7

Spitzer's Condition

7.1 Introduction

We have seen that Spitzer's condition

$$\frac{1}{t}\int_0^t \mathbb{P}\{X_s > 0\}ds \to \rho \in (0,1) \text{ as } t \to \infty \textbf{ or } \text{as } t \to 0+ \tag{7.1.1}$$

is important, essentially because it is equivalent to the ladder time subordinators being asymptotically stable, and hence to the Arc-sine laws holding. Obviously (7.1.1) is implied by

$$\mathbb{P}\{X_t > 0\} \to \rho, \tag{7.1.2}$$

and in 40 years no-one was able to give an example of (7.1.1) holding and (7.1.2) failing, either in the Lévy process or random walk context. What we will see is that they are in fact equivalent, and this equivalence also extends to the degenerate cases $\rho = 0, 1$.

Theorem 23. *For any Lévy process X and for any $0 \le \rho \le 1$, the statements* (7.1.1) *and* (7.1.2) *are equivalent (as $t \to \infty$, or as $t \to 0+$).*

Since the case $t \to \infty$ can be deduced from the random walk results in Doney [33], we will deal here with the case $t \to 0+$. Following Bertoin and Doney [18], we treat the case $\rho = 0, 1$, first, and then give two different proofs for $0 < \rho < 1$. The first is the simplest; it is based on a duality identity for the ladder time processes and does not use any local limit theorem. The second is essentially an adaptation of my method for random walks; in particular it requires a version of the local limit theorem for small times, and a Wiener–Hopf result from Chapter 5.

7.2 Proofs

The purpose of this section is to prove Theorem 23 when $t \to 0+$. The case when the Lévy process $X = (X_t, t \ge 0)$ is a compound Poisson process with drift is of no interest, since in this case $\rho(t) \to 0$ or 1 according as the drift is positive or non-positive, so we will exclude this case. It then follows that $\mathbb{P}\{X_t = 0\} = 0$ for all $t > 0$, and that the mapping $t \to \rho(t) = \mathbb{P}\{X_t > 0\}$ is continuous on $(0, \infty)$ (because X is continuous in probability).

7.2.1 The Case $\rho = 0, 1$

The argument relies on a simple measure-theoretic fact.

Lemma 6. *Let $B \subset [0, \infty)$ be measurable set such that*

$$\lim_{t \to 0+} t^{-1} m(B \cap [0, t]) = 1,$$

where m denotes Lebesgue measure. Then $B + B \supset (0, \varepsilon)$ for some $\varepsilon > 0$.

Proof. Pick $c > 0$ such that $t^{-1} m(B \cap [0, t]) > 3/4$ for all $t \le c$. Then

$$m(B \cap [t, 2t]) \ge \frac{1}{2} t \quad \text{for all} \quad t < \frac{1}{2} c. \tag{7.2.1}$$

Suppose now that there exists $t < \frac{1}{2} c$ such that $2t \notin B + B$. Then for every $s \in [0, t] \cap B$, $2t - s \in B^c \cap [t, 2t]$ and therefore

$$\begin{aligned} m(B \cap [t, 2t]) &= t - m(B^c \cap [t, 2t]) \\ &\le t - m(2t - B \cap [0, t]) \\ &\le t - m(B \cap [0, t]) < \frac{1}{4} t, \end{aligned}$$

and this contradicts (7.2.1). ■

We are now able to complete the proof of Theorem 23 (as $t \to 0+$) for $\rho = 0, 1$. Obviously it suffices to consider the case $\rho = 1$, so assume $t^{-1} \int_0^t \rho(s) ds \to 1$, and for $\delta \in (0, 1)$ consider $B = \{t : \rho(t) \ge \delta\}$. Then B satisfies the hypothesis of Lemma 6 and we have that $B + B \supset (0, \varepsilon)$ for some $\varepsilon > 0$. For any $t \in (0, \varepsilon)$ choose $s \in (0, t) \cap B$ with $t - s \in B$, so that $\rho(s) \ge \delta$ and $\rho(t - s) \ge \delta$. Then by the Markov property

$$\rho(t) = \mathbb{P}\{X_t > 0\} \ge \mathbb{P}\{X_s > 0\} \mathbb{P}\{X_{t-s} > 0\} \ge \delta^2.$$

Since δ can be chosen arbitrarily close to 1, we conclude that $\lim_{t \to 0+} \rho(t) = 1$.

7.2.2 A First Proof for the Case $0 < \rho < 1$

Recall that the ladder time subordinator $\tau = L^{-1}$ is the inverse local time at the supremum, and has Laplace exponent

$$\Phi(q) = \exp\left\{ \int_0^\infty \left(e^{-t} - e^{-qt} \right) t^{-1} \rho(t) dt \right\}, \qquad q \ge 0. \tag{7.2.2}$$

Also from Corollary 3 in Chapter 4 we know that, with an appropriate choice of the normalisation of local time, the Laplace exponent Φ^* corresponding to the dual Lévy process $X^* = -X$ satisfies

$$\Phi(q)\Phi^*(q) = q.$$

So differentiating (7.2.2) we see that

$$\int_0^\infty e^{-qt}\rho(t)dt = \Phi'(q)/\Phi(q) = \Phi'(q)\Phi^*(q)/q\,. \tag{7.2.3}$$

Suppose now that (7.1.1) holds as $t \to 0+$. By results discussed in Chapter 2, this implies that Φ is regularly varying at ∞ with index ρ, and hence also that Φ^* is regularly varying at ∞ with index $1-\rho$. Because Φ and Φ^* are Laplace exponents of subordinators with zero drift, we obtain from the Lévy–Khintchine formula that

$$\Phi'(q) = \int_0^\infty e^{-qx}xd\left(-T(x)\right), \quad \Phi^*(q)/q = \int_0^\infty e^{-qx}T^*(x)dx\,,$$

where T (respectively, T^*) is the tail of the Lévy measure of the ladder time process of X (respectively, of X^*). We now get from (7.2.3)

$$\rho(t) = \int_{(0,t)} T^*(t-s)sd\left(-T(s)\right) \qquad \text{for a.e. } t>0\,. \tag{7.2.4}$$

By a change of variables, the right-hand-side can be re-written as

$$t\int_{(0,1)} T^*(t(1-u))ud\left(-T(tu)\right) = \int_{(0,1)} \frac{T^*(t(1-u))}{\Phi^*(1/t)}ud\left(-\frac{T(tu)}{\Phi(1/t)}\right).$$

Now, apply a Tauberian theorem, the monotone density theorem and the uniform convergence theorem (see Theorems 1.7.1, 1.7.2 and 1.5.2 in [20]). For every fixed $\varepsilon \in (0,1)$, we have, uniformly on $u \in [\varepsilon, 1-\varepsilon]$ as $t \to 0+$,

$$\frac{T(tu)}{\Phi(1/t)} \to \frac{u^{-\rho}}{\Gamma(1-\rho)}, \quad \frac{T^*(t(1-u))}{\Phi^*(1/t)} \to \frac{(1-u)^{(1-\rho)}}{\Gamma(\rho)}\,.$$

Recall $\rho(t)$ depends continuously on $t>0$. We deduce from (7.2.4) that

$$\liminf_{t\to 0+} \rho(t) \geq \frac{\rho}{\Gamma(\rho)\Gamma(1-\rho)}\int_\varepsilon^{1-\varepsilon} (1-u)^{\rho-1}u^{-\rho}du\,,$$

and as ε can be picked arbitrarily small, $\liminf_{t\to 0+}\rho(t) \geq \rho$. The same argument for the dual process gives $\liminf_{t\to 0+}\mathbb{P}\{X_t<0\} \geq 1-\rho$, and this completes the proof.

7.2.3 A Second Proof for the Case $0 < \rho < 1$

Here we will use one of the Wiener–Hopf results we discussed in Chapter 5, specifically

Lemma 7. *We have the following identity between measures on* $(0,\infty) \times (0,\infty)$*:*

$$\mathbb{P}\{X_t \in dx\}dt = t \int_0^\infty \mathbb{P}\{L^{-1}(u) \in dt, H(u) \in dx\}u^{-1}du.$$

We next give a local limit theorem which is more general than we need.

Proposition 10. *Suppose that* $Y = (Y_t, t \geq 0)$ *is a real-valued Lévy process and there exists a measurable function* $r : (0,\infty) \to (0,\infty)$ *such that* $Y_t/r(t)$ *converges in distribution to some law which is not degenerate at a point as* $t \to 0+$*. Then*

(i) *r is regularly varying of index $1/\alpha, 0 < \alpha \leq 2$, and the limit distribution is strictly stable of index α;*
(ii) *for each $t > 0$, Y_t has an absolutely continuous distribution with continuous density function $p_t(\cdot)$;*
(iii) *uniformly for $x \in \mathbb{R}$, $\lim_{t\to 0+} r(t)p_t(xr(t)) = p^{(\alpha)}(x)$, where $p^{(\alpha)}(\cdot)$ is the continuous density of the limiting stable law.*

Proof. (i) This is proved in exactly the same way as the corresponding result for $t \to \infty$. (ii) If $\Psi(\lambda)$ denotes the characteristic exponent of Y, so that

$$\mathbb{E}(\exp\{i\lambda Y_t\}) = \exp\{-t\Psi(\lambda)\}\,, \qquad t \geq 0, \lambda \in \mathbb{R},$$

then we have $t\Psi(\lambda/r(t)) \to \Psi^{(\alpha)}(\lambda)$ as $t \to 0+$, where $\Psi^{(\alpha)}$ is the characteristic exponent of a strictly stable law of index α. Because we have excluded the degenerate case, $\mathrm{Re}(\Psi(\lambda))$, the real part of the characteristic exponent (which is an even function of λ), is regularly varying of index α at $+\infty$. It follows that for each $t > 0$, $\exp -t\Psi(\cdot)$ is integrable over $\mathbb{R}$. Consequently (ii) follows by Fourier inversion, which also gives

$$r(t)p_t(xr(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\{i\lambda x + t\Psi(\lambda/r(t))\}d\lambda$$

and

$$p^{(\alpha)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\{i\lambda x + \Psi^{(\alpha)}(\lambda)\}d\lambda.$$

(iii) In view of the above formulae, it suffices to show that

$$|\exp -t\Psi(\lambda/r(t))| = \exp -tRe\Psi(\lambda/r(t))$$

is dominated by an integrable function on $|\lambda| \geq K$ for some $K < \infty$ and all small enough λ. But this follows easily from Potter's bounds for regularly varying functions. (See [20], Theorem 1.5.6.) ∎

We assume from now on that (7.1.1) holds as $t \to 0+$, so that $\Phi(\lambda)$, the Laplace exponent of the subordinator τ, is regularly varying at ∞ with index ρ. It follows that if we denote by a the inverse function of $1/\Phi(1/\cdot)$, then a is regularly varying with index $1/\rho$ and $\tau(t)/a(t)$ converges in distribution to a non-negative stable law of index ρ as $t \to 0+$. In view of Proposition 10, τ_t has a continuous density which we denote by $g_t(\cdot)$, and $a(t)g_t(a(t)\cdot)$ converges uniformly to the continuous stable density, which we denote by $g^{(\rho)}(\cdot)$. Applying Lemma 7, we obtain the following expression for $\rho(t)$ that should be compared with (7.2.4):

$$\rho(t) = t \int_0^\infty g_u(t) u^{-1} du \qquad \text{for a.e. } t > 0\,. \tag{7.2.5}$$

We are now able to give an alternative proof of Theorem 23 for $0 < \rho < 1$ and $t \to 0+$. By a change of variable,

$$t \int_0^\infty g_u(t) u^{-1} du \;=\; t \int_0^\infty g_{su}(t) u^{-1} du,$$

for any $s > 0$. We now choose $s = 1/\Phi(1/t)$, so that $a(s) = t$, and note that

$$t g_{su}(t) = \frac{a(s)}{a(su)} \cdot a(su) g_{su}\left(a(su) \cdot \frac{a(s)}{a(su)}\right).$$

When $t \to 0+$, $s \to 0+$ and since a is regularly varying with index $1/\rho$, $a(s)/a(su)$ converges pointwise to $u^{-1/\rho}$. It then follows from Proposition 10 that

$$\lim_{t \to 0+} t g_{su}(t) = u^{-1/\rho} g^{(\rho)}(u^{-1/\rho}).$$

Recall that $\rho(t)$ depends continuously on $t > 0$, so that (7.2.5) and Fatou's lemma give

$$\liminf_{t \to 0+} \rho(t) \geq \int_0^\infty g^{(\rho)}(u^{-\frac{1}{\rho}}) u^{-\frac{1}{\rho}-1} du = \rho \int_0^\infty g^{(\rho)}(v) dv = \rho.$$

Replacing X by $-X$ gives $\limsup_{t \to 0+} \mathbb{P}\{X_t \geq 0\} \leq \rho$, and the result follows.

7.3 Further Results

The ultimate objective is to find a necessary and sufficient condition, in terms of the characteristics of X, for Spitzer's condition to hold. Current knowledge can be summarised as follows.

(i) If X is symmetric it holds with $\rho = 1/2$, both at 0 and ∞.
(ii) If $\sigma \neq 0$ it holds with $\rho = 1/2$ at 0.

(iii) If X is in the domain of attraction of a strictly stable process with positivity parameter ρ either as $t \to \infty$ or as $t \downarrow 0$ it holds correspondingly at ∞ or at 0.
(iv) It holds with $\rho = 1/2$ at ∞ in some situations where X has an almost symmetric distribution, but is not in the domain of attraction of any symmetric stable process: see Doney [28] for the random-walk case.
(v) It holds if Y is strictly stable with positivity parameter ρ and $X = Y(\tau)$ is a subordinated process, τ being an arbitrary independent subordinator; the point here is that τ can be chosen so that X is not in any domain of attraction. (This observation is due to J. Bertoin.)

The only obvious examples where it doesn't hold is in the spectrally one-sided case; this was pointed out in the random-walk case more than 40 years ago by Spitzer! See [94], p. 227.

Again for random walks the only situation where a necessary and sufficient condition is known is the special case $\rho = 1$. This can be extended to the Lévy process case at ∞, the most efficient way of doing this being to use the stochastic bounds from Chapter 4; see Doney [36]. The result there suggests:

Proposition 11. *For any Lévy process X we have $\rho_t = \mathbb{P}(X_t > 0) \to 1$ as $t \to 0$ if and only if $\pi_x := \mathbb{P}(X$ exits $[-x, x]$ at the top$) \to 1$ as $x \to 0$.*

We now have two possible lines of attack: we could try to find the necessary and sufficient condition for $\rho_t \to 1$ directly, and then Proposition 11 says we have also solved the corresponding exit problem; this progamme is carried out in Doney [37]. But instead we will tackle the exit problem, using material from Andrew [6]. We need some notation; we use the functions (all on $x > 0$)

$$N(x) = \Pi((x, \infty)), \quad M(x) = \Pi((-\infty, -x)),$$

$$L(x) = N(x) + M(x), \quad D(x) = N(x) - M(x),$$

$$A(x) = \gamma + D(1) - \int_x^1 D(y)dy = \gamma + \int_{(x,1]} ydD(y) + xD(x),$$

and

$$U(x) = \sigma^2 + 2\int_0^x yL(y)dy.$$

(It might help to observe that $A(x)$ and $U(x)$ are respectively the mean and variance of $\tilde{X}_1^x$, where $\tilde{X}^x$ is the Lévy process we get by replacing each jump in X which is bigger than x, (respectively less than $-x$) by a jump equal to x, (respectively $-x$).)

Note that always $\lim_{x\to 0} U(x) = \sigma^2$ and $\lim_{x\to 0} xA(x) = 0$, and if X is of bounded variation, $\lim_{x\to 0} A(x) = \delta$, the true drift of X. Also we always have $\lim_{x\to\infty} U(x) = VarX_1 \le \infty$ and $\lim_{x\to\infty} x^{-1}A(x) = 0$, and if $\mathbb{E}|X_1| < \infty$, $\lim_{x\to\infty} A(x) = \mathbb{E}X_1$.

In any study of exits from 2-sided intervals the following quantity is of crucial importance:

$$k(x) = x^{-1}|A(x)| + x^{-2}U(x), \ x > 0.$$

For Lévy processes, its importance stems from the following bounds, which are due to Pruitt [83], although he uses a function which is slightly different from k.

Let

$$\overline{\overline{X}}(t) = \sup_{0 \le s \le t} |X(s)|$$

and write

$$T_r = \inf(t : \overline{\overline{X}}(t) > r\}.$$

Lemma 8. *There are positive constants c_1, c_2, c_3, c_4 such that, for all Lévy processes and all $r > 0$, $t > 0$,*

$$\mathbb{P}\{\overline{\overline{X}}(t) \ge r\} \le c_1 t k(r), \ \ \mathbb{P}\{\overline{\overline{X}}(t) \le r\} \le \frac{c_2}{tk(r)}, \tag{7.3.1}$$

and

$$\frac{c_3}{k(r)} \le \mathbb{E}(T(r)) \le \frac{c_4}{k(r)}. \tag{7.3.2}$$

Moreover

$$\frac{1}{\lambda^3} \le \frac{k(\lambda x)}{k(x)} \le 3 \textit{ for all } x > 0 \textit{ and } \lambda > 1. \tag{7.3.3}$$

Proof of Proposition 11. We start by assuming $\rho_t = \mathbb{P}(X_t > 0) \to 1$ as $t \to 0$, and suppose that $t = l/k(r)$, where $l \in \mathbb{N}$. (Note that with this choice, the bounds in (7.3.1) are $O(1)$.) Take $\tau_0^r = 0$ and for $j = 0, 1, \cdots$ define

$$\tau_{j+1}^r = \inf\{s > \tau_j : |X_s - X_{\tau_j}| > r\}.$$

Suppose now that the event A_j^r occurs for each $0 \le j < l^2$, where

$$A_j^r = \left(\frac{1}{lk(r)} \le \tau_{j+1}^r - \tau_j^r \le \frac{l}{k(r)} \text{ and } X_{\tau_{j+1}^r} \le X_{\tau_j^r} - r \right);$$

then $X_s \le 0$ for $s \in [\tau_1^r, \tau_{l^2}^r]$. Moreover $t = l/k(r) \in [\tau_1^r, \tau_{l^2}^r]$ and

$$\begin{aligned}
\mathbb{P}(X_t \le 0) &\ge \mathbb{P}\left(\bigcap_{j=1}^{l^2} A_j^r \right) = (\mathbb{P}A_1^r)^{l^2} \\
&\ge \left(\left[\mathbb{P}\{X_{\tau_1^r} < 0\} - \mathbb{P}\left\{ \tau_1^r > \frac{l}{k(r)} \right\} - \mathbb{P}\left\{ \tau_1^r < \frac{1}{lk(r)} \right\} \right]^+ \right)^{l^2} \\
&= \left(\left[\mathbb{P}\{X_{T_r} < 0\} - \mathbb{P}\left\{ \overline{\overline{X}}\left(\frac{l}{k(r)} \right) \le r \right\} - \mathbb{P}\left\{ \overline{\overline{X}}\left(\frac{1}{lk(r)} \right) \ge r \right\} \right]^+ \right)^{l^2}.
\end{aligned}$$

Using Lemma 8, we conclude that:

$$\text{when } t = \frac{l}{k(r)},\ \mathbb{P}(X_t \le 0) \ge \left(\left[\mathbb{P}\{X_{T_r} < 0\} - \frac{c}{l} \right]^+ \right)^{l^2}. \tag{7.3.4}$$

It is easy to check that $k(r) \to \infty$ as $r \to 0$, unless $X_t \equiv 0$, a case we implicitly exclude. Therefore if we fix l and let $r \downarrow 0$ then $t(r) = l/k(r) \downarrow 0$, so (7.3.4) gives

$$\limsup_{r \downarrow 0} \mathbb{P}\{X_{T_r} < 0\} \le \frac{c}{l},$$

and the result follows since l is arbitrary. A somewhat similar argument establishes

$$\text{when } t = \frac{l}{k(r)},\ \mathbb{P}(X_t \ge 0) \ge [\mathbb{P}\{X_{T_r} > 0\}]^{l^2} - \frac{c}{l},$$

which leads quickly to the converse implication, but we omit the details. ■

We will use Lemma 8 in conjunction with the following straight-forward consequence of the compensation formula: let

$$\begin{aligned} U_r(dy) &= \int_0^\infty \mathbb{P}\left\{ \sup_{0 \le r < t} |X(u)| \le r, X(t) \in dy \right\} dt \\ &= \int_0^\infty \mathbb{P}\{T_r > t, X(t) \in dy\} dt. \end{aligned}$$

Then:

Lemma 9. *For* $0 \le |y| \le r < |z|$ *we have*

$$\mathbb{P}\{X(T(r)-) \in dy, X(T(r)) \in dz\} = U_r(dy)\Pi(dz - y). \tag{7.3.5}$$

In what follows, it is convenient to focus on the situation where $\pi_x \to 0$; of course the results for $\pi_x \to 1$ follow by considering $-X$. It is not difficult to guess that any necessary and sufficient condition for $\pi_x \to 0$ must involve some control over the sizes of the **positive** jumps which occur before T_r, so let us write $\Delta(T_r) = X_{T_r} - X_{T_r-}$ for the jump which takes X out of $[-r, r]$, and

$$\overline{\Delta}(T_r) = \sup\{(\Delta_t)^+ : t \le T_r\}$$

for the size of the largest positive jump before T_r. Then since

$$\mathbb{E}T_r = \int_{-r}^{r} U_r(dy),$$

an immediate consequence of Lemma 9 is that for all $r > 0, \delta > 0$

$$N((\delta + 2)r)\mathbb{E}T_r \le \mathbb{P}\left\{\Delta_{T_r} > \delta r\right\} \le N(\delta r)\mathbb{E}T_r. \tag{7.3.6}$$

Thus, by Lemma 8,

$$\frac{c_3 N((\delta+2)r)}{k(r)} \le \mathbb{P}\{\Delta_{T_r} > \delta r\} \le \frac{c_4 N(\delta r)}{k(r)},$$

and using (7.3.3) we conclude that

$$\frac{(\Delta_{T_r})^+}{r} \xrightarrow{P} 0 \text{ as } r \to 0 \text{ if and only if } \frac{N(r)}{k(r)} \to 0 \text{ as } r \to 0.$$

By another application of the compensation formula we see that

$$\begin{aligned} \mathbb{P}\left\{\overline{\Delta}_{T_r} > \delta r\right\} &= \mathbb{P}\left\{\sum_{0\le t\le T_r} \mathbf{1}_{\{\Delta X_t > \delta r\}} \ge 1\right\} \le \mathbb{E}\left\{\sum_{0\le t\le T_r} \mathbf{1}_{\{\Delta X_t > \delta r\}}\right\} \\ &= N(\delta r)\mathbb{E}T_r \le \frac{c_4 N(\delta r)}{k(r)}, \end{aligned}$$

and of course $\mathbb{P}\left\{\overline{\Delta}_{T_r} > \delta r\right\} \ge \mathbb{P}\left\{\Delta_{T_r} > \delta r\right\}$. Finally we see that if $r^{-1}(\Delta_{T_r})^+ \overset{P}{\nrightarrow} 0$, there exists $\delta, \varepsilon > 0$, $r_n \downarrow 0$ with

$$\mathbb{P}\{X(T_{r_n}) > 0\} \ge \mathbb{P}\{\Delta(T_{r_n}) > \varepsilon r_n\} \ge \delta,$$

and since $r + \Delta(T_r) \ge X_{T_r} \ge r$ on $\{X_{T_r} > 0\}$ we see that

$$\mathbb{P}\{\Delta(T_{r_n}) > \frac{\varepsilon}{1+\varepsilon} X(T_{r_n}) > 0\} \ge \mathbb{P}\{\Delta(T_{r_n}) > \varepsilon r_n\} \ge \delta,$$

so that $\overline{\Delta}_{T_r}/X_{T_r} \overset{P}{\nrightarrow} 0$. Since $|X_{T_r}| \ge r$, the reverse implication is obvious, and we have shown the following:

Proposition 12. *The following are equivalent as* $r \downarrow 0$:

$$\text{(i) } \frac{N(r)}{k(r)} \to 0; \text{ (ii) } \frac{(\Delta_{T_r})^+}{r} \xrightarrow{P} 0; \text{ (iii) } \frac{\overline{\Delta}_{T_r}}{r} \xrightarrow{P} 0; \text{ (iv) } \frac{\overline{\Delta}_{T_r}}{X_{T_r}} \xrightarrow{P} 0.$$

Before formulating the final conclusion, we need an intermediate result.

Proposition 13. *A necessary and sufficient condition for* $\pi_x \to 0$ *as* $x \to 0$ *is*

$$\lim_{r\to 0} \frac{N(r)}{k(r)} = 0 \textit{ and } \limsup_{r\to 0} \frac{A(r)}{rk(r)} < 0. \tag{7.3.7}$$

Remark 2. *In the spectrally negative case we have* N *identically zero, so the first part of (7.3.7) is automatic. It is not difficult to show the second part is actually equivalent to*

$$\sigma = 0 \textit{ and } A(r) \le 0 \textit{ for all small enough } r. \tag{7.3.8}$$

In particular, in this case $A(r) = \gamma - M(1) + \int_r^1 M(y)dy$. *So when (7.3.8) holds,* $\int_0^1 M(y)dy$ *is finite, and* X *is of bounded variation with drift* $\delta = \gamma - M(1) + \int_0^1 M(y)dy \leq 0$. *Thus* $-X$ *is a subordinator, and hence* $\pi_x \equiv 0$. *(In fact, in analogy with later results in Chapter 9, the only possible limits for* π_x *in the case that* X *is spectrally negative and* $-X$ *is not a subordinator lie in* $[1/2, 1]$.*)*

Proof of Proposition 13. We will write $\tilde{\mathbb{P}}^x$ for the measure under which X has the distribution of the truncated process $\tilde{X}^x$ under $\mathbb{P}$, and note that the corresponding Lévy tails are given by

$$\begin{aligned}
&\tilde{M}(y) = M(y),\ \tilde{N}(y) = N(y) \text{ for } y < x,\\
&\tilde{M}(y) = \tilde{N}(y) = 0,\ \text{ for } y \geq x.
\end{aligned}$$

As previously observed, $\tilde{\mathbb{E}}^x X_1 = A(x)$, so $X_t - tA(x)$ is a $\tilde{\mathbb{P}}^x-$martingale, and optional stopping gives

$$\tilde{\mathbb{E}}^x X_{T_r} = A(x)\tilde{\mathbb{E}}^x T_r$$

We will work with $x = \lambda r$, and note, from the fact that under $\tilde{\mathbb{P}}^{\lambda r}$ no jumps exceed λr in absolute value, that

$$\begin{aligned}
\tilde{\mathbb{E}}^{\lambda r} X_{T_r} &\geq r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} > 0\} - (\lambda+1)r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\}\\
&= r - (\lambda+2)r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbb{E}}^{\lambda r} X_{T_r} &\leq (\lambda+1)r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} > 0\} - r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\}\\
&= (\lambda+1)r - (\lambda+2)r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\}.
\end{aligned}$$

Thus

$$\frac{1 - r^{-1}A(\lambda r)\tilde{\mathbb{E}}^{\lambda r}X_{T_r}}{(\lambda+2)} \leq \tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} \leq \frac{(\lambda+1) - r^{-1}A(\lambda r)\tilde{\mathbb{E}}^{\lambda r}X_{T_r}}{(\lambda+2)}. \tag{7.3.9}$$

If we now choose $\lambda = 2$ we will have X and $\tilde{X}^{2r}$ agreeing up to time $\tilde{T}_r = T_r$, so this gives

$$\mathbb{P}\{X_{T_r} < 0\} = \tilde{\mathbb{P}}^{2r}\{X_{T_r} < 0\} \leq \frac{3}{4} - \frac{r^{-1}A(2r)\mathbb{E}X_{T_r}}{4},$$

and hence, using Lemma 8 again

$$\frac{cA(2r)}{rk(r)} \leq \frac{3}{4} - \mathbb{P}\{X_{T_r} < 0\}.$$

Thus

$$\pi_r \to 0 \Longrightarrow \limsup_{r\to 0} \frac{A(r)}{rk(r)} \le -\frac{1}{4}.$$

But also $\pi_r \to 0$ implies $r^{-1}(\Delta_{T_r})^+ \xrightarrow{P} 0$, and by Proposition 12 this implies $\lim_{r\to 0} N(r)/k(r) = 0$. To reverse the argument, we will assume that (7.3.7) holds and prove

$$\lim_{\lambda\to 0} \liminf_{r\to 0} \tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} = 1; \tag{7.3.10}$$

then the result follows from

$$\begin{aligned}\lim_{\lambda\to 0} \liminf_{r\to 0} \tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} &\le \lim_{\lambda\to 0} \liminf_{r\to 0} \left(\mathbb{P}\{X_{T_r} < 0\} - \mathbb{P}\{\overline{\Delta}_{T_r} \ge \lambda r\}\right)\\ &\le \liminf_{r\to 0} \mathbb{P}\{X_{T_r} < 0\},\end{aligned}$$

where we have used Proposition 12. We do this in two stages; the first step is to deduce from (7.3.9) that $\exists c > 0$ such that

$$\lim_{\lambda\to 0} \liminf_{r\to 0} \tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} \ge \frac{1+c}{2}. \tag{7.3.11}$$

By considering the sequence defined by

$$\tau_0 = 0, \tau_{j+1} = \inf\{t > \tau_j : |X_t - X_{\tau_j}| > \lambda r\},$$

it is not difficult to show that for any $r > 0$ and $0 < \lambda < 1/2$

$$\mathbb{E}T_{\lambda r} \le 3\lambda \tilde{\mathbb{E}}^{\lambda r} T_r.$$

Using the left-hand side of (7.3.9) and Lemma 8 gives

$$\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} \ge \frac{1 - \frac{cA(\lambda r)}{\lambda r k(\lambda r)}}{\lambda + 2},$$

and letting $r \to 0$ then $\lambda \to 0$ we get (7.3.11).

Now define $p = (2-c)/4$, where c is the constant in (7.3.11), and denote by $\{S_n, n \ge 0\}$ a simple random walk with $P(S_1 = 1) = p, P(S_1 = -1) = q = 1-p$. Put $\sigma_N = \min\{n : |S_n| > N\}, N \in \mathbb{N}$, so that, since $p < 1/2$, we have $P(S_{\sigma_N} < 0) \to 1$ as $N \to \infty$. Thus given $\varepsilon > 0$ we can choose N, K with $P(S_{\sigma_N} < 0, \sigma_N \le K) \ge 1 - \varepsilon$. Take r and λ sufficiently small so that

$$\tilde{q} := \tilde{\mathbb{P}}^{\lambda r}\{X(T_{r/2N} < 0\} \ge q;$$

then, in the obvious notation

$$\begin{aligned}&\tilde{\mathbb{P}}^{\lambda r}\{X \text{ leaves } [-r/2 + \lambda r K, r/2 + \lambda r K] \text{ downwards}\}\\ &\ge \tilde{P}(S_{\sigma_N} < 0, \sigma_N \le K) \ge P(S_{\sigma_N} < 0, \sigma_N \le K) \ge 1 - \varepsilon.\end{aligned}$$

It follows that

$$\lim_{\lambda\to 0}\liminf_{r\to 0}\tilde{\mathbb{P}}^{\lambda r}\{X \text{ leaves } [-r/3, 2r/3] \text{ downwards}\} \geq 1-\varepsilon,$$

and hence

$$\lim_{\lambda\to 0}\liminf_{r\to 0}\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} \geq (1-\varepsilon)^3.$$

Since ε is arbitrary, (7.3.10) follows. ∎

Remark 3. *This proof shows that it is impossible for*

$$-\frac{1}{4} < \limsup_{r\to 0}\frac{A(r)}{rk(r)} < 0$$

to occur; this phenomenom was first observed in the random-walk case in Griffin and McConnell [53].

We can now state our main result.

Theorem 24. *Assume X is not a compound Poisson process: then (i) if $N(0+) > 0$ the following are equivalent;*

$$\pi_x \to 0 \text{ as } x \to 0; \tag{7.3.12}$$

$$\rho_t \to 0 \text{ as } t \to 0; \tag{7.3.13}$$

$$\frac{X_{T_r}}{\overline{\Delta}_{T_r}} \xrightarrow{P} -\infty \text{ as } r \to 0; \tag{7.3.14}$$

$$\frac{X_t}{\overline{\Delta}_t} \xrightarrow{P} -\infty \text{ as } t \to 0; \tag{7.3.15}$$

$$\sigma = 0,\ \frac{A(x)}{xN(x)} \to -\infty \text{ as } x \to 0; \tag{7.3.16}$$

(ii) if $N(0+) = 0$ then (7.3.12)$\Longleftrightarrow$(7.3.13)$\Longleftrightarrow$

$$A(x) \leq 0 \text{ for all small enough } x. \tag{7.3.17}$$

Proof. (i) First we need the fact that (7.3.16) is equivalent to (7.3.7) from Proposition 13, which we recall is

$$\lim_{x\to 0}\frac{N(x)}{k(x)} = 0 \text{ and } \limsup_{x\to 0}\frac{A(x)}{xk(x)} < 0. \tag{7.3.18}$$

If this holds, clearly

$$\lim_{x\to 0}\frac{A(x)}{xN(x)} = \lim_{x\to 0}\frac{A(x)}{xk(x)}\frac{k(x)}{N(x)} = -\infty,$$

and if $\sigma^2 > 0$ we would have $k(x) \geq \sigma^2/x^2$ and hence

$$\limsup_{x\to 0} \frac{|A(x)|}{xk(x)} \leq \limsup_{x\to 0} x|A(x)| = 0;$$

thus $\sigma = 0$ and (7.3.16) holds. So assume (7.3.16) and note first that

$$\frac{k(x)}{N(x)} = \frac{|A(x)|}{xN(x)} + \frac{U(x)}{x^2 N(x)} \geq \frac{|A(x)|}{xN(x)},$$

so $N(x)/k(x) \to 0$. Also

$$\frac{xk(x)}{|A(x)|} = 1 + \frac{U(x)}{x^2 k(x)},$$

so since (7.3.16) implies that $A(x) < 0$ for all small x, we see by writing

$$\frac{U(x)}{xA(x)} = \frac{U(x)}{x^2 k(x)} \frac{xk(x)}{A(x)}$$

that

$$\limsup_{x\to 0} \frac{A(x)}{xk(x)} < 0 \text{ if and only if } \liminf_{x\to 0} \frac{U(x)}{xA(x)} > -\infty.$$

Now given $\varepsilon > 0$ we have $yN(y) \leq -\varepsilon A(y)$ for all $y \leq x_0$. Also integration by parts gives

$$\int_0^x A(y)dy = xA(x) - \int_0^x yN(y)dy + \int_0^x yM(y)dy.$$

So for $x \leq x_0$

$$\int_0^x yN(y)dy \leq -\varepsilon xA(x) + \varepsilon \int_0^x yN(y)dy - \varepsilon \int_0^x yM(y)dy. \tag{7.3.19}$$

This implies that

$$(1-\varepsilon)\int_0^x yN(y)dy \leq -\varepsilon xA(x),$$

and also, putting $\varepsilon = 1$ in (7.3.19), that $\int_0^x yM(y)dy \leq -xA(x)$. Thus

$$U(x) = 2\int_0^x y(N(y)+M(y))dy \leq -xA(x)\frac{2\varepsilon}{1-\varepsilon},$$

for all $x \leq x_0$, and the result (7.3.18) follows. The equivalence of (7.3.12), (7.3.13), (7.3.14) and (7.3.16) now follows from Propositions 11, 12, and 13, bearing in mind that

$$\pi_x \to 0 \text{ and } \frac{\Delta_{T_r}}{X_{T_r}} \xrightarrow{P} 0 \Longrightarrow \frac{X_{T_r}}{\Delta_{T_r}} \xrightarrow{P} -\infty.$$

Since (7.3.15) obviously implies (7.3.13), we are left to prove that

$$\mathbb{P}\{X_t < 0\} \to 1 \Longrightarrow \frac{X_t}{\overline{\Delta}_t} \xrightarrow{P} -\infty \text{ as } t \to 0.$$

The argument here proceeds by contradiction; so assume $\exists\, t_j \downarrow 0$ with $\mathbb{P}C_j \geq 8\varepsilon > 0$ for all j, where $C_j = \{X_{t_j} > -2k\overline{\Delta}_{t_j}\}$ and k is a fixed integer. Then for each j we can choose c_j such that

$$\mathbb{P}\{(\overline{\Delta}_{t_j} \leq c_j) \cap C_j\} \geq 2\varepsilon \text{ and } \mathbb{P}\{(\overline{\Delta}_{t_j} \geq c_j) \cap C_j\} \geq 6\varepsilon. \tag{7.3.20}$$

It follows that for each j at least one of the following must hold:

$$\mathbb{P}\{(\overline{\Delta}_{t_j} > 2c_j) \cap C_j\} \geq 2\varepsilon \tag{7.3.21}$$

or

$$\mathbb{P}\{(c_j \leq \overline{\Delta}_{t_j} \leq 2c_j) \cap C_j\} \geq 4\varepsilon. \tag{7.3.22}$$

Suppose (7.3.21) holds for infinitely many j. Then write N_t^j for the number of jumps exceeding $2c_j$ which occur by time t, Z_t^j for the sum of these jumps, and $Y_t^j = X_t - Z_t^j$. Of course $N_{t_j}^j$ has a Poisson distribution, and we denote its parameter by p_j. Note that we have

$$\mathbb{P}\{N_{t_j}^j = 0\} \geq \mathbb{P}\{(\overline{\Delta}_{t_j} \leq c_j) \cap C_j\} \geq 2\varepsilon \text{ and}$$

$$\mathbb{P}\{N_{t_j}^j > 0\} \geq \mathbb{P}\{(\overline{\Delta}_{t_j} > 2c_j) \cap C_j\} \geq 2\varepsilon,$$

so p_j is bounded uniformly away from 0 and ∞. It follows that $\exists \nu > 0$ with

$$\mathbb{P}\{N_{t_j}^j \geq k\} > e^{-p_j} \frac{p_j^k}{k!} > \nu \text{ for all } j.$$

Also

$$\mathbb{P}\{Z_{t_j}^j = 0, Y_{t_j}^j \in (-2kc_j, 0)\} \geq \mathbb{P}\{C_j \cap (X_{t_j} < 0) \cap (\overline{\Delta}_{t_j} \leq c_j)\} \geq \varepsilon$$

for all large j, by (7.3.20) and the fact that $\mathbb{P}(X_{t_j} < 0) \to 1$. So, as Y and Z are independent, the contradiction follows from

$$\liminf_{j\to\infty} \mathbb{P}(X_{t_j} > 0) \geq \liminf_{j\to\infty} \mathbb{P}\{N_{t_j}^j \geq k, Y_{t_j}^j \in (-2kc_j, 0)\} \geq \nu\varepsilon.$$

The second case, when (7.3.22) holds for infinitely many j, can be dealt with in a similar way; see [6] for the details.

(ii) This follows from Propositions 11 and 13, and Remark 2. ■

Some comments on this result are in order.

- The condition (7.3.16) can be shown to be equivalent to

$$\frac{A(x)}{\sqrt{U(x)N(x)}} \to -\infty. \tag{7.3.23}$$

- There are other conditions we can add to the equivalences in Theorem 24. In particular,

$$\exists \text{ a slowly varying } l \text{ such that } \frac{X_t}{tl(t)} \xrightarrow{P} -\infty. \tag{7.3.24}$$

(This is demonstrated in [37].) Note that this implies $t^{-\alpha}X_t \xrightarrow{P} -\infty$ for any $\alpha > 1$.

- At the cost of considerable extra work, it is possible to give analogous results for sequential limits; see Andrew [6] for the Lévy-process case and Kesten and Maller [62] for the random-walk case.
- Remarkably, the equivalences stated in Theorem 24, and their equivalence to (7.3.23) and (7.3.24), remain valid if limits at zero are replaced by limits at infinity throughout, with only one exception: the large time version of (7.3.16) places no restriction on σ, since the Brownian component is irrelevant for large t. One further difference is that one can add one further equivalence in the $t \to \infty$ case, which is

$$X_t \xrightarrow{P} -\infty \text{ as } t \to \infty.$$

- Suppose X is spectrally positive, so that

$$\frac{A(x)}{xN(x)} = \frac{\gamma + N(1) - \int_x^1 N(y)dy}{xN(x)}.$$

If X is of bounded variation, i.e. $\int_0^1 N(y)dy < \infty$, then $xN(x) \to 0$ and (7.3.16) is equivalent to $d = \gamma + N(1) - \int_0^1 N(y)dy < 0$. Otherwise, it is equivalent to

$$\frac{\int_x^1 N(y)dy}{xN(x)} \to \infty,$$

and this happens if and only if $\int_x^1 N(y)dy$ is slowly varying, so that X is "almost" of bounded variation. Note also that a variation of the above shows that in all cases $\int_x^1 N(y)dy$ being slowly varying is **necessary** in order that (7.3.16) holds; of course this includes the case $\int_0^1 N(y)dy < \infty$.

7.4 Tailpiece

None of this helps in finding the necessary and sufficient condition for Spitzer's condition when $0 < \rho < 1$; if anything it suggests how difficult this problem is. This is reinforced by the following results, taken from Andrew [7].

(i) Given any $0 < \alpha \leq \beta < 1$ there are Lévy processes with

$$\alpha = \liminf \pi_x, \ \beta = \limsup \pi_x,$$

and other Lévy processes with

$$\alpha = \liminf \rho_t, \ \beta = \limsup \rho_t.$$

(ii) For any $0 < \alpha < 1$ there is a Lévy process with

$$\alpha = \lim \pi_x = \lim \rho_t.$$

(Non-symmetric stable processes are examples where the two limits exist, but differ.)

(iii) For any $0 < \alpha < \beta < 1$ there is a Lévy process with $\alpha = \lim \rho_t$ and such that π_x fluctuates between α and β for small x.
In conclusion; **every** type of limit behaviour seems to be **possible**.