6 Traffic Plans and Distances between Measures

In this chapter, we consider the irrigation and who goes where problems for the cost functional E^α introduced at the end of Chapter 3. We prove in Section 6.1 that for $\alpha > 1 - \frac{1}{N}$ where N is the dimension of the ambient space, the optimal cost to transport μ^+ to μ^- is finite. More precisely, if μ^+ and μ^- are two nonnegative measures on a domain X with the same total mass M and $\alpha > 1 - 1/N$, set

$$E^\alpha(\mu^+, \mu^-) := \min_{\chi \in \mathrm{TP}(\mu^+,\mu^-)} E^\alpha(\chi). \tag{6.1}$$

Then $E^\alpha(\mu^+, \mu^-)$ can be bounded by

$$E^\alpha(\mu^+, \mu^-) \leq C_{\alpha,N} M^\alpha \mathrm{diam}(X).$$

The proof of this property, first proven in [94], follows from the explicit construction of a dyadic tree connecting any probability measure on X to a Dirac mass. If α is under this threshold it may happen that the infimum is in fact $+\infty$.

Section 6.3 compares E^α with the so called Wasserstein distance associated with the Monge-Kantorovich model. The sharp quantitative estimate that is obtained takes the form

$$W_1(\mu^+, \mu^-) \leq E^\alpha(\mu^+, \mu^-) \leq c W_1(\mu^+, \mu^-)^\beta$$

for some $\beta > 0$. The question of the existence of such an equality was raised by Cédric Villani and its proof given in [63]. This inequality gives a quantitative proof of the fact that E^α and W_1 induce the same topology on the set $\mathcal{P}(X)$ of probability measures on X. This topology is the weak convergence topology.

Because the topology induced by E^α induces the weak topology for $\alpha > 1 - \frac{1}{N}$, we have $E^\alpha(\nu_n, \nu) \to 0$ when ν_n is a sequence of probability measures on the compact $X \subset \mathbb{R}^N$ weakly converging to ν. As a consequence the limit of a converging sequence of optimal traffic plans for E^α is still optimal. This settles the stability of optima with respect to μ^+ and μ^-.

Lemma 6.1. *Let us denote W_1 the Wasserstein distance of order 1 and let μ^+ and μ^- be two probability measures. We have $W_1(\mu^+, \mu^-) \leq E^\alpha(\mu^+, \mu^-)$ for all $\alpha \in [0, 1]$.*

Proof. Indeed,

$$E^\alpha(\mu^+, \mu^-) := \inf_{\chi \in \mathrm{TP}(\mu^+,\mu^-)} \int_\Omega \int_t |\chi(\omega, t)|_\chi^{\alpha-1} |\dot\chi(\omega, t)| d\omega dt,$$

M. Bernot et al., *Optimal Transportation Networks*. Lecture Notes in Mathematics 1955.

where the infimum is taken over all parameterizations transporting μ^+ to μ^-. In particular,

$$E^1(\mu^+,\mu^-) := \inf_{\chi \in \mathrm{TP}(\mu^+,\mu^-)} \int_\Omega \int_t |\dot{\chi}(\omega,t)| d\omega dt$$

is precisely $W_1(\mu^+,\mu^-)$. Since $|\chi(\omega,t)|_\chi^{\alpha-1} \geq 1$, we obtain

$$W_1(\mu^+,\mu^-) \leq E^\alpha(\mu^+,\mu^-).$$

Proposition 6.2. *E^α is a pseudo-distance on the space of probability measures on X.*

Proof. Because of Lemma 6.1, we have $E^\alpha(\nu_1,\nu_2)=0$ if and only if $\nu_1=\nu_2$. Next, the triangular inequality is easily proved as follows: let $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ be optimal traffic plans respectively from ν_1 to ν_2 and from ν_2 to ν_3. By definition of E^α, we have

$$E^\alpha(\nu_1,\nu_3) \leq E^\alpha(\boldsymbol{P}),$$

where $\boldsymbol{P}$ is the concatenation of $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ defined in Lemma 5.5. Thus

$$E^\alpha(\nu_1,\nu_3) \leq E^\alpha(\boldsymbol{P}_1)+E^\alpha(\boldsymbol{P}_2) = E^\alpha(\nu_1,\nu_2)+E^\alpha(\nu_2,\nu_3).$$

6.1 All Measures can be Irrigated for $\alpha > 1 - \frac{1}{N}$

Let C be a cube with edge length L and center c. Let ν be a probability measure on $X \subset C$. One can approximate ν by atomic measures as follows. For each i, let

$$C_j^i : j \in \mathbb{Z}^N \cap [0,2^i)^N$$

be a partition of C into cubes of edge length $\frac{L}{2^i}$. For $j \in \mathbb{Z}^N \cap [0,2^i)^N$ call x_j^i the center of C_j^i and let $m_j^i = \nu(C_j^i)$ be the ν-mass of the cube C_j^i.

Definition 6.3. *With the above notation we call dyadic approximation of a measure ν supported by a cube the atomic measure*

$$\mu_i = \mu_i(\nu) = \sum_{j \in \mathbb{Z}^N \cap [0,2^i)^N} m_j^i \delta_{x_j^i}.$$

The following lemma is very classical.

Lemma 6.4. *The atomic measures μ_i weakly converge to ν.*

Lemma 6.5. *Let ν be a probability measure on a cube C of edge length L. Then for $n > m$,*

$$E^\alpha(\mu_m,\mu_n) \leq \mathcal{E}^\alpha(\boldsymbol{P}_{m,n}) \leq \frac{\sqrt{N}L}{2} \frac{2^{m(N(1-\alpha)-1)}}{2^{1-N(1-\alpha)}-1}.$$

Proof. This a direct application of Corollary 5.9. The number k_i of collectors at scale 2^{-i} is equal to 2^{Ni} and the length of the segments connecting them to the collectors at scale 2^{-i+1} is equal to $l_i = L\sqrt{N}2^{-i-1}$. Thus (see Figure 6.1),

$$\begin{aligned}\mathcal{E}^\alpha(\boldsymbol{P}_{m,n}) &\leq \sum_{i=m+1}^{n} k_i^{1-\alpha} l_i \\ &\leq \sum_{i=m+1}^{\infty} \frac{L\sqrt{N}}{2} 2^{i(N(1-\alpha)-1)} \\ &= \frac{L\sqrt{N}}{2} \frac{2^{m(N(1-\alpha)-1)}}{2^{1-N(1-\alpha)}-1}.\end{aligned}$$

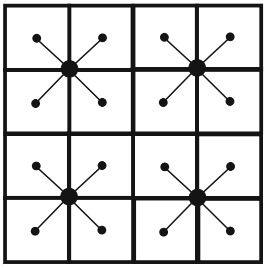

Fig. 6.1. To transport μ_i to μ_{i+1}, all the mass at the center of a cube with edge length $\frac{L}{2^{i-1}}$ is transported to the centers of its sub-cubes with edge length $\frac{L}{2^i}$.

Proposition 6.6. *Let $\alpha \in (1-\frac{1}{N}, 1]$. Let ν be a probability measure supported in a cube centered at c with edge length L. Then*

$$E^\alpha(\mu_n(\nu), \nu) \leq \frac{2^{n(N(1-\alpha)-1)}}{2^{1-N(1-\alpha)}-1} \frac{\sqrt{N}L}{2}.$$

In particular, $E^\alpha(\mu_n(\nu), \nu) \to 0$ uniformly for all ν when $n \to \infty$

Proof. By construction, the traffic plan $\boldsymbol{P}_{m,n}$ converges to a traffic plan $\boldsymbol{P}^m$ irrigating the measure ν from μ_m. (All fibers of $\boldsymbol{P}_{m,n}$ converge uniformly to fibers whose length is less than $\sqrt{N}L$.) Thus by Lemma 6.5 and Proposition 3.40,

$$E^\alpha(\mu_m, \nu) \leq \liminf_n \mathcal{E}^\alpha(\boldsymbol{P}_{m,n}) \leq \frac{2^{n(N(1-\alpha)-1)}}{2^{1-N(1-\alpha)}-1} \frac{\sqrt{N}L}{2}. \tag{6.2}$$

Since $\mu_0 = \delta_c$, we obtain directly from the previous proposition applied with $n = 0$ the following uniform bound on the energy required to irrigate a measure. Notice that a set with diameter L is contained in a cube with edge $2L$.

Corollary 6.7. *Let $\alpha \in (1 - \frac{1}{N}, 1]$ and ν be a probability measure on a set X with diameter L. There exists $\boldsymbol{P} \in \mathrm{TP}(\delta_c, \nu)$ such that*

$$E^\alpha(\boldsymbol{P}) \leq \frac{1}{2^{1-N(1-\alpha)} - 1}\sqrt{N}L.$$

Remark 6.8. The work of Devillanova and Solimini [78] refines widely the result of Corollary 6.7 by giving precise conditions on ν to be $\alpha-$irrigable (see chapter 10).

Finally, combining a transport from μ^+ to δ_c with a transport from δ_c to μ^-, it is possible to obtain any transference plan, so that the who goes where problem has a solution at finite cost in the case $\alpha > 1 - \frac{1}{N}$.

Corollary 6.9. *Let $\alpha \in (1 - \frac{1}{N}, 1]$. Let μ^+ and μ^- be probability measures on X, and π a prescribed transference plan with marginals μ^+ and μ^-. There exists $\boldsymbol{P} \in \mathrm{TP}(\pi)$ such that*

$$E^\alpha(\boldsymbol{P}) \leq \frac{1}{2^{1-N(1-\alpha)} - 1}2\sqrt{N}L.$$

Proof. Indeed, we can find a traffic plan $\boldsymbol{P}_1$ transporting μ^+ to δ_c and a traffic plan $\boldsymbol{P}_2$ transporting δ_c to μ^- such that

$$E^\alpha(\boldsymbol{P}_1) + E^\alpha(\boldsymbol{P}_2) \leq \frac{2}{2^{1-N(1-\alpha)} - 1}\sqrt{N}L.$$

By concatenating $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ one obtains a traffic plan $\boldsymbol{P}$ with a transference plan $\pi_{\tilde{\boldsymbol{P}}}$ that can be any transference plan with marginal laws μ^+ and μ^-. Since $|x|_{\tilde{\boldsymbol{P}}} \leq |x|_{\boldsymbol{P}_1} + |x|_{\boldsymbol{P}_2}$, we have

$$E^\alpha(\tilde{\boldsymbol{P}}) \leq E^\alpha(\boldsymbol{P}_1) + E^\alpha(\boldsymbol{P}_2) \leq \frac{2}{2^{1-N(1-\alpha)} - 1}\sqrt{N}L.$$

Corollary 6.10. *If the transported measure has mass M, the uniform bounds obtained in Corollaries 6.7 and 6.9 scale as M^α and we have*

$$E^\alpha(\mu^+, \mu^-) \leq C_{\alpha,N} M^\alpha \mathrm{diam}(X) \tag{6.3}$$

6.2 Stability with Respect to $\boldsymbol{\mu}^+$ and $\boldsymbol{\mu}^-$

In this section we partially answer the stability question, i.e. "is the limit of a sequence of optimal traffic plans optimal?". The property of the E^α pseudo-distance in the case $\alpha \in (1 - \frac{1}{N}, 1]$ permits to answer yes (Proposition 6.12). However, in the case $\alpha \leq 1 - \frac{1}{N}$ this stability is conjectural.

Lemma 6.11. *Let $\alpha \in (1 - \frac{1}{N}, 1]$. If ν_n is a sequence of probability measures on the compact $X \subset \mathbb{R}^N$ weakly converging to ν, then $E^\alpha(\nu_n, \nu) \to 0$ when $n \to \infty$.*

Proof. Let us adopt the notation of Definition 6.3 and Proposition 6.6 and let us assume that X is contained in a cube with edge length L subdivided into dyadic cubes C_j^i with edge length $2^{-i}L$. The weak convergence of ν_n to ν applied to the characteristic functions of the cubes C_j^i implies that $m_j^i(\nu_n)$ converges to $m_j^i(\nu)$ when $n \to \infty$, where $m_j^i(\nu)$ denotes the mass of ν contained in the cube C_j^i. Thus for any $\varepsilon > 0$ and for n large enough,

$$\sum_j |m_j^i(\nu_n) - m_j^i(\nu)| < \varepsilon.$$

By Proposition 6.6, $E^\alpha(\mu_i(\nu), \nu) \leq \varepsilon$ and $E^\alpha(\mu_i(\nu_n), \nu_n) \leq \varepsilon$ for i large enough, independently of n. We are left to evaluate $E^\alpha(\mu_i(\nu_n), \mu_i(\nu))$. Since these measures are concentrated at the centers of cubes C_j^i, this amounts to transport in the whole cube a mass less than $\sum_j |m_j^i(\nu_n) - m_j^i(\nu)| < \varepsilon$. By (6.3), we deduce that $E^\alpha(\mu_i(\nu), \mu_i(\nu_n)) \leq C\varepsilon^\alpha$ for a constant C depending only on X and α. The triangular inequality for E^α yields

$$E^\alpha(\nu, \nu_n) \leq E^\alpha(\nu, \mu_i(\nu)) + E^\alpha(\mu_i(\nu), \mu_i(\nu_n)) + E^\alpha(\mu_i(\nu_n), \nu_n) \leq 2\varepsilon + C\varepsilon^\alpha.$$

Proposition 6.12. *Let $\alpha \in (1 - \frac{1}{N}, 1]$. If $\boldsymbol{P}_n$ is a sequence of optimal traffic plans for the irrigation problem and $\boldsymbol{P}_n$ is converging to $\boldsymbol{P}$, then $\boldsymbol{P}$ is optimal.*

Proof. Since $E^\alpha(\boldsymbol{P}_n) = \mathcal{E}^\alpha(\boldsymbol{P}_n)$ and $E^\alpha(\boldsymbol{P}) \leq \mathcal{E}^\alpha(\boldsymbol{P})$, using the lower semi-continuity of $\mathcal{E}^\alpha$, we have

$$\begin{aligned} E^\alpha(\boldsymbol{P}) &\leq \liminf_n E^\alpha(\boldsymbol{P}_n) = \liminf_n E^\alpha(\mu_n^+, \mu_n^-) \\ &\leq \liminf_n \left(E^\alpha(\mu_n^+, \mu^+) + E^\alpha(\mu^+, \mu^-) + E^\alpha(\mu^-, \mu_n^-)\right) \\ &\leq E^\alpha(\mu^+, \mu^-) \text{ since } \mu_n^+ \to \mu^+ \text{ and } \mu_n^+ \to \mu^+. \end{aligned}$$

Thus, $\boldsymbol{P}$ is optimal.

Remark 6.13. In the case $\alpha < 1 - \frac{1}{N}$, the stability of optimal traffic plans remains an open question (see Chapter 15). Of course, only the case when $\boldsymbol{P}_n$ is a sequence of optimal traffic plans with $E^\alpha(\boldsymbol{P}_n) < \infty$ is of interest. The stability in the case of the who goes where problem is also an open problem.

6.3 Comparison of Distances between Measures

Proposition 6.12 implies that the topology induced by the distance E^α on $\mathcal{P}(X)$ is exactly the weak-* topology.

Proposition 6.14. *If $\alpha \in (1-\frac{1}{N}, 1]$, E^α is a metric of the weak-* topology of probability measures $\mathcal{P}(X)$.*

Proof. Indeed, Proposition 6.12 asserts that if ν_n weakly converges to ν then $E^\alpha(\nu_n,\nu) \to 0$. Conversely, if $E^\alpha(\nu_n,\nu) \to 0$, then Lemma 6.1 asserts that $W_1(\nu_n,\nu) \to 0$, so that ν_n weakly converges to ν.

Remark 6.15. If $\alpha \leq 1-\frac{1}{N}$, then it is no longer true that if ν_n weakly converges to ν then $E^\alpha(\nu_n,\nu) \to 0$. Indeed, let us consider $\nu_n := \frac{1}{v_n}1\!\!1_{B(0,\frac{1}{n})}$, where v_n is the volume of a ball with radius $\frac{1}{n}$. In that case $\nu_n \rightharpoonup \delta_0$ but, by Theorem 10.26, $E^\alpha(\nu_n,\delta_0) = \infty$ if $\alpha \leq 1-\frac{1}{N}$.

The following proposition gives a quantitative version of Proposition 6.14. To fix ideas, we consider two probability measures μ^+ and μ^- with support in an N-dimensional cube C with edge 1, say $C = [0,1]^N$. It is not difficult to scale the result to any bounded domain in $\mathbb{R}^N$.

Proposition 6.16. *Let $\alpha \in (1-\frac{1}{N}, 1]$, then*

$$E^\alpha(\mu^+,\mu^-) \leq cW_1(\mu^+,\mu^-)^{N(\alpha-(1-1/N))},$$

where c denotes a suitable constant depending only on N and α.

Proof. Let us denote X^+ and X^- the two projections from $C\times C$ onto C, so that $X^+(x,y)=x$, $X^-(x,y)=y$.

Let π_0 be an optimal transport plan between μ^+ and μ^-, i.e. a probability measure on $C\times C$ such that $X^\pm_\sharp \pi_0 = \mu^\pm$ and with cost $w := W_1(\mu^+,\mu^-)$. We denote by

$$\Lambda_i = \left\{(x,y) \in C\times C : (2^i-1)\frac{w}{2} \leq |x-y| < (2^{i+1}-1)\frac{w}{2}\right\}.$$

We can limit ourselves to consider those indices i which are not too large, i.e. up to $(2^i-1)\frac{w}{2} \leq \sqrt{N}$ (where $\sqrt{N}$ is the diameter of C). Let I be the maximal index i so that this inequality is satisfied. The set $\Lambda = \cup_{i=0}^I \Lambda_i$ is a disjoint union and

$$\sum_{i=0}^{I}(2^i-1)\frac{w}{2}\pi_0(\Lambda_i) \leq W_1(\mu^+,\mu^-) = w \leq \sum_{i=0}^{I}(2^{i+1}-1)\frac{w}{2}\pi_0(\Lambda_i). \quad (6.4)$$

We call cube with edge e any translate of $[0,e[^N$. For each $i=0,\cdots,I$, using a regular grid in $\mathbb{R}^N$, one can cover C with disjoint cubes $C_{i,k}$ with edge $(2^{i+1}-1)w$. The number of the cubes in the $i-$th covering may be easily estimated by

$$\left(\frac{1}{(2^{i+1}-1)w}+1\right)^N \leq \left(\frac{c}{(2^{i+1}-1)w}\right)^N = K(i). \quad (6.5)$$

For each index i, C is included in the disjoint union $\subset \cup_{k=1}^{K(i)} C_{i,k}$. Let us set

$$\Lambda_{i,k} = (C_{i,k} \times C) \cap \Lambda_i, \quad \mu_{i,k}^+ = X_\sharp^+(\pi_0 \mathbb{1}_{\Lambda_{i,k}}) \text{ and } \mu_{i,k}^- = X_\sharp^-(\pi_0 \mathbb{1}_{\Lambda_{i,k}}).$$

We have just cut μ^+ and μ^- into pieces. Let us call informally μ_i^+ the pieces of μ^+ for which the Wasserstein distance to the corresponding part μ_i^- of μ^- is of order $2^i \frac{w}{2}$. Then $\mu_{i,k}^+$ is the part of μ_i^+ whose support is in the cube $C_{i,k}$. What we have now gained is that each $\mu_{i,k}^+$ has a specified diameter of order $2^i w$ and is at a distance to its corresponding $\mu_{i,k}^-$ which is of the same order $2^i w$ (see picture 6.2). Let us be a bit more precise. The support of $\mu_{i,k}^+$ is a cube with edge $(2^i - 1)w$. By definition of Λ_i, the maximum distance of a point of $\mu_{i,k}^-$ to a point of $\mu_{i,k}^+$ is less than $(2^{i+1} - 1)\frac{w}{2}$. Thus the supports of $\mu_{i,k}^-$ and $\mu_{i,k}^+$ are both contained in a same cube with edge $6 \cdot 2^i w$.

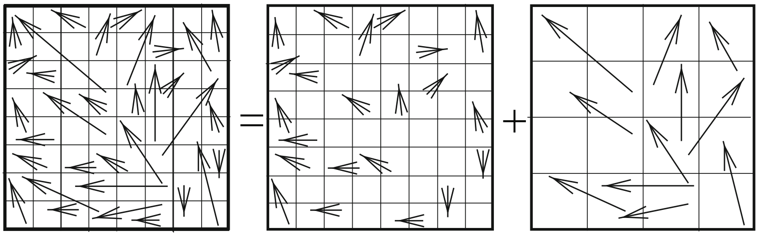

Fig. 6.2. Decomposition of Monge's transportation into the sets $\Lambda_{i,k}$.

By the scaling property of the E^α distance (6.3), we deduce that for some constant c, depending only on α and N, holds:

$$E^\alpha(\mu_{i,k}^+, \mu_{i,k}^-) \le c 2^i w \pi_0(\Lambda_{i,k})^\alpha.$$

From this last relation, the sub-additivity of E^α, Hölder inequality, (6.4) and the bound on $K(i)$ given in (6.5), one obtains in turn

$$\begin{aligned}
E^\alpha(\mu^+, \mu^-) &\le \sum_{i,k} E^\alpha(\mu_{i,k}^+, \mu_{i,k}^-) \\
&\le \sum_{i,k} c 2^i w \pi_0(\Lambda_{i,k})^\alpha = c \sum_{i,k} (2^i w \pi_0(\Lambda_{i,k}))^\alpha (2^i w)^{1-\alpha} \\
&\le c \left(\sum_{i,k} (2^i w \pi_0(\Lambda_{i,k})) \right)^\alpha \left(\sum_{i,k} 2^i w \right)^{1-\alpha} \\
&\le c \left(\sum_{i} (2^i w \pi_0(\Lambda_{i})) \right)^\alpha \left(\sum_{i=0}^{I} K(i) 2^i w \right)^{1-\alpha} \\
&\le c w^\alpha \left(\sum_{i=0}^{I} \left(\frac{c}{(2^{i+1} - 1)w} \right)^N 2^i w \right)^{1-\alpha}
\end{aligned}$$

$$\leq cw^{\alpha+(1-N)(1-\alpha)}\left(\sum_{i=0}^{I}2^{i(1-N)}\right)^{1-\alpha}$$
$$\leq cw^{\alpha N-(N-1)} = cW_1(\mu^+,\mu^-)^{\alpha N-(N-1)},$$

where c denotes various constants depending only on N and α and where the last two inequalities are valid if $N \geq 2$ so that the series $\sum_{i=0}^{\infty} 2^{i(1-N)}$ is convergent.

In the case $N = 1$ a different proof is needed. In this case we know how does an optimal transportation for $E^\alpha(\mu^+,\mu^-)$ look like. In the one-dimensional setting, we have

$$E^\alpha(\mu^+,\mu^-) = \int_0^1 |\theta(x)|^\alpha dx.$$

The function θ plays the role of the multiplicity and it is given by

$$\theta(x) = \mu([0,x]), \quad \mu := \mu^+ - \mu^-,$$

as a consequence of its constraint on the derivative. Hence we have

$$E^\alpha(\mu^+,\mu^-) = \int_0^1 |\mu([0,x])|^\alpha dx \leq \left[\int_0^1 |\mu([0,x])|dx\right]^\alpha,$$

where the inequality comes from Jensen's inequality. Then we set

$$A = \{x \in [0,1] : \mu([0,x]) > 0\}$$

and $h(x) = \mathbb{1}_A(x) - \mathbb{1}_{[0,1]\setminus A}(x)$ and we have

$$\int_0^1 |\mu([0,x])|dx = \int_0^1 \mu([0,x])h(x)dx = \int_0^1 h(x)dx \int_0^1 \mathbb{1}\{t \leq x\}\mu(dt)$$
$$= \int_0^1 \mu(dt)\int_t^1 h(x)dx = \int_0^1 u(t)\mu(dt) \leq W_1(\mu^+,\mu^-),$$

where $u(t) = \int_t^1 h(x)dx$ is a Lipschitz continuous function whose Lipschitz constant does not exceed 1 as a consequence of $|h(x)| \leq 1$. Thus the last inequality is justified by the duality formula (see [86], Theorem 1.14, page 34):

$$W_1(\mu^+,\mu^-) = \sup_{v\in Lip_1}\int_0^1 v\, d(\mu^+ - \mu^-).$$

Hence it follows easily $E^\alpha(\mu^+,\mu^-) \leq W_1(\mu^+,\mu^-)^\alpha$, which is the thesis for the one dimensional case.

Remark 6.17. The assumption $\alpha > 1 - 1/N$ cannot be removed since, for $N \geq 2$, if we remove this assumption, the quantity E^α could be infinite while W_1 is always finite. In dimension 1 the only uncovered case is $\alpha = 0$. In this case E^α is in fact always finite but, for instance if $\mu^+ = w_0$ and $\mu^- = (1-\varepsilon)w_0 + \varepsilon w_1$ one has $E^\alpha(\mu^+,\mu^-) = 1$ while $W_1(\mu^+,\mu^-) = \varepsilon$. As ε is as small as we want, this excludes any desired inequality.

Remark 6.18. The exponent $N(\alpha - (1 - 1/N))$ cannot be improved as can be seen from the following example.

Example 6.19. There exists a sequence (μ_n^+, μ_n^-) of pairs of probability measures on the cube C such that

$$E^\alpha(\mu_n^+, \mu_n^-) = cn^{-N(\alpha-(1-1/N))} \text{ and } W_1(\mu_n^+, \mu_n^-) = c/n.$$

Proof. It is sufficient to divide the cube C into n^N small cubes of edge $1/n$ and to set $\mu_n^+ = \sum_{i=1}^{n^N} \frac{1}{n^N}\delta_{x_i}$ and $\mu_n^- = \sum_{i=1}^{n^N} \frac{1}{n^N}\delta_{y_i}$, where each x_i is a vertex of one of the n^N cubes (let us say the one with minimal sum of the N coordinates) and the corresponding y_i is the center of the same cube. In this way y_i realizes the minimal distance to x_i among the y_j's. Thus the optimal configuration both for E^α and W_1 is given by linking any x_i directly to the corresponding y_i. In this way we have

$$E^\alpha(\mu_n^+, \mu_n^-) = n^N \left(\frac{1}{n^N}\right)^\alpha \frac{c}{n} = cn^{-N(\alpha-(1-1/N))}$$
$$W_1(\mu_n^+, \mu_n^-) = n^N \frac{1}{n^N}\frac{c}{n} = \frac{c}{n},$$

where $c = \frac{\sqrt{N}}{2}$.

Remark 6.20. One can deduce easily inequalities between E^α and W_p by using standard inequalities between W_1 and W_p, namely $cW_p^p \leq E^\alpha \leq cW_p^{N(\alpha-(1-1/N))}$. The right hand inequality is sharp by using again example 6.19. It is not clear instead whether the left-hand inequality is optimal.

Remark 6.21. Since the W_1 distance between two probability measures is always finite, Proposition 6.16 gives another proof of the fact that there is a traffic plan at finite cost for $\alpha \in (1 - \frac{1}{N}, 1]$.