Chapter 7

Forward-Backward SDEs with Reflections

In this chapter we study FBSDEs with boundary conditions. In the simplest case when the FBSDE is decoupled, it is reduced to a combination of a well-understood (forward) reflected diffusion and a newly developed reflected backward SDE. However, the extension of such FBSDEs to the general coupled case is quite delicate. In fact, none of the methods that we have seen in the previous chapters seems to be applicable, due to the presence of the reflecting process. Therefore, the route we take in this chapter to reach the existence and uniqueness of the adapted solution is slightly different from those we have seen before.

§1. Forward SDEs with Reflections

Let $\mathcal{O}$ be a closed convex domain in $\mathbb{R}^n$. Define for any $x \in \partial\mathcal{O}$ the set of inward normals to $\mathcal{O}$ at x by

$$\mathcal{N}_x = \{\gamma : |\gamma| = 1, \text{ and } \langle \gamma, x - y \rangle \le 0, \quad \forall y \in \mathcal{O}\}. \tag{1.1}$$

It is clear that if the boundary $\partial\mathcal{O}$ is smooth (say, C^1), then for any $x \in \partial\mathcal{O}$, the set $\mathcal{N}_x$ contains only one vector, that is, the unit inner normal vector at x. We denote $BV([0,T];\mathbb{R}^n)$ to be the set of all $\mathbb{R}^n$-valued functions of bounded variation; and for $\eta \in BV([0,T];\mathbb{R}^n)$, we denote $|\eta|(T)$ to be the total variation of η on $[0,T]$.

A general form of (forward) SDEs with reflection (FSDER, for short) is the following:

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s) + \eta(t). \tag{1.2}$$

Here the b and σ are functions of $(t,x,\omega) \in [0,T] \times \mathbb{R}^n \times \Omega$ (with ω being suppressed, as usual); and $\eta \in BV_{\mathcal{F}}([0,T];\mathbb{R}^m)$, the set of all $\{\mathcal{F}_t\}_{t\ge 0}$-adapted processes η with paths in $BV([0,T];\mathbb{R}^m)$.

Definition 1.1. A pair of continuous, $\{\mathcal{F}_t\}_{t\ge0}$-adapted processes $(X,\eta) \in L^2_{\mathcal{F}}([0,T];\mathbb{R}^n) \times BV_{\mathcal{F}}([0,T];\mathbb{R}^n)$ is called a solution to the FSDER (1.2) if

1) $X(t) \in \mathcal{O}$, $\forall t \in [0,T]$, a.s.;

2) $\eta(t) = \int_0^t 1_{\{X(s)\in\partial\mathcal{O}\}}\gamma(s)d|\eta|(s)$, where $\gamma(s) \in \mathcal{N}_{X(s)}$, $0 \le s \le t \le T$, $d|\eta|$-a.e.;

3) equation (1.2) is satisfied almost surely.

A widely used tool for solving an FSDER is the following (deterministic) function-theoretic technique known as the *Skorohod Problem*: Let the domain $\mathcal{O}$ be given,

Problem $SP(\cdot\,;\mathcal{O})$**:** Let $\psi \in C([0,T];\mathbb{R}^n)$ with $\psi(0) \in \mathcal{O}$ be given. Find a pair $(\varphi,\eta) \in C([0,T];\mathbb{R}^n) \times BV([0,T];\mathbb{R}^n)$ such that

1) $\varphi(t) = \psi(t) + \eta(t)$, $\forall t \in [0,T]$, and $\varphi(0) = \psi(0)$;

2) $\varphi(t) \in \mathcal{O}$, for $t \in [0,T]$;

3) $|\eta|(t) = \int_0^t 1_{\{\varphi(s)\in\partial\mathcal{O}\}} d|\eta|(s)$;

4) there exists a measurable function $\gamma : [0,T] \mapsto \mathbb{R}^n$, such that $\gamma(t) \in \mathcal{N}_{\varphi(t)}$ ($d|\eta|$ a.s.) and $\eta(t) = \int_0^t \gamma(s)d|\eta|(s)$.

A pair (φ,η) satisfying the above 1)–4) is called a *solution* of the $SP(\psi;\mathcal{O})$.

It is known that under various technical conditions on the domain $\mathcal{O}$ and its boundary, for any $\psi \in C([0,T];\mathbb{R}^n)$ there exists a unique solution to $SP(\psi;\mathcal{O})$. In particular, these conditions are satisfied when $\mathcal{O}$ is convex and with smooth boundary, which will be the case considered throughout this chapter. Therefore we can consider a well-defined mapping $\Gamma : C([0,T];\mathbb{R}^n) \mapsto C([0,T];\mathbb{R}^n)$ such that $\Gamma(\psi)(t) = \varphi(t)$, $t \in [0,T]$, where (φ,η) is the (unique) solution to $SP(\psi;\mathcal{O})$. We will call Γ the *solution mapping* of the $SP(\cdot\,;\mathcal{O})$.

An elegant feature of the solution mapping Γ is that it may have a *Lipschitz property*: for some constant $K > 0$ that is independent of T, such that for $\psi_i \in C([0,T],\mathbb{R}^n)$, $i = 1,2$, it holds that

$$|\Gamma(\psi_1)(\cdot) - \Gamma(\psi_2)(\cdot)|_T^* \le K|\psi_1(\cdot) - \psi_2(\cdot)|_T^*, \tag{1.3}$$

where $|\xi|_t^*$ denotes the sup-norm on $[0,t]$ for $\xi \in C([0,T];\mathbb{R}^n)$. Consequently, if (φ_i,η_i), $i = 1,2$ are solutions to $SP(\psi_i;\mathcal{O})$, $i = 1,2$, respectively, then for some constant K independent of T,

$$|\varphi_1(\cdot) + \varphi_2(\cdot)|_T^* + |\eta_1(\cdot) - \eta_2(\cdot)|_T^* \le K|\psi_1(\cdot) - \psi_2(\cdot)|_T^*. \tag{1.4}$$

In what follows we call a (convex) domain $\mathcal{O} \subseteq \mathbb{R}^n$ *regular* if the solution mapping of the corresponding $SP(\cdot\,;\mathcal{O})$ satisfies (1.3). The simplest but typical example of a regular domain is the "half space" $\mathcal{O} = \mathbb{R}^n_+ \stackrel{\Delta}{=} \{(x_1,\cdots,x_n) \in \mathbb{R}^n : x_n \ge 0\}$. With a standard localization technique, one can show that a convex domain with smooth boundary is also regular. A much deeper result of Dupuis and Ishii [1] shows that a convex polyhedron is regular, which can be extended to a class of convex domains with piecewise smooth boundaries. We should note that proving the regularity of a given domain is in general a formidable problem with independent interest of its own. To simplify presentation, however, in this chapter we consider only the case when the domains are regular, although the result we state below should hold true for a much larger class of (convex) domains, with proofs more complicated than what we present here.

We shall make use of the following assumptions.

(A1) (i) for fixed $x \in \mathbb{R}^n$, $b(\cdot,x,\cdot)$ and $\sigma(\cdot,x,\cdot)$ are $\{\mathcal{F}_t\}_{t\ge0}$-progressively measurable;

(ii) there exists constant $K > 0$, such that for all $(t, \omega) \in [0, T] \times \Omega$ and $x, x' \in \mathbb{R}^n$, it holds that

$$
\begin{aligned}
|b(t, x, \omega) - b(t, x', \omega)| &\le K|x - x'|; \\
|\sigma(t, x, \omega) - \sigma(t, x', \omega)| &\le K|x - x'|.
\end{aligned}
\tag{1.5}
$$

Theorem 1.2. *Suppose that $\mathcal{O} \subseteq \mathbb{R}^n$ is a regular, convex domain; and that (A1) holds. Then the SDER (1.2) has a unique strong solution.*

Proof. Let Γ be the solution mapping to $SP(\cdot\,; \mathcal{O})$. Consider the following SDE (without reflection):

$$
\widetilde{X}(t) = x + \int_0^t \widetilde{b}(s, \widetilde{X}(\cdot))ds + \int_0^t \widetilde{\sigma}(s, \widetilde{X}(\cdot))dW(s),
\tag{1.6}
$$

where for $y(\cdot) \in C([0, T]; \mathbb{R}^n)$,

$$
\widetilde{b}(t, y(\cdot), \omega) = b(t, \Gamma(y)(t), \omega); \quad \widetilde{\sigma}(t, y(\cdot), \omega) = \sigma(t, \Gamma(y)(t), \omega).
$$

Note that for any $\{\mathcal{F}_t\}_{t\ge 0}$-adapted, continuous process Y, the processes $\widetilde{b}(\cdot, Y(\cdot), \cdot)$ and $\widetilde{\sigma}(\cdot, Y(\cdot), \cdot)$, are all $\{\mathcal{F}_t\}_{t\ge 0}$-progressively measurable. Further, the regularity of the domain $\mathcal{O}$ implies that there exists a constant $K_0 > 0$ depending only on the Lipschitz constant of Γ and K in (A1), such that for any $\{\mathcal{F}_t\}_{t\ge 0}$-adapted, continuous processes Y and Y', it holds that

$$
\begin{aligned}
|\widetilde{b}(s, Y(\cdot, \omega), \omega) - \widetilde{b}(s, Y'(\cdot, \omega), \omega)|_t^* &\le K_0|Y(s, \omega) - Y'(s, \omega))|_t^*; \\
|\widetilde{\sigma}(s, Y(\cdot, \omega), \omega) - \widetilde{\sigma}(s, Y'(\cdot, \omega), \omega)|_t^* &\le K_0|Y(s, \omega) - Y'(s, \omega))|_t^*,
\end{aligned}
$$

for all $(t, \omega) \in [0, T] \times \mathcal{O}$. Therefore, by the standard theory of SDEs (cf. e.g., Protter [1]), we know that the SDE (1.6) has a unique strong solution $\widetilde{X}$.

Next, we define a process $X(t) = \Gamma(\widetilde{X})(t)$, $t \in [0, T]$. Then by definition of the Skorohod problem, we see that there exists a process η such that (X, η) satisfies the conditions 1)–3) of Definition 1.1. Consequently, for all $t \in [0, T]$, we have

$$
\begin{aligned}
X(t) &= \widetilde{X}(t) + \eta(t) \\
&= x + \int_0^t \widetilde{b}(s, \widetilde{X}(\cdot))ds + \int_0^t \widetilde{\sigma}(s, \widetilde{X}(\cdot))dW(s) + \eta(t) \\
&= x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s) + \eta(t).
\end{aligned}
$$

In other words, (X, η) is a solution to the SDER (1.5). The uniqueness follows easily from the construction of the solution and the Lipschitz property (1.3) and (1.4). The proof is complete. $\square$

§2. Backward SDEs with Reflections

In this section we study the reflected BSDEs (BSDERs, for short). For clearer notation we will call the domain in which a BSDE lives by $\mathcal{O}_2$, to distinguish it from those in the previous section. A slight difference is that we shall allow $\mathcal{O}_2$ to "move" when time varies, and even randomly. Namely, we shall consider a family of closed, convex domains $\{\mathcal{O}_2(t,\omega) : (t,\omega) \in [0,T] \times \Omega\}$ in $\mathbb{R}^m$ satisfying certain conditions. Let $\xi \in \mathcal{O}_2(T,\omega)$ be given, we consider the following SDE:

$$Y(t) = \xi + \int_t^T h(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s) + \zeta(T) - \zeta(t). \tag{2.1}$$

Analogous to the FSDER, we define the adapted solution to a BSDER as follows:

Definition 2.1. A triplet of processes $(Y, Z, \zeta) \in L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R}^m)) \times L^2_{\mathcal{F}}(0,T; \mathbb{R}^{m\times d}) \times BV_{\mathcal{F}}([0,T]; \mathbb{R}^m)$ is called a solution to (2.1) if

(1) $Y(t,\omega) \in \mathcal{O}_2(t,\omega)$, for all $t \in [0,T]$, $\mathbf{P}$-a.e. ω;

(2) for any $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, RCLL process $V(t)$ such that $V(t) \in \mathcal{O}_2(t,\cdot)$, $\forall t \in [0,T]$, a.s., it holds that $\langle\, Y(t) - V(t), d\zeta(t)\,\rangle \leq 0$, as a signed measure.

We note that Definition 2.1 more or less requires that the domains $\{\mathcal{O}_2(\cdot,\cdot)\}$ be "measurable" (or even "progressively measurable") in (t,ω) in a certain sense, which we now describe. Let $y \in \mathbb{R}^m$ and $A \subseteq \mathbb{R}^m$ be any closed set, we define the *projection operator* Pr with respect to A, denoted $Pr(\cdot\,; A)$, by

$$Pr(y; A) = y - \frac{1}{2}\nabla_y d^2(y, A), \qquad y \in \mathbb{R}^m; \tag{2.2}$$

where $d(\cdot,\cdot)$ is the usual distance function:

$$d(y, A) \stackrel{\Delta}{=} \inf\{|y - x| : x \in A\}. \tag{2.3}$$

For each $y \in \mathbb{R}^m$, we define $\beta(t, y, \omega) = Pr(y; \mathcal{O}_2(t,\omega))$. Throughout this chapter we shall assume the following technical condition.

(A2) (i) For every fixed $y \in \mathbb{R}^m$, the process $(t,\omega) \mapsto \beta(t, y, \omega)$ is $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable;

(ii) for fixed $y \in \mathbb{R}^m$, it holds that

$$E\int_0^T |\beta(t, y, \cdot)|^2 dt < \infty. \tag{2.4}$$

Before we go any further, let us look at some examples.

Example 2.2. Let $\mathcal{H}_m$ be the collection of all compact subsets of $\mathbb{R}^m$, endowed with the *Hausdorff* metric d^*, that is,

$$d^*(A, B) = \max\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\}, \qquad \forall A, B \in \mathcal{H}_m. \tag{2.5}$$

It is well-known that $(\mathcal{H}_m, d^*)$ is a complete metric space. Now suppose that $\mathcal{O}_2 \stackrel{\Delta}{=} \{\mathcal{O}_2(t,\omega) : (t,\omega) \in [0,T] \times \Omega\} \subseteq (\mathcal{H}_m, d^*)$, then we can view $\mathcal{O}_2$ as an

$(\mathcal{H}_m, d^*)$-valued process, and thus assume that it is $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable. Noting that for fixed $y \in \mathbb{R}^m$, the mapping $A \mapsto d(y, A)$ is a continuous mapping from $(\mathcal{H}_m, d^*)$ to $\mathbb{R}$, as

$$|d(y, A) - d(y, B)| \leq d^*(A, B), \qquad \forall y \in \mathbb{R}^m,\ \forall A, B \in \mathcal{H}_m,$$

the composition function $(t, \omega) \mapsto d^2(y, \mathcal{O}(t, \omega))$ is $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable as well, which then renders $\nabla_y d^2(y, \mathcal{O}_2(\cdot, \cdot))$ an $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable process, for any fixed $y \in \mathbb{R}^m$. Consequently, $\mathcal{O}_2$ satisfies (A2)-(i).

Next, using elementary inequality $|d(z_1, A) - d(z_2, A)| \leq |z_1 - z_2|$, $\forall z_1, z_2 \in \mathbb{R}^m$, $\forall A \subseteq \mathbb{R}^m$ one shows that

$$|\nabla_y d^2(y, \mathcal{O}_2(t, \omega))| \leq 2d(y, \mathcal{O}_2(t, \omega)).$$

Assumption (A2)-(ii) is easily satisfied provided $d(y, \mathcal{O}_2(\cdot, \cdot)) \in L^2([0, T] \times \Omega)$, which is always the case if, for example, $0 \in \mathcal{O}_2(t, \omega)$ for all (t, ω), or, more generally, $\mathcal{O}_2(t, \omega)$ has a selection in $L^2_{\mathcal{F}}(0, T; \Omega)$. □

Example 2.3. As a special case of Example 2.2, the following moving domains are often seen in applications. Let $\{\mathcal{O}(t, x) : (t, x) \in [0, T] \times \mathbb{R}^n\}$ be a family of convex, compact domains in $\mathbb{R}^m$ such that

(i) the mapping $(t, x) \mapsto \mathcal{O}(t, x)$ is continuous as a function from $[0, T] \times \mathbb{R}^n$ to $(\mathcal{H}_m, d^*)$.

(ii) for each (t, x), $0 \in \mathcal{O}(t, x)$; and there exists a constant $C > 0$ such that

$$\sup_{t\in[0,T]} d^*(\mathcal{O}(t, x), \mathcal{O}(t, 0)) \leq C|x|.$$

Let $X \in L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^n))$, and define $\mathcal{O}_2(t, \omega) \stackrel{\Delta}{=} \mathcal{O}(t, X(t, \omega))$, $(t, \omega) \in [0, T] \times \Omega$. We leave it to the readers to check that $\mathcal{O}_2$ satisfies (A2). □

Example 2.4. Continuing from the previous examples, let us assume that $m = 1$ and $\mathcal{O}(t, x) = [L(t, x), U(t, x)]$, where $-\infty < L(t, x) < 0 < U(t, x) < \infty$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$. Suppose that the functions L and U are both uniformly Lipschitz in x, uniformly in $t \in [0, T]$. Then a simple calculation using the definition of the Hausdorff metric shows that

$$d^*(\mathcal{O}(t, 0), \mathcal{O}(t, x)) = \max\{|L(t, x) - L(t, 0)|,\ |U(t, x) - U(t, 0)|\} \leq C|x|.$$

Thus $\mathcal{O}_2$ satisfies (A2), thanks to the previous example. □

Let us now turn our attention to the well-posedness of the BSDER (2.1). We shall make use of the following standing assumptions on coefficient $h : [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \Omega \mapsto \mathbb{R}^m$ and the domain $\{\mathcal{O}_2(t, \omega)\}$.

(**A3**) (i) for each $(y, z) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}$, $h(\cdot, y, z, \cdot)$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable process; and for fixed $(t, z) \in [0, T] \times \mathbb{R}^{m\times d}$ and a.e. $\omega \in \Omega$, $h(t, \cdot, z, \omega)$ is continuous;

(ii) $E\int_0^T |h(t,0,0)|^2 dt < \infty$;

(iii) there exist $\alpha \in \mathbb{R}$ and $k_2 > 0$, such that for all $t \in [0,T]$, $y, y' \in \mathbb{R}^m$, and $z, z' \in \mathbb{R}^{m\times d}$, it holds $\mathbf{P}$-a.s. that

$$\begin{cases} \langle\, y - y', h(t,y,z) - h(t,y',z)\,\rangle \le \alpha|y-y'|^2; \\ |h(t,y,z) - h(t,y,z')| \le k_2|z-z'|; \\ |h(t,y,z) - h(t,0,z)| \le k_2(1+|y|). \end{cases}$$

(iv) The domains $\{\mathcal{O}_2(t,\cdot)\}$ is "non-increasing". In other words, it holds that

$$\mathcal{O}(t,\omega) \subseteq \mathcal{O}(s,\omega), \qquad \forall t \ge s, \text{ a.s.}$$

Our main result of this section is the following theorem.

Theorem 2.5. *Suppose that (A2) and (A3) are in force. Then the BSDER (2.1) has a unique (strong) solution. Furthermore, the process ζ_t is absolutely continuous with respect to Lebesgue measure, and for any process V_t such that $V_t(\omega) \in \mathcal{O}_2(t,\omega)$, $\forall t \in [0,T]$, a.s., it holds that*

$$\langle\, \frac{d\zeta_t}{dt}, Y_t - V_t\,\rangle \le 0, \qquad \forall t \in [0,T], \text{ a.s.} \tag{2.6}$$

Remark 2.6. Suppose $m = 1$ and $\mathcal{O}_2 = [L, U]$, for appropriate processes L and U. Denote by $\zeta = \zeta^+ - \zeta^-$, $\zeta_0^+ = \zeta_0^- = 0$, the minimal decomposition of ζ as a difference of two non-decreasing processes. By replacing V in (2.6) by

$$\begin{aligned} V_t^L &= L_t \mathbf{1}_{\{\frac{d\zeta_t}{dt} \ge 0\}} + Y_t \mathbf{1}_{\{\frac{d\zeta_t}{dt} < 0\}}, \\ V_t^U &= U_t \mathbf{1}_{\{\frac{d\zeta_t}{dt} \le 0\}} + Y_t \mathbf{1}_{\{\frac{d\zeta_t}{dt} > 0\}}, \qquad t \in [0,T], \end{aligned}$$

respectively, we obtain

$$\langle\, Y_t - L_t, d\zeta_t^+\,\rangle = 0, \quad \langle\, Y_t - U_t, d\zeta_t^-\,\rangle = 0, \quad \forall t \in [0,T], \text{ a.s.} \tag{2.7}$$

Proof of Theorem 2.5. Since the proof is quite lengthy, we shall split it into several lemmas. To begin with, let us first recall the notion of *Yosida approximation*, which is another typical route of attacking the existence and uniqueness of an SDE with reflection other than using Skorohod problem.

Let φ be any *proper*, lower semicontinuous (l.s.c., for short), convex function (by *proper* we mean that φ is not identically equal to $+\infty$). Let $\mathcal{D}(\varphi) = \{x : \varphi(x) < \infty\}$. We define the *subdifferential* of φ, denoted by $\partial\varphi$, as

$$\partial\varphi(y) \stackrel{\Delta}{=} \{x^* \in \mathbb{R}^m : \langle\, x^*, y - x\,\rangle \ge 0, \ \forall x \in \overline{\mathcal{D}(\varphi)}\}.$$

In what follows we denote $A \stackrel{\Delta}{=} \partial\varphi$. Define, for each $\varepsilon > 0$, a function

$$\varphi_\varepsilon(y) \stackrel{\Delta}{=} \inf_{x\in\mathbb{R}^m} \Big\{\frac{1}{2\varepsilon}|y-x|^2 + \varphi(x)\Big\}. \tag{2.8}$$

Since $\mathbb{R}^m$ is a Hilbert space, and φ is a l.s.c. proper convex mapping, the the following result can be found in standard text (cf. Barbu [1, Chapter II]):

Lemma 2.7. *(i) The function φ_ε is (Fréchet) differentiable.*

(ii) The Fréchet differential of φ_ε, denoted by $D\varphi_\varepsilon$, satisfies $D\varphi_\varepsilon = A_\varepsilon$, where A_ε is the Yosida approximation of A, define by

$$A_\varepsilon(y) = \frac{1}{\varepsilon}(y - J_\varepsilon(y)), \qquad \text{where } J_\varepsilon(y) = (I + \varepsilon A)^{-1}(y). \tag{2.9}$$

(iii) $|J_\varepsilon(x) - J_\varepsilon(y)| \le |x - y|$; $|A_\varepsilon(x) - A_\varepsilon(y)| \le \frac{1}{\varepsilon}|x - y|$,

(iv) $A_\varepsilon(y) \in \partial\varphi(J_\varepsilon(y))$.

(v) $|A_\varepsilon(y)| \nearrow_{\varepsilon\to 0} \begin{cases} |A^0(x)|, & \text{if } x \in \mathcal{O}; \\ +\infty, & \text{otherwise,} \end{cases}$ where $A^0(y) \triangleq Pr_{\partial\varphi(y)}(0)$, $y \in \mathbb{R}^m$. □

Let us now specify a l.s.c. proper convex function to fit our discussion. For any convex, closed subset $\mathcal{O} \subseteq \mathbb{R}^m$, we define its *indicator* function, denoted by $\varphi := I_{\mathcal{O}}$ to be

$$\varphi(y) \triangleq \begin{cases} 0 & y \in \mathcal{O}; \\ +\infty & y \notin \mathcal{O}, \end{cases}$$

In this case, $\mathcal{D}(\varphi) = \mathcal{O}$. Now by definitions (2.8) and (2.9), we have

$$\begin{aligned} \varphi_\varepsilon(y) &= \inf_{x\in\mathcal{O}} \frac{1}{2\varepsilon}|y - x|^2 = \frac{1}{2\varepsilon} d^2(y, \mathcal{O}), \\ A_\varepsilon(y) &= D\varphi_\varepsilon(y) = \frac{1}{2\varepsilon}\nabla d^2(y, \mathcal{O}) = \frac{1}{\varepsilon}(y - Pr(y, \mathcal{O})), \end{aligned}$$

Consequently, we have

$$\begin{cases} J_\varepsilon(y) = Pr(y; \mathcal{O}), & \forall \varepsilon > 0; \\ A_\varepsilon(y) = 0, & \forall y \in \mathcal{O}, \ \forall \varepsilon > 0; \\ A^0(y) = 0, & \forall y \in \mathcal{O}. \end{cases} \tag{2.10}$$

Further, we replace $\mathcal{O}$ by the $(\mathcal{H}_m, d^*)$-valued process $\{\mathcal{O}_2\}$, then

$$\begin{aligned} \varphi_\varepsilon(t, y, \omega) &= \frac{1}{2\varepsilon} I_{\mathcal{O}_2(t,\omega)}(y), \quad \forall \varepsilon > 0; \\ J_\varepsilon(t, y, \omega) &= (I + \varepsilon A(t, \cdot, \omega))^{-1}(y); \\ A_\varepsilon(t, y, \omega) &= \frac{1}{\varepsilon}(y - J_\varepsilon(t, y, \omega)). \end{aligned} \tag{2.11}$$

By (2.10) we know that $J_\varepsilon(t, y, \omega) = Pr(y, \mathcal{O}_2(t, \omega))$, and by assumption (A2) we have that for every $\varepsilon > 0$, $J_\varepsilon(\cdot, y, \cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ for all $y \in \mathbb{R}^m$.

Let us now consider the following approximation of (2.1):

$$
\begin{aligned}
Y^\varepsilon(t) = \xi + \int_t^T h(s, Y^\varepsilon(s), Z^\varepsilon(s))ds - \int_t^T Z^\varepsilon(s)dW(s) \\
- \int_t^T A_\varepsilon(Y^\varepsilon(s))ds,
\end{aligned}
\tag{2.12}
$$

where A_ε is the Yosida approximation of $A(t,\omega) = \partial I_{\mathcal{O}_2(t,\omega)}$ defined by (2.11). Since A_ε is uniform Lipschitz for each fixed ε, by Lemma 2.7-(iii) and by slightly modifying the arguments in Chapter 1, §4 to cope with the current situation where α in (A3) is allowed to be negative, one shows that (2.12) has a unique strong solution $(Y^\varepsilon, Z^\varepsilon)$ satisfying

$$
E\Big\{ \sup_{0\le t\le T} |Y^\varepsilon(t)|^2 + \int_0^T \|Z^\varepsilon(t)\|^2 dt \Big\} < \infty. \tag{2.13}
$$

We will first show that as $\varepsilon \to 0$, $(Y^\varepsilon, Z^\varepsilon)$ converges in a certain sense, then show that the limit will give the solution of (2.12). To begin with, we need some elementary estimates.

Lemma 2.8. *Suppose that condition (A3) holds, and that $\xi \in L^2_{\mathcal{F}_T}(\Omega)$. Then there exists a constant $C > 0$, independent of ε, such that the following estimates hold*

$$
\begin{cases}
E\Big\{ \sup\limits_{t\in[0,T]} |Y^\varepsilon(t)|^2 + \displaystyle\int_0^T |Z^\varepsilon(t)|^2 dt \Big\} \le C; \\
E\Big\{ \displaystyle\int_0^T |A_\varepsilon(t, Y^\varepsilon(t))|^2 dt \Big\} \le C.
\end{cases}
\tag{2.14}
$$

Proof. The proof of the first inequality is quite similar to those we have seen many times before, with the help of the properties of Yosida approximations listed in §2.2, we only prove the second one. First note that since $\mathcal{O}_2$ is convex, so is $\varphi_\varepsilon(t,\cdot,\omega)$ (recall (2.11)). We have the following inequality (suppressing ω):

$$
\varphi_\varepsilon(t,y) + \langle D\varphi_\varepsilon(t,y), \widetilde{y} - y \rangle \le \varphi_\varepsilon(t, \widetilde{y}), \qquad \forall (t,y), \text{ a.s.} \tag{2.15}
$$

Now let $t = t_0 < t_1 < \cdots < t_n = T$ be any partition of $[t,T]$. Then (2.15) leads to that

$$
\begin{aligned}
\varphi_\varepsilon(t_i, Y^\varepsilon(t_i)) + \langle D\varphi_\varepsilon(t_i, Y^\varepsilon(t_i)), Y^\varepsilon(t_{i+1}) - Y^\varepsilon(t_i) \rangle \\
\le \varphi_\varepsilon(t_i, Y^\varepsilon(t_{i+1})) \le \varphi_\varepsilon(t_{i+1}, Y^\varepsilon(t_{i+1})), \qquad \text{a.s.},
\end{aligned}
\tag{2.16}
$$

where the last inequality is due to Assumption (A3)-iv). Summing both sides of (2.16) up and letting the mesh size of the partition $\max_i |t_{i+1} - t_i| \to 0$ we obtain that

$$
\varphi_\varepsilon(t, Y^\varepsilon(t)) + \int_t^T \langle D\varphi_\varepsilon(s, Y^\varepsilon(s)), dY^\varepsilon(s) \rangle \le \varphi_\varepsilon(T, \xi) = 0. \tag{2.17}
$$

Thus, recall the equation for Y^ε we have

$$
\begin{aligned}
&\varphi_\varepsilon(t, Y^\varepsilon(t)) + \frac{1}{\varepsilon}\int_t^T |D\varphi_\varepsilon(Y^\varepsilon(s))|^2 ds \\
&\qquad \le \varphi_\varepsilon(T,\xi) + \int_t^T \langle\, D\varphi_\varepsilon(s, Y^\varepsilon(s)), h(s, Y^\varepsilon(s), Z^\varepsilon(s)) \,\rangle\, ds \\
&\qquad\quad - \int_t^T \langle\, D\varphi_\varepsilon(Y^\varepsilon(s)), Z^\varepsilon dW_s \,\rangle\,.
\end{aligned}
\tag{2.18}
$$

By Cauchy-Schwartz inequality and (A3)-(iii),

$$
\langle\, D\varphi_\varepsilon(t,y), h(t,y,z)\,\rangle \le \frac{1}{2\varepsilon}|D\varphi_\varepsilon(t,y)|^2 + \varepsilon C(1 + \|z\|^2 + |y|^2), \quad \forall (t,y,z).
$$

We now recall that $\varphi_\varepsilon \ge 0$; $\xi \in \mathcal{O}_2(T,\cdot)$ (i.e., $\varphi_\varepsilon(T,\xi) = 0$); and $A_\varepsilon(t,y,\omega) = D\varphi_\varepsilon(t,y,\omega)$. Using the first inequality of this lemma we obtain that

$$
\begin{aligned}
E\int_t^T |A_\varepsilon(t, Y^\varepsilon(s))|^2 ds &= E\int_t^T |D\varphi_\varepsilon(Y^\varepsilon(s))|^2 ds \\
&\le C\Big(1 + E\sup_{t\in[0,T]} |Y^\varepsilon(t)|^2 + E\int_0^T \|Z^\varepsilon(t)\|^2 dt\Big) \le \widetilde{C},
\end{aligned}
$$

where $\widetilde{C} > 0$ is some constant independent of ε. Thus, by a slightly abuse of notations on the constant C, we obtain the desired estimate. □

Lemma 2.9. *Suppose that the assumptions of Lemma 2.8 hold. Then there exists a constant $C > 0$, such that for any $\varepsilon, \delta > 0$, it holds that*

$$
E\Big\{\sup_{t\in[0,T]} |Y^\varepsilon(t) - Y^\delta(t)|^2 + \int_0^T |Z^\varepsilon(t) - Z^\delta(t)|^2 dt\Big\} \le (\varepsilon + \delta) C.
\tag{2.19}
$$

Proof. Applying Itô's formula we get

$$
\begin{aligned}
&|Y^\varepsilon(t) - Y^\delta(t)|^2 + \int_t^T \|Z^\varepsilon(s) - Z^\delta(s)\|^2 ds \\
&\qquad + 2\int_t^T \langle\, A_\varepsilon(s, Y^\varepsilon(s)) - A_\delta(s, Y^\delta(s)), Y^\varepsilon(s) - Y^\delta(s) \,\rangle\, ds \\
&= 2\int_t^T \langle\, h(s, Y^\varepsilon(s), Z^\varepsilon(s)) - h(s, Y^\delta(s), Z^\delta(s)), Y^\varepsilon(s) - Y^\delta(s)\,\rangle ds \\
&\qquad - 2\int_t^T \langle\, Y^\varepsilon(s) - Y^\delta(s), [Z^\varepsilon(s) - Z^\delta(s)] dW(s) \,\rangle\,.
\end{aligned}
\tag{2.20}
$$

Since $A_\varepsilon(t,y,\omega) \in \partial\varphi(J_\varepsilon(y))$, we have by definition that

$$
\langle\, A_\varepsilon(t,y,\omega), J_\varepsilon(t,y,\omega) - x \,\rangle \ge 0, \qquad \forall\, x \in \mathcal{O}_2(t,\omega).
$$

In particular for any $\widetilde{y} \in \mathbb{R}^m$, and any $\delta > 0$, $J_\delta(t,\widetilde{y},\omega) \in \mathcal{O}_2(t,\omega)$ and therefore

$$\langle A_\varepsilon(t,y,\omega), J_\varepsilon(t,y,\omega) - J_\delta(t,\widetilde{y},\omega) \rangle \ge 0, \qquad \forall\, \widetilde{y} \in \mathbb{R}^m, \text{ a.e. } \omega \in \Omega.$$

Similarly,

$$\langle A_\delta(t,\widetilde{y},\omega), J_\delta(t,\widetilde{y},\omega) - J_\varepsilon(t,y,\omega) \rangle \ge 0, \qquad \forall\, y \in \mathbb{R}^m, \text{ a.e. } \omega \in \Omega.$$

Consequently, we have (suppressing ω)

$$\begin{aligned} &\langle A_\varepsilon(t,y) - A_\delta(t,\widetilde{y}), y - \widetilde{y} \rangle \\ &= \langle A_\varepsilon(t,y), [y - J_\varepsilon(t,y)] + [J_\varepsilon(t,y) - J_\delta(t,\widetilde{y})] + J_\delta(t,\widetilde{y}) - \widetilde{y} \rangle \\ &\quad + \langle A_\delta(t,\widetilde{y}), [\widetilde{y} - J_\delta(t,\widetilde{y})] + [J_\delta(t,\widetilde{y}) - J_\varepsilon(t,y)] + J_\varepsilon(t,y) - y \rangle \\ &\ge - \langle A_\varepsilon(t,y), \delta A_\delta(t,\widetilde{y}) \rangle - \langle A_\delta(t,\widetilde{y}), \varepsilon A_\varepsilon(t,y) \rangle \\ &= -(\varepsilon+\delta) \langle A_\varepsilon(t,y), A_\delta(t,\widetilde{y}) \rangle . \end{aligned} \tag{2.21}$$

Also, some standard arguments using Schwartz inequality lead to that

$$2 \langle h(t,y,z) - h(t,\widetilde{y},\widetilde{z}), y - \widetilde{y} \rangle \rangle \le \frac{1}{2}\|z - \widetilde{z}\|^2 + C|y - \widetilde{y}|^2. \tag{2.22}$$

Combining (2.20)—(2.22) and using the Burkholder and Gronwall inequalities we obtain, for some constant $C > 0$,

$$\begin{aligned} &E\Big\{ \sup_{t\in[0,T]} |Y^\varepsilon(t) - Y^\delta(t)|^2 + \int_0^T \|Z^\varepsilon(t) - Z^\delta(t)\|^2 dt \Big\} \\ &\le (\varepsilon+\delta) E \int_0^T \Big| \langle A_\varepsilon(t, Y^\varepsilon(t)), A_\delta(t, Y^\delta(t)) \rangle \Big| dt \\ &\le (\varepsilon+\delta) \Big\{ E \int_0^T |A_\varepsilon(Y^\varepsilon(t))|^2 dt \cdot E \int_0^T |A_\delta(Y^\delta(t))|^2 dt \Big\}^{\frac{1}{2}} \le (\varepsilon+\delta) C, \end{aligned}$$

thanks to (2.14). This proves the Lemma. □

As a direct consequence of Lemma 2.8, we see that if we send ε to zero along an arbitrary sequence $\{\varepsilon_n\}$, then there exist processes $Y \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^m))$, $Z \in L^2_{\mathcal{F}}(\Omega \times [0,T];\mathbb{R}^m))$, independent of the choice of the sequence $\{e_n\}$ chosen, such that

$$(Y^n, Z^n) \stackrel{\Delta}{=} (Y^{\varepsilon_n}, Z^{\varepsilon_n}) \to (Y, Z), \qquad \text{as } n \to \infty,$$

strongly in $L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^m)) \times L^2_{\mathcal{F}}(\Omega \times [0,T];\mathbb{R}^m)$.

Furthermore, by Lemma 2.8 and the equation (2.12), it follows that for some $\eta \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, $\zeta \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^m))$, and possibly along a subsequence which we still denote by $\{\varepsilon_n\}$, it holds that

$$\begin{cases} A_{\varepsilon_n}(Y^{\varepsilon_n}(\cdot)) \to -\eta(\cdot), \qquad \text{weakly in } L^2_{\mathcal{F}}(0,T;\mathbb{R}^m); \\ E\Big\{ \sup_{0\le t\le T} \Big| \int_0^t A_{\varepsilon_n}(Y^{\varepsilon_n}(s)) ds + \zeta(t) \Big|^2 \Big\} \to 0, \qquad \text{as } n \to \infty. \end{cases}$$

Here, we use $-\eta$ and $-\zeta$ to match the signs in (2.1) and (2.12). Obviously, we see that the limiting processes Y, Z, and ζ will satisfy the SDE (2.1), and the proof of Theorem 2.6 will be complete after we prove the following lemma.

Lemma 2.10. *Suppose that the process (Y, Z), η, and ζ are defined as before. Then (Y, Z, ζ) satisfies (2.11), such that*

(i) $E|\zeta|(T) = E\int_0^T |\eta(t)|dt < \infty$;

(ii) $Y(t) \in \mathcal{O}_2(t, \cdot)$, $\forall t \in [0, T]$, a.s.;

(iii) for any RCLL, $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process V, $\langle Y(t) - V(t), \eta(t) \rangle \leq 0$, a.s., as a signed measure.

Proof. (i) We first show that ζ has absolutely continuous paths almost surely and that $\dot{\zeta} = \eta$. To see this, note that η is the weak limit of $A_{\varepsilon_n}(Y^{\varepsilon_n})$'s. By Mazur's theorem, there exists an convex combination of $A_{\varepsilon_n}(Y^{\varepsilon_n})$'s, denoted by $\widetilde{A}_{\varepsilon_n}(Y^{\varepsilon_n})$, such that $\widetilde{A}_{\varepsilon_n}(Y^{\varepsilon_n}) \to \eta$, strongly in $L^2_{\mathcal{F}}(\Omega \times [0, T]; \mathbb{R}^m))$. Note that for this sequence of convex combinations of the sequence $A_{\varepsilon_n}(Y^{\varepsilon_n})$, we also have

$$E\Big\{ \sup_{0\leq t\leq T} \Big| \int_0^t \widetilde{A}_{\varepsilon_n}(Y^{\varepsilon_n}(s))ds + \zeta(t)\Big|^2 \Big\} \to 0, \qquad \text{as } n \to \infty.$$

Thus the uniqueness of the limit implies that $\zeta(t) = \int_0^t \eta(s)ds$, $\forall t \in [0, T]$. Furthermore, since $L^2_{\mathcal{F}}(\Omega) \subseteq L^1(\Omega)$, we derive (i) immediately.

(ii) In what follows we denote $d(y, t, \omega) = d(y, \mathcal{O}_2(t, \omega))$. Since $\mathcal{O}_2(t, \omega)$ is convex for fixed (t, ω), $d(\cdot, \mathcal{O}_2(t, \omega))$ is a convex function. Further, since $\mathcal{O}_2$ has smooth boundary, one derives from (2.9) that

$$d(y, t, \omega) = |y - Pr(y, \mathcal{O}_2(t, \omega))| = |y - J_\varepsilon(y)| = \varepsilon|A_\varepsilon(y, t, \omega)|.$$

for all $y \in \mathbb{R}^m$, and $t \in [0, T]$, P-a.s.. Hence by part (i), we see that

$$\begin{aligned} E\int_0^T d(Y^\varepsilon(t), t, \omega)dt &\leq \varepsilon E\int_0^T |A_\varepsilon(Y^\varepsilon(t))|dt \\ &\leq \varepsilon\sqrt{T} E\Big\{\int_0^T |A_\varepsilon(Y^\varepsilon(t))|^2 dt\Big\}^{\frac{1}{2}} \to 0. \end{aligned} \tag{2.23}$$

Next, define for each $(t, \omega) \in [0, T] \times \Omega$ the *conjugate* function of $d(\cdot, t, \omega)$ by

$$G(z, t, \omega) \stackrel{\Delta}{=} \inf_y \{d(y, t, \omega) - \langle z, y \rangle\}, \tag{2.24}$$

and define the *effective domain* of G by

$$\mathcal{D}^G(t, \omega) = \{z \in \mathbb{R} : \ G(z, t, \omega) > -\infty\}. \tag{2.25}$$

Since $d(\cdot, t, \omega)$ is convex and continuous everywhere, it must be identical to its biconjugate function, or equivalently, its closed convex hull (see

Hiriart-Urruty-Lemaréchal [1]). Consequently, the following conjugate relation holds:

$$d(y,t,\omega)=\sup_{z\in\mathcal{D}^G(t,\omega)}\{G(z,t,\omega)+\langle\, z,y\,\rangle\}; \tag{2.26}$$

and both the infimum of (2.24) and the supremum of (2.26) are achieved for every fixed (t,ω). Now for fixed (t,ω), and any $z_0\in\mathcal{D}^G(t,\omega)$, we let $y_0=y_0(t,\omega)$ be the minimizer in (2.24). Then

$$d(y_0,t,\omega)-\langle\, y_0,z_0\,\rangle=G(z_0,t,\omega)\le d(y,t,\omega)-\langle\, y,z_0\,\rangle,\qquad \forall y\in\mathbb{R}^n,$$

and hence

$$\langle\, y-y_0,z_0\,\rangle\le d(y,t,\omega)-d(y_0,t,\omega),\qquad \forall y\in\mathbb{R}^n.$$

Since it is easily checked that $d(\cdot,t,\omega)$ is uniformly Lipschitz with Lipschitz constant 1, we deduce from above that $|z_0|\le 1$. Namely $\mathcal{D}^G(t,\omega)\subseteq[-1,1]$.

Now let Y be the limit process of Y^{ε_n}, we apply a measurable selection theorem to obtain a (bounded) $\{\mathcal{F}_t\}_{t\ge0}$-adapted process R, such that $R(t,\omega)\in\mathcal{D}^G(t,\omega)\subseteq[-1,1]$, $\forall t$, a.s.; and

$$\begin{cases} d(Y(t,\omega),t,\omega)=G(R(t,\omega),t,\omega)+\langle\, R(t,\omega),Y(t,\omega)),\\ d(Y^{\varepsilon_n}(t,\omega),t,\omega)\ge G(R(t,\omega),t,\omega)+\langle\, R(t,\omega),Y^{\varepsilon_n}(t,\omega)),\end{cases} \tag{2.27}$$

Therefore, recall that $Y^{\varepsilon_n}\to Y$, we have

$$\begin{aligned} E\int_0^T d(Y(t),t,\cdot)dt&=E\int_0^T \Big\{G(Y(t),t,\cdot)+\langle\, R(t),Y(t)\,\rangle\, dt\\ &=\lim_{n\to\infty}E\int_0^T\Big\{G(Y(t),t,\cdot)+\langle\, R(t),Y^n(t)\,\rangle\, dt\\ &\le\lim_{n\to\infty}E\int_0^T d(Y^{\varepsilon_n}(t),t,\cdot)dt=0,\end{aligned}$$

thanks to (2.23). That is, $E\int_0^T d(Y(t),t,\cdot)dt=0$, which implies that $Y(t,\omega)\in\mathcal{O}_2(t,\omega)$, $dt\times d\mathbf{P}$-a.e. Thus the conclusion follows from the continuity of the paths of Y.

(iii) Let $V(t)$ be any $\{\mathcal{F}_t\}_{t\ge0}$-adapted process such that $V(t,\omega)\in\mathcal{O}_2(t,\omega)$, $\forall t\in[0,T]$, P-a.s. For every $\varepsilon>0$, and $t\in[0,T]$, consider

$$\Lambda_\varepsilon(t)=E\int_0^t\langle\, J_\varepsilon(Y^\varepsilon(s))-V(s),A_\varepsilon(Y^\varepsilon(s))\,\rangle\, ds. \tag{2.28}$$

Since $V(t)\in\mathcal{O}_2(t,\cdot)$, for all t, and $A_\varepsilon(Y^\varepsilon(t))\in\partial I_{\mathcal{O}_2(t,\cdot)}(J_\varepsilon(Y^\varepsilon(t))$ (see Lemma 2.7-(iv)), we have

$$\langle\, J_\varepsilon(Y^\varepsilon(t))-V(t),A_\varepsilon(t,Y^\varepsilon(t))\,\rangle\ge 0.$$

Namely, $\Lambda_\varepsilon(t) \geq 0$, $\forall \varepsilon > 0$ and $t \in [0,T]$. On the other hand, since

$$\begin{aligned}\Lambda_\varepsilon(t) &= E\int_0^t \Big\{ \langle J_\varepsilon(Y^\varepsilon(s)) - Y^\varepsilon(s), A_\varepsilon(s, Y^\varepsilon(s)) \rangle \\ &\qquad + \langle Y^\varepsilon(s) - V(s), A_\varepsilon(s, Y^\varepsilon(s)) \rangle \Big\} ds \\ &= E\int_0^t \Big\{ -\varepsilon |A_\varepsilon(s, Y^\varepsilon(s))|^2 + \langle Y^\varepsilon(s) - V(s), A_\varepsilon(s, Y^\varepsilon(s)) \rangle \Big\} ds\end{aligned} \tag{2.29}$$

Now using the uniform boundedness (2.14) and the weak convergence of $\{A_{\varepsilon_n}(\cdot, Y^{\varepsilon_n})(\cdot))\}$, and the fact that Y^{ε_n} converges to Y strongly in $L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R}^m))$, one derives easily by sending $n \to \infty$ in (2.29) that

$$0 \leq E\int_0^t \langle Y(s) - V(s), -\eta(s) \rangle \, ds, \quad \forall t \in [0,T].$$

Or equivalently,

$$\langle Y(t) - V(t), \eta(t) \rangle = \langle Y(t) - V(t), \frac{d\zeta}{dt}(t) \rangle \leq 0, \quad \forall t \in [0,T], \quad \text{a.s.}$$

as a (random) signed measure. Thus completes the proof of Lemma 2.10.

□

§3. Reflected Forward-Backward SDEs

We are now ready to formulate forward-backward SDEs with reflection (FBSDER, for short). Let $\mathcal{O}_1$ be a closed, convex domain in $\mathbb{R}^n$, and $\mathcal{O}_2 = \{\mathcal{O}_2(t,\omega) : (t,\omega) \in [0,T] \times \mathbb{R}^n \times \Omega\}$ be a family of closed, convex domains in $\mathbb{R}^m$. Let $x \in \mathcal{O}_1$, and $g : \mathbb{R}^n \times \Omega \mapsto \mathbb{R}^m$ be a given $\mathcal{F}_T$-measurable random field satisfying

$$g(x,\omega) \in \mathcal{O}_2(T,\omega), \qquad \forall (x,\omega). \tag{3.1}$$

Consider the following FBSDER:

$$\begin{cases} X_t = x + \displaystyle\int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dW_s + \eta_t; \\ Y_t = g(X_T) + \displaystyle\int_t^T h(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + \zeta_T - \zeta_t; \end{cases} \tag{3.2}$$

Definition 3.1. A quintuple of processes (X, Y, Z, η, ζ) is called an adapted solution of the FBSDER (3.2) if

1) $(X,Y) \in L^2_{\mathcal{F}}(\Omega, C(0,T;\mathbb{R}^n \times \mathbb{R}^m))$, $Z \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m\times d})$, $(\eta,\zeta) \in BV_{\mathcal{F}}(0,T;\mathbb{R}^n \times \mathbb{R}^m)$;
2) $X_t \in \mathcal{O}_1$, $Y_t \in \mathcal{O}_2(t,\cdot)$, $\forall t \in [0,T]$, a.s.;
3) $|\eta|_t = \int_0^t \mathbf{1}_{\{X_s \in \partial\mathcal{O}_1\}} d|\eta|_s$; $\eta_t = \int_0^t \gamma_s d|\eta|_s$, $\forall t \in [0,T]$, a.s., for some progressively measurable process γ such that $\gamma_s \in \mathcal{N}_{X_s}(\mathcal{O}_1)$, $d|\eta|$-a.e.;
4) for all RCLL and progressively measurable processes U such that $U_t \in \mathcal{O}_2(t,\cdot)$, $\forall t \in [0,T]$, a.s., one has $\langle Y_t - U_t, d\zeta_t \rangle \leq 0$, $\forall t \in [0,T]$, a.s.;

5) (X, Y, Z, η, ζ) satisfies the SDE (3.2) almost surely.

In light of assumptions (A1)-(A3), we will assume the following

(A4) (i) $\mathcal{O}_1$ has smooth boundary;

(ii) $\mathcal{O}_2(t,\omega) \subseteq \mathcal{O}_2(s,\omega)$, $\forall t \geq s$, a.s.; and for fixed $y \in \mathbb{R}^m$, the mapping $(t,\omega) \mapsto \beta(t,y,\omega) \stackrel{\Delta}{=} Pr(y; \mathcal{O}_2(t,\omega)$ belongs to $L^2_{\mathcal{F}}([0,T];\mathbb{R}^m)$.

(iii) The coefficients b, h, σ, and g are random fields defined on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ such that for fixed (x,y,z), the processes $b(\cdot,x,y,z,\cdot)$, $h(\cdot,x,y,z,\cdot)$, and $\sigma(\cdot,x,y,z,\cdot)$ are $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable, and $g(x,\cdot)$ is $\mathcal{F}_T$-measurable.

(iv) For fixed (t,x,z) and a.e. ω, $h(t,x,\cdot,z,\omega)$ is continuous, and there exists a constant $K>0$ such that $|h(t,x,y,z,\omega)| \leq K(1+|x|+|y|)$, for all (t,x,y,z,ω). Moreover,

$$E\int_0^T |b(t,0,0,0)|^2 dt + E\int_0^T |\sigma(t,0,0,0)|^2 dt + E|g(0)|^2 < \infty.$$

(v) There exist constants $k_i \geq 0, i=1,2$ and $\gamma \in \mathbb{R}$ such that for all $(t,\omega) \in [0,T]\times\Omega$ and $\mathbf{x} \stackrel{\Delta}{=} (x,y,z), \mathbf{x}_i \stackrel{\Delta}{=} (x_i,y_i,z_i) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, $i=1,2$, and $\mathbf{x}^0 \stackrel{\Delta}{=} (x,y)$ for $\mathbf{x} = (x,y,z)$.

- $|b(t,\mathbf{x}_1,\omega) - b(t,\mathbf{x}_2,\omega)| \leq K|\mathbf{x}_1 - \mathbf{x}_2|$;
- $\langle\, h(t,x,y_1,z,\omega) - h(t,x,y_2,z,\omega), y_1 - y_2 \,\rangle \leq \gamma|y_1-y_2|^2$;
- $|h(t,x_1,y,z_1,\omega) - h(t,x_2,y,z_2,\omega)| \leq K(|x_1-x_2| + \|z_1-z_2\|)$;
- $\|\sigma(t,\mathbf{x}_1,\omega) - \sigma(t,\mathbf{x}_2,\omega)\|^2 \leq K^2|\mathbf{x}_1^0 - \mathbf{x}_2^0|^2 + k_1^2\|z_1-z_2\|^2$;
- $|g(x_1,\omega) - g(x_2,\omega)| \leq k_2|x_1-x_2|$.

We should note that if $k_1 = k_2 = 0$, then σ and g are independent of z, just as the many cases we considered before. Therefore, the FBSDE considered in this chapter is more general. We note also that the method presented here should also work when there is no reflection involved (e.g., $\mathcal{O}_1 = \mathbb{R}^n$, $\mathcal{O}_2 \equiv \mathbb{R}^m$).

§3.1. A priori estimates

We first establish a new type of a priori estimates that is different from what we have seen in the previous chapters. To simplify notations we shall denote, for $t\in[0,T)$, $\mathbf{H}(t,T) = L^2_{\mathcal{F}}(t,T;\mathbb{R})$, and let $\mathbf{H^c}(t,T)$ be the subset of $\mathbf{H}(t,T)$ consisting of all continuous processes. For any $\lambda \in \mathbb{R}$, define an equivalent norm on $\mathbf{H}(t,T)$ by:

$$\|\xi\|_{t,\lambda} \stackrel{\Delta}{=} \Big\{E\int_t^T e^{-\lambda s}|\xi(s)|^2 ds\Big\}^{\frac{1}{2}}.$$

Then $\mathbf{H}_\lambda(t,T) \stackrel{\Delta}{=} \{\xi \in \mathbf{H}(t,T) : \|\xi\|_{t,\lambda} < \infty\} = \mathbf{H}(t,T)$. We shall also use the following norm on $\mathbf{H^c}(t,T)$:

$$[\![\xi]\!]_{t,\lambda,\beta} \stackrel{\Delta}{=} e^{-\lambda T}E|\xi_T|^2 + \beta\|\xi\|^2_{t,\lambda}, \qquad \xi \in \mathbf{H^c}(t,T),\ \lambda\in\mathbb{R}, \beta > 0,$$

and denote $\mathbf{H}_{\lambda,\beta}(t,T)$ to be the completion of $\mathbf{H^c}(t,T)$ under norm $[\![\cdot]\!]_{t,\lambda,\beta}$. Then for any λ and β, $\mathbf{H}_{\lambda,\beta}(t,T)$ is a Banach space. Further, if $t=0$, we simply denote $\|\cdot\|_\lambda \stackrel{\Delta}{=} \|\cdot\|_{0,\lambda}$; $[\![\cdot]\!]^2_{\lambda,\beta} \stackrel{\Delta}{=} [\![\cdot]\!]^2_{0,\lambda,\beta}$; $\mathbf{H} = \mathbf{H}(0,T)$; $\mathbf{H^c} = \mathbf{H^c}(0,T)$; $\mathbf{H}_\lambda = \mathbf{H}_\lambda(t,T)$, and $\mathbf{H}_{\lambda,\beta} = \mathbf{H}_{\lambda,\beta}(t,T)$.

Moreover, the following functions will be frequently used in this section: for $\lambda \in \mathbb{R}$ and $t \in [0,T]$,

$$A(\lambda,t) = e^{-(\lambda\wedge 0)t}; \quad B(\lambda,t) = \frac{1-e^{-\lambda t}}{\lambda} = t\int_0^1 e^{-\lambda t\theta}d\theta. \tag{3.3}$$

It is easy to see that, for all $\lambda \in \mathbb{R}^n$, $B(\lambda,\cdot)$ is a nonnegative, increasing function, $A(\lambda,t) \ge 1$; and $B(\lambda,0)=0$, $A(\lambda,0)=1$.

Lemma 3.2. *Let (A4) hold. Let (X,Y,Z,η,ζ) and (X',Y',Z',η',ζ') be two solutions to the FBSDER (3.2), and let $\widehat{\xi} \stackrel{\Delta}{=} \xi - \xi'$, where $\xi = X,Y,Z,\eta,\zeta$, respectively.*

(i) Let $\lambda \in \mathbb{R}, C_1, C_2 > 0$, and let $\bar\lambda_1 = \lambda - K(2 + C_1^{-1} + C_2^{-1}) - K^2$. Then, for all $\lambda' \in \mathbb{R}$,

$$\begin{aligned} &e^{-\lambda t}E|\widehat{X}_t|^2 + (\bar\lambda_1 - \lambda')\int_0^t e^{-\lambda\tau}e^{-\lambda'(t-\tau)}E|\widehat{X}_\tau|^2 d\tau \\ &\le \int_0^t e^{-\lambda\tau}e^{-\lambda'(t-\tau)}\{K(C_1+K)E|\widehat{Y}_\tau|^2 + (KC_2 + k_1^2)E|\widehat{Z}_\tau|^2\}d\tau. \end{aligned} \tag{3.4}$$

(ii) Let $\lambda \in \mathbb{R}$ and $C_3, C_4 > 0$, and let $\bar\lambda_2 = -\lambda - 2\gamma - K(C_3^{-1} + C_4^{-1})$. Then, for all $\lambda' \in \mathbb{R}$,

$$\begin{aligned} &e^{-\lambda t}E|\widehat{Y}_t|^2 + (\bar\lambda_2 - \lambda')\int_t^T e^{-\lambda\tau}e^{-\lambda'(\tau-t)}E|\widehat{Y}_\tau|^2 d\tau \\ &\qquad + (1 - KC_4)\int_t^T e^{-\lambda\tau}e^{-\lambda'(\tau-t)}E|\widehat{Z}_\tau|^2 d\tau \\ &\le k_2^2 e^{-\lambda T}e^{-\lambda'(T-t)}E|\widehat{X}_T|^2 + KC_3\int_t^T e^{-\lambda\tau}e^{-\lambda'(\tau-t)}|\widehat{X}_\tau|^2 d\tau \end{aligned} \tag{3.5}$$

Consequently, if $KC_4 = 1-\alpha$ for some $\alpha \in (0,1)$, then

$$e^{-\lambda T}E|\widehat{X}_T|^2 + \bar\lambda_1\|\widehat{X}\|^2_\lambda \le K(C_1+K)\|\widehat{Y}\|^2_\lambda + (KC_2 + k_1^2)\|\widehat{Z}\|^2_\lambda. \tag{3.6}$$

$$\|\widehat{X}\|^2_\lambda \le B(\bar\lambda_1,T)[K(C_1+K)\|\widehat{Y}\|^2_\lambda + (KC_2+k_1^2)\|\widehat{Z}\|^2_\lambda]. \tag{3.7}$$

$$\|\widehat{Y}\|^2_\lambda \le B(\bar\lambda_2,T)[k_2^2 e^{-\lambda T}E|\widehat{X}_T|^2 + KC_3\|\widehat{X}\|^2_\lambda], \tag{3.8}$$

$$\|\widehat{Z}\|^2_\lambda \le \frac{A(\bar\lambda_2,T)}{\alpha}[k_2^2 e^{-\lambda T}E|\widehat{X}_T|^2 + KC_3\|\widehat{X}\|^2_\lambda]. \tag{3.9}$$

Proof. We first show (3.4). Let $t \in (0,T]$, λ,λ' be arbitrarily given, and consider the function $F_t(s,x) \stackrel{\Delta}{=} e^{-\lambda s}e^{-\lambda'(t-s)}|x|^2$, for $(s,x) \in [0,t]\times\mathbb{R}^n$.

Applying Itô's formula to $F_t(s,\widehat{X}_s)$ from 0 to t, and then taking expectation we have

$$\begin{aligned}
&e^{-\lambda t}E|\widehat{X}_t|^2+(\lambda-\lambda')E\int_0^t e^{-\lambda\tau}e^{-\lambda'(t-\tau)}|\widehat{X}_\tau|^2d\tau\\
&=\int_0^t e^{-\lambda\tau}e^{-\lambda'(t-\tau)}\Big\{2\,\langle\,\widehat{X}_\tau,b(\tau,X_\tau,Y_\tau,Z_\tau)-b(\tau,X'_\tau,Y'_\tau,Z'_\tau)\,\rangle\\
&\quad+\|\sigma(\tau,X_\tau,Y_\tau,Z_\tau)-\sigma(\tau,X'_\tau,Y'_\tau,Z'_\tau)\|^2\Big\}d\tau\\
&\quad+2E\int_0^t e^{-\lambda\tau}e^{-\lambda'(t-\tau)}\,\langle\,\widehat{X}_\tau,d\widehat{\eta}_\tau\,\rangle\,.
\end{aligned}$$

Since $X_t,X'_t\in\overline{\mathcal{O}}_1$, $\forall t\in[0,T]$, a.s., we derive from Definition 3.1-(3) that $e^{-\lambda t}e^{-\lambda'(t-\tau)}\,\langle\,\widehat{X}_t,d\widehat{\eta}_\tau\,\rangle\le 0$ (as a signed measure), $\forall s\in[0,T]$, a.s.. Therefore, repeatedly applying the Schwartz inequality and the inequality $2ab\le ca^2+c^{-1}b^2$, $\forall c>0$, using the definition of $\bar\lambda_1$, together with some elementary computation with the help of (A4), we derive (3.4).

To prove (3.5), we let $\widetilde{F}_t(s,x)=e^{-\lambda s}e^{-\lambda'(s-t)}|x|^2$, and apply Itô's formula to $\widetilde{F}_t(s,Y_s)$ from t to T to get

$$\begin{aligned}
&e^{-\lambda t}E|\widehat{Y}_t|^2+(\lambda'+\lambda)E\int_t^T e^{-\lambda\tau}e^{-\lambda'(\tau-t)}|\widehat{Y}_\tau|^2d\tau\\
&\qquad+E\int_t^T e^{-\lambda\tau}e^{-\lambda'(\tau-t)}\|\widehat{Z}_\tau\|^2d\tau\\
&=e^{-\lambda T}e^{-\lambda'(T-t)}E|g(X_T)-g(X'_T)|^2\\
&\quad+2\int_t^T e^{-\lambda'(\tau-t)}e^{-\lambda\tau}\,\langle\,\widehat{Y}_\tau,h(\tau,X_\tau,Y_\tau,Z_\tau)-h(\tau,X'_\tau,Y'_\tau,Z'_\tau)\,\rangle\,d\tau\\
&\quad+2E\int_t^T e^{-\lambda'(\tau-t)}e^{-\lambda\tau}\,\langle\,\widehat{Y}_\tau,d\widehat{\zeta}_\tau\,\rangle\,.
\end{aligned}$$

Again, since $Y(t,\cdot),Y'(t,\cdot)\in\mathcal{O}_2(t,\cdot)$, P-a.s., by Definition 3.1-(4) we have $\langle\,\widehat{Y}_t(\omega),d\widehat{\zeta}_t(\omega)\,\rangle\le 0$, $dt\times dP$-a.s.. Thus, by using the similar argument as before, and using the definition of $\bar\lambda_2$, we obtain (3.5).

Now, letting $\lambda'=0$ and $t=T$ in (3.4) yields (3.6); letting $\lambda'=\bar\lambda_1$ in (3.4) and then integrating both sides from 0 to T yields (3.7), since $B(\lambda_1,\cdot)$ is increasing; letting $\lambda'=\bar\lambda_2$ in (3.5) and integrating from 0 to T yields (3.8). Finally, note that if $\bar\lambda_2\le 0$, then letting $\lambda'=\bar\lambda_2$ and $t=0$ in (3.5) one has (remember $KC_4=1-\alpha$)

$$\begin{aligned}
\|\widehat{Z}\|_\lambda^2&\le\int_0^T e^{-\lambda\tau}e^{-\bar\lambda_2(\tau-t)}\|\widehat{Z}_\tau\|^2d\tau\\
&\le\frac{e^{|\bar\lambda_2|T}}{\alpha}\Big\{k_2^2e^{-\lambda T}E|\widehat{X}_T|^2+KC_3\|\widehat{X}\|_\lambda^2\Big\};
\end{aligned}$$

while if $\bar{\lambda}_2 > 0$, then let $\lambda' = 0$ in (3.4) one has

$$\|\widehat{Z}\|_\lambda^2 \le \frac{1}{\alpha}\Big\{k_2^2 e^{-\lambda T} E|\widehat{X}_T|^2 + KC_3\|\widehat{X}\|_\lambda^2\Big\}.$$

Combining the above we obtain (3.9). □

We now present another set of useful a priori estimates for the adapted solution to FBSDER (3.2). Denote $\sigma^0(t,\omega) = \sigma(s,0,0,0,\omega)$, $f^0(t,\omega) = f(s,0,0,0,\omega)$, $h^0(t,\omega) = h(t,0,0,0,\omega)$, and $g^0(\omega) = g(0,\omega)$.

Lemma 3.3. *Assume (A4). Let (X,Y,Z,η,ζ) be an adapted solution to the FBSDER (3.2). For any $\lambda, \lambda' \in \mathbb{R}, \varepsilon > 0, C_1, C_2, C_3, C_4 > 0$, we define $\bar{\lambda}_1^\varepsilon = \bar{\lambda}_1 - (1+K^2)\varepsilon$ and $\bar{\lambda}_2^\varepsilon = \bar{\lambda}_2 - \varepsilon$, where $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are those defined in Lemma 3.2. Then*

$$\begin{aligned}
&e^{-\lambda t}E|X_t|^2 + (\bar{\lambda}_1^\varepsilon - \lambda')\int_0^t e^{-\lambda'(t-\tau)}e^{-\lambda s}E|X_\tau|^2 d\tau \le e^{-\lambda' t}|x|^2 \\
&\quad + \int_0^t e^{-\lambda'(t-\tau)}e^{-\lambda\tau}\Big\{\frac{1}{\varepsilon}E|f(\tau,0,0,0)|^2 + \Big(1+\frac{1}{\varepsilon}\Big)|\sigma(\tau,0,0,0)|^2 \\
&\quad + K(C_1 + K(1+\varepsilon))E|Y_\tau|^2 + (KC_2 + k_1^2(1+\varepsilon))E|Z_\tau|^2\Big\}d\tau.
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&e^{-\lambda t}E|Y_t|^2 + (\bar{\lambda}_2^\varepsilon - \lambda')\int_t^T e^{-\lambda'(\tau-t)}e^{-\lambda\tau}E|Y_\tau|^2 d\tau \\
&\qquad + (1 - k_4C_4)\int_t^T e^{-\lambda'(\tau-t)}e^{-\lambda\tau}E|Z_\tau|^2 d\tau \\
&\le k_2^2(1+\varepsilon)e^{-\lambda'(T-t)}e^{-\lambda T}E|X_T|^2 + \Big(1+\frac{1}{\varepsilon}\Big)e^{-\lambda'(T-t)}e^{-\lambda T}E|g(0)|^2 \\
&\quad + \int_t^T e^{-\lambda'(\tau-t)}e^{-\lambda\tau}\Big\{KC_3E|X_\tau|^2 + \frac{1}{\varepsilon}E|h(\tau,0,0,0)|^2\Big\}d\tau.
\end{aligned} \tag{3.11}$$

Consequently, if $C_4 = \frac{1-\alpha}{K}$, for some $\alpha \in (0,1)$, we have

$$\begin{aligned}
&e^{-\lambda T}E|X_T|^2 + \bar{\lambda}_1^\varepsilon\|X\|_\lambda^2 \le \Big[|x|^2 + K(C_1 + K(1+\varepsilon))\|Y\|_\lambda^2 \\
&\qquad + (KC_2 + k_1^2(1+\varepsilon))\|Z\|_\lambda^2 + \frac{1}{\varepsilon}\|f^0\|_\lambda^2 + \Big(1+\frac{1}{\varepsilon}\Big)\|\sigma^0\|_\lambda^2\Big].
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\|X\|_\lambda^2 \le B(\bar{\lambda}_1^\varepsilon, T)\Big[|x|^2 + K(C_1 + K(1+\varepsilon))\|Y\|_\lambda^2 \\
&\qquad + (KC_2 + k_1^2(1+\varepsilon))\|Z\|_\lambda^2 + \frac{1}{\varepsilon}\|f^0\|_\lambda^2 + \Big(1+\frac{1}{\varepsilon}\Big)\|\sigma^0\|_\lambda^2\Big].
\end{aligned} \tag{3.13}$$

$$\|Y\|_\lambda^2 \le B(\bar\lambda_2^\varepsilon, T)\Big[k_2^2(1+\varepsilon)e^{-\lambda T}E|X_T|^2 + KC_3\|X\|_\lambda^2 + \Big(1+\frac{1}{\varepsilon}\Big)e^{-\lambda T}E|g^0|^2 + \frac{1}{\varepsilon}\|h^0\|_\lambda^2\Big] \tag{3.14}$$

$$\|Z\|_\lambda^2 \le \frac{A(\bar\lambda_2^\varepsilon, T)}{\alpha}\Big[k_2^2(1+\varepsilon)e^{-\lambda T}E|X_T|^2 + KC_3\|X\|_\lambda^2 + \Big(1+\frac{1}{\varepsilon}\Big)e^{-\lambda T}E|g^0|^2 + \frac{1}{\varepsilon}\|h^0\|^2\Big]. \tag{3.15}$$

§3.2. Existence and uniqueness of the adapted solutions

We are now ready to study the well-posedness of the FBSDER (3.2). To begin with we introduce a mapping $\Gamma : \mathbf{H^c} \mapsto \mathbf{H^c}$ defined as follows: for fixed $x \in \mathbb{R}^n$, let $\overline{X} \stackrel{\Delta}{=} \Gamma(X)$ be the solution to the FSDER:

$$\overline{X}_t = x + \int_0^t b(s, \overline{X}_s, Y_s, Z_s)ds + \int_0^t \sigma(s, \overline{X}_s, Y_s, Z_s)dW_s + \overline{\eta}_t, \tag{3.16}$$

where the processes Y and Z are the solution to the following BSDER:

$$Y_t = g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + \zeta_T - \zeta_t. \tag{3.17}$$

Clearly, the assumption (A4) enables us to apply Theorem 2.5 to conclude that the BSDER (3.17) has a unique solution (Y, Z, ζ), which in turn guarantees the existence and uniqueness of the adapted solution $\overline{X}$ to the FSDER (3.16), thanks to Theorem 1.2. Furthermore, by definition of λ_1^ε (Lemma 3.3) we see that if λ is chosen so that $\bar\lambda_1 > 0$, then it is always possible to choose $\varepsilon > 0$ small enough so that $\bar\lambda_1^\varepsilon > 0$ as well; and (3.12) will lead to $\overline{X} \in \mathbf{H}_{\lambda,\bar\lambda_1}$ (since $\bar\lambda_1 > 0$ and $\bar\lambda_1^\varepsilon > 0$). Let us try to find a suitable $\bar\lambda_1 > 0$ so that Γ is a contraction on $\mathbf{H}_{\lambda,\bar\lambda_1}$, which will lead to the existence and uniqueness of the adapted solution to the FBSDER (3.2) immediately.

To this end, let $X^1, X^2 \in \mathbf{H^c}$; and let (Y^i, Z^i, ζ^i) and $(\overline{X}^i, \overline{\eta}^i)$, $i = 1, 2$, be the corresponding solutions to (3.17) and (3.16), respectively. Denote $\Delta\xi = \xi^1 - \xi^2$, for $\xi = X, Y, Z, \overline{X}$. Applying (3.6)–(3.9) (with $C_4 = \frac{1-\alpha}{K}$) we easily deduce that

$$e^{-\lambda T}E|\Delta\overline{X}_T|^2 + \bar\lambda_1\|\Delta\overline{X}\|_\lambda^2 \le \mu(\alpha, T)\{k_2^2 e^{-\lambda T}E|\Delta X_T|^2 + KC_3\|\Delta X\|_\lambda^2\}. \tag{3.18}$$

where

$$\mu(\alpha, T) \stackrel{\Delta}{=} K(C_1 + K)B(\bar\lambda_2, T) + \frac{A(\bar\lambda_2, T)}{\alpha}(KC_2 + k_1^2); \tag{3.19}$$

and (recall Lemma 3.2)

$$\bar\lambda_1 = \lambda - K(2 + C_1^{-1} + C_2^{-1}) - K^2; \quad \bar\lambda_2 = -\lambda - 2\gamma - K(C_3^{-1} + C_4^{-1}). \tag{3.20}$$

Clearly, the function $\mu(\cdot,\cdot)$ depends on the constants K, k_1, k_2, γ, the duration $T > 0$, and the choice of C_1–C_4 as well as λ, α. To compensate the generality of the coefficients, we shall impose the following *compatibility conditions*.

(**C-1**) $0 \le k_1 k_2 < 1$;

(**C-2**) $k_2 = 0$; $\exists \alpha \in (0,1)$ such that $\mu(\alpha, T)KC_3 < \bar{\lambda}_1$,

(**C-3**) $k_2 > 0$; $\exists \alpha_0 \in (k_1 k_2, 1)$, such that $\mu(\alpha_0^2, T)k_2^2 < 1$ and $\bar{\lambda}_1 = \frac{KC_3}{k_2^2}$.

We remark here that the compatibility condition (C-1) is not a surprise. We already saw it in Chapter 1 (Theorem 1.5.1). In fact, in Example 1.5.2 we showed that such a condition is almost necessary for the solvability of an FBSDE with general coefficients, even in non-reflected cases with small duration. The first existence and uniqueness result for FBSDER (3.2) is the following.

Theorem 3.4. *Assume (A4) and fix $C_4 = \frac{1-\alpha_0^2}{K}$. Assume that the compatibility conditions (C-1), and either (C-2) or (C-3) hold for some choices of constants λ, α, and C_1–C_3. Then the FBSDER (3.2) has a unique adapted solution over $[0,T]$.*

Proof. Fix $C_4 = \frac{1-\alpha_0^2}{K}$. First assume that (C-1) and (C-2) hold. Since $k_2 = 0$, (3.18) leads to that

$$\|\Delta \overline{X}\|_\lambda^2 \le \frac{\mu(\alpha, T)KC_3}{\bar{\lambda}_1}\|\Delta X\|_\lambda^2,$$

Since we can find C_1—C_3 and $\alpha \in (0,1)$ so that $\mu(\alpha,T)KC_3 < 1$, Γ is a contraction mapping on $(H, \|\cdot\|_\lambda)$. The theorem follows.

Similarly, if (C-1) and (C-3) hold, then we can solve λ from (3.20) and $\bar{\lambda}_1 = KC_3/k_2^2$, and then derive from (3.18) that

$$|\!|\!| \Delta \overline{X} |\!|\!|_{\lambda^0, \bar{\lambda}_1}^2 \le \mu(\alpha_0^2, T)k_2^2 |\!|\!| \Delta X |\!|\!|_{\lambda^0, \bar{\lambda}_1}^2,$$

Let C_i, $i = 1,2,3$ and $\alpha_0 \in (k_1 k_2, 1)$ be such that $\mu(\alpha_0^2, T)k_2^2 < 1$, the mapping Γ is again a contraction, but on the space $\mathbf{H}_{\lambda, \bar{\lambda}_1}$, proving the theorem again. □

A direct consequence of Theorem 3.4 is the following.

Corollary 3.5. *Assume (A4) and the compatibility condition (C-1). Then there exists $T_0 > 0$ such that for all $T \in (0, T_0]$, the FBSDER (3.2) has a unique adapted solution.*

In particular, if either $k_1 = 0$ or $k_2 = 0$, then the FBSDER (3.2) is always uniquely solvable on $[0,T]$ for T small.

Proof. First assume $k_2 = 0$. In light of Theorem 3.4 we need only show that there exists $T_0 = T_0(C_1, C_2, C_3, \lambda, \alpha)$ such that (C-2) holds for some choices of C_1–C_3 and λ, α, for all $T \in (0, T_0]$.

For fixed C_1, C_2, C_3, λ, and $\alpha \in (0,1)$ we have from (3.19) that

$$\mu(\alpha, 0)KC_3 = \frac{(KC_2 + k_1^2)KC_3}{\alpha}.$$

Therefore, let C_1–C_3 and α be fixed we can choose λ large enough so that $\mu(\alpha, 0)KC_3 < \bar{\lambda}_1$ holds. Then, by the continuity of the functions $A(\alpha, \cdot)$ and $B(\alpha, \cdot)$, for this fixed λ we can find $T_0 > 0$ such that $\mu(\alpha, T)KC_3 < \bar{\lambda}_1$ for all $T \in (0, T_0]$. Thus (C-2) holds for all $T \in (0, T_0]$ and the conclusion follows from Theorem 3.4.

Now assume that $k_2 > 0$. In this case we pick an $\alpha_0 \in (k_1 k_2, 1)$, and define

$$\delta \stackrel{\Delta}{=} \frac{1}{k_2^2} - \frac{k_1^2}{\alpha_0^2} > 0. \tag{3.21}$$

Now let $C_2 = \frac{\alpha_0^2 \delta}{2K}$, $C_4 = \frac{1-\alpha_0^2}{K}$, and choose λ so that $\bar{\lambda}_1 = (k_3 C_3)/k_2^2 > 0$. Since in this case we have

$$\mu(\alpha_0^2, 0) = \frac{KC_2 + k_1^2}{\alpha_0^2} = \frac{1}{2k_2^2} + \frac{k_1^2}{2\alpha_0^2} < \frac{1}{k_2^2},$$

thanks to (3.21). Using the continuity of $\mu(\alpha_0^2, \cdot)$ again, for any $C_1, C_3 > 0$ we can find $T_0(C_1, C_3) > 0$ such that $\mu(\alpha_0^2, T)k_2^2 < 1$ for all $T \in (0, T_0]$. In other words, the compatibility condition (C-3) holds for all $T \in (0, T_0]$, proving our assertion again.

Finally if $k_1 = 0$, then (C-1) becomes trivial, thus the corollary always holds. $\square$

From the proofs above we see that there is actually room for one to play with constant C_1–C_3 to improve the "maximum existence interval" $[0, T_0)$. A natural question is then *is there any possibility that $T_0 = \infty$ so that the FBSDER (3.2) is solvable over arbitrary duration $[0, T]$?* Unfortunately, so far we have not seen an affirmative answer for such a question, even in the non-reflecting case, under this general setting. Furthermore, in the reflecting case, even if we assume all the coefficients are deterministic and smooth, it is still far from clear that we can successfully apply the method of optimal control or Four Step Scheme (Chapters 3 and 4) to solve an FBSDER, because the corresponding PDE will become a quasilinear variational inequality, thus seeking its classical solution becomes a very difficult problem in general.

We nevertheless have the following result that more or less covers a class of FBSDERs that are solvable over arbitrary durations.

Theorem 3.6. *Assume (A4) and the compatibility condition (C-1). Then there exists a constant $\Lambda > 0$, depending only on the constants K, k_1, k_2, such that whenever $\gamma < -\Lambda$, the FBSDER (3.2) has a unique adapted solution for all $T > 0$.*

Proof. We shall prove that either (C-2) or (C-3) will hold for all $T > 0$ provided γ is negative enough, and we shall determine the constant Λ in each case, separately.

First assume $k_2 = 0$. In this case let us consider the following minimization problem with constraints:

$$\min_{\substack{C_i>0,\ i=1,2,3;\bar\lambda_1>0,0<\alpha<1,\\ \bar\lambda_1-2K(KC_2+k_1^2)C_3>0}} F(C_1,C_2,C_3,\bar\lambda_1,\alpha), \tag{3.22}$$

where

$$\begin{aligned} F(C_1,C_2,C_3,\bar\lambda_1,\alpha) &\stackrel{\Delta}{=} \frac{(C_1+K)K^2C_3}{\bar\lambda_1-2(KC_2+k_1^2)KC_3}+\bar\lambda_1 \\ &\quad + K(2+C_1^{-1}+C_2^{-1}+C_3^{-1})+K^2\Big(\frac{2-\alpha}{1-\alpha}\Big). \end{aligned} \tag{3.23}$$

Let Λ be the value of the problem (3.22) and (3.23). We show that if $\gamma < -\Lambda/2$, then (C-2) holds for all $T>0$.

Indeed, if $\gamma<-\Lambda/2$, then we can find $C_1,C_2,C_3,\bar\lambda_1>0$ and $\alpha\in(0,1)$, such that $\bar\lambda_1-2(KC_2+k_1^2)KC_3>0$, and

$$\begin{aligned} -2\gamma &> \frac{(C_1+K)K^2C_3}{\bar\lambda_1-2(KC_2+k_1^2)KC_3}+\bar\lambda_1 \\ &\quad + K(2+C_1^{-1}+C_2^{-1}+C_3^{-1})+K^2\Big(\frac{2-\alpha}{1-\alpha}\Big). \end{aligned} \tag{3.24}$$

On the other hand, eliminating λ in the expressions of $\bar\lambda_1$ and $\bar\lambda_2$ in (3.20), and letting $C_4=\frac{(1-\alpha)}{K}$ we have

$$\bar\lambda_2 = -\Big(\bar\lambda_1+K(2+C_1^{-1}+C_2^{-1}+C_3^{-1})+\frac{K^2}{1-\alpha}+K^2\Big)-2\gamma.$$

Thus (3.24) is equivalent to

$$\frac{1}{\bar\lambda_1}\Big\{\frac{K(C_1+K)}{\bar\lambda_2}+\frac{(KC_2+k_1^2)}{\alpha}\Big\}KC_3<1, \tag{3.25}$$

and $\bar\lambda_2>0$. Consequently, $A(\bar\lambda_2,T)=1$ and $B(\bar\lambda_2,T)\le\bar\lambda_2^{-1}$ (recall (3.3)); and (3.25) implies that $\mu(\alpha,T)KC_3<\bar\lambda_1$, i.e., (C-2) holds for all $T>0$.

Now assume $k_2>0$. Following the arguments in Corollary 3.5 we choose $\bar\lambda_1=\frac{KC_3}{k_2^2}>0$, $C_4=\frac{1-\alpha_0^2}{K}$, and $\alpha_0\in(k_1k_2,1)$. Let $\delta>0$ be that defined by (3.21), and consider the minimization problem:

$$\min_{\substack{C_i>0,\ i=1,2,3;\\ \delta\alpha_0^2-KC_2>0}} \widetilde F(C_1,C_2,C_3), \tag{3.26}$$

where

$$\begin{aligned} \widetilde F(C_1,C_2,C_3) &\stackrel{\Delta}{=} \frac{\alpha_0^2K(C_1+K)}{\delta\alpha_0^2-KC_2}+K(2+C_1^{-1}+C_2^{-1}+C_3^{-1}) \\ &\quad + \frac{KC_3}{k_2^2}+K^2\Big(\frac{2-\alpha_0^2}{1-\alpha_0^2}\Big). \end{aligned} \tag{3.27}$$

Let Λ be the value of the problem (3.26) and (3.27), one can show as in the previous case that if $\gamma < -\Lambda/2$, then $\bar{\lambda}_2 > 0$ (hence $A(\bar{\lambda}_2, T) = 1$ and $B(\bar{\lambda}_2, T) \le \bar{\lambda}_2^{-1}$), and $\mu(\alpha_0^2, T)k_2^2 < 1$. Namely (C-3) holds for all $T > 0$. Combining the above we proved the theorem. $\square$

§3.3. A continuous dependence result

In many applications one would like to study the dependence of the adapted solution of an FBSDE on the initial data. For example, suppose that there exists a constant $T > 0$ such that the FBSDER (3.2) is uniquely solvable over any duration $[t, T] \subseteq [0, T]$, and denote its adapted solution by $(X^{t,x}, Y^{t,x}, Z^{t,x}, \eta^{t,x}, \zeta^{t,x})$. Then an interesting question would be how the random field $(t, x) \mapsto (X^{t,x}, Y^{t,x}, Z^{t,x}, \eta^{t,x}, \zeta^{t,x})$ behaves. Such a behavior is particularly useful when one wants to relate an FBSDE to a partial differential equation, as we shall see in the next chapter.

In what follows we consider only the case when $m = 1$, namely, the BSDER is one dimensional. We shall also make use of the following assumption:

(A5) (i) The coefficients b, h, σ, g are deterministic;

(ii) The domains $\{\mathcal{O}_2(\cdot,\cdot)\}$ are of the form $\mathcal{O}(s, \omega) = \mathcal{O}_2(s, X^{t,x}(s, \omega))$, $(s, \omega) \in [t, T] \times \mathbb{R}^n$, where $\mathcal{O}_2(t, x) = (L(t, x), U(t, x))$, where $L(\cdot,\cdot)$ and $U(\cdot,\cdot)$ are smooth deterministic functions of (t, x).

We note that the part (ii) of assumption (A5) does not cover, and is not covered by, the assumption (A4) with $m = 1$. This is because when $m = 1$ the domain $\mathcal{O}_2$ is simply an interval, and can be handled differently from the way we presented in §2 (see, e.g., Cvitanic & Karatzas [1]). Note also that if we can bypass §2 to derive the solvability of BSDERs, then the method we presented in the current section should always work for the solvability for FBSDERs. Therefore in what follows we shall discuss the continuous dependence in an a priori manner, without going into the details of existence and uniqueness again. Next, observe that under (A5) FBSDER (3.2) becomes "Markovian", we can apply the standard technique of "time shifting" to show that the process $\{Y^{t,x}(s)\}_{s \ge t}$ is $\mathcal{F}_t^s$-adapted, where $\mathcal{F}_s^t = \sigma\{W_r, t \le r \le s\}$. Consequently an application of the Blumenthal 0-1 law leads to that the function $u(t, x) = Y_t^{t,x}$ is always deterministic!

In what follows we use the convention that $X^{t,x}(s) \equiv x$, $Y^{t,x}(s) \equiv Y^{t,x}(t)$, and $Z^{t,x}(s) \equiv 0$, for $s \in [0, t]$. Our main result of this subsection is the following.

Theorem 3.7. *Assume (A5) as well as (A4)-(iii)–(v). Assume also that the compatibility conditions (C-1) and either (C-2) or (C-3) hold. Let* $u(t, x) \stackrel{\Delta}{=} Y_t^{t,x}$, $(t, x) \in [0, T] \times \mathcal{O}_1$. *Then* u *is continuous on* $[0, T] \times \mathcal{O}$ *and there exists* $C > 0$ *depending only on* T, b, h, g, *and* σ, *such that the following estimate holds:*

$$
(3.28)\quad |u(t_1, x_1) - u(t_2, x_2)|^2 \le C\Big(|x_1 - x_2|^2 + (1 + |x_1|^2 \vee |x_2|^2)|t_1 - t_2|\Big).
$$

Proof. The proof is quite similar to that of Theorem 3.4, so we only sketch it.

Let (t_1, x_1) and (t_2, x_2) be given, and let $\widehat{X} = X^{t_1,x_1} - X^{t_2,x_2}$. Assume first $t_1 \ge t_2$, and recall the norms $\|\cdot\|_{t,\lambda}$ and $|\!|\!|\cdot|\!|\!|_{t,\lambda,\beta}$ at the beginning of §3.1. Repeating the arguments of Theorem 3.4 over the interval $[t_2, T]$, we see that (3.8) and (3.9) will look the same, with $\|\cdot\|_\lambda$ being replaced by $\|\cdot\|_{t_2,\lambda}$; but (3.6) and (3.7) become

$$\begin{aligned} &e^{-\lambda T} E|\widehat{X}_T|^2 + \bar{\lambda}_1 \|\widehat{X}\|^2_{t_1,\lambda} \\ \le &K(C_1+K)\|\widehat{Y}\|^2_{t_2,\lambda} + (KC_2 + k_1^2)\|\widehat{Z}\|^2_{t_2,\lambda} + E|\widehat{X}(t_2)|^2. \end{aligned} \tag{3.6$'$}$$

$$\begin{aligned} \|\widehat{X}\|^2_{t_2,\lambda} \le &\widetilde{B}(\bar{\lambda}_1, T)[K(C_1+K)\|\widehat{Y}\|^2_{t_2,\lambda} \\ &+ (KC_2 + k_1^2)\|\widehat{Z}\|^2_{t_2,\lambda} + E|\widehat{X}(t_2)|^2], \end{aligned} \tag{3.7$'$}$$

where $\widetilde{B}(\lambda, T) \stackrel{\Delta}{=} \frac{e^{-\lambda t_2} - e^{-\lambda T}}{\lambda}$. Now similar to (3.18), one shows that

$$\begin{aligned} &e^{-\lambda T} E|\widehat{X}_T|^2 + \bar{\lambda}_1 \|\widehat{X}\|^2_{t_2,\lambda} \\ \le &\mu(\alpha, T)\{k_2^2 e^{-\lambda T} E|\widehat{X}_T|^2 + KC_3\|\widehat{X}\|^2_{t_2,\lambda}\} + E|\widehat{X}(t_2)|^2. \end{aligned} \tag{3.18$'$}$$

Arguing as in the proof of Theorem 3.4 and using compatibility conditions (C-1)–(C-3), we can find a constant $C > 0$ depending only on $T > 0$ and K, k_1, k_2 such that

$$|\!|\!|\widehat{X}|\!|\!|^2_{t_2,\lambda,\beta} \le CE|\widehat{X}(t_2)|^2 = CE|x_2 - X^{t_1,x_1}(t_2)|^2, \tag{3.29}$$

where $\beta = \bar{\lambda}_1 - \mu(\alpha, T)KC_3$ if $k_2 = 0$; and $\beta = \mu(\alpha, T)k_2^2$ if $k_2 > 0$.

From now on by slightly abuse of notations we let $C > 0$ be a generic constant depending only on T, K, k_1 and k_2, and be allowed to vary from line to line. Applying standard arguments using Burkholder-Davis-Gundy inequality we obtain that

$$E \sup_{t_2 \le s \le T} |X^1(s)|^2 + E \sup_{t_2 \le s \le T} |Y^1(s)|^2 \le CE|\widehat{X}(t_2)|^2, \tag{3.30}$$

To estimate $E|\widehat{X}(t_2)|^2$ let us recall the parameters λ_1^ε and λ_2^ε defined in Lemma 3.3. For each $\varepsilon > 0$ define

$$\mu^\varepsilon(\alpha, T) \stackrel{\Delta}{=} K(C_1 + K(1+\varepsilon))B(\lambda_2^\varepsilon, T) + \frac{A(\lambda_2^\varepsilon, T)}{1 - KC_4} KC_2.$$

Since $\lambda_1^\varepsilon \to \lambda_1$, $\lambda_2^\varepsilon \to \lambda_2$, and $\mu^\varepsilon(\alpha, T) \to \mu(\alpha, T)$, as $\varepsilon \to 0$, if the compatibility condition (C-1) and either (C-2) or (C-3) hold, then we can choose $\varepsilon > 0$ such that $\mu^\varepsilon(\alpha, T)k_2^2(1+\varepsilon) < 1$ when $k_2 = 0$ and $\mu^\varepsilon(\alpha, T)KC_3 < \lambda_1^\varepsilon$ when $k_2 \ne 0$. For this fixed $\varepsilon > 0$ we can then repeat the argument of Theorem 3.4 by using (3.12)—(3.15) to derive that

$$\Big(1 - \frac{\mu^\varepsilon(\alpha, T)KC_3}{\bar{\lambda}_1^\varepsilon}\Big)\|X^1\|^2_\lambda \le C(\varepsilon)\left[|x_1|^2 + \Big(1 + \frac{1}{\varepsilon}\Big)\right], \quad k_2 = 0;$$

or

$$\Big(1-\mu^{\varepsilon}(\alpha,T)k_2^2\Big)|\!|\!| X^1 |\!|\!|^2_{\lambda,\beta}\le C(\varepsilon)\left[|x_1|^2+\Big(1+\frac{1}{\varepsilon}\Big)\right],\quad k_2\neq 0,$$

where $C(\varepsilon)$ is some constant depending on T, K, k_1, k_2, and ε. Since $\varepsilon>0$ is now fixed, in either case we have, for a generic constant $C>0$,

$$\|X^1\|_\lambda^2\le C(1+|x_1|^2),$$

which in turn shows that, in light of (3.12)–(3.15) $\|Y^1\|_\lambda^2\le C(1+|x_1|^2)$, and $\|Z\|_\lambda^2\le C(1+|x_1|^2)$. Again, applying the Burkholder and Hölder inequalities we can then derive

$$E\Big\{\sup_{t_1\le s\le T}|X^1(t)|^2\Big\}+E\Big\{\sup_{t_1\le s\le T}|Y^1(t)|^2\Big\}\le C(1+|x_1|^2). \tag{3.31}$$

Now, note that on the interval $[t_1,t_2]$ the process $(\widehat X,\widehat Y,\widehat Z)$ satisfies the following SDE:

$$\begin{cases}\widehat X(s)=(x_1-x_2)+\displaystyle\int_{t_1}^s b^1(r)dr+\int_{t_1}^s\sigma^1(r)dW(r),\\ \widehat Y(s)=\hat Y(t_2)+\displaystyle\int_s^{t_2}h^1(r)dr+\int_s^{t_2}Z^1(r)dW(r),\end{cases}\qquad s\in[t_1,t_2], \tag{3.32}$$

where $b^1(r)=b(r,X^2(r),Y^1(r),Z^1(r))$, $\sigma^1(r)=\sigma(r,X^1(r),Y^1(r),Z^1(r))$, and $h^1(r)=h(r,X^1(r),Y^1(r),Z^1(r))$. Now from the first equation of (3.32) we derive easily that

$$E\{\sup_{t_1\le s\le t_2}|\widehat X(s)|^2\}\le C\{|x_1-x_2|^2+(1+|x_1|^2)|t_1-t_2|\}.$$

Combining this with (3.30), (3.31), as well as the assumption (A4-iv), we derive from the second equation of (3.32) that

$$\begin{aligned}E|\widehat Y(t_1)|^2&\le E|\widehat Y(t_2)|^2+C(1+|x_1|^2\vee|x_2|^2)|t_1-t_2|\\&\le C\{|x_1-x_2|^2+(1+|x_1|^2\vee|x_2|^2)|t_1-t_2|\}.\end{aligned}$$

Since $\widehat Y(t_1)=u(t_1,x_1)-u(t_2,x_2)$ is deterministic, (3.28) follows. The case when $t_1\le t_2$ can be proved by symmetry, the proof is complete. □