_a(f),$$

which together with $r_a = R - d_a$ implies (6.5.2).

2. Sharpness of the constant in (6.5.2). Let $C(r, r_a)$ stand for the best constant in

$$\sum_{n=1}^{\infty} |c_n(a)(z-a)^n| \le C(r, r_a) \mathcal{Q}_a(f). \tag{6.5.3}$$

We showed in the first part of the proof that

$$C(r, r_a) \le \frac{2rR}{(R+r_a)(R-r_a-r)}. \tag{6.5.4}$$

We now derive the reverse inequality in (6.5.4). Let $a = r_a e^{i\vartheta}$. Consider the families $f_\xi(z), g_\xi(z), h_\xi(z)$ of analytic functions in D_R given by (5.7.8). By (5.7.9)

$$|f_\xi^{(n)}(a)| = |g_\xi^{(n)}(a)| = |h_\xi^{(n)}(a)| = \frac{n!\rho}{(\rho - r_a)^{n+1}},$$

which implies

$$\sum_{n=1}^{\infty} \frac{|f_\xi^{(n)}(a)|}{n!} r^n = \sum_{n=1}^{\infty} \frac{|g_\xi^{(n)}(a)|}{n!} r^n = \sum_{n=1}^{\infty} \frac{|h_\xi^{(n)}(a)|}{n!} r^n = \frac{\rho r}{(\rho - r_a)(\rho - r_a - r)}.$$

Combining this with (6.5.3) and (5.7.10)-(5.7.12) we find

$$C(r, r_a) \ge \frac{r(\rho + R)}{(R+r_a)(\rho - r_a - r)}.$$

Passing here to the limit as $\rho \downarrow R$, we conclude that

$$C(r, r_a) \ge \frac{2rR}{(R+r_a)(R-r_a-r)} = \frac{2rR}{(2R-d_a)(d_a-r)},$$

which together with (6.5.4) proves sharpness of the constant in (6.5.2). □

From one hand, estimate (6.5.2) is a refinement of Hadamard-Borel-Carathéodory inequality (1.7.8) for non-concentric circles and its consequences

$$\max_{|z-a|=r} |f(z) - f(a)| \le \frac{2Rr}{(2R-d_a)(d_a-r)} \mathcal{Q}_a(f)$$

with $\mathcal{Q}_a(f)$ given by (ii)-(iv), from the other hand, (6.5.2) is a generalization of Theorems 6.1-6.4 with $q = 1, m = 1$ for any point $a \in D_R$.

The next assertion follows from Theorem 6.5 and is a generalization of Corollaries 6.1-6.3 with $q = m = 1$ for the series (6.5.1) with an arbitrary $a \in D_R$.

Corollary 6.4. *Let f be analytic on D_R, and let* (6.5.1) *be its Taylor expansion in a neighbourhood of $a \in D_R$. Then for any z in the disk*

$$|z - a| \leq \frac{d_a(2R - d_a)}{4R - d_a} \tag{6.5.5}$$

the inequality

$$\sum_{n=0}^{\infty} |c_n(a)(z-a)^n| \leq \mathcal{N}(f) \tag{6.5.6}$$

holds, where $\mathcal{N}(f)$ is each of the following expressions:

(i) $\sup\limits_{|\zeta|<R} |f(\zeta)|$,

(ii) $\sup\limits_{|\zeta|<R} \Re\{e^{-i \arg f(a)} f(\zeta)\}$,

(iii) $\sup\limits_{|\zeta|<R} |\Re\{e^{-i \arg f(a)} f(\zeta)\}|$,

(iv) $2|f(a)|$, *if* $\Re\{e^{-i \arg f(a)} f(\zeta)\} > 0$ *on* D_R.

Moreover, $d_a(2R - d_a)/(4R - d_a)$ is the radius of the largest disk centered at a in which (6.5.6) *takes place for all f. Here* $\arg f(a)$ *is a real number if* $f(a) = c_0(a) = 0$.

Proof. Obviously, the condition

$$\frac{2rR}{(2R - d_a)(d_a - r)} \leq 1$$

for the sharp constant in (6.5.2) holds for all r satisfying (6.5.5). Therefore, the disk of radius $d_a(2R - d_a)/(4R - d_a)$ centered at $a \in D_R$ is the largest disk, where the inequality

$$\sum_{n=1}^{\infty} |c_n(a)(z-a)^n| \leq \mathcal{Q}_a(f) \tag{6.5.7}$$

holds for all f.

Suppose first that $f(a) \neq 0$. Setting $e^{-i \arg f(a)} f$ in place of f in (6.5.7) and noting that the coefficients $|c_n(a)|$ in the left-hand side of (6.5.7) do not change, when $\Re f(a)$ is replaced by $|f(a)| = |c_0(a)|$, we arrive at (6.5.6). In the case $f(0) = c_0$ we choose the value α of $\arg f(0)$ so that $\mathcal{Q}_a(e^{-i\alpha} f) < \infty$. Replacing f by $f e^{-i\alpha}$ in (6.5.7), we obtain (6.5.6). $\square$

Passing to the limit as $R \to \infty$ in Corollary 6.4, we arrive at the following Bohr's type theorem for a half-plane.

Corollary 6.5. *Let f be analytic in a half-plane $\mathbb{H} \subset \mathbb{C}$, and let* (6.5.1) *be its Taylor expansion in a neighbourhood of $a \in \mathbb{H}$, $d_a = \mathrm{dist}(a, \partial\mathbb{H})$. Then for any z in the disk*

$$|z - a| \leq d_a/2$$

the inequality

$$\sum_{n=0}^{\infty} |c_n(a)(z-a)^n| \leq \mathcal{S}(f) \tag{6.5.8}$$

holds, where $\mathcal{S}(f)$ is each of the following expressions:

(i) $\sup_{\zeta \in \mathbb{H}} |f(\zeta)|$,

(ii) $\sup_{\zeta \in \mathbb{H}} \Re\{e^{-i \arg f(a)} f(\zeta)\}$,

(iii) $\sup_{\zeta \in \mathbb{H}} |\Re\{e^{-i \arg f(a)} f(\zeta)\}|$,

(iv) $2|f(a)|$, *if* $\Re\{e^{-i \arg f(a)} f(\zeta)\} > 0$ *on* $\mathbb{H}$.

Moreover, $d_a/2$ is the radius of the largest disk centered at a in which (6.5.8) *takes place for all f. Here $\arg f(a)$ is a real number if $f(a) = c_0(a) = 0$.*