3

A General Approach to Stabilization

The first Almighty Cause acts not by partial,
but by gen'ral laws.
A. Pope

In this short chapter we develop a general approach for stabilized evaluation of a closed unbounded operator using von Neumann's theorem and the spectral theory of *bounded* self-adjoint operators as basic tools. Recall that the essential problem of evaluating an unbounded linear operator $L : \mathcal{D}(L) \subseteq H_1 \to H_2$ at a vector $x \in \mathcal{D}(L)$ given only an approximation $x^\delta \in H_1$ satisfying $\|x - x^\delta\| \le \delta$ is that in general $x^\delta \notin \mathcal{D}(L)$, and even if $x^\delta \in \mathcal{D}(L)$ one can not guarantee that $Lx^\delta \to Lx$ as $\delta \to 0$, since L is discontinuous. Our goal in this chapter is to develop general approximations to Lx of the form $S_\alpha x$, where S_α is a bounded linear operator depending on a parameter $\alpha > 0$. Since S_α is bounded the vector $S_\alpha x^\delta$ is defined for all $x^\delta \in H_1$ and for each $\alpha > 0$ the mapping $x^\delta \mapsto S_\alpha x^\delta$ is stable. The next ingredient in the stabilization strategy is a parameter choice scheme $\alpha = \alpha(\delta)$ that ensures the regularity of the approximations, that is, so that

$$\|S_{\alpha(\delta)} x^\delta - Lx\| \to 0 \quad \text{as} \quad \delta \to 0.$$

3.1 A General Method

The key to our development is von Neumann's theorem. Recall that this states that if $L : \mathcal{D}(L) \subseteq H_1 \to H_2$ is a closed densely defined linear operator, then the operator $\widetilde{L} : H_1 \to H_2$ defined by

$$\widetilde{L} = (I + L^* L)^{-1}$$

is bounded and self-adjoint with $\sigma(\widetilde{L}) \subseteq [0, 1]$ and the operator $L\widetilde{L} : H_1 \to H_2$ is bounded with $\|L\widetilde{L}\| \le 1$. In the same way the operator $\widehat{L} : H_2 \to H_2$ defined by

$$\widehat{L} = (I + LL^*)^{-1}$$

is bounded and self-adjoint and the operator $L^*\widehat{L} : H_2 \to H_1$ is bounded.

Before proceeding we notice a simple property of these operators.

Theorem 3.1. *If $f \in C[0,1]$ and $x \in \mathcal{D}(L)$, then $f(\widetilde{L})x \in \mathcal{D}(L)$ and $f(\widehat{L})Lx = Lf(\widetilde{L})x$.*

Proof. For any $x \in H_1$ we have $\widetilde{L}x \in \mathcal{D}(L^*L)$ and $(I+L^*L)\widetilde{L}x = x$. Therefore, if $x \in \mathcal{D}(L)$, then

$$Lx = L(I + L^*L)\widetilde{L}x = (I + LL^*)L\widetilde{L}x.$$

We then have

$$\widehat{L}Lx = (I + LL^*)^{-1}Lx = L\widetilde{L}x$$

and from this it follows that

$$p(\widehat{L})Lx = Lp(\widetilde{L})x$$

for any polynomial p. If $f \in C[0,1]$, then by the Weierstrass approximation theorem there is a sequence of polynomials $\{p_n\}$ that converges uniformly to f. For $x \in \mathcal{D}(L)$ we then have

$$f(\widehat{L})Lx = \lim_n p_n(\widehat{L})Lx = \lim_n Lp_n(\widetilde{L})x.$$

For any $y \in \mathcal{D}(L^*)$ it follows that

$$\begin{aligned} \langle f(\widehat{L})Lx, y\rangle &= \lim_n \langle Lp_n(\widetilde{L})x, y\rangle \\ &= \lim_n \langle p_n(\widetilde{L})x, L^*y\rangle \\ &= \langle f(\widetilde{L})x, L^*y\rangle. \end{aligned}$$

Therefore, using Theorem 2.7, we find that

$$f(\widetilde{L})x \in \mathcal{D}(L^{**}) = \mathcal{D}(L)$$

and

$$Lf(\widetilde{L})x = f(\widehat{L})Lx. \quad \square$$

In the same way one finds that $f(\widetilde{L})L^*x = L^*f(\widehat{L})x$ for all $x \in \mathcal{D}(L^*)$.

Corollary 3.2. *If $f(t) = \sqrt{t}g(t)$ where $g \in C[0,1]$, then $Lf(\widetilde{L})$ is bounded and*

$$\|Lf(\widetilde{L})\| \le \|g\|_\infty$$

Proof. If $x, z \in \mathcal{D}(L)$, then

$$\begin{aligned}\langle (Lf(\widetilde{L}))^*(Lf(\widetilde{L}))x, z\rangle &= \langle Lf(\widetilde{L})x, Lf(\widetilde{L})z\rangle \\ &= \langle f(\widehat{L})Lx, f(\widehat{L})Lz\rangle \\ &= \langle \widehat{L}^{1/2}g(\widehat{L})Lx, \widehat{L}^{1/2}g(\widehat{L})Lz\rangle \\ &= \langle \widehat{L}g(\widehat{L})Lx, \widehat{L}g(\widehat{L})Lz\rangle \\ &= \langle \widehat{L}Lg(\widetilde{L})x, Lg(\widetilde{L})z\rangle \\ &= \langle L^*\widehat{L}Lg(\widetilde{L})x, g(\widetilde{L})z\rangle.\end{aligned}$$

But

$$L^*\widehat{L}L = I - \widetilde{L}$$

and therefore,

$$\langle (Lf(\widetilde{L}))^*(Lf(\widetilde{L}))x, z\rangle = \langle (I - \widetilde{L})g(\widetilde{L})x, g(\widetilde{L})z\rangle.$$

But, since $I - \widetilde{L}$ and $g(\widetilde{L})$ are bounded linear operators, this identity extends to all $x, z \in H_1$, and hence

$$\|Lf(\widetilde{L})\|^2 \leq \|I - \widetilde{L}\|\|g(\widetilde{L})\|^2 \leq (\|g\|_\infty)^2. \quad \square$$

A general stabilization procedure is suggested by the formal identity $Lx = L\widetilde{L}\widetilde{L}^{-1}x$. Stable approximations $\{y_\alpha\}$ to Lx will be formed in the following way:

$$y_\alpha = L\widetilde{L}T_\alpha(\widetilde{L})x \tag{3.1}$$

where $T_\alpha \in C[0,1]$ for each $\alpha > 0$ and the family of functions is shaped to approximate t^{-1} in the following sense:

$$T_\alpha(t) \to t^{-1} \quad \text{as} \quad \alpha \to 0 \quad \text{for each} \quad t \in (0,1] \tag{3.2}$$

and

$$|tT_\alpha(t)| \quad \text{is uniformly bounded for} \quad \alpha > 0, \quad t \in [0,1]. \tag{3.3}$$

Note that, since $L\widetilde{L}$ and $T_\alpha(\widetilde{L})$ are both (by von Neumann's theorem) bounded linear operators, the approximations y_α given by (3.1) are defined for *all* $x \in H_1$, not just for $x \in \mathcal{D}(L)$, and the mapping $x \mapsto y_\alpha$ is stable. In particular this means that the approximations

$$y_\alpha^\delta = L\widetilde{L}T_\alpha(\widetilde{L})x^\delta$$

are defined and stable for approximations $x^\delta \in H_1$ to $x \in \mathcal{D}(L)$ satisfying $\|x - x^\delta\| \leq \delta$ even if $x^\delta \notin \mathcal{D}(L)$. A general stable approximation scheme for Lx then consists of a choice of a family $\{T_\alpha\}$ satisfying (3.2) and (3.3) matched with a parameter choice strategy $\alpha = \alpha(\delta)$ designed to ensure that $y^\delta_{\alpha(\delta)} \to Lx$ as $\delta \to 0$. Before treating some specific cases we establish basic convergence and stabilization results. We consider first the case of error-free data x.

Theorem 3.3. *Suppose $L : \mathcal{D}(L) \subseteq H_1 \to H_2$ is a closed densely defined linear operator and $\{T_\alpha\}$ is a family of continuous real-valued functions defined on $[0,1]$ satisfying (3.2) and (3.3).*

(i) For all $x \in H_1$, $x_\alpha = \widetilde{L}T_\alpha(\widetilde{L})x \to x$ as $\alpha \to 0$.

(ii) If $x \in \mathcal{D}(L)$, then $y_\alpha = Lx_\alpha \to Lx$ as $\alpha \to 0$.

(iii) If $x \notin \mathcal{D}(L)$, then $\|y_\alpha\| \to \infty$ as $\alpha \to 0$.

Proof. First note that

$$x - x_\alpha = (I - \widetilde{L}T_\alpha(\widetilde{L}))x$$

But by (3.2) and (3.3), the function $1 - tT_\alpha(t)$ converges in a pointwise and uniformly bounded manner to the function

$$\varphi(t) = \begin{cases} 1, & t = 0 \\ 0, & t \in (0,1] \end{cases}$$

The spectral theorem applied to the bounded self-adjoint operator $\widetilde{L}$ then gives

$$x - x_\alpha \to P_{N(\widetilde{L})}x = 0, \quad \text{as} \quad \alpha \to 0.$$

By Lemma (3.1), if $x \in \mathcal{D}(L)$, then

$$Lx - y_\alpha = L(I - \widetilde{L}T_\alpha(\widetilde{L}))x = (I - \widehat{L}T_\alpha(\widehat{L}))Lx.$$

The spectral theorem applied to the bounded self-adjoint operator $\widehat{L}$ then gives

$$Lx - y_\alpha \to P_{N(\widehat{L})}Lx = 0, \quad \text{as} \quad \alpha \to 0.$$

To establish the final assertion, note that if $\{\|y_\alpha\|\}$ has a bounded sequence, then it has a weakly convergent subsequence, say $y_{\alpha_n} \rightharpoonup w$, for some sequence $\alpha_n \to 0$. Now $y_{\alpha_n} = Lx_n$ where $x_n = \widetilde{L}T_{\alpha_n}(\widetilde{L})x$ and by the properties of $\{T_\alpha\}$ and the Spectral Theorem,

$$x_n \to x - P_{N(\widetilde{L})}x = x \quad \text{as} \quad n \to \infty$$

But since the graph of L is closed and convex, and hence weakly closed, $x_n \to x$ and $Lx_n \rightharpoonup w$, and we have $x \in \mathcal{D}(L)$ and $Lx = w$. So if $x \notin \mathcal{D}(L)$, then $\{\|y_\alpha\|\}$ must be unbounded. □

Suppose now that $x^\delta \in H_1$ is an approximation to $x \in \mathcal{D}(L)$ satisfying $\|x - x^\delta\| \le \delta$. The stability error

$$y_\alpha - y_\alpha^\delta := L\widetilde{L}T_\alpha(\widetilde{L})x - L\widetilde{L}T_\alpha(\widetilde{L})x^\delta$$

may be estimated as follows:

$$\begin{aligned}\|y_\alpha - y_\alpha^\delta\|^2 &= \langle L^*L\widetilde{L}T_\alpha(\widetilde{L})(x-x^\delta), \widetilde{L}T_\alpha(\widetilde{L})(x-x^\delta)\rangle \\ &= \langle (I-\widetilde{L})T_\alpha(\widetilde{L})(x-x^\delta), \widetilde{L}T_\alpha(\widetilde{L})(x-x^\delta)\rangle \\ &\le \delta^2 ||(I-\widetilde{L})T_\alpha(\widetilde{L})\|\|\widetilde{L}T_\alpha(\widetilde{L})\|.\end{aligned}$$

Therefore, if $r(\alpha)$ is a function satisfying

$$|(1-t)T_\alpha(t)| \le r(\alpha) \quad \text{for} \quad t \in [0,1] \tag{3.4}$$

then, since $\|\widetilde{L}T_\alpha(\widetilde{L})\|$ is uniformly bounded, we find that

$$\|y_\alpha - y_\alpha^\delta\| \le \delta O(\sqrt{r(\alpha)}).$$

Putting these results together we obtain the following general stabilization result:

Theorem 3.4. *If $x \in \mathcal{D}(L)$ and $\|x - x^\delta\| \le \delta$ then*

$$y_\alpha^\delta = L\widetilde{L}T_\alpha(\widetilde{L})x^\delta \to Lx \quad \textit{as} \quad \delta \to 0$$

if $\alpha = \alpha(\delta) \to 0$ as $\delta \to 0$ in such a way that $\delta\sqrt{r(\alpha(\delta))} \to 0$.

Under appropriate conditions an actual rate of convergence can be obtained in terms of a function $\omega(\alpha, \nu)$ satisfying

$$\max_{t\in[0,1]} |(1 - tT_\alpha(t))t^\nu| \le \omega(\alpha, \nu) \tag{3.5}$$

Theorem 3.5. *Suppose $x \in \mathcal{D}(L)$ and $Lx \in R(\widehat{L}^\nu)$ for some $\nu > 0$. If $\{T_\alpha\}$ satisfies (3.5), then*

$$\|Lx - L\widetilde{L}T_\alpha(\widetilde{L})x\| = O(\omega(\alpha, \nu)).$$

Proof. This is immediate for if $Lx = \widehat{L}^\nu w$, then

$$\begin{aligned}Lx - L\widetilde{L}T_\alpha(\widetilde{L})x &= \widehat{L}^\nu w - \widehat{L}T_\alpha(\widehat{L})\widehat{L}^\nu w \\ &= (I - \widehat{L}T_\alpha(\widehat{L}))\widehat{L}^\nu w. \quad \square\end{aligned}$$

Note that in the particular case $\nu = 1$ the requirement on x in the previous theorem may be expressed simply as $x \in \mathcal{D}(LL^*L)$ since by Lemma 2.8

$$\mathcal{D}(LL^*L) = \{x \in \mathcal{D}(L) : Lx \in R(\widehat{L})\}.$$

Also, the relaxed assumption that $x \in \mathcal{D}(L^*L)$ leads to a special convergence rate.

Theorem 3.6. *If* $x \in \mathcal{D}(L^*L)$*, then* $\|Lx - L\widetilde{L}T_\alpha(\widetilde{L})x\| = O(\omega(\alpha, 1/2))$.

Proof. Write

$$Lx - L\widetilde{L}T_\alpha(\widetilde{L})x = LS_\alpha(\widetilde{L})x$$

where $S_\alpha(t) = 1 - tT_\alpha(t)$. Then, on setting $w = x + L^*Lx$, we find that

$$\begin{aligned}
\|Lx - L\widetilde{L}T_\alpha(\widetilde{L})x\|^2 &= \langle L^*LS_\alpha(\widetilde{L})x, S_\alpha(\widetilde{L})x\rangle \\
&= \langle -S_\alpha(\widetilde{L})x + S_\alpha(\widetilde{L})w, S_\alpha(\widetilde{L})x\rangle \\
&\leq \langle S_\alpha(\widetilde{L})w, S_\alpha(\widetilde{L})\widetilde{L}w\rangle \\
&= \|S_\alpha(\widetilde{L})\widetilde{L}^{1/2}w\|^2 \leq \omega(\alpha, 1/2)^2\|w\|^2. \quad \square
\end{aligned}$$

3.2 Some Cases

We now illustrate the general results of the previous section on some specific stable approximate evaluation methods. In some of the examples the stabilization parameter has a continuous range of positive values, while in other iterative methods the role of the stabilization parameter is played by a discrete iteration number that tends to infinity. Depending on the particular method under consideration, the stabilization parameter may take values which approach either 0 or ∞.

3.2.1 The Tikhonov-Morozov Method

The best known stabilization procedure is what we call the Tikhonov-Morozov method in which $y = Lx$ is approximated by

$$y_\alpha = L(I + \alpha L^*L)^{-1}x \tag{3.6}$$

where α is a positive stabilization parameter converging to zero.

Before taking up general results for the Tikhonov-Morozov method, we illustrate the method on a couple of simple examples of unbounded operators that were discussed in Chapter 1. For example, if L is the Dirichlet-to-Neumann map on the unit disk with domain

$$\mathcal{D}(L) = \left\{ f \in L^2(\partial D) : \sum_{n\in\mathbf{Z}} |n|^2|\hat{f}(n)|^2 < \infty \right\}$$

defined by

$$(Lf)(e^{i\theta}) = \sum_{n\in\mathbf{Z}} |n|\hat{f}(n)\exp(in\theta).$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, dt \, .$$

then one sees easily that

$$y_\alpha^\delta(e^{i\theta}) = L(I + \alpha L^* L)^{-1} f^\delta(\exp i\theta) = \sum_{n \in \mathbf{Z}} \left(\frac{|n|}{1 + \alpha n^2} \right) \widehat{f^\delta}(n) \exp{(in\theta)}.$$

As another illustration consider the problem of reconstructing a source distribution g in the heat problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x), \quad 0 < x < \pi, \quad 0 < t,$$

where $u(x,t)$ is subject to the boundary conditions

$$u(0,t) = u(\pi,t) = 0, \qquad 0 < t$$

and initial condition

$$u(x,0) = 0, \qquad 0 \le x \le \pi,$$

from knowledge of the temperature distribution $f(x) = u(x,1)$ at time $t = 1$. We found in Chapter 1 that $g = Lf$ where

$$\mathcal{D}(L) = \left\{ h : \sum_{m=1}^{\infty} m^4 a_m^2 < \infty, \quad a_m = \frac{2}{\pi} \int_0^\pi h(s) \sin msds \right\}$$

and

$$Lf(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n^2}{1 - e^{-n^2}} \sin nx \int_0^\pi f(s) \sin nsds.$$

In this case the Tikhonov-Morozov approximation to g is found to be

$$\begin{aligned} g_\alpha^\delta &= L(I + \alpha L^* L)^{-1} f^\delta \\ &= \frac{\pi}{2} \sum_{n=1}^{\infty} \alpha_n \sin nx \int_0^\pi f^\delta(s) \sin nsds \end{aligned}$$

where

$$\alpha_n = \left(\frac{n^2}{1 - e^{-n^2}} \right) \Big/ \left(1 + \alpha \left(\frac{n^2}{1 - e^{-n^2}} \right)^2 \right).$$

Returning to the general situation, we find that since

$$(I + \alpha L^* L)^{-1} = ((1-\alpha)I + \alpha(I + L^* L))^{-1} = \widetilde{L}((1-\alpha)\widetilde{L} + \alpha I)^{-1}$$

the Tikhonov-Morozov method fits into our general scheme if we take

$$T_\alpha(t) = (\alpha + (1-\alpha)t)^{-1}, \quad t \in [0,1].$$

Note that this class of functions satisfies criteria (3.2) and (3.3) and hence

$$L(I + \alpha L^*L)^{-1}x \to Lx \quad \text{as} \quad \alpha \to 0$$

for each $x \in \mathcal{D}(L)$. Also, since

$$\max_{t\in[0,1]} |(1-t)T_\alpha(t)| = \frac{1}{\alpha}$$

we may take $r(\alpha) = 1/\alpha$ in Theorem 3.4. That is, if we set

$$x_\alpha = (I + \alpha L^*L)^{-1}x \quad \text{and} \quad x_\alpha^\delta = (I + \alpha L^*L)^{-1}x^\delta$$

then we have:

Corollary 3.7. *If $x \in \mathcal{D}(L)$ and $\|x - x^\delta\| \le \delta$, then*

$$\|Lx_\alpha - Lx_\alpha^\delta\| \le \delta/\sqrt{\alpha}.$$

Therefore, if $x^\delta \in H_1$ is some approximation to $x \in \mathcal{D}(L)$ satisfying $\|x - x^\delta\| \le \delta$ and if $\delta = \circ(\sqrt{\alpha(\delta)})$, then by Theorem 3.4

$$L(I + \alpha(\delta)L^*L)^{-1}x^\delta \to Lx \quad \text{as} \quad \delta \to 0.$$

In order to apply Theorem 3.5 and obtain a convergence rate we require an upper bound for

$$(1 - tT_\alpha(t))t^\nu = \frac{\alpha(1-t)t^\nu}{\alpha + (1-\alpha)t}.$$

For $0 < \nu \le 1$ we claim that

$$\frac{\alpha(1-t)t^\nu}{\alpha + (1-\alpha)t} \le \alpha^\nu \quad \text{for} \quad t \in [0,1]. \tag{3.7}$$

This is the same as

$$\frac{(1-t)(t/\alpha)^\nu}{1 + (1-\alpha)(t/\alpha)} \le 1.$$

But setting $z = t/\alpha$, this is seen to be equivalent to

$$z^\nu - \alpha z^{\nu+1} \le 1 + (1-\alpha)z, \quad \text{for} \quad z \in [0, 1/\alpha]. \tag{3.8}$$

The function on the left of the inequality above has a maximum of

$$\left(\frac{\nu}{\nu+1}\right)^\nu \frac{1}{\nu+1}\alpha^{1-\nu}$$

which is clearly not greater than 1 for $0 < \nu \le 1$ and hence for $0 < \nu \le 1$ and $0 < \alpha < 1$ inequality (3.8) holds and hence so does the bound (3.7). We may therefore take

$$\omega(\alpha, \nu) = \alpha^{\nu}$$

for the Tikhonov-Morozov method. An immediate application of Theorems 3.5 and 3.6 gives:

Corollary 3.8. *If $x \in \mathcal{D}(L)$ and $Lx \in R(\widehat{L})$ (equivalently, $x \in \mathcal{D}(LL^*L)$), then*

$$\|Lx - y_\alpha\| = O(\alpha).$$

If $x \in \mathcal{D}(L^*L)$, then

$$\|Lx - y_\alpha\| = O(\sqrt{\alpha}).$$

A converse of the first rate in Corollary 3.8 may be of interest. We provide two proofs of such a converse as each is instructive in its own way. In the first converse we assume that $\widehat{L}$ is compact, that is, that LL^* has compact resolvent. For operators LL^* with compact resolvent, we show that if $x \in \mathcal{D}(L)$ and $\|Lx - Lx_\alpha\| = O(\alpha)$ as $\alpha \to 0$, then $x \in \mathcal{D}(LL^*L)$. To see this suppose the self-adjoint bounded operator $\widehat{L}$ is compact and let $\{u_j; \lambda_j\}$ be a complete orthonormal eigensystem for $\widehat{L}$. Note that the eigenvalues $\{\lambda_j\}$ lie in the interval $(0, 1]$. Suppose they are ordered as:

$$0 < \ldots \le \lambda_{n+1} \le \lambda_n \le \ldots \le \lambda_2 \le \lambda_1.$$

Then

$$\begin{aligned} Lx_\alpha &= L\widetilde{L}(\alpha I + (1-\alpha)\widetilde{L})^{-1}x \\ &= \widehat{L}(\alpha I + (1-\alpha)\widehat{L})^{-1}Lx \\ &= \textstyle\sum_{j=1}^{\infty} \displaystyle\frac{\lambda_j}{\alpha + (1-\alpha)\lambda_j}\langle Lx, u_j\rangle u_j \end{aligned}$$

and hence

$$\begin{aligned} \|Lx - Lx_\alpha\|^2 &= \alpha^2 \textstyle\sum_{j=1}^{\infty} \displaystyle\frac{(1-\lambda_j)^2}{(\alpha + (1-\alpha)\lambda_j)^2}|\langle Lx, u_j\rangle|^2 \\ &\ge \alpha^2(1-\lambda_1)^2 \textstyle\sum_{j=1}^{\infty}(\alpha + (1-\alpha)\lambda_j)^{-2}|\langle Lx, u_j\rangle|^2. \end{aligned}$$

Therefore, if $\|Lx - Lx_\alpha\| = O(\alpha)$, we have

$$\sum_{j=1}^{\infty}(\alpha + (1-\alpha)\lambda_j)^{-2}|\langle Lx, u_j\rangle|^2 \le C$$

for some constant C and all $\alpha \in (0, 1]$. In particular, all of the partial sums of the above series are uniformly bounded by C. Letting $\alpha \to 0^+$ in each of the individual partial sums shows that

$$\sum_{j=1}^{n} \lambda_j^{-2} |\langle Lx, u_j \rangle|^2 \leq C$$

for each n and hence the series $\sum_{j=1}^{\infty} \lambda_j^{-2} |\langle Lx, u_j \rangle|^2$ is convergent. The vector

$$z = \sum_{j=1}^{\infty} \lambda_j^{-1} \langle Lx, u_j \rangle u_j$$

is therefore well-defined and

$$\widehat{L}z = \sum_{j=1}^{\infty} \langle Lx, u_j \rangle u_j = Lx,$$

that is, $Lx \in R(\widehat{L})$ and hence $x \in \mathcal{D}(LL^*L)$ in light of Lemma 2.8. We now drop the assumption that $\widetilde{L}$ is compact.

Theorem 3.9. *If $x \in \mathcal{D}(L)$ and $\|Lx - Lx_\alpha\| = O(\alpha)$, then $x \in \mathcal{D}(LL^*L)$.*

Proof. First note that

$$x_\alpha = (I + \alpha L^*L)^{-1} x \in \mathcal{D}(L^*L)$$

and hence

$$Lx_\alpha = (I + \alpha LL^*)^{-1} Lx \in \mathcal{D}(LL^*).$$

and

$$Lx_\alpha - Lx = -\alpha LL^*Lx_\alpha.$$

Therefore, by the hypothesis

$$\|LL^*Lx_\alpha\| = O(1). \tag{3.9}$$

By Theorem 2.11 we know that LL^* is closed. This can be seen in another way by using Theorem 2.7. Indeed, if $\{y_n\} \subseteq \mathcal{D}(LL^*)$ satisfies, $y_n \to y$ and $LL^*y_n \to p$, then, using Theorem (2.7), we have for any $u \in \mathcal{D}(LL^*) = \mathcal{D}(L^{**}L^*)$

$$\langle LL^*u, y \rangle = \lim_{n\to\infty} \langle LL^*u, y_n \rangle = \lim_{n\to\infty} \langle u, L^{**}L^*y_n \rangle$$

$$= \lim_{n\to\infty} \langle u, LL^*y_n \rangle = \langle u, p \rangle.$$

Therefore, $y \in \mathcal{D}(LL^*)$ and $LL^*y = p$, that is, LL^* is closed. Hence the graph of LL^* is closed and convex, and hence weakly closed. By (3.9) there is then a sequence $\alpha_n \to 0$ with

$$LL^*Lx_{\alpha_n} \rightharpoonup w$$

for some w. But $Lx_{\alpha_n} \to Lx$, by Theorem 3.3 (ii). Since the graph of LL^* is weakly closed, it follows that $Lx \in \mathcal{D}(LL^*)$ and $LL^*Lx = w$. In particular, $x \in \mathcal{D}(LL^*L)$ as claimed. □

An application of Theorem 3.5 and Theorem 3.4 gives:

Theorem 3.10. *Suppose $x \in \mathcal{D}(L)$ and $Lx \in R(\widehat{L}^\nu)$ for some $0 < \nu \leq 1$. If $\|x - x^\delta\| \leq \delta$ and the stabilization parameter is chosen in the form $\alpha = C\delta^{2/(2\nu+1)}$, then*

$$\|y^\delta_{\alpha(\delta)} - Lx\| = O(\delta^{2\nu/(2\nu+1)}).$$

Note that this theorem gives a best rate of $O(\delta^{2/3})$ for the case $\nu = 1$. In the next chapter we shall show that this rate is essentially best possible.

3.2.2 The Iterated Tikhonov-Morozov Method

In this section we briefly investigate an iterative stabilization method related to the Tikhonov-Morozov method. The simplest iterative stabilization method is suggested by the approximation

$$\frac{1}{t} = \frac{1}{1-(1-t)} \approx \sum_{j=0}^{n-1}(1-t)^j = \frac{1-(1-t)^n}{t} =: T_n(t).$$

It is easy to see that the family $\{T_n(t)\}$ satisfies (3.2) and (3.3). Furthermore,

$$tT_n(t) = (1-t)tT_{n-1}(t) + t, \quad T_0(t) = 0$$

and hence one is led via the general spectral approach to the iterative method

$$x_n = (I - \widetilde{L})x_{n-1} + \widetilde{L}x$$

or equivalently

$$(I + L^*L)x_n = L^*Lx_{n-1} + x, \quad x_0 = 0 \tag{3.10}$$

This gives, for each n, a stable approximation $y_n = Lx_n$ to the value Lx. The method (3.10) is a special case (for $\alpha = 1$, see below) of the iterated Tikhonov-Morozov method.

Approximation orders that are arbitrarily near to the order of error in the data, $O(\delta)$, are achievable by use of iteration methods. One such method is the iterated Tikhonov-Morozov method. A different iterative stabilization method is studied in the next section. In the iterated Tikhonov-Morozov method a positive parameter α is fixed and the job of stabilization is assumed by an iteration number n (so in this case the stabilization parameter $n \to \infty$). In the ordinary Tikhonov-Morozov method with parameter α the value Lx is approximated by a stable approximation Lx_α where x_α satisfies

$$(I + \alpha L^* L)x_\alpha = x.$$

In the iterated method approximations $x_0, x_1, \ldots$ are given by $x_0 = 0$ and

$$(I + \alpha L^* L)x_n = x + \alpha L^* L x_{n-1} \tag{3.11}$$

where α is a *fixed* positive parameter.

In the case when only an approximate data vector x^δ is available satisfying $\|x - x^\delta\| \leq \delta$, the approximations generated in the same way with x replaced by x^δ will be denoted $\{x_n^\delta\}$. We emphasize that in this case the stabilization parameter is the iteration number n which satisfies $n \to \infty$, rather than the parameter $\alpha \to 0$, as in the general discussion of the first section, and we trust that this trivial modification will cause no confusion.

The iterated method (3.11) may be expressed in terms of the operator $\widetilde{L} = (I + L^* L)^{-1}$ by

$$(\alpha I + (1 - \alpha)\widetilde{L})x_n = \widetilde{L}x + \alpha(I - \widetilde{L})x_{n-1}.$$

In other words the stabilized approximations to Lx are given by $y_n = Lx_n$ where $x_n = \widetilde{L}T_n(\widetilde{L})x$ and the functions $\{T_n\}$ are defined iteratively by $T_0(t) = 0$ and

$$T_n(t) = \frac{1}{(1-\alpha)t + \alpha}\left[1 + \frac{\alpha(1-t)}{t}T_{n-1}(t)\right], \quad n = 1, 2, \ldots$$

for $t \in (0, 1]$, or equivalently

$$T_n(t) = \frac{1}{t}\left(1 - \left(\frac{\alpha(1-t)}{(1-\alpha)t + \alpha}\right)^n\right).$$

The definition $T_n(0) = 0$ extends these functions continuously to $[0, 1]$. Note that $|tT_n(t)| \leq 1$ for all n and $T_n(t) \to 1/t$ as $n \to \infty$ for each $t \in (0, 1]$, therefore $\{T_n\}$ satisfies (3.2) and (3.3) and hence the approximations satisfy $Lx_n \to Lx$ as $n \to \infty$.

The general stability estimate requires a bound $r(n)$ for the function

$$(1 - t)T_n(t) = \frac{1}{\alpha}\frac{1 - s}{s}(1 - (1 - s)^n)$$

where $s = t/(\alpha + (1-\alpha)t) \in [0,1]$. But note that

$$\frac{1-s}{s}(1-(1-s)^n) = \sum_{j=1}^{n}(1-s)^j \le n$$

and hence we may take $r(n) = n/\alpha$. We therefore have

Corollary 3.11. *Suppose $x \in \mathcal{D}(L)$ and $\|x - x^\delta\| \le \delta$. If $n = n(\delta) \to \infty$ as $n \to \infty$ while $\delta\sqrt{n(\delta)} \to 0$, then $Lx^\delta_{n(\delta)} \to Lx$.*

We shall give another proof of this result from an entirely different perspective in the next chapter.

To obtain an error bound we notice that

$$t^\nu(1 - tT_n(t)) = \alpha^\nu(1-s)^n\left(\frac{s}{(1-s)+s\alpha}\right)^\nu$$

where again $s = t/(\alpha + (1-\alpha)t) \in [0,1]$. We may assume that $\alpha < 1$ and then

$$\alpha^\nu(1-s)^n\left(\frac{s}{(1-s)+s\alpha}\right)^\nu \le (1-s)^n s^\nu$$

$$\le \left(\frac{n}{n+\nu}\right)^n\left(\frac{\nu}{n+\nu}\right)^\nu = O(n^{-\nu}).$$

If $x \in \mathcal{D}(L)$ and $Lx \in R(\widehat{L}^\nu)$ for some $\nu > 0$, we then find that

$$\|y_n - Lx\| = O(n^{-\nu}).$$

Combining this with the previous result gives

Corollary 3.12. *If $x \in \mathcal{D}(L)$ and $Lx \in R(\widehat{L}^\nu)$ for some $\nu > 0$, then for $\|x - x^\delta\| \le \delta$ if the iteration parameter $n = n(\delta)$ is chosen so that $n(\delta) \sim \delta^{-2/(2\nu+1)}$ we have*

$$\|y^\delta_{n(\delta)} - Lx\| = O(\delta^{2\nu/(2\nu+1)}).$$

Note that the restriction $\nu \le 1$ is not imposed here and hence rates arbitrarily close to $O(\delta)$ are in principle achievable by the iterated Tikhonov-Morozov method. In the next chapter we will study the nonstationary Tikhonov-Morozov method in which the value of the constant α may change from one iteration to the next.

3.2.3 An Interpolation-Based Method

Of course there are many other possible choices for the family $\{T_\alpha\}$ that lead to stabilization methods. For example, another arises from interpolatory function theory. Let T_n be the polynomial of degree not greater than n that interpolates the function $f(t) = 1/t$ at $t = \beta_1^{-1}, \beta_2^{-1}, \ldots, \beta_n^{-1}$, where $\{\beta_j\}$ are positive numbers satisfying $1 \geq \beta_n \to 0$ as $n \to \infty$ and $\sum \beta_j = \infty$, that is

$$T_n(t) = \frac{1}{t}\left(1 - \prod_{j=1}^{n}(1 - \beta_j t)\right).$$

Note that $\{T_n\}$ is given iteratively by $T_1(t) = \beta_1$ and

$$T_{n+1}(t) = \beta_{n+1} + (1 - \beta_{n+1} t)\, T_n(t), \quad n = 1, 2, \ldots. \tag{3.12}$$

It is evident that $|tT_n(t)| \leq 1$ and therefore $\{T_n\}$ satisfies (3.3). Also,

$$0 \leq \prod_{j=1}^{n}(1 - \beta_j t) \leq \prod_{j=1}^{n} e^{-t\beta_j} = \exp\left(-\sum_{j=1}^{n}\beta_j t\right) \to 0 \quad \text{as} \quad n \to \infty$$

for $t \in (0, 1]$, and hence $\{T_n\}$ satisfies (3.2). We therefore immediately obtain from Theorem 3.3 that the iteratively defined sequence

$$y_{n+1} = \beta_{n+1} L\widetilde{L}x + \left(I - \beta_{n+1}\widehat{L}\right) y_n, \quad n = 0, 1, \ldots \qquad y_0 = 0 \tag{3.13}$$

converges to Lx for each $x \in \mathcal{D}(L)$. In order to obtain a stability result we need a candidate for the function $r(n)$ in equation (3.4) (again in this instance the iteration number n plays the role of a stability parameter). This is easily had from the iterative formulation (3.12):

$$|(1-t)T_{n+1}(t)| \leq \beta_{n+1} + |(1-t)T_n(t)|$$

and hence

$$\max_{t\in[0,1]} |(1-t)T_n(t)| \leq \sum_{j=1}^{n}\beta_j =: \sigma_n.$$

From the general stability result of the previous section we then obtain

$$\|y_n^\delta - y_n\| \leq \delta \sigma_n^{1/2}.$$

Therefore, if the iteration parameter $n = n(\delta)$ grows at a rate controlled so that $\delta^2 \sigma_{n(\delta)} \to 0$ as $\delta \to 0$, then Theorem 3.4 guarantees that

$$y_{n(\delta)}^\delta \to Lx, \qquad \text{as} \quad \delta \to 0$$

where y_n^δ is defined as in (3.13) with x replaced by x^δ satisfying $\|x - x^\delta\| \leq \delta$.

For convergence rates we need a function $\omega(n, \nu)$ satisfying (3.5). But note that

$$0 \leq (1 - tT_n(t))t^\nu \leq t^\nu \prod_{j=1}^{n} e^{-t\beta_j} = t^\nu e^{-\sigma_n t}.$$

However, the function on the right of the inequality above achieves for $t \in [0,1]$ a maximum value of $\nu^\nu(e\sigma_n)^{-\nu}$ and hence we may use

$$\omega(n,\nu) = \left(\frac{\nu}{e}\right)^\nu \sigma_n^{-\nu}.$$

From Theorem 3.5 we obtain:

Corollary 3.13. *If $x \in \mathcal{D}(L)$ and $Lx \in R(\widehat{L}^\nu)$ for some $\nu > 0$, then*

$$\|y_n - Lx\| = O(\sigma_n^{-\nu}).$$

Combining the results of Theorem (3.4) and Theorem (3.5) we therefore obtain:

Corollary 3.14. *If $x \in \mathcal{D}(L)$ and $Lx \in R(\widehat{L}^\nu)$ for some $\nu > 0$, then for $\|x - x^\delta\| \le \delta$ if the iteration parameter $n = n(\delta)$ is chosen so that $\sigma_{n(\delta)} \sim \delta^{-2/(2\nu+1)}$ we have*

$$\|y^\delta_{n(\delta)} - Lx\| = O(\delta^{2\nu/(2\nu+1)}).$$

Under appropriate conditions one can achieve rates that are arbitrarily close to optimal for this method by use of an *a posteriori* choice of the iteration parameter rather than with an *a priori* choice as in Theorem 3.14. In fact, note that

$$y_n = L\left(I - \prod_{j=1}^n \left(I - \beta_j \widetilde{L}\right)\right) x = Lx_n \tag{3.14}$$

where

$$x_n = \left(I - \prod_{j=1}^n \left(I - \beta_j \widetilde{L}\right)\right) x, \quad x_0 = 0 \tag{3.15}$$

and $\{x_n^\delta\}$ is defined in the same way with x replaced by x^δ. The approximations $\{x_n^\delta\}$ can be compared with the available data x^δ in order to monitor the convergence of $\{y_n^\delta\}$ to Lx. First note that $x_n^\delta \to x^\delta$ as $n \to \infty$ and

$$\|x^\delta - x_n^\delta\| = \left\|\left(I - \beta_n \widetilde{L}\right)(x^\delta - x^\delta_{n-1})\right\| \le \|x^\delta - x^\delta_{n-1}\|.$$

We assume that for a given constant $\tau > 1$, the signal-to-noise ratio is not less than τ, i.e., we assume that

$$\|x^\delta - x_0^\delta\| = \|x^\delta\| \ge \tau\delta.$$

There is then a first value $n = n(\delta) \ge 1$ of the iteration index for which

$$\|x^\delta - x^\delta_{n(\delta)}\| < \tau\delta. \tag{3.16}$$

Note that this iteration number is chosen in an *a posteriori* manner as the computation proceeds.

Lemma 3.15. *If $x_{n(\delta)}$ is given by (3.15) where $n(\delta)$ satisfies (3.16), then*

$$\|x - x_{n(\delta)}\| \leq (\tau + 1)\delta.$$

Proof. From (3.15), using the approximate data x^δ in one case and "clean" data $x \in \mathcal{D}(L)$ in the other, we have

$$x^\delta_{n(\delta)} - x_{n(\delta)} = \left(I - \prod_{j=1}^{n(\delta)} (I - \beta_j \widetilde{L}) \right) (x^\delta - x).$$

Now

$$\begin{aligned} x - x_{n(\delta)} &= x^\delta - x^\delta_{n(\delta)} + x - x^\delta + x^\delta_{n(\delta)} - x_{n(\delta)} \\ &= x^\delta - x^\delta_{n(\delta)} + \left(\prod_{j=1}^{n(\delta)} (I - \beta_j \widetilde{L}) \right) (x - x^\delta). \end{aligned}$$

Since $\|I - \beta_j \widetilde{L}\| \leq 1$, we have by (3.16)

$$\|x - x_{n(\delta)}\| \leq \tau\delta + \|x - x^\delta\| \leq (\tau + 1)\delta. \quad \square$$

We now need an inequality.

Lemma 3.16. *For $\mu > 0$, $\|\widehat{L}^\mu z\| \leq \|z\|^{1/(2\mu+1)} \|\widehat{L}^{\mu+1/2} z\|^{2\mu/(2\mu+1)}$.*

Proof. Let $\{E_\lambda\}$ be a resolution of the identity for H_2 generated by the bounded self-adjoint operator $\widehat{L} : H_2 \to H_2$. By Hölder's inequality

$$\begin{aligned} \|\widehat{L}^\mu z\|^2 &= \int_0^1 1 \cdot \lambda^{2\mu} d\|E_\lambda z\|^2 \\ &\leq \left(\int_0^1 1 d\|E_\lambda z\|^2 \right)^{1/(2\mu+1)} \left(\int_0^1 \lambda^{2\mu+1} d\|E_\lambda z\|^2 \right)^{2\mu/(2\mu+1)} \\ &= \|z\|^{2/(2\mu+1)} \left(\|\widehat{L}^{\mu+1/2} z\|^2 \right)^{2\mu/(2\mu+1)}. \quad \square \end{aligned}$$

Lemma 3.17. *If $x \in \mathcal{D}(L)$ and $x = \widetilde{L}^\mu w$ for some $w \in \mathcal{D}(L)$, then*

$$\|Lx - Lx_{n(\delta)}\| = O(\delta^{\mu/(\mu+1)}).$$

Proof. From (3.14) and lemma 3.1 we find

$$\begin{aligned} Lx - Lx_{n(\delta)} &= L \left(\prod_{j=1}^{n(\delta)} (I - \beta_j \widetilde{L}) \right) \widetilde{L}^\mu w \\ &= \left(\prod_{j=1}^{n(\delta)} (I - \beta_j \widehat{L}) \right) \widehat{L}^\mu L w = \widehat{L}^\mu z_{n(\delta)} \end{aligned}$$

where $z_{n(\delta)} = \left(\prod_{j=1}^{n(\delta)} (I - \beta_j \widehat{L}) \right) Lw$ and hence $\|z_{n(\delta)}\| \leq \|Lw\|$.

Applying the previous lemma, we find

$$\begin{aligned}\|Lx - Lx_{n(\delta)}\| &\le \|Lw\|^{1/(2\mu+1)}\|\widehat{L}^{\mu+1/2}z_{n(\delta)}\|^{2\mu/(2\mu+1)} \\ &= \|Lw\|^{1/(2\mu+1)}\|\widehat{L}^{1/2}(Lx - Lx_{n(\delta)})\|^{2\mu/(2\mu+1)}.\end{aligned} \tag{3.17}$$

However, since $\|x - x_{n(\delta)}\| \le (\tau+1)\delta$ and $\|L\widetilde{L}\| \le 1$,

$$\begin{aligned}\|\widehat{L}^{1/2}(Lx - Lx_{n(\delta)})\|^2 &= \langle \widehat{L}(Lx - Lx_{n(\delta)}), Lx - Lx_{n(\delta)}\rangle \\ &= \langle L\widetilde{L}(x - x_{n(\delta)}), Lx - Lx_{n(\delta)}\rangle \\ &\le (\tau+1)\delta\|Lx - Lx_{n(\delta)}\|.\end{aligned}$$

Therefore (3.17) gives

$$\|Lx - Lx_{n(\delta)}\| = O(\delta^{\mu/(2\mu+1)})\|Lx - Lx_{n(\delta)}\|^{\mu/(2\mu+1)}$$

that is,

$$\|Lx - Lx_{n(\delta)}\| = O(\delta^{\mu/(\mu+1)}). \quad \square$$

Theorem 3.18. *Suppose that $x \in \mathcal{D}(L)$ and $x = \widetilde{L}^\mu w$ for some $w \in \mathcal{D}(L)$ and $\mu > 1/2$. If $x^\delta \in H_1$ satisfies $\|x - x^\delta\| \le \delta$ and $n(\delta)$ is chosen by (3.16), then*

$$\|Lx^\delta_{n(\delta)} - Lx\| = O(\delta^{\min((2\mu-1)/(2\mu),\mu/(\mu+1))}).$$

Proof. First note that

$$(x_{n-1} - x^\delta_{n-1}) - (x - x^\delta) = -\left(\prod_{j=1}^{n-1}\left(I - \beta_j\widetilde{L}\right)\right)(x - x^\delta)$$

and hence by (3.16)

$$\begin{aligned}\|x_{n(\delta)-1} - x\| &= \|x^\delta_{n(\delta)-1} - x^\delta + (x_{n(\delta)-1} - x^\delta_{n(\delta)-1}) - (x - x^\delta)\| \\ &\ge \|x^\delta_{n(\delta)-1} - x^\delta\| - \|\left(\prod_{j=1}^{n(\delta)-1}\left(I - \beta_j\widetilde{L}\right)\right)(x - x^\delta)\| \\ &\ge \tau\delta - \delta = (\tau-1)\delta\end{aligned} \tag{3.18}$$

If $x = \widetilde{L}^\mu w$, then

$$\|x_{n-1} - x\| = \left\|\widetilde{L}^\mu \prod_{j=1}^{n-1}\left(I - \beta_j\widetilde{L}\right)w\right\| = O(\sigma_{n-1}^{-\mu}).$$

However, $\sigma_n/\sigma_{n-1} \to 1$ as $n \to \infty$ and hence $\sigma_{n-1}^{-\mu} = O(\sigma_n^{-\mu})$, therefore

$$\|x_{n(\delta)-1} - x\| = O(\sigma_{n(\delta)}^{-\mu}).$$

In light of (3.18), we then have

$$\sigma_{n(\delta)} = O(\delta^{-1/\mu}). \tag{3.19}$$

By the general stability estimate

$$\|Lx_{n(\delta)}^{\delta} - Lx_{n(\delta)}\| = \|y_{n(\delta)}^{\delta} - y_{n(\delta)}\| \le \delta\sqrt{\sigma_{n(\delta)}} = O(\delta^{(2\mu-1)/(2\mu)}).$$

Combining this with the previous lemma gives the result. □

We note that this result says that rates arbitrarily close to optimal may in principle be obtained by use of the iteration number choice criterion (3.16).

3.2.4 A Method Suggested by Dynamical Systems

We begin with some heuristics and then we develop a theoretical method for stable approximation of Lx. This method will then be used to give another motivation for the iterated Tikhonov-Morozov method at the end of this section.

Given data $x^\delta \in H_1$ our goal is to produce a smoothed approximation $z(\alpha) \in \mathcal{D}(L)$ to x^δ with $Lz(\alpha) \to Lx$ as $\alpha \to \infty$ (again in this section we find it convenient to reverse the direction of the stabilization parameter). That is, if $w(\alpha) = x^\delta - z(\alpha)$ we want $w(0) = x^\delta$ and $w(\alpha) \to 0$ as $\alpha \to \infty$ in an appropriate way as $\delta \to 0$. We are only concerned with unbounded operators L and in this case the positive unbounded operator L^*L has an unbounded spectrum. On the other hand the operator $(L^*L)^\dagger$ typically has positive eigenvalues that converge to 0. In order to suppress high frequency components in w one might then seek the long term trend in the solution of

$$\frac{dw}{d\alpha} = -(L^*L)^\dagger w, \qquad w(0) = x^\delta. \tag{3.20}$$

This has the formal solution

$$x^\delta - z(\alpha) = w(\alpha) = \exp\left(-(L^*L)^\dagger \alpha\right)x^\delta$$

or equivalently

$$z(\alpha) = \left(I - \exp\left(-(L^*L)^\dagger \alpha\right)\right)x^\delta$$

which in light of Theorem 2.13 suggests the definition

$$z^\delta(\alpha) = \widetilde{L}T_\alpha(\widetilde{L})x^\delta$$

where

$$T_\alpha(t) = \begin{cases} \frac{1}{t}\left(1 - e^{-\alpha t/(1-t)}\right), & t \in (0,1) \\ 0 & , t = 0, 1. \end{cases}$$

Note that these functions are continuous on $[0,1]$ and satisfy conditions (3.2) and (3.3) (of course, with the modification that $\alpha \to \infty$). While this does not result in a computable method, the theory of the previous section applies nevertheless. Setting $s = t/(1-t)$ we find that

$$\max_{t\in[0,1]} |(1-t)T_\alpha(t)| = \max_{s\in[0,\infty)} (1 - e^{-\alpha s})/s = \alpha$$

Therefore, we may set $r(\alpha) = \alpha$ in Theorem 3.4 and hence if $\alpha = \alpha(\delta) \to \infty$ and $\delta^2\alpha(\delta) \to 0$, then $Lz^\delta(\alpha(\delta)) \to Lx$ as $\delta \to 0$.

With the same transformation $t \to s$ we find that for $\nu > 0$

$$(1 - tT_\alpha(t))t^\nu = \left(\frac{s}{s+1}\right)^\nu e^{-\alpha s} \le s^\nu e^{-\alpha s} \le \left(\frac{\nu}{e}\right)^\nu \alpha^{-\nu}.$$

Therefore we may take $\omega(\alpha, \nu) = O(\alpha^{-\nu})$ in Theorem 3.5. Combining these two results we obtain

Corollary 3.19. *Suppose $x \in \mathcal{D}(L)$ and $\|x - x^\delta\| \le \delta$. If $Lx \in R(\widetilde{L}^\nu)$ for some $\nu > 0$ and $\alpha(\delta) = C\delta^{-2/(2\nu+1)}$ then*

$$\|Lz^\delta(\alpha(\delta)) - Lx\| = O(\delta^{2\nu/(2\nu+1)}).$$

Working formally, (3.20) suggests

$$L^*L\frac{dw}{d\alpha} = -w$$

or since $w(\alpha) = x^\delta - z(\alpha)$

$$L^*L\frac{dz}{d\alpha} = x^\delta - z(\alpha). \tag{3.21}$$

If we approximate the solution of the differential equation by the simple implicit forward difference method

$$L^*L\frac{z_n - z_{n-1}}{h} = x^\delta - z_n, \qquad z_0 = 0$$

with step size h, we find on setting $\beta = 1/h$ and rearranging that

$$(I + \beta L^*L)z_n = \beta L^*Lz_{n-1} + x^\delta$$

which is the iterated Tikhonov-Morozov method. An entirely different motivation and convergence proof for this method will be given in the next chapter.

The iterated Tikhonov-Morozov method may be motivated by equation (3.21) in a slightly different way. Specifically, from (3.21), we have

$$L^*L\int_{\alpha_n}^{\alpha_{n+1}}\frac{dz}{d\tau}d\tau = (\alpha_{n+1}-\alpha_m)x^\delta - \int_{\alpha_n}^{\alpha_{n+1}} z(\tau)d\tau$$

or, on setting $\beta = 1/(\alpha_{n+1}-\alpha_n)$, and using the right hand rule on the last integral, we are led to

$$z_{n+1} + \beta L^*Lz_{n+1} = x^\delta + \beta L^*Lz_n$$

which is the iterated Tikhonov-Morozov method. This approach suggests the use of other closed quadrature rules on the integral above. While we do not suggest that such rules will lead to methods that offer any computational advantage over the Tikhonov-Morozov method, it is instructive to see how the general theory applies to another method. For example, if the trapezoidal rule is used we are led to the approximation

$$\beta L^*L(z_{n+1}-z_n) = x^\delta - (z_{n+1}+z_n)/2$$

or, setting $\gamma = 2\beta > 0$,

$$(I+\gamma L^*L)z_{n+1} = 2x^\delta + \gamma L^*Lz_n - z_n.$$

Equivalently,

$$z_n = \widetilde{L}T_n(\widetilde{L})x^\delta$$

where $T_0(t) = 0$ and

$$T_{n+1}(t) = \frac{2}{\gamma(1-t)+t} + \frac{\gamma(1-t)-t}{\gamma(1-t)+t}T_n(t), \quad n = 0, 1, \ldots.$$

One finds immediately that

$$T_n(t) = \frac{1}{t}\left(1 - \left(\frac{\gamma(1-t)-t}{\gamma(1-t)+t}\right)^n\right)$$

and from this it follows that

$$|(1-t)T_n(t)| \le \frac{2}{\gamma}n.$$

Therefore, one may take $r(n) = O(n)$ in the general stability estimate. However, for this method the convergence analysis given previously does not automatically apply since

$$-1 \le \frac{\gamma(1-t)-t}{\gamma(1-t)+t} < 1$$

for $t \in (0,1]$ with the equality holding at $t = 1$. Therefore, the convergence of the spectral approximation may fail if the resolution of the identity generated by $\widetilde{L}$ has a jump discontinuity at $t = 1$. This is equivalent to the condition that $\{0\} \ne N(I-\widetilde{L}) = N(L)$ (see Lemma 2.9).

3.3 Notes

Lardy [34] was the first to exploit von Neumann's theorem in applications to series representations for the Moore-Penrose inverse of a closed unbounded linear operator. The general spectral approach to stabilized approximate evaluation based on von Neumann's theorem was introduced in [15]. Another approach to general stabilization theory, based on the theory of regularization and Theorem 2.10 is suggested in [21].

The best known specific instance of the general method is the Tikhonov-Morozov method. This method was developed by V.A. Morozov and his co-workers and is summarized in [39]. A much more extensive treatment of this method, based on our spectral approach, emerges in the following chapters. See also [40] and [22] for further developments. The line of reasoning in the proof of Theorem 3.9 is inspired by an argument of Neubauer [41]. The iterative stabilization method that is motivated by functional interpolation appears in [18]; the techniques of that paper owe a lot to [27]. It would appear that other stabilization methods based on numerical integration techniques for initial value problems for ordinary differential equations could be developed.

In appropriate circumstances the Tikhonov-Morozov method can be adapted to stably evaluate certain nonlinear operators A. We outline the theory of Al'ber [1] for accomplishing this. Suppose that $A : \mathcal{D}(A) \subseteq H \to H$ is a nonlinear monotone operator defined on a subset $\mathcal{D}(A)$ of a real Hilbert space H, that is

$$\langle Ax - Ay, x - y \rangle \geq 0$$

for all $x, y \in \mathcal{D}(A)$. Given $x^\delta \in H$ and $x \in \mathcal{D}(A)$ with

$$\|x - x^\delta\| \leq \delta$$

we wish to stably approximate Ax using the data x^δ. We assume that A is discontinuous in the usual sense, but satisfies a weak condition called hemicontinuity, namely that A is weakly continuous along rays, that is

$$A(u + tv) \rightharpoonup Au \quad \text{as} \quad t \to 0^+$$

when $u + tv \in \mathcal{D}(A)$ for sufficiently small nonnegative t.

Under these conditions it can be shown, by use of a fundamental result on maximal monotone operators, that for $\alpha > 0$ the nonlinear operator $I + \alpha A$ has a single valued continuous inverse defined on all of H (see, e.g., [49]). The mapping

$$x^\delta \mapsto (I + \alpha A)^{-1} x^\delta$$

is therefore a stable operation. Also,

$$A(I + \alpha A)^{-1} = \frac{1}{\alpha}(I - (I + \alpha A)^{-1})$$

and hence, for fixed $\alpha > 0$, the operation

$$x^\delta \mapsto A(I+\alpha A)^{-1}x^\delta = Ax^\delta_\alpha$$

where x^δ_α be the unique solution of

$$x^\delta_\alpha + \alpha A x^\delta_\alpha = x^\delta$$

is a stable operation. Let x_α be the solution of the same equation using the "clean" data x:

$$x_\alpha + \alpha A x_\alpha = x.$$

The goal is to show that $Ax^\delta_\alpha \to Ax$ if $\alpha = \alpha(\delta) \to 0$ in some appropriate sense as $\delta \to 0$. First, we show that $Ax_\alpha \to Ax$ as $\alpha \to 0$.

Note that, by the monotonicity of A

$$\begin{aligned} 0 &\le \langle Ax_\alpha - Ax, x_\alpha - x\rangle \\ &= -\alpha\langle Ax_\alpha - Ax, Ax_\alpha\rangle \\ &\le -\alpha\|Ax_\alpha\|^2 + \alpha\|Ax\|\|Ax_\alpha\| \end{aligned}$$

and hence

$$\|Ax_\alpha\| \le \|Ax\|.$$

From the definition of x_α we then have

$$\|x_\alpha - x\| = \alpha\|Ax_\alpha\| = O(\alpha)$$

and hence $x_\alpha \to x$ as $\alpha \to 0$.

Suppose $v \in H$ is arbitrary and $t \ge 0$. Then

$$\begin{aligned} 0 &\le \langle Ax_\alpha - A(x+tv), x_\alpha - (x+tv)\rangle \\ &= \langle Ax_\alpha, x_\alpha - x\rangle - t\langle Ax_\alpha, v\rangle - \langle A(x+tv), x_\alpha - x - tv\rangle. \end{aligned}$$

Since $\|Ax_\alpha\|$ is bounded, for any sequence $\alpha_n \to 0$, there is a subsequence, which we again denote by α_n, and a $y \in H$ such that

$$Ax_{\alpha_n} \rightharpoonup y.$$

Therefore, taking limits as $\alpha_n \to 0$ above, and using the fact that $x_{\alpha_n} \to x$, we arrive at

$$0 \le -\langle y, v\rangle + \langle A(x+tv), v\rangle = \langle A(x+tv) - y, v\rangle.$$

By the hemicontinuity of A we then have

$$0 \le \langle Ax - y, v\rangle$$

for any $v \in H$. Therefore, $Ax = y$, that is,

$$Ax_\alpha \rightharpoonup Ax \quad \text{as} \quad \alpha \to 0.$$

However, since $\|Ax_\alpha\| \leq \|Ax\|$, it follows from the weak lower semicontinuity of the norm that $\|Ax_\alpha\| \to \|Ax\|$ and hence

$$Ax_\alpha \to Ax \quad \text{as} \quad \alpha \to 0.$$

The convergence of $\{Ax_\alpha^\delta\}$ will now be established. First, we have

$$\|Ax_\alpha^\delta - Ax\| \leq \|Ax_\alpha^\delta - Ax_\alpha\| + \|Ax_\alpha - Ax\|$$

and

$$\begin{aligned} \|Ax_\alpha^\delta - Ax_\alpha\| &= \alpha^{-1}\|(x^\delta - x_\alpha^\delta) + (x_\alpha - x)\| \\ &\leq \delta/\alpha + \alpha^{-1}\|x_\alpha - x_\alpha^\delta\|. \end{aligned} \tag{3.22}$$

But, since

$$x_\alpha^\delta - x_\alpha + \alpha(Ax_\alpha^\delta - Ax_\alpha) = x^\delta - x,$$

one finds, using the monotonicity of A,

$$\begin{aligned} \|x_\alpha^\delta - x_\alpha\|^2 &\leq \|x_\alpha^\delta - x_\alpha\|^2 + \langle Ax_\alpha^\delta - Ax_\alpha, x_\alpha^\delta - x_\alpha\rangle \\ &= \langle x^\delta - x, x_\alpha^\delta - x_\alpha\rangle \leq \delta\|x_\alpha^\delta - x_\alpha\| \end{aligned}$$

and hence $\|x_\alpha^\delta - x_\alpha\| \leq \delta$. From (3.22), we then obtain

$$\|Ax_\alpha^\delta - Ax_\alpha\| \leq 2\delta/\alpha.$$

Therefore, if $\alpha = \alpha(\delta) \to 0$ in such a way that $\delta/\alpha \to 0$, then

$$Ax_\alpha^\delta - Ax_\alpha \to 0.$$

But, as previously established, $Ax_\alpha \to Ax$ and hence

$$Ax_\alpha^\delta \to Ax.$$