7

Value-Distribution in the Complex Plane

> Une fonction entière, qui ne devient jamais ni à a ni à b est nécessairement une constante. Emile Picard

Many beautiful results on the value-distribution of L-functions follow from the general theory of Dirichlet series like the Big Picard theorem (see Boas [26] and Mandelbrojt [234]), but more advanced statements can only be proved by exploiting the characterizing properties (the functional equation and the Euler product). In this chapter, we study the distribution of values of Dirichlet series satisfying a Riemann-type functional equation. These results are due to Steuding [346, 347] and their proofs follow in the main part the methods of Levinson [217], Levinson and Montgomery [218], and Nevanlinna theory.

7.1 Sums Over c-Values

Let c be any complex number. Levinson [217] proved that all but $\ll N(T)(\log\log T)^{-1}$ of the roots of $\zeta(s) = c$ in $T < t < 2T$ lie in

$$\left|\sigma - \frac{1}{2}\right| < \frac{(\log\log T)^2}{\log T}.$$

Thus, the c-values of the zeta-function are clustered around the critical line. In particular, we see that the density estimate (1.13) alone does not indicate the truth of the Riemann hypothesis. As we shall show in this chapter, this distribution of c-values is typical for Dirichlet series satisfying a Riemann-type functional equation.

Throughout this chapter, we shall assume that

$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

satisfies the axioms (1)–(3) from the definition of the Selberg class $\mathcal{S}$, and so we may define the degree $d_{\mathcal{L}}$ of $\mathcal{L}$ by (6.2); we shall not make use of axiom (4) (neither do we use the condition on the real parts of the complex numbers μ_j in the Gamma-factors of the functional equation), however, for simplicity we suppose that $a(1) = 1$. In some places we shall assume the Lindelöf hypothesis for $\mathcal{L}(s)$; by that we mean the estimate (6.18).

We give an example of a function satisfying these axioms which does not have an Euler product. The Davenport–Heilbronn zeta-function is given by

$$L(s) = \frac{1 - i\kappa}{2} L(s, \chi) + \frac{1 + i\kappa}{2} L(s, \overline{\chi}), \tag{7.1}$$

where

$$\kappa := \frac{\sqrt{10 - 2\sqrt{5}} - 2}{\sqrt{5} - 1}$$

and χ is the character mod 5 with $\chi(2) = i$. It is easily seen that the Davenport–Heilbronn zeta-function satisfies the functional equation

$$\left(\frac{5}{\pi}\right)^{s/2} \Gamma\left(\frac{s+1}{2}\right) L(s) = \left(\frac{5}{\pi}\right)^{(1-s)/2} \Gamma\left(1 - \frac{s}{2}\right) L(1 - s).$$

Davenport and Heilbronn [65] introduced this function as an example for a Dirichlet series having zeros in the half-plane $\sigma > 1$ although $L(s)$ satisfies a Riemann-type functional equation; see Balanzario [14] for more examples of a similar type.

The c-values of $\mathcal{L}(s)$ are the roots of the equation

$$\mathcal{L}(s) = c, \tag{7.2}$$

which we denote by $\varrho_c = \beta_c + i\gamma_c$. Our first aim is to prove estimates for sums taken over c-values, weighted with respect to their real parts.

Theorem 7.1. *Assume that $\mathcal{L}(s)$ satisfies the axioms (1)–(3) with $a(1) = 1$ and let $c \neq 1$. Then, for any $b > \max\{\frac{1}{2}, 1 - \frac{1}{d_{\mathcal{L}}}\}$,*

$$\sum_{\substack{\beta_c > b \\ T < \gamma_c \leq 2T}} (\beta_c - b) \ll T.$$

Assuming the truth of Lindelöf's hypothesis for $\mathcal{L}(s)$,

$$\sum_{\substack{\beta_c > \frac{1}{2} \\ T < \gamma_c \leq 2T}} \left(\beta_c - \frac{1}{2}\right) = O(T \log T).$$

The case $c = 1$ is exceptional since $1 = a(1)$ is the limit of $\mathcal{L}(s)$ as $\sigma \to \infty$:

$$\mathcal{L}(s) = 1 + \mathrm{O}(2^{-\sigma}). \tag{7.3}$$

We will briefly discuss this case at the end of Sect. 7.2.

Proof. In view of (7.3) there exists a positive real number A depending on c such that all real parts β_c of c-values satisfy $\beta_c < A$. Put

$$\ell(s) = \frac{\mathcal{L}(s) - c}{1 - c}.$$

Obviously, the zeros of $\ell(s)$ correspond exactly to the c-values of $\mathcal{L}(s)$. Next we will apply Littlewood's lemma which relates the zeros of an analytic function $f(s)$ with a contour integral over $\log f(s)$.

Lemma 7.2 (Littlewood). *Let $b < a$ and let $f(s)$ be analytic on $\mathcal{R} := \{s \in \mathbb{C} : b \le \sigma \le a, |t| \le T\}$. Suppose that $f(s)$ does not vanish on the right edge $\sigma = a$ of $\mathcal{R}$. Let $\mathcal{R}'$ be $\mathcal{R}$ minus the union of the horizontal cuts from the zeros of f in $\mathcal{R}$ to the left edge of $\mathcal{R}$, and choose a single-valued branch of* $\log f(s)$ *in the interior of $\mathcal{R}'$. Denote by $\nu(\sigma, T)$ the number of zeros $\varrho = \beta + \mathrm{i}\gamma$ of $f(s)$ inside the rectangle with $\beta > \sigma$ including zeros with $\gamma = T$ but not those with $\gamma = -T$. Then*

$$\int_{\partial\mathcal{R}} \log f(s)\,\mathrm{d}s = -2\pi\mathrm{i} \int_b^a \nu(\sigma, T)\,\mathrm{d}\sigma.$$

This is an integrated version of the principle of the argument. We give a sketch of the simple proof. Cauchy's theorem implies $\int_{\partial\mathcal{R}'} \log f(s)\,\mathrm{d}s = 0$, and so the left-hand side of the formula of the lemma, $\int_{\partial\mathcal{R}}$, is minus the sum of the integrals around the paths hugging the cuts. Since the function $\log f(s)$ jumps by $2\pi\mathrm{i}$ across each cut (assuming for simplicity that the zeros of f in $\mathcal{R}$ are simple and have different height; the general case is no harder), $\int_{\partial\mathcal{R}}$ is $-2\pi\mathrm{i}$ times the total length of the cuts, which is the right-hand side of the formula in the lemma. For more details we refer to Titchmarsh [353, Sect. 9.9], or Littlewood's original paper [224].

Let $\nu(\sigma, T)$ denote the number of zeros ϱ_c of $\ell(s)$ with $\beta_c > \sigma$ and $T < \gamma_c \le 2T$ (counting multiplicities). Now let a be a parameter with $a > \max\{A + 1, b\}$. Then Littlewood's Lemma 7.2, applied to the rectangle $\mathcal{R}$ with vertices $a + \mathrm{i}T, a + 2\mathrm{i}T, b + \mathrm{i}T, b + 2\mathrm{i}T$, gives

$$\int_{\mathcal{R}} \log \ell(s)\,\mathrm{d}s = -2\pi\mathrm{i} \int_b^a \nu(\sigma, T)\,\mathrm{d}\sigma.$$

Since

$$\int_b^a \nu(\sigma, T)\,\mathrm{d}\sigma = \sum_{\substack{\beta_c > b \\ T < \gamma \le 2T}} \int_b^{\beta_c} \mathrm{d}\sigma = \sum_{\substack{\beta_c > b \\ T < \gamma_c \le 2T}} (\beta_c - b) \tag{7.4}$$

and this quantity is real-valued, we get

$$2\pi \sum_{\substack{\beta_c > b \\ T < \gamma_c \leq 2T}} (\beta_c - b) = \int_T^{2T} \log |\ell(b + \mathrm{i}t)| \,\mathrm{d}t - \int_T^{2T} \log |\ell(a + \mathrm{i}t)| \,\mathrm{d}t + \\ - \int_b^a \arg \ell(\sigma + \mathrm{i}T) \,\mathrm{d}\sigma + \int_b^a \arg \ell(\sigma + 2\mathrm{i}T) \,\mathrm{d}\sigma \\ = \sum_{j=1}^{4} I_j, \tag{7.5}$$

say. To define $\log \ell(s)$ and $\log \mathcal{L}(s)$ we may choose the principal branch of the logarithm on the real axis, as $\sigma \to \infty$; for other points s the value of the logarithm is obtained by continuous variation along line segments (this is in agreement with Lemma 7.2).

We start with the vertical integrals. Obviously,

$$I_1(T, b) := I_1 = \int_T^{2T} \log |\mathcal{L}(b + \mathrm{i}t) - c| \,\mathrm{d}t - T \log |1 - c|. \tag{7.6}$$

By Jensen's inequality the integral is

$$\leq \frac{T}{2} \log \left(\frac{1}{T} \int_T^{2T} |\mathcal{L}(b + \mathrm{i}t)|^2 \,\mathrm{d}t \right) + \mathrm{O}(T).$$

By Corollary 6.11 (which also applies to L-functions satisfying just axioms (1)–(3) as already remarked) this is $\ll T$ for $b > \max\{\frac{1}{2}, 1 - \frac{1}{\mathrm{d}_\mathcal{L}}\}$. Thus we get $I_1(T, b) \ll T$ unconditionally. An immediate consequence of Lindelöf's hypothesis is

$$\int_T^{2T} \left| \mathcal{L}\left(\frac{1}{2} + \mathrm{i}t \right) \right|^2 \,\mathrm{d}t \ll T^{1+\epsilon}$$

for any positive ϵ. Thus, assuming the truth of Lindelöf's hypothesis we get

$$I_1\left(T, \frac{1}{2}\right) \ll \epsilon T \log T.$$

Next we consider I_2. Since $a > 1$ we have

$$\ell(a + \mathrm{i}t) = 1 + \frac{1}{1 - c} \sum_{n=2}^{\infty} \frac{a(n)}{n^{a + \mathrm{i}t}}, \tag{7.7}$$

and in view of (7.3) the absolute value of the series is less than 1 for sufficiently large a. Therefore, we find by the Taylor expansion of the logarithm

$$\log |\ell(a + \mathrm{i}t)| = \mathrm{Re} \sum_{k=1}^{\infty} \frac{(-1)^k}{k(1 - c)^k} \sum_{n_1=2}^{\infty} \cdots \sum_{n_k=2}^{\infty} \frac{a(n_1) \cdots a(n_k)}{(n_1 \cdots n_k)^{a + \mathrm{i}t}}.$$

This leads to the estimate

$$I_2 = \operatorname{Re} \sum_{k=1}^{\infty} \frac{(-1)^k}{k(1-c)^k} \sum_{n_1=2}^{\infty} \cdots \sum_{n_k=2}^{\infty} \frac{a(n_1)\cdots a(n_k)}{(n_1\cdots n_k)^a} \int_T^{2T} \frac{\mathrm{d}t}{(n_1\cdots n_k)^{\mathrm{i}t}}$$

$$\ll \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{n=2}^{\infty} \frac{1}{n^{a-\epsilon}} \right)^k \ll 1 \tag{7.8}$$

for sufficiently large a. It remains to estimate the horizontal integrals I_3, I_4.

Suppose that $\operatorname{Re}\ell(\sigma+\mathrm{i}T)$ has N zeros for $b \le \sigma \le a$. Divide the interval $[b,a]$ into at most $N+1$ subintervals in each of which $\operatorname{Re}\ell(\sigma+\mathrm{i}T)$ is of constant sign. Then

$$|\arg \ell(\sigma+\mathrm{i}T)| \le (N+1)\pi. \tag{7.9}$$

To estimate N let

$$g(z) = \frac{1}{2}\left(\ell(z+\mathrm{i}T) + \overline{\ell(\overline{z}+\mathrm{i}T)} \right).$$

Then we have $g(\sigma) = \operatorname{Re}\ell(\sigma+\mathrm{i}T)$. Let $R = a-b$ and choose T so large that $T > 2R$. Now, $\operatorname{Im}(z+\mathrm{i}T) > 0$ for $|z-a| < T$. Thus $\ell(z+\mathrm{i}T)$, and hence $g(z)$ is analytic for $|z-a| < T$. Let $n(r)$ denote the number of zeros of $g(z)$ in $|z-a| \le r$. Obviously, we have

$$\int_0^{2R} \frac{n(r)}{r}\,\mathrm{d}r \ge n(R) \int_R^{2R} \frac{\mathrm{d}r}{r} = n(R)\log 2.$$

With Jensen's formula (see for example, Titchmarsh [353, Sect. 3.61]),

$$\int_0^{2R} \frac{n(r)}{r}\,\mathrm{d}r = \frac{1}{2\pi} \int_0^{2\pi} \log\left| g\left(a+2R e^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta - \log|g(a)|, \tag{7.10}$$

we deduce

$$n(R) \le \frac{1}{2\pi\log 2} \int_0^{2\pi} \log\left| g\left(a+2R e^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta - \frac{\log|g(a)|}{\log 2}.$$

By (7.7) it follows that: $\log|g(a)|$ is bounded. By Theorem 6.8, in any vertical strip of bounded width,

$$\mathcal{L}(s) \ll |t|^B$$

as $|t| \to \infty$ with a certain positive constant B. Obviously, the same estimate holds for $g(z)$. Thus, the integral above is $\ll \log T$, and $n(R) \ll \log T$. Since the interval (b,a) is contained in the disc $|z-a| \le R$, the number N is less than or equal to $n(R)$. Therefore, with (7.9), we get

$$|I_4| \le \int_b^a |\arg \ell(\sigma+\mathrm{i}T)|\,\mathrm{d}\sigma \ll \log T.$$

Obviously, I_3 can be bounded in the same way.

Collecting all estimates, the assertions of the theorem follow. □

Now we want to include *most* of the c-values into our observations. In view of Lemma 6.7 and Theorem 6.8 there exist positive constants C', T' such that there are no c-values in the region $\sigma < -C', t \geq T'$. Therefore, assume that $b < -C' - 1$ and $T \geq T' + 1$. By the functional equation in the form (6.13),

$$\log|\mathcal{L}(s) - c| = \log|\Delta_{\mathcal{L}}(s)| + \log|\overline{\mathcal{L}}(1-\overline{s})| + \mathrm{O}\left(\frac{1}{|\Delta_{\mathcal{L}}(s)\overline{\mathcal{L}(1-\overline{s})}|}\right).$$

In view of Lemma 6.7

$$\log|\Delta_{\mathcal{L}}(s)| = \left(\frac{1}{2} - \sigma\right)(\,\mathrm{d}_{\mathcal{L}} \log t + \log(\lambda Q^2)) + O\left(\frac{1}{t}\right).$$

Thus

$$\begin{aligned}
&\int_T^{2T} \log|\mathcal{L}(b+\mathrm{i}t) - c|\,\mathrm{d}t \\
&= \left(\frac{1}{2} - b\right)\int_T^{2T}(\,\mathrm{d}_{\mathcal{L}} \log t + \log(\lambda Q^2))\,\mathrm{d}t \\
&\quad + \int_T^{2T} \log|\mathcal{L}(1-b-\mathrm{i}t)|\,\mathrm{d}t + \mathrm{O}(\log T).
\end{aligned}$$

Now suppose that $c \neq 1$. The first integral on the right-hand side is easily calculated by elementary methods. The second integral is small if $-b$ is chosen sufficiently large (see (7.8)). Together with (7.6) we get

$$I_1 = \left(\frac{1}{2} - b\right)\left(\mathrm{d}_{\mathcal{L}} T \log\frac{4T}{e} + T\log(\lambda Q^2)\right) - T\log|1-c| + \mathrm{O}(\log T).$$

By (7.5) and with the estimates for the I_j's from the proof of Theorem 7.1, we obtain

Theorem 7.3. *Assume that $\mathcal{L}(s)$ satisfies the axioms (1)–(3) with $a(1) = 1$ and let $c \neq 1$. Then, for sufficiently large negative b,*

$$\begin{aligned}
2\pi \sum_{T<\gamma_c\leq 2T}(\beta_c - b) = &\left(\frac{1}{2} - b\right)\left(\mathrm{d}_{\mathcal{L}} T\log\frac{4T}{e} + T\log(\lambda Q^2)\right) \\
&- T\log|1-c| + \mathrm{O}(\log T).
\end{aligned}$$

7.2 Riemann–von Mangoldt-Type Formulae

We can rewrite the sum over c-values from Sect. 7.1 as follows:

$$\sum_{\beta_c}(\beta_c - b) = \left(\frac{1}{2} - b\right)\sum_{\beta_c} 1 + \sum_{\beta_c}\left(\beta_c - \frac{1}{2}\right).$$

The first sum on the right counts the number of c-values and the second sum measures the distances of the c-values from the critical line. Let $\mathcal{N}^c(T)$ count the number of c-values of $\mathcal{L}(s)$ with $T < \gamma_c \leq 2T$. Then, subtracting the formula of Theorem 7.3 with $b+1$ instead of b from the formula with b, we obtain

Corollary 7.4. *Assume that $\mathcal{L}(s)$ satisfies the axioms (1)–(3) with $a(1) = 1$. Then, for $c \neq 1$,*

$$\mathcal{N}^c(T) = \frac{\mathrm{d}_{\mathcal{L}}}{2\pi} T \log \frac{4T}{e} + \frac{T}{2\pi} \log(\lambda Q^2) + \mathrm{O}(\log T).$$

Furthermore,

Corollary 7.5. *Assume that $\mathcal{L}(s)$ satisfies the axioms (1)–(3) with $a(1) = 1$. Then, for $c \neq 1$,*

$$\sum_{T<\gamma_c\leq 2T} \left(\beta_c - \frac{1}{2}\right) = -\frac{T}{2\pi} \log|1-c| + \mathrm{O}(\log T).$$

Thus, for $c \neq 1$ satisfying $|1-c| \neq 1$, the c-values, weighted with respect to their distance to the critical line, lie asymmetrically distributed. Nevertheless, our next aim is to show that *most* of the c-values lie close to the critical line. Unfortunately, for this purpose we have to assume the Lindelöf hypothesis. Define the counting functions (according multiplicities)

$$\mathcal{N}_+^c(\sigma, T) = \sharp\{\varrho_c : T < \gamma_c \leq 2T, \beta_c > \sigma\},$$

and

$$\mathcal{N}_-^c(\sigma, T) = \sharp\{\varrho_c : T < \gamma_c \leq 2T, \beta_c < \sigma\}.$$

Then

Theorem 7.6. *Assume that $\mathcal{L}(s)$ satisfies the axioms (1)–(3) with $a(1) = 1$ and let $c \neq 1$. Then, for any $\sigma > \max\{\frac{1}{2}, 1 - \frac{1}{\mathrm{d}_{\mathcal{L}}}\}$,*

$$\mathcal{N}_+^c(\sigma, T) \ll T, \tag{7.11}$$

and assuming the truth of the Lindelöf hypothesis, for any $\delta > 0$,

$$\mathcal{N}_-^c\left(\frac{1}{2} - \delta, T\right) + \mathcal{N}_+^c\left(\frac{1}{2} + \delta, T\right) \ll \delta T \log T.$$

Proof. First of all, let $\sigma > \max\{\frac{1}{2}, 1 - \frac{1}{\mathrm{d}_{\mathcal{L}}}\}$ and fix $\sigma_1 \in (\max\{\frac{1}{2}, 1 - \frac{1}{\mathrm{d}_{\mathcal{L}}}\}, \sigma)$. Then

$$\mathcal{N}_+^c(\sigma, T) \leq \frac{1}{\sigma - \sigma_1} \sum_{\substack{\beta_c > \sigma \\ T<\gamma_c\leq 2T}} (\beta_c - \sigma_1).$$

The sum on the right hand-side is less than or equal to

$$\sum_{\substack{\beta_c > \sigma_1 \\ T<\gamma_c\leq 2T}} (\beta_c - \sigma_1) \ll \int_T^{2T} \log|\ell(\sigma_1 + \mathrm{i}t)| \, \mathrm{d}t + \mathrm{O}(\log T),$$

where we used Littlewood's Lemma 7.2 and the techniques from Sect. 7.1 for the latter inequality. In view of the unconditional estimate for (7.6) in the proof of Theorem 7.1 we obtain (7.11). Assuming the truth of the Lindelöf hypothesis, we get analogously

$$\mathcal{N}_+^c\left(\frac{1}{2}+\delta, T\right) \ll \frac{\epsilon}{\delta} T \log T \tag{7.12}$$

for any positive ϵ.

Next we consider $\mathcal{N}_-^c$. Let b be a sufficiently large constant. We have

$$\sum_{\substack{\beta_c \geq \frac{1}{2}-\delta \\ T<\gamma_c\leq 2T}} (\beta_c - b) \leq \left(\frac{1}{2}-b\right) \sum_{\substack{\beta_c \geq \frac{1}{2}-\delta \\ T<\gamma_c\leq 2T}} 1 + \sum_{\substack{\beta_c \geq \frac{1}{2} \\ T<\gamma_c\leq 2T}} \left(\beta_c - \frac{1}{2}\right).$$

Hence

$$\begin{aligned}\sum_{T<\gamma_c\leq 2T} (\beta_c - b) &= \sum_{\substack{\beta_c < \frac{1}{2}-\delta \\ T<\gamma_c\leq 2T}} \left(\frac{1}{2} - b + \beta_c - \frac{1}{2}\right) + \sum_{\substack{\beta_c \geq \frac{1}{2}-\delta \\ T<\gamma_c\leq 2T}} (\beta_c - b) \\ &\leq \left(\frac{1}{2}-b\right)\mathcal{N}^c(T) + \sum_{\substack{\beta_c < \frac{1}{2}-\delta \\ T<\gamma_c\leq 2T}} \left(\beta_c - \frac{1}{2}\right) \\ &\quad + \sum_{\substack{\beta_c > \frac{1}{2} \\ T<\gamma_c\leq 2T}} \left(\beta_c - \frac{1}{2}\right).\end{aligned}$$

By Theorem 7.1, the second sum on the right is bounded by $\epsilon T \log T$. Since any term in the first sum on the right is $< -\delta$, we obtain

$$-\delta \mathcal{N}_-^c\left(\frac{1}{2}-\delta, T\right) \geq \sum_{T<\gamma_c\leq 2T} (\beta_c - b) - \left(\frac{1}{2}-b\right)\mathcal{N}^c(T) + \mathrm{O}(\epsilon T \log T).$$

In view of Theorem 7.3 and Corollary 7.4 we get

$$\mathcal{N}_-^c\left(\frac{1}{2}-\delta, T\right) \ll \frac{\epsilon}{\delta} T \log T.$$

This is the same bound as for $\mathcal{N}_+^c$ in (7.12). Putting $\epsilon = \delta^2$ we obtain the assertion of the theorem. □

Thus, subject to the truth of the Lindelöf hypothesis, we get by comparing Corollary 7.4 and Theorem 7.6, for any positive ϵ,

$$\mathcal{N}_-^c\left(\frac{1}{2}-\epsilon, T\right) + \mathcal{N}_+^c\left(\frac{1}{2}+\epsilon, T\right) \ll \epsilon \mathcal{N}^c(T),$$

so the c-values are clustered around the critical line for any c. This extraordinary value distribution shows that if the Lindelöf hypothesis for $\mathcal{L}(s)$ is true, the critical line is a so-called Julia line from the classical theory of functions. Julia [151] improved the Big Picard theorem by showing: if the analytic function f has an essential singularity at a, then there exist a real θ_0 and at most one complex number z such that for every sufficiently small $\epsilon > 0$

$$\mathbb{C} \setminus \{z\} \subset f(\{a + r\exp(\mathrm{i}\theta) : |\theta - \theta_0| < \epsilon, 0 < r < \epsilon\});$$

the ray $\{a + r\exp(\mathrm{i}\theta_0) : r > 0\}$ is called a Julia line. For more details on Julia's theorem we refer to Burckel [48, Sect. XII.4].

The distribution of the c-values close to the real axis is quite regular. It can be shown that there is always a c-value in some neighbourhood of any trivial zero of $\mathcal{L}(s)$ with sufficiently large negative real part, and with finitely many exceptions there are no other in the left half-plane. The main ingredients for the proof are Rouché's theorem (Theorem 8.1) and Stirling's formula (2.17). With regard to (6.6), thus the number of these c-values having real part in $[-R, 0]$ is asymptotically $\frac{1}{2}\,\mathrm{d}_{\mathcal{L}}R$. On the other side, by (7.3) the behaviour nearby the positive real axis is very regular. Note that all results from above hold as well with respect to c-values from the lower half-plane.

Now let $N_{\mathcal{L}}^c(\sigma, T)$ count the number of c-values $\varrho_c = \beta_c + \mathrm{i}\gamma_c$ of $\mathcal{L}(s)$ satisfying $\beta_c > \sigma, |\gamma_c| \le T$. Using Corollary 7.4 with $2^{-n}T$ for $n \in \mathbb{N}$ instead of T and adding up, we get, for fixed $\sigma \le 0$,

$$\begin{aligned} N_{\mathcal{L}}^c(\sigma, T) &= 2\sum_{n=1}^{\infty} \mathcal{N}^c(\sigma, 2^{-n}T) \\ &= \left(\frac{\mathrm{d}_{\mathcal{L}}}{\pi} T \log\frac{T}{e} + \frac{T}{\pi}\log(\lambda Q^2)\right)\sum_{n=1}^{\infty}\frac{1}{2^n} \\ &\quad + \frac{\mathrm{d}_{\mathcal{L}}}{\pi} T \sum_{n=1}^{\infty} \frac{\log 4 - n\log 2}{2^n} + \mathrm{O}(\log T). \end{aligned}$$

The appearing infinite series are equal to 1 and 0, respectively. Hence, this summation removes the factor 4 in the logarithmic term, and we have proved

Theorem 7.7. *Assume that $\mathcal{L}(s)$ satisfies the axioms (1)–(3) with $a(1) = 1$. For any fixed $\sigma \le 0$ and any complex $c \ne 1$,*

$$N_{\mathcal{L}}^c(\sigma, T) = \frac{\mathrm{d}_{\mathcal{L}}}{\pi} T \log\frac{T}{e} + \frac{T}{\pi}\log(\lambda Q^2) + \mathrm{O}(\log T).$$

The case $c = \sigma = 0$ (the non-trivial zeros of $\mathcal{L}(s)$) is a precise Riemann–von Mangoldt formula (1.5). Similar results were obtained by Perelli [289] and Lekkerkerker [214] for other classes of Dirichlet series. It should be noticed that Tsang [355] investigated the number of c-values of $\zeta(s)$ with respect to short intervals for the imaginary parts. Let $\sigma < \frac{1}{2}$, $T^{(1/2)+\epsilon} \le H \le T$, and

c be a complex number satisfying $\epsilon \leq |1-c| \leq \frac{1}{\epsilon}$ with sufficiently small ϵ. Assuming the truth of the Riemann hypothesis, Tsang proved

$$N_\zeta^c(\sigma, T+H) - N_\zeta^c(\sigma, T) \sim \frac{H}{\pi} \log \frac{T}{2\pi}$$

with an explicit error term depending on ϵ, H and T; his result holds unconditionally provided $\sigma \leq 0$.

An immediate consequence of Theorem 7.7 is that the multiplicity of nontrivial zeros ϱ of $\mathcal{L}(s)$ is bounded by $1 + \log|\gamma|$. More advanced results on the multiplicities of the zeros were obtained by Ivić [142] in the case of the Riemann zeta-function.

We conclude with another result from Selberg [323] for L-functions from $\mathcal{S}$. Assuming the truth of the Riemann hypothesis and of conjecture A, he obtained for $c \neq 1$ the asymptotic formula

$$\sum_{\substack{\beta_c > \frac{1}{2} \\ 0 < \gamma_c < T}} \left(\beta_c - \frac{1}{2}\right) = \frac{\sqrt{n_\mathcal{L}}}{4\pi^{3/2}} T \sqrt{\log\log T} + \frac{T}{4\pi} \log \frac{|c|}{1-|c|^2} + \\ + \mathrm{O}\left(T \frac{(\log\log\log T)^3}{\sqrt{\log\log T}}\right).$$

Furthermore, for

$$\sigma(T) := \frac{1}{2} - \nu \frac{\sqrt{\log\log T}}{\log T} \qquad \text{and} \qquad \xi := \frac{d_\mathcal{L}\nu}{2\sqrt{\pi n_\mathcal{L}}}$$

with positive ν, he proved

$$\sum_{\substack{\beta_c > \sigma(T) \\ 0 < \gamma_c < T}} (\beta_c - \sigma(T)) \\ = \frac{1}{2}\sqrt{\frac{n_\mathcal{L}}{\pi}} \left(\frac{\exp(-\pi\xi^2)}{2\pi} + \xi - \xi \int_\xi^\infty \exp(-\pi x^2)\,\mathrm{d}x\right) T\sqrt{\log\log T} \\ + \left(\log|c| \int_\xi^\infty \exp(-\pi x^2)\,\mathrm{d}x - \log|1-c|\right) \frac{T}{2\pi} \\ + \mathrm{O}\left(T \frac{(\log\log\log T)^3}{\sqrt{\log\log T}}\right).$$

From these results Selberg deduced that about half of the c-values lie to the left of the critical line, statistically well distributed at distances of order

$$\frac{\sqrt{\log\log T}}{\log T}$$

off $\sigma = \frac{1}{2}$, and that

$$N_\mathcal{L}^c(\sigma(T), T) \sim N_\mathcal{L}^c(T) \int_{-\xi}^\infty \exp(-\pi x^2)\,\mathrm{d}x.$$

Most of the remaining c-values lie rather close to the critical line at distances of order not exceeding

$$\frac{(\log\log\log T)^3}{\log T\sqrt{\log\log T}}.$$

This improves some previous results of Selberg (unpublished) and Joyner [150] and gives a much more detailed description of the clustering of the c-values around the critical line.

In the exceptional case $c = 1$ one has to consider the function

$$\ell(s) = \frac{q^s}{a(q)}(\mathcal{L}(s) - 1),$$

where $q \geq 1$ is the least integer such that $a(q) \neq 0$. Then, by a similar reasoning as in the proof of Theorem 7.7, one gets analogous results. For the special case of the zeta-function this is carried out in Steuding [348, 349] where Levinson's method is applied to Epstein zeta-functions. These methods also allow to drop the condition $a(1) = 1$.

7.3 Nevanlinna Theory

Nevanlinna theory was created by Nevanlinna [281] in the 1920's to tackle the value-distribution of meromorphic functions in general. We recall some basic facts which, for example, can be found in Nevanlinna's monograph [281, Chaps. VI and IX].

Let f be a meromorphic function and denote the number of poles of $f(s)$ in $|s| \leq r$ by $\mathbf{n}(f, \infty, r)$ (counting multiplicities). The number of c-values of f is given by

$$\mathbf{n}(f, c, r) = \mathbf{n}\left(\frac{1}{f - c}, \infty, r\right).$$

The integrated counting function is

$$\mathbf{N}(f, c, r) = \int_0^r (\mathbf{n}(f, c, \varrho) - \mathbf{n}(f, c, 0))\frac{\mathrm{d}\varrho}{\varrho} + \mathbf{n}(f, c, 0)\log r.$$

The proximity function is defined by

$$\mathbf{m}(f, r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r\exp(\mathrm{i}\theta))|\,\mathrm{d}\theta,$$

and, for $c \in \mathbb{C}$, by

$$\mathbf{m}(f, c, r) = \mathbf{m}\left(\frac{1}{f - c}, r\right),$$

where $\log^+ x := \max\{0, \log x\}$. The function $\mathbf{m}(f, c, r)$ indicates how close $f(s)$ is to the value c on the circle $|s| = r$. The characteristic function of f is defined by

$$\mathbf{T}(f, r) = \mathbf{N}(f, \infty, r) + \mathbf{m}(f, r).$$

Furthermore, let

$$\mathbf{T}(f,c,r)=\mathbf{N}(f,c,r)+\mathbf{m}(f,c,r)$$

for $c\in\mathbb{C}$. The first main theorem in Nevanlinna theory states states that $\mathbf{T}(f,c,r)$ differs from the characteristic function by a bounded quantity:

Theorem 7.8. *Let f be a meromorphic function and let c be any complex number. Then*

$$\mathbf{T}(f,c,r)=\mathbf{T}(f,r)+\mathrm{O}(1),$$

where the error term depends on f and c.

The proof relies on Jensen's formula (7.10).

Thus, $\mathbf{T}(f,c,r)$ for different values of c is invariant up to additive terms that are bounded. The invariant, the characteristic function $\mathbf{T}(f,r)$, encodes information about the analytic behaviour of f.

The quantity

$$\delta(f,c):=1-\limsup_{r\to\infty}\frac{\mathbf{N}(f,c,r)}{\mathbf{T}(f,r)}$$

is called the deficiency of the value c of f. This deficiency is positive only if there are *relatively few* c-values. The second main theorem in Nevanlinna theory implies the so-called deficiency relation which states that

$$\sum_{c\in\mathbb{C}\cup\{\infty\}}\delta(f,c)\le 2\,;$$

note that only for countably many values of c the deficiency can differ from zero. Another consequence is the Big Picard theorem.

Only recently Ye [372] computed the Nevanlinna functions for the Riemann zeta-function. Without big effort we can extend his results to the class of Dirichlet series under investigation. The Nevanlinna functions for those $\mathcal{L}(s)$ are determined by the Gamma-factors in the functional equation.

First, let $\sigma_0>1$ be fixed. We write $s=r\exp(\mathrm{i}\theta)$, so $\sigma=r\cos\theta$. It is easily seen that

$$\frac{1}{2\pi}\int_{\{\theta:r\cos\theta>\sigma_0\}}\log^+|\mathcal{L}(r\exp(\mathrm{i}\theta))|\,\mathrm{d}\theta\ll 1.$$

Further, in view of Theorem 6.8,

$$\frac{1}{2\pi}\int_{\{\theta:1-\sigma_0\le r\cos\theta\le\sigma_0\}}\log^+|\mathcal{L}(r\exp(\mathrm{i}\theta))|\,\mathrm{d}\theta\ll\log r;$$

note that the Lebesgue measure of the set

$$\{\theta\in[0,2\pi]:\sigma=r\cos\theta\in[1-\sigma_0,\sigma_0]\}$$

is bounded by $\frac{1}{r}$. Finally, for $\sigma\le 1-\sigma_0$ we deduce from the functional equation in the form (6.13) that

$$\log^+ |\mathcal{L}(r\exp(\mathrm{i}\theta))| \leq \sum_{j=1}^{f} \Big\{ \log^+ |\Gamma(\lambda_j(1 - r\exp(\mathrm{i}\theta)) + \overline{\mu_j})| + \log^+ |\Gamma(\lambda_j r\exp(\mathrm{i}\theta) + \mu_j)| \Big\} + \mathrm{O}(r).$$

Now we shall use Ye's decomposition of the Gamma-function. For any $z = r\exp(\mathrm{i}\theta)$, there is an integer n_0 with $n_0 < r \leq n_0 + 1$ such that

$$\frac{1}{\Gamma(z)} = F_1(z)\,F_2(z) \quad \text{with} \quad F_1(z) := z\left(\gamma z - \sum_{n=1}^{2n_0} \frac{z}{n}\right),$$

where γ is the Euler–Mascheroni constant, and $F_2(z)$ is an entire function with $\mathbf{m}(F_2, r) \ll r$. The order of growth of $\Gamma(z)$ is ruled by the order of growth of $F_1(z)$. Ye computed

$$\log |F_1(z)| = -r\log r\cos\theta + \mathrm{O}(r).$$

If λ is a positive real number and μ an arbitrary complex number, Ye's estimate leads to

$$\frac{1}{2\pi}\int_{\{\theta : r\cos\theta < 1-\sigma_0\}} \log^+ |\Gamma(\lambda(1 - r\exp(\mathrm{i}\theta)) + \mu)|\,\mathrm{d}\theta \leq \frac{\lambda}{2\pi}\int_{-\pi/2}^{\pi/2} r\log r\cos\theta\,\mathrm{d}\theta + \mathrm{O}(r) = \frac{\lambda}{\pi} r\log r + \mathrm{O}(r),$$

and, similarly,

$$\frac{1}{2\pi}\int_{\{\theta : r\cos\theta < 1-\sigma_0\}} \log^+ |\Gamma(\lambda r\exp(\mathrm{i}\theta)) + \mu)|\,\mathrm{d}\theta \leq \frac{\lambda}{\pi} r\log r + \mathrm{O}(r).$$

Thus, we get

$$\frac{1}{2\pi}\int_{\{\theta : r\cos\theta < 1-\sigma_0\}} \log^+ |\mathcal{L}(r\exp(\mathrm{i}\theta))|\,\mathrm{d}\theta \leq \frac{\mathrm{d}_{\mathcal{L}}}{\pi} r\log r + \mathrm{O}(r).$$

Adding the estimates for the other cases, we obtain for the proximity function of $\mathcal{L}(s)$

$$\mathbf{m}(\mathcal{L}, r) \leq \frac{\mathrm{d}_{\mathcal{L}}}{\pi} r\log r + \mathrm{O}(r).$$

Since $\mathcal{L}(s)$ is regular except for at most a pole at $s = 1$,

$$\mathbf{N}(\mathcal{L}, \infty, r) \ll \int_1^r \frac{\mathrm{d}\varrho}{\varrho} = \log r. \tag{7.13}$$

Thus, we get

$$\mathbf{T}(\mathcal{L}, r) \leq \frac{\mathrm{d}_{\mathcal{L}}}{\pi} r\log r + \mathrm{O}(r). \tag{7.14}$$

It follows from Theorem 7.7 that:

$$\mathbf{N}(\mathcal{L},0,r)=\frac{\mathrm{d}_{\mathcal{L}}}{\pi}r\log r+\mathrm{O}(r). \tag{7.15}$$

The first main Theorem 7.8 implies

$$\mathbf{N}(\mathcal{L},0,r)\le\mathbf{T}(\mathcal{L},0,r)=\mathbf{T}(\mathcal{L},r)+\mathrm{O}(1).$$

In view of (7.14) and (7.15) we get an asymptotic formula for the characteristic function:

Theorem 7.9. *For $\mathcal{L}$ satisfying axioms (1)–(3) with $a(1)=1$,*

$$\mathbf{T}(\mathcal{L},r)=\frac{\mathrm{d}_{\mathcal{L}}}{\pi}r\log r+\mathrm{O}(r).$$

We deduce from this and (7.13) for the deficiency value of infinity:

$$\delta(\mathcal{L},\infty)=1-\limsup_{r\to\infty}\frac{\mathbf{N}(\mathcal{L},\infty,r)}{\mathbf{T}(\mathcal{L},r)}=1.$$

In view of Theorem 7.7 the deficiency values for $c\neq 1,\infty$ are equal to zero.

In combination with Theorem 7.7 the asymptotic formula of the theorem shows that the counting function $\mathbf{N}(\mathcal{L},c,r)$ *dominates* the proximity function $\mathbf{m}(\mathcal{L},c,r)$, at least for any complex value $c\neq 1$. In the exceptional case $c=1$, by the first main Theorem 7.8, we may deduce from Theorem 7.9 that

$$N_{\mathcal{L}}^{1}(T)\le\frac{\mathrm{d}_{\mathcal{L}}}{\pi}T\log T+\mathrm{O}(T).$$

A more sophisticated analysis would show that this is actually an equality. However, we do not go into the details. In Sect. 9.7 we return to the distribution of c-values in the half-plane of absolute convergence.

We conclude with a description of the analytic behaviour of the Dirichlet series $\mathcal{L}$ under investigation in terms of the notion of finite order. A positive function $t(r)$ is said to be of finite order λ if

$$\limsup_{r\to\infty}\frac{\log t(r)}{\log r}=\lambda;$$

$t(r)$ is of maximum, mean or minimum type of order λ if the upper limit

$$\limsup_{r\to\infty}\frac{t(r)}{r^{\lambda}}$$

is infinite, finite and positive, or zero. A meromorphic function is defined to be of the same order and the same type as its characteristic function $\mathbf{T}(r,f)$. Thus, by Theorem 7.9, we get

Corollary 7.10. *Every $\mathcal{L}$ satisfying the axioms (1)–(3) with $a(1)=1$ is of order one and of maximum type.*

7.4 Uniqueness Theorems

We say that two meromorphic functions f and g share a value $c \in \mathbb{C} \cup \{\infty\}$ if the sets of pre-images of the value c under f and under g are equal, for short

$$f^{-1}(c) := \{s \in \mathbb{C} : f(s) = c\} = g^{-1}(c). \tag{7.16}$$

We say that f and g share the value c counting multiplicities (CM) if (7.16) holds and if the roots of the equations

$$f(s) = c \quad \text{and} \quad g(s) = c$$

have the same multiplicities; if there is no restriction on the multiplicities, f and g are said to share the value c ignoring multiplicities (IM). Nevanlinna [280] proved two fundamental results on shared values. His remarkable five-point theorem states that any two non-constant meromorphic functions which share five distinct values are equal. Since $f(s) = \exp(s)$ and $g(s) = \exp(-s)$ share the four values $0, \pm 1, \infty$, the number 5 in Nevanlinna's statement is best possible. If multiplicities are taken into account, Nevanlinna proved that if two meromorphic functions f and g share four distinct values $c_1, \ldots, c_4$ CM, then either $f \equiv g$ or there exists a linear fractional transformation M such that $g \equiv M \circ f$ and

$$M(c_1) = c_1, \quad M(c_2) = c_2, \quad M(c_3) = c_4, \quad \text{and} \quad M(c_4) = c_3;$$

in the latter case f and g do not assume the values c_3 and c_4. Also the number 4 for the upper bound of shared values CM is best possible. The result can be sharpened if two of the four values are allowed to be shared IM (see [111]). In [347], Steuding, investigated how many values L-functions can share. In this special case better estimates are possible than those which Nevanlinna's theorems provide. It is expected that *independent* L-functions cannot share any complex value which is actually taken.

First of all, we trivially note that two L-functions from the Selberg class share the value ∞ CM if and only if both are entire or if they both have a pole at $s = 1$ of the same order (other poles cannot occur), e.g., the Riemann zeta-function $\zeta(s)$ and a Dedekind zeta-function to a quadratic number field. If the orders of the poles differ, they share the value ∞ IM. Further, we observe that two different L-functions in the Selberg class cannot share the value zero CM. This follows immediately from a theorem of M.R. Murty and V.K. Murty [268]. To see that denote the non-trivial zeros of $\mathcal{L} \in \mathcal{S}$ by ϱ and let $m_{\mathcal{L}}(\varrho)$ be the multiplicity of ϱ. Further, define for $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}$ the function

$$D_{\mathcal{L}_1,\mathcal{L}_2}(T) = \sum_{\varrho} |m_{\mathcal{L}_1}(\varrho) - m_{\mathcal{L}_2}(\varrho)|,$$

where the summation is taken over all non-trivial zeros ϱ of $\mathcal{L}_1$ and $\mathcal{L}_2$ (counting multiplicities). Then M.R. Murty and V.K. Murty proved that $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}$ are either equal or

$$\liminf_{T\to\infty} \frac{1}{T} D_{\mathcal{L}_1,\mathcal{L}_2}(T) > 0$$

(see also the related result of Bombieri and Perelli (6.8)). However, the trivial example $\zeta(s)$ and $\zeta(s)^2$ shows that different elements of $\mathcal{S}$ can share the value zero IM.

Concerning CM-shared values we shall prove that two different Dirichlet series satisfying our axioms do not share any complex value CM. For sharing values IM we shall only obtain an improvement of the five-point theorem under an additional assumption on the number of distinct c-values. For this purpose let $\tilde{N}_{\mathcal{L}}^{c}(T)$ count the number of distinct roots ϱ_c of the equation $\mathcal{L}(s) = c$ lying in the rectangle $0 \leq \sigma \leq 1, |t| \leq T$.

Theorem 7.11. *Assume that $\mathcal{L}_1, \mathcal{L}_2$ satisfy the axioms (1)–(3) with $a(1) = 1$.*

(i) If $\mathcal{L}_1, \mathcal{L}_2$ share a value $c \neq \infty$ CM, then $\mathcal{L}_1 \equiv \mathcal{L}_2$.
(ii) If $\mathcal{L}_1, \mathcal{L}_2$ satisfy the same functional equation and share two distinct values $c_1, c_2 \neq \infty$ IM such that

$$\liminf_{T\to\infty} \frac{\tilde{N}_{\mathcal{L}_j}^{c_1}(T) + \tilde{N}_{\mathcal{L}_j}^{c_2}(T)}{N_{\mathcal{L}_j}^{c_1}(T) + N_{\mathcal{L}_j}^{c_2}(T)} > \frac{1}{2} + \epsilon \tag{7.17}$$

for some positive ϵ with either $j = 1$ or 2, then $\mathcal{L}_1 \equiv \mathcal{L}_2$.

We briefly discuss the second assertion of the theorem. Condition (7.17) reflects that more than 50% of the c_1- and c_2-values of $\mathcal{L}_j(s)$ are supposed to be distinct. It should be noted that such conditions are very difficult to verify. For instance, Farmer [78] proved that more than 63% of the zeros of $\zeta(s)$ are distinct; however, any extension to L-functions of larger degree seems to be hard to realize.

Proof. We start with the first assertion. In view of Theorem 7.7 two L-functions satisfying the axioms (1)–(3) can only share a value $c \neq \infty$ CM if they have the same degree, d say. First of all assume that $\mathcal{L}_1, \mathcal{L}_2$ are both entire functions and share the value $c \neq \infty$ CM. Define the function

$$\ell(s) = \frac{\mathcal{L}_1(s) - c}{\mathcal{L}_2(s) - c}.$$

Since $\mathcal{L}_1(s)$ assumes the value c if and only if $\mathcal{L}_2(s) = c$ and since for any such root the multiplicities coincide, $\ell(s)$ is a non-vanishing entire function. In view of the first main Theorem 7.8 and Theorem 7.9

$$\begin{aligned}\mathbf{T}(\mathcal{L}_2, c, r) = \mathbf{T}(\mathcal{L}_2, r) + \mathrm{O}(1) &= \frac{\mathrm{d}}{\pi} r \log r + \mathrm{O}(r)\\ &= \mathbf{T}(\mathcal{L}_1 - c, r) + \mathrm{O}(r).\end{aligned}$$

For a meromorphic function f denote its order by $\lambda(f)$. Then it follows that:

$$\lambda\left(\frac{1}{\mathcal{L}_1 - c}\right) = \lambda(\mathcal{L}_2 - c) = \lambda(\mathcal{L}_2) = 1.$$

It is easily seen that the order of a finite product of functions of finite order is less than or equal to the maximum of the order of the factors. Thus $\lambda(\ell) \leq 1$. By Hadamard's factorization theorem (see [281, Sect. VIII.2]) this implies that $\ell(s)$ is of the form

$$\ell(s) = \exp(P(s)),$$

where P is a polynomial of degree at most $\lambda(\ell) \leq 1$. Since $\mathcal{L}_j(s)$ tends to one as $s \to \infty$ for $j = 1, 2$, we have

$$\lim_{s\to\infty} \ell(s) = \frac{1-c}{1-c} = 1.$$

This implies that the polynomial P is vanishing identically, which implies $\mathcal{L}_1 \equiv \mathcal{L}_2$.

If $\mathcal{L}_1(s)$ or $\mathcal{L}_2(s)$ has a pole at $s = 1$ of order k, we may replace $\mathcal{L}_j(s)$ by $(s-1)^k \mathcal{L}_j(s)$ and repeat the argument from above. This proves the first assertion.

Now we shall prove the second statement. If $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy the same functional equation, they both have the same degree, d say. Now consider the function

$$\ell(s) := \mathcal{L}_1(s) - \mathcal{L}_2(s).$$

Obviously, also $\ell(s)$ satisfies the common functional equation for the $\mathcal{L}_j$'s. Then the number $N_\ell(T)$ of zeros of $\ell(s)$ in the rectangle $0 \leq \sigma \leq 1, |t| \leq T$ (counting multiplicities) is asymptotically given by

$$N_\ell(T) \sim \frac{\mathrm{d}}{\pi} T \log T. \tag{7.18}$$

Now suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ share two distinct complex values c_1, c_2 IM. Then $\ell(s)$ vanishes also for the pre-images of the c_k's. Hence, we obtain a lower bound for the number of zeros of $\ell(s)$ in terms of the c_1- and c_2-value counting functions, namely

$$N_\ell(T) \geq \tilde{N}^{c_1}_{\mathcal{L}_j}(T) + \tilde{N}^{c_2}_{\mathcal{L}_j}(T), \tag{7.19}$$

where we can take $j = 1$ or $j = 2$. Taking into account Theorem 7.7 and (7.18) we can replace $N_\ell(T)$ by

$$\frac{1}{2}\left(N^{c_1}_{\mathcal{L}_j}(T) + N^{c_2}_{\mathcal{L}_j}(T)\right) + \mathrm{O}(T).$$

Thus we can rewrite (7.19) as

$$\frac{\tilde{N}^{c_1}_{\mathcal{L}_j}(T) + \tilde{N}^{c_2}_{\mathcal{L}_j}(T)}{N^{c_1}_{\mathcal{L}_j}(T) + N^{c_2}_{\mathcal{L}_j}(T)} \leq \frac{1}{2} + o(1).$$

This contradicts (7.17). Hence $\mathcal{L}_1$ and $\mathcal{L}_2$ can share at most one value $c \in \mathbb{C}$. Theorem 7.11 is proved. □

The functions $\mathcal{L}(s)$ and $\mathcal{L}(s)^2$ share the value zero IM. This is a special example for two reasons. First, these functions are *not* independent in the sense that they have the same primitive functions in their factorizations. Second, they share the zeros. Bombieri and Hejhal [40] proved, assuming some widely believed but yet unproved hypotheses, that almost all zeros of pairwise independent L-functions are distinct. Of course, we expect the same to hold for other c-values too. With respect to condition (7.17) this leads us to conjecture that zero is the only possible shared value and that this happens only in cases of *dependent* L-functions.

We conclude with a few words about the significance of such studies. Some problems in arithmetic (see Chap. 13.7) could be solved if one could show that, given distinct primitive L-functions $\mathcal{L}_1(s), \ldots, \mathcal{L}_m(s)$ (in the sense of the Selberg class), then $\mathcal{L}_j(\varrho_k) = 0$ holds only for $j = k$, where the ϱ_k denote the non-trivial zeros of $\mathcal{L}_k(s)$. Clearly, this would also imply the unique factorization into primitive elements.