2

Subordinators

2.1 Introduction

It is not difficult to see, by considering what happens near time 0, that a Lévy process which starts at 0 and only takes values in $[0, \infty)$ must have $\sigma = \Pi\{(-\infty, 0)\} = 0$, bounded variation and drift coefficient $\delta \geq 0$. Clearly such a process has monotone, non-decreasing paths. These processes, which are the continuous analogues of renewal processes, are called **subordinators**. (The name comes from the fact that whenever X is a Lévy process and T is an independent subordinator, the *subordinated* process defined by $Y_t = X_{T_t}$ is also a Lévy process.) Apart from the interest in subordinators as a sub-class of Lévy processes, we will see that they play a crucial rôle in fluctuation theory of general Lévy processes, just as renewal processes do in random-walk theory.

2.2 Basics

For subordinators it is possible, and convenient, to work with Laplace transforms rather than Fourier transforms. Since

$$\int_0^\infty (1 \wedge x)\Pi(dx) < \infty, \tag{2.2.1}$$

we can write the Lévy exponent in the form

$$\Psi(\lambda) = -i\delta\lambda + \int_0^\infty \{1 - e^{i\lambda x}\}\Pi(dx),$$

and it is clear from (2.2.1) that the integral converges on the upper half of the complex λ plane. So we can define the *Laplace exponent* by

$$\Phi(\lambda) = -\log \mathbb{E}\{e^{-\lambda X_1}\} = \Psi(i\lambda) = \delta\lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx), \tag{2.2.2}$$

and have

$$\mathbb{E}(e^{-\lambda X_t}) = \exp\{-t\Phi(\lambda)\},\ \lambda \geq 0.$$

It is also useful to observe that, by integration by parts, we can rewrite (2.2.2) in terms of the Lévy tail, $\overline{\Pi}(x) = \Pi\{(x,\infty)\}$, as

$$\frac{\Phi(\lambda)}{\lambda} = \delta + \int_0^\infty \overline{\Pi}(x)e^{-\lambda x}dx. \tag{2.2.3}$$

A further integration by parts gives

$$\frac{\Phi(\lambda)}{\lambda^2} = \int_0^\infty e^{-\lambda x}\left\{\delta + I(x)\right\}dx, \tag{2.2.4}$$

where $I(x) = \int_0^x \overline{\Pi}(y)dy$ denotes the integrated tail of the Lévy measure.

One reason why subordinators are interesting is that they often turn up whilst studying other processes: for example, the first passage process in Brownian motion is a subordinator with $\delta = 0$ and $\Pi(dx) = cx^{-\frac{3}{2}}\mathbf{1}_{\{x>0\}}dx$, $\Phi(\lambda) = c'\lambda^{\frac{1}{2}}$. This is a stable subordinator of index 1/2. For $\alpha \in (0,1)$ **a stable subordinator of index** α has Laplace exponent

$$\Phi(\lambda) = c\lambda^\alpha = \frac{c\alpha}{\Gamma(1-\alpha)}\int_0^\infty (1-e^{-\lambda x})x^{-1-\alpha}dx.$$

The c here is just a scale factor, and the restriction on α comes from condition (2.2.1). Poisson processes are also subordinators, and the Gamma process we met earlier is a representative of the class of **Gamma subordinators**. These have

$$\Phi(\lambda) = a\log(1+b^{-1}\lambda) = \int_0^\infty (1-e^{-\lambda x})ax^{-1}e^{-bx}dx;$$

where $a, b > 0$ are parameters. (The second equality here is an example of the **Frullani integral**: see [20], Section 1.6.4.) This family is noteworthy because we also have an explicit expression for the distribution of X_t, viz

$$\mathbb{P}(X_t \in dx) = \frac{b^{at}}{\Gamma(at)}x^{at-1}e^{-bx}dx.$$

2.3 The Renewal Measure

Just as in the discrete case, an important object in the study of a subordinator is the associated renewal measure. Because X is transient, its potential measure

$$U(dx) = \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{X_t\in dx\}}dt\right) = \int_0^\infty \mathbb{P}(X_t \in dx)dt$$

is a Radon measure, and its distribution function, which we denote by $U(x)$, is called the renewal function of X. If $T_x = T_{(x,\infty)}$ we can also write

$$U(x) = U([0,x]) = \mathbb{E}T_x. \tag{2.3.1}$$

Let us first point out why the name is appropriate.

Lemma 1. *Let $Y = X_{\boldsymbol{e}}$, where $\boldsymbol{e}$ is an independent, Exp(1) random variable, and with $Y_1, Y_2 \cdots$ independent and identically distributed copies of Y, put $S_0 = 0$ and $S_n = \sum_1^n Y_j$ for $n \geq 1$. Write V for the renewal function of the renewal process S, viz $V(x) = \sum_0^\infty P(S_n \leq x)$. Then*

$$V(x) = 1 + U(x),\ \ x \geq 0.$$

Proof. Since

$$\begin{aligned} E(e^{-\lambda Y}) &= \int_0^\infty \int_0^\infty e^{-\lambda x} e^{-t} \mathbb{P}(X_t \in dx) dt \\ &= \int_0^\infty e^{-t} e^{-t\Phi(\lambda)} dt = \frac{1}{1+\Phi(\lambda)} \end{aligned}$$

we see that

$$\int_0^\infty e^{-\lambda x} V(dx) = (1 - E(e^{-\lambda Y}))^{-1} = 1 + \frac{1}{\Phi(\lambda)}.$$

But

$$\begin{aligned} \int_0^\infty e^{-\lambda x} U(dx) &= \int_0^\infty e^{-\lambda x} \int_0^\infty \mathbb{P}(X_t \in dx) dt \\ &= \int_0^\infty e^{-t\phi(\lambda)} dt = \frac{1}{\Phi(\lambda)}. \end{aligned}$$

■

This tells us that asymptotic results such as the Renewal Theorem have analogues for subordinators: note in this context that Y has the same mean as X_1. Also, it is easy to see that, in essence, we don't need to worry about the difference between the lattice and non-lattice cases: the only time the support of U is contained in a lattice is when X is a compound Poisson process whose step distribution is supported by a lattice. If X is not compound Poisson, then the measure U is diffuse, and $U(x)$ is continuous; this is also true in the case of a compound Poisson process whose step distribution is diffuse, except that there is a Dirac mass at zero.

Another property which goes over to the continuous case is that of subadditivity, since the useful inequality

$$U(x+y) \leq U(x) + U(y),\ \ x, y \geq 0,$$

can be seen directly from (2.3.1). The behaviour of U for both large and small x is of interest, and in this the following lemma, which is slightly more general than we need, is useful.

Lemma 2. *Suppose that for $\lambda > 0$*

$$f(\lambda) = \lambda \int_0^\infty e^{-\lambda y} W(y)dy = \int_0^\infty e^{-y} W(y/\lambda)dy, \tag{2.3.2}$$

where W is non-negative, non-decreasing, and such that there is a positive constant c with

$$W(2x) \leq cW(x) \text{ for all } x > 0. \tag{2.3.3}$$

Then

$$W(x) \approx f(1/x), \tag{2.3.4}$$

where $\approx$ means that the ratio of the two sides is bounded above and below by positive constants for all $x > 0$.

Proof. It is immediate from (2.3.2) that for any $k > 0, \lambda > 0$,

$$W(k/\lambda) = e^k W(k/\lambda) \int_k^\infty e^{-y} dy \leq e^k \int_k^\infty e^{-y} W(y/\lambda)dy \leq e^k f(\lambda), \tag{2.3.5}$$

and with $k = 1$ this is one of the required bounds. Next, condition (2.3.3) gives

$$f(\lambda/2) = \int_0^\infty e^{-y} W(2y/\lambda)dy \leq c \int_0^\infty e^{-y} W(y/\lambda)dy = cf(\lambda).$$

Using this and rewriting (2.3.5) as

$$W(y/\lambda) = W((y/2)/(\lambda/2)) \leq e^{y/2} f(\lambda/2)$$

gives, for any $x > 0$,

$$\begin{aligned} f(\lambda) &\leq W(x/\lambda) \int_0^x e^{-y} dy + f(\lambda/2) \int_x^\infty e^{y/2} e^{-y} dy \\ &= (1 - e^{-x}) W(x/\lambda) + 2f(\lambda/2) e^{-x/2} \\ &\leq (1 - e^{-x}) W(x/\lambda) + 2cf(\lambda) e^{-x/2}. \end{aligned}$$

Assuming, with no loss of generality, that $c > 1/4$, and choosing $x = x_0 := 2 \log 4c$ and an integer n_0 with $2^{n_0} \geq x_0$ we deduce, using (2.3.3) again, that

$$f(\lambda) \leq 2 \left(1 - \frac{1}{16c^2}\right) W(x_0/\lambda) \leq 2c^{n_0} \left(1 - \frac{1}{16c^2}\right) W(1/\lambda),$$

and this is the other bound. ■

For some applications, it is important that the constants in the upper and lower bounds only depend on W through the constant c in (2.3.3). For example, when $c = 2$, as it does in the special case that W is subadditive, we can take them to be $8/63$ and e.

Corollary 1. *Let X be any subordinator, and write $I(x) = \int_0^x \overline{\Pi}(y)dy$. Then*

$$U(x) \approx \frac{1}{\Phi(1/x)} \quad and \quad \frac{\Phi(x)}{x} \approx I(1/x) + \delta.$$

Proof. Recall (2.2.4) and the fact that $\int_0^\infty e^{-\lambda x} U(x)dx = \lambda/\phi(\lambda)$ and check that the conditions of the previous lemma are satisfied. ■

These estimates can of course be refined if we assume more. If either of U or Φ is in $RV(\alpha)$ (i.e. is regularly varying with index α; see [20] for details) with $\alpha \in [0,1]$ at $0+$ or ∞, then the other is in $RV(\alpha)$ at ∞, respectively $0+$; in fact

$$\Gamma(1+\alpha)U(x) \sim \frac{1}{\Phi(1/x)}.$$

Similarly we have

$$\Gamma(2-\alpha)\{I(x)+\delta\} \sim x\Phi(1/x),$$

and moreover when this happens with $\alpha < 1$, the monotone density theorem applies and

$$\Gamma(1-\alpha)\overline{\Pi}(x) \sim \frac{1}{\Phi(1/x)}.$$

2.4 Passage Across a Level

We will be interested in the undershoot and overshoot when the subordinator crosses a positive level x, but in continuous time we have to consider the possibility of continuous passage, i.e. that T_x is not a time at which X jumps. We start with our first example of the use of the compensation formula.

Theorem 2. *If X is a subordinator we have*

(i) for $0 \le y \le x$ and $z > x$

$$\mathbb{P}(X_{T_x-} \in dy, X_{T_x} \in dz) = U(dy)\Pi(dz-y):$$

(ii) for every $x > 0$,

$$\mathbb{P}(X_{T_x-} < x = X_{T_x}) = 0.$$

Proof. (i) Recall that the process of jumps Δ is a Poisson point process on $\mathbb{R}\times[0,\infty)$ with characteristic measure Π, so

$$\begin{aligned}\mathbb{P}(X_{T_x-} \in dy, X_{T_x} \in dz) &= \mathbb{E}\left(\sum_{t\ge 0} \mathbf{1}_{(X_{t-}\in dy, X_t \in dz)}\right)\\ &= \mathbb{E}\left(\sum_{t\ge 0} \mathbf{1}_{(X_{t-}\in dy, \Delta_t \in dz-y)}\right)\end{aligned}$$

$$= \int_0^\infty dt \mathbb{E}\left(\mathbf{1}_{(X_{t-}\in dy)} \int_{-\infty}^\infty \Pi(ds)\mathbf{1}_{(s\in dz-y)}\right)$$
$$= \int_0^\infty dt \mathbb{P}(X_t \in dy)\Pi(dz-y) = U(dy)\Pi(dz-y).$$

(ii) The statement is clearly true if X is a compound Poisson process, since then the values of X form a discrete set, and otherwise we know that U is diffuse. In this case the above argument gives

$$\mathbb{P}(X_{T_x-} < x = X_{T_x}) = \int_{[0,x)} U(dy)\Pi(\{x-y\}) = 0,$$

since $\Pi(\{z\}) = 0$ off a countable set. ■

Observe that a similar argument gives the following extension of (i):

$$\mathbb{P}(X_{T_x-} \in dy, X_{T_x} \in dz, T_x \le t) = \int_0^t \mathbb{P}(X_s \in dy)ds\Pi(dz-y).$$

From this we deduce the following equality of measures:

$$\mathbb{P}(X_{T_x-} \in dy, X_{T_x} \in dz, T_x \in dt) = \mathbb{P}(X_t \in dy)\Pi(dz-y)dt$$
$$\text{for } 0 \le y \le x, z > x \text{ and } t > 0.$$

Part (ii) says that if a subordinator crosses a level by a jump, then a.s. that jump takes it over the level.

It turns out that the question of continuous passage (or “creeping”) of a subordinator is quite subtle, and was only resolved in [58], and we refer to that paper, [22] or [12], Section III.2 for a proof of the following.

Theorem 3. *If X is a subordinator with drift δ,*

(i) if $\delta = 0$ then $\mathbb{P}(X_{T_x} = x) = 0$ for **all** $x > 0$,
(ii) if $\delta > 0$ then U has a strictly positive and continuous density u on $(0,\infty)$,

$$\mathbb{P}(X_{T_x} = x) = \delta u(x) \text{ for } \textbf{all } x > 0, \tag{2.4.1}$$

and $\lim_{x\downarrow 0} u(x) = 1/\delta$.

Parts of this are easy; for example, by applying the strong Markov property at time T_x we get

$$U(dw) = \int_{[x,w]} U(dw-z)\mathbb{P}(X_{T_x} \in dz),\ w \ge x,$$

and taking Laplace transforms gives

$$\int_{[x,\infty)} e^{-\lambda w}U(dw) = \int_{[0,\infty)} e^{-\lambda w}U(dw) \int_{[x,\infty)} e^{-\lambda z}\mathbb{P}(X_{T_x} \in dz)$$
$$= \frac{\mathbb{E}(e^{-\lambda X_{T_x}})}{\Phi(\lambda)}.$$

This leads quickly to

$$\int_0^\infty e^{-qx}\mathbb{E}\left(e^{-\lambda(X_{T_x}-x)}\right)dx = \frac{\Phi(\lambda)-\Phi(q)}{(\lambda-q)\Phi(q)}, \tag{2.4.2}$$

and since, by Proposition 4, Chapter 1, $\lambda^{-1}\Phi(\lambda) \to \delta$ as $\lambda \to \infty$, we arrive at the conclusion that

$$\int_0^\infty e^{-qx}\mathbb{P}(X_{T_x}=x)dx = \frac{\delta}{\Phi(q)} = \delta\int_0^\infty e^{-qx}U(dx).$$

If $\delta = 0$ this tells us that $\mathbb{P}(X_{T_x}=x)=0$ for a.e. Lebesgue x. Also, if $\delta > 0$, then a simple Fourier-analytic estimate shows that U is absolutely continuous, and hence statement (2.4.1) holds a.e. The proof of the remaining statements in [12], Section III.2 is based on clever use of the inequalities:

$$\mathbb{P}(X_{T_{x+y}}=x+y) \geq \mathbb{P}(X_{T_x}=x)\mathbb{P}(X_{T_y}=y)$$
$$\mathbb{P}(X_{T_{x+y}}=x+y) \leq \mathbb{P}(X_{T_x}=x)\mathbb{P}(X_{T_y}=y) + 1 - \mathbb{P}(X_{T_x}=x).$$

Further results involving creeping of a general Lévy process will be discussed in Chapter 6.

2.5 Arc-Sine Laws for Subordinators

Our interest here is in the analogue of the "arc-sine theorem for renewal processes", see e.g. [20], Section 8.6. Apart from the interest in the results for subordinators per se, we will see that, just as in the case of random walks, it enables us to derive arc-sine theorems for general Lévy processes.

Note that the the random variable $x - X_{T_x-}$, which we have referred to as the undershoot, is the analogue of the quantity referred to in Renewal theory as, "unexpired lifetime" or "backward recurrence time", but we will phrase our results in terms of X_{T_x-}. First we use an argument similar to that leading to (2.4.2) to see that

$$\int_0^\infty e^{-qx}\mathbb{E}\left(e^{-\lambda X_{T_x-}}\right)dx = \frac{\Phi(q)}{q\Phi(q+\lambda)},$$

and hence, writing $A_t(x) = x^{-1}X(T_{tx}-)$

$$\int_0^\infty e^{-qt}\mathbb{E}\left(e^{-\lambda A_t(x)}\right)dt = \frac{\Phi(q/x)}{q\Phi((q+\lambda)/x)}.$$

Now if X is a stable subordinator with index $0 < \alpha < 1$, we see that the right-hand side does not depend on x, and equals $q^{\alpha-1}(q+\lambda)^{-\alpha}$. By checking that

$$\int_0^\infty e^{-qt}\int_0^t e^{-\lambda s}\frac{s^{\alpha-1}(t-s)^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}ds = q^{\alpha-1}(q+\lambda)^{-\alpha}$$

we see that for each $t, x > 0$, $A_t(x) \stackrel{D}{=} A_t(1) \stackrel{D}{=} A_1(1)$, and this last has the generalised arc-sine law with parameter α. As a general subordinator X is in the domain of attraction of a standard stable subordinator of index α (i.e. $\exists$ a norming function $b(t)$ such that the process $\{X_{ts}/b(t), s \geq 0\}$ converges weakly to it), as $t \to \infty$ or $t \to 0+$, if and only if its exponent $\Phi \in RV(\alpha)$ (at 0 or ∞, respectively), the following should not be a surprise. For a proof we again refer to [12], Section III.3.

Theorem 4. *The following statements are equivalent.*

(i) The random variables $x^{-1}X(T_x-)$ converge in distribution as $x \to \infty$ (respectively as $x \to 0+$).
(ii) $\lim x^{-1}\mathbb{E}(X(T_x-)) = \alpha \in [0,1]$ as $x \to \infty$ (respectively as $x \to 0+$).
(iii) The Laplace exponent $\Phi \in RV(\alpha)$ (at 0 or ∞, respectively) with $\alpha \in [0,1]$.

When this happens the limit distribution is the arc-sine law with parameter α if $0 < \alpha < 1$, and is degenerate at 0 or 1 if $\alpha = 0$ or 1.

2.6 Rates of Growth

The following fundamental result shows that strong laws of large numbers hold, both at infinity and zero.

Proposition 5. *For any subordinator X*

$$\lim_{t\to\infty} \frac{X_t}{t} \stackrel{a.s.}{=} \mathbb{E}X_1 = \delta + \int_0^\infty \overline{\Pi}(x)dx \leq \infty, \qquad \lim_{t\to 0+} \frac{X_t}{t} \stackrel{a.s.}{=} \delta \geq 0.$$

Proof. The first result follows easily by random-walk approximation, and the second follows because we know from $\lim_{t\to 0+} t\Phi(\lambda/t) = \delta\lambda$ that we have convergence in distribution, and ([12], Section III.4) we can also show that $(t^{-1}X_t, t > 0)$ is a reversed martingale. ■

There are many results known about rates of growth of subordinators, both for large and small times. Just to give you an indication of their scope I will quote a couple of results from [12], Section III.4.

Theorem 5. *Assume that $\delta = 0$ and $h : [0,\infty) \to [0,\infty)$ is a non-decreasing function such that $t^{-1}h(t)$ is also non-decreasing. Then*

$$\limsup_{t\to 0+} \frac{X_t}{h(t)} = \infty \ a.s.$$

if and only if

$$\int_0^1 \overline{\Pi}(h(x))dx < \infty,$$

and if these fail,

$$\lim_{t\to 0+} \frac{X_t}{h(t)} = 0 \ a.s.$$

Notice that in the situation of this result, the lim sup has to be either 0 or ∞; this contrasts with the behaviour of the lim inf, as we see from the following.

Theorem 6. *Suppose that $\Phi \in RV(\alpha)$ at ∞, and Φ has inverse ϕ. Define*

$$f(t) = \frac{\log|\log t|}{\phi(t^{-1}\log|\log t|)}, \ 0 < t < 1/e.$$

Then

$$\liminf \frac{X_t}{f(t)} = \alpha(1-\alpha)^{(1-\alpha)/\alpha} \ a.s. \ .$$

There are exactly analogous statements for large t.

2.7 Killed Subordinators

It is important, particularly in connection with the ladder processes, to treat subordinators with a possibly finite lifetime. In order for the Markov property to hold, the lifetime has to be exponentially distributed, say with parameter k. It is also easy to see that if $\tilde{X}$ is such a subordinator, then it can be considered as a subordinator X with infinite lifetime killed at an independent exponential time, and that the corresponding exponents are related by

$$\tilde{\Phi}(\lambda) = k + \Phi(\lambda), \ \lambda \geq 0.$$

So the **characteristics** of a (possibly killed) subordinator are its Lévy measure Π, its drift coefficient δ, and its killing rate $k \geq 0$.