1

Introduction

Dynamical systems describe the time evolution of the various states $z \in \mathcal{P}$ in a given state space. When this description includes both (the complete) past and future this leads to a *group action*[1]

$$\begin{array}{rcccl} \varphi & : & \mathbb{R} \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ & & (t, z) & \mapsto & \varphi_t(z) \end{array}$$

of the time axis $\mathbb{R}$ on $\mathcal{P}$, i.e. $\varphi_0 = \mathrm{id}$ (the present) and $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all times $s, t \in \mathbb{R}$. Immediate consequences are $\varphi_s \circ \varphi_t = \varphi_t \circ \varphi_s$ and $\varphi_t^{-1} = \varphi_{-t}$. In case φ is differentiable one can define the vector field

$$X(z) \quad = \quad \frac{\mathrm{d}}{\mathrm{d}t} \, \varphi_t(z) \Big|_{t=0}$$

on $\mathcal{P}$ and if e.g. $\mathcal{P}$ is a differentiable manifold then φ can be reconstructed from X as its flow. Note that

$$\dot{z} \quad = \quad X(z) \tag{1.1}$$

defines an autonomous ordinary differential equation on $\mathcal{P}$.

Given a state $z \in \mathcal{P}$ the set $\{ \, \varphi_t(z) \; \big| \; t \in \mathbb{R} \, \}$ is called the orbit of z. Particularly simple orbits are equilibria, $\varphi_t(z) = z$ for all $t \in \mathbb{R}$, and periodic orbits which satisfy $\varphi_T(z) = z$ for some period $T > 0$ and hence $\varphi_{t+T}(z) = \varphi_t(z)$ for all $t \in \mathbb{R}$. All other orbits define injective immersions $t \mapsto \varphi_t(z)$ of $\mathbb{R}$ in $\mathcal{P}$. By definition unions of orbits form sets $M \subseteq \mathcal{P}$ that are invariant under φ, and if M is a differentiable manifold we call M an invariant manifold.

A complete understanding of a dynamical system φ is equivalent to finding (and understanding) all solutions of (1.1) whence one often concentrates on the long time behaviour as $t \to \pm\infty$. One approach is to determine all attractors[2]

[1] *Technical terms* are explained in a glossary preceeding the references.

[2] Since there are no attractors in Hamiltonian dynamical systems we do not give a formal definition.

in $\mathcal{P}$, compact invariant subsets A satisfying $\varphi_t(z) \stackrel{t\to+\infty}{\longrightarrow} A$ for all z near A, that are minimal with this property. Such attractors can be equilibria, periodic orbits, invariant manifolds, or even more general invariant sets. If A is an invariant manifold without equilibrium, then the Euler characteristic of A vanishes and the simplest such manifolds are the n-tori T, submanifolds of $\mathcal{P}$ that are diffeomorphic to $\mathbb{T}^n = \mathbb{R}^n/_{\mathbb{Z}^n}$. Where we speak of n-tori we always assume $n \geq 2$ in these notes.

The flow φ on a torus T is parallel or *conditionally periodic* if there is a global chart

$$\begin{array}{ccc} \mathsf{T} & \longrightarrow & \mathbb{T}^n \\ z & \mapsto & x \end{array}$$

and a frequency vector $\omega \in \mathbb{R}^n$ such that[3]

$$\bigwedge_{x\in\mathbb{T}^n} \quad \bigwedge_{t\in\mathbb{R}} \quad \varphi_t(x) \;=\; x + \omega t \;.$$

In case there are no resonances $\langle k, \omega \rangle = 0$, $k \in \mathbb{Z}^n$ every orbit on T is dense. If there are $n-1$ independent resonances then ω is a multiple of an integer vector and all orbits on T are periodic. For $m \leq n-2$ independent resonances the motion is *quasi-periodic* and spins densely around invariant $(n-m)$-tori into which T decomposes. The flow on a given invariant torus may be much more complicated, this is often accompanied by a loss of differentiability. However, if the flow is equivariant with respect to the $\mathbb{T}^n$-action $x \mapsto x + \xi$ then all motions are necessarily conditionally periodic. Our starting point is therefore a family of tori carrying parallel flow, and we hope for persistence under small perturbations for the measure-theoretically large subfamily where the frequency vector satisfies a strong non-resonance condition.

Considering the long time behaviour for $t \to -\infty$ attractors are replaced by repellors and more generally one is interested in "minimal" invariant sets M. Where the dynamics on M itself is understood – for equilibria, periodic orbits and invariant tori with conditionally periodic flow – one concentrates on the dynamics nearby. Equilibria and periodic orbits are (under quite weak conditions) *structurally stable* with respect to small perturbations of the dynamical system, while invariant tori and more complicated, strange invariant sets may disintegrate. This makes it preferable to study parametrised families of such invariant sets.

In applications the equations of motion are known only to finite precision of the coefficients. Giving these coefficients the interpretation of parameters leads to a whole family of dynamical systems. Under variation of the parameters the invariant sets may then bifurcate. Bifurcations of equilibria are fairly well understood, at least for low co-dimension, cf. [129, 173] and references therein. Since these bifurcations concern a small neighbourhood of the equilibrium, we speak of *local bifurcations*. Using a *Poincaré mapping*, periodic orbits can be

[3] We use the same letter φ for the flow in the chart as well.

studied as fixed points of a discrete dynamical system. In addition to the analogues of bifurcations of equilibria, periodic orbits may undergo period doubling bifurcations, cf. [223, 58].

For a family of invariant n-tori with conditionally periodic flow the frequency vector ω varies in general with the parameter; let us therefore now consider $\omega \in \mathbb{R}^n$ itself as the parameter. Clearly both the resonant and the non-resonant tori are dense in the family. Under an arbitrary small perturbation (breaking the $\mathbb{T}^n$-symmetry that forces the toral flows to be conditionally periodic) the situation changes drastically. Using KAM-techniques one can formulate conditions under which most invariant tori survive the perturbation, together with their quasi-periodic flow; the families of tori are parametrised over a Cantor set of large n-dimensional (Hausdorff)-measure, see [159, 56, 55]. Within the gaps of the Cantor set completely new dynamical phenomena emerge; the dynamics on the torus may cease to be conditionally periodic[4] even in case there are circumstances like *normal hyperbolicity* that force the torus to persist. Note that the union of the gaps of a Cantor set is open and dense in $\mathbb{R}^n$. This is an exemplary instance of coexisting complementary sets, one of which is measure-theoretically large and the other topologically large, cf. [231].

It turns out that the bifurcations of equilibria and periodic orbits have quasi-periodic counterparts, see [34, 284] and references therein. In the integrable case where the perturbation respects the $\mathbb{T}^n$-action this is an immediate consequence of the behaviour of the reduced system obtained after reducing the torus symmetry. In the nearly integrable case where the torus symmetry is broken by a small perturbation one can use KAM theory to show that the bifurcation persists on Cantor sets. Notably the bifurcating torus has to be in *Floquet form*. In the same way the higher topological complexity of periodic orbits leads to period doubling bifurcations, tori that are not in Floquet form can bifurcate in a skew Hopf bifurcation, see [282, 60].

Bifurcations of invariant tori have a *semi-local* character, they concern a neighbourhood of the invariant torus which need not be confined to a small region of $\mathcal{P}$. Exceptions are bifurcations subordinate to local bifurcations and these were in fact the motivating examples for the above results. In contrast, global bifurcations lead to new interactions of different parts of $\mathcal{P}$ not present before or after the bifurcation. Examples are *connection bifurcations* involving heteroclinic orbits (these also exist subordinate to local or semi-local bifurcations).

The quasi-periodic persistence results in [159, 56, 55] are formulated and proven in terms of Lie algebras of vector fields and this allowed for a generalization to volume-preserving, Hamiltonian and reversible dynamical systems,

[4] For instance, if $\omega \in \omega_0 \cdot \mathbb{Z}^n$ only finitely many periodic orbits are expected to survive and the perturbed flow may consist of asymptotic motions between these. The structural stability of surviving periodic orbits is in turn the reason why a simple resonant frequency vector opens a whole gap of the Cantor set.

see also [216]. We will henceforth speak of dissipative systems when there is no such structure preserved. A dynamical system is Hamiltonian if the vector fields derives[5] from a single "Hamiltonian" function by means of a *Poisson structure*, a bilinear and alternating composition on $\mathcal{A} \subseteq C(\mathcal{P})$ that satisfies the Jacobi identity and Leibniz' rule. An important feature of integrable Hamiltonian systems is that the torus symmetry yields conjugate actions by Noether's theorem. Accordingly, invariant n-tori in integrable Hamiltonian systems with d degrees of freedom, $d \geq n$, occur as "intrinsic" n-parameter families, without the need for external parameters.

In particular, periodic orbits form 1-parameter families, or 2-dimensional cylinders (while equilibria remain in general as isolated as in the dissipative case). Thus, periodic orbits in (single) integrable Hamiltonian systems may undergo co-dimension one bifurcations, without the need of an external parameter. The ensuing possibilities were analysed in [205, 207], see also [208, 38, 232, 227, 228]. This yields transparent explanations for common phenomena like the gyroscopic stabilization of a sleeping top, cf. [13, 84, 81, 147].

Interestingly, results on bifurcations of invariant n-tori (which form n-parameter families in a Hamiltonian system) were first derived in the dissipative context (where external parameters are needed), see again [34] and references therein. Our aim is to detail the Hamiltonian part of the theory, extending the results in [139, 50] to more general bifurcations. At the same time we seize the occasion to put the well-known results on Hamiltonian bifurcations of equilibria, which are scattered throughout the literature, into a systematic framework. See also [75, 76, 45, 44] for recent progress concerning torus bifurcations in the reversible context.

1.1 Hamiltonian systems

A Hamiltonian system is defined by a Hamiltonian function on a phase space. The latter is a *symplectic manifold*, or, more generally a *Poisson space*, where the Hamiltonian H determines the vector field

$$X_H \ : \ \dot{z} \ = \ \{z, H\} \ .$$

If all solutions of X_H exist for all times, the flow φ^H is a group action

$$\begin{array}{rccc} \varphi^H \ : & \mathbb{R} \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ & (t, z) & \mapsto & \varphi^H_t(z) \end{array} \tag{1.2}$$

on the phase space $\mathcal{P}$ – in case there are orbits that leave $\mathcal{P}$ in finite time (1.2) is only a *local group action*.

Despite this simple construction where a single real valued function defines a whole vector field, the study of Hamiltonian systems is a highly non-trivial

[5] Similar to gradient vector fields defined by means of a Riemannian structure.

task. The first systems that were successfully treated were integrable and the study of Hamiltonian systems still starts with the search for the integrals of motion. Since $\{H, H\} = 0$ the Hamiltonian is always[6] an integral of motion, whence all systems with one *degree of freedom* are integrable.

However, already in two degrees of freedom integrable systems are the exception rather than the rule, cf. [239, 117, 26]. This led to the so-called *ergodic* hypothesis that the flow of a Hamiltonian system is "in general" ergodic on the *energy shell*. That this hypothesis does not hold for *generic* Hamiltonian systems, see [191], is one of the consequences of KAM theory.

KAM theory deals with small perturbations of integrable systems and may in fact be thought of as a theory on the integrable systems themselves. Indeed, in applications the special circumstances that render a Hamiltonian system integrable may not be satisfied with absolute precision and only properties that remain valid under the ensuing small perturbations have physical relevance.

An integrable Hamiltonian system with, say, compact energy shells gives the phase space $\mathcal{P}$ the structure of a *ramified torus bundle*. The regular fibres of this bundle are the maximal invariant tori of the system. The singular fibres define a whole hierarchy of lower dimensional tori, in case of (dynamically) unstable tori together with their *(un)stable manifolds*. In this way there are two types of "least degenerate" singular fibres: the *elliptic* tori with one *normal frequency* and the *hyperbolic* tori T with stable and unstable manifolds of the form $\mathsf{T} \times \mathbb{R}$. These two types of singular fibres determine the distribution of the regular fibres. Different families of maximal tori are separated by (un)stable manifolds of hyperbolic tori and may shrink down to elliptic tori.

On the next level of the hierarchy of singular fibres of the ramified torus bundle we can distinguish four or five different types. Lowering the dimension of the torus once more we are led to elliptic tori with two normal frequencies, to *hypo-elliptic* tori and to hyperbolic tori with four *Floquet exponents*. For these latter we might want to distinguish between the focus-focus case of a quartet $\pm\Re\pm\mathrm{i}\Im$ of complex exponents and the saddle-saddle case of two pairs of real exponents. This decision would relegate hyperbolic tori with a double pair of real exponents to the next level of the hierarchy of singular fibres. We can do the same with elliptic tori with two resonant normal frequencies. Where the two normal frequencies are in $1{:}-1$ resonance, the torus may undergo a quasi-periodic *Hamiltonian Hopf bifurcation* and we *always* relegate these elliptic tori to the third level of the hierarchy of singular fibres of the ramified torus bundle.

The last type of second level singular fibres consists of invariant tori (and their (un)stable manifolds) of the same dimension as the first level tori, but with *parabolic* normal behaviour. Such tori may for instance undergo a quasi-periodic *centre-saddle bifurcation*. We see that the kth level singular fibres determine the distribution of the $(k-1)$th level singular fibres (where we could abuse language and address the regular fibres as 0th level singular fibres).

[6] Our Hamiltonians are autonomous, there is no explicit time dependence.

Notably all invariant n-tori of the ramified torus bundle are *isotropic*, having a (commuting) set $y_1, \ldots, y_n$ of actions conjugate to the toral angles. Locally these may be used to parametrise the various families of n-tori. There is a branch of KAM theory that explores non-isotropic (in particular *co-isotropic*) invariant tori. In such a situation, the symplectic structure is necessarily non-exact and it is moreover the symplectic structure that should satisfy certain non-resonance conditions. For more information see [262] and references therein.

The aim of KAM theory is to study the fate of this ramified torus bundle under small perturbations of the integrable Hamiltonian system. Traditionally, this has been done on phase spaces that are symplectic manifolds where the perturbation of the phase space may be neglected and only the Hamiltonian gets perturbed (but see also [175]). Furthermore, a non-degeneracy condition forces the maximal tori to be *Lagrangean*, whence their dimension equals the number d of degrees of freedom. Consequently, for *superintegrable systems*[7] one uses part of the perturbation to construct from the unperturbed ramified torus bundle a non-degenerate ramified torus bundle, see [6, 196, 268, 116].

Persistence of Lagrangean tori under small perturbations was first proven in [166] under the condition that the (internal) frequencies satisfy *Diophantine conditions*, a strong form of non-resonance. This allows to solve the "homological equation" at every step of an iteration scheme, the convergence of which is ensured by the superlinear convergence of a Newton-like approximation. This set-up was modified in [5], restricting to only finitely many resonances in the homological equation by means of an *ultraviolet cut-off* (which is in turn increased at every iteration step). This allowed to successfully treat perturbations of superintegrable systems that *remove the degeneracy* in [6].

The above results were obtained for analytic Hamiltonians. In an attempt to verify the statement of [166] the validity was extended in [215] to Hamiltonians that are only finitely often differentiable. Subsequently the necessary order of differentiability could be brought down in [250]. A lower bound was provided by a counterexample in [270], sharper bounds are discussed in [109]. The machinery of the KAM iteration was condensed in [298, 299] to abstract theorems. In [121, 71, 108] convergence of the KAM iteration scheme was directly proven, without the need for a Newton-like approximation.

While (Lebesgue)-almost all frequency vectors are non-resonant, the complement of Diophantine frequency vectors is an open and dense set. Still, the relative measure of Diophantine frequency vectors is close to 1. In [72, 240] the local structure of persistent tori was shown to inherit the Cantor-like structure of Diophantine frequency vectors. The local *conjugacies* that relate the persistent tori to their unperturbed counterparts are patched together in [46] to form a global conjugacy. This should allow to recover the geometry of the bundle of maximal tori in the perturbed system.

[7] In the literature these are also called properly degenerate systems.

The first proof of persistence of elliptic tori in [216] only addressed the case of a single normal frequency. A more general result had already been announced in [204], but proofs appeared much later; see [55] for an extensive bibliography. In case of hyperbolic tori one can always resort to a centre manifold, cf. [211, 160], although this generally results in finite differentiability. For a direct approach see [249] and references therein. Hypo-elliptic tori can either be treated directly, cf. [159, 56, 251], or by first getting rid of the hyperbolic part by means of a centre manifold. As pointed out in [162, 55, 279, 163] the latter approach may yield additional tori that are not in Floquet form.

Parabolic tori are generically involved in quasi-periodic bifurcations and may in particular cease to exist. Correspondingly, one cannot expect persistence of the "isolated" family of parabolic tori; but the whole bifurcation scenario has a chance to persist, in this way including the bifurcating (parabolic) tori. A first such persistence result appeared in [139], which was generalized in [50] to all parabolic tori one can generically encounter in Hamiltonian systems with finitely many degrees of freedom. Additional hyperbolicity may again be dealt with by means of a centre manifold, while additional normal frequencies can be successfully carried through the KAM iteration scheme, cf. [296].

KAM theory does not predict the fate of close-to-resonant tori under perturbations. For fully resonant tori the phenomenon of frequency locking leads to the destruction of the torus under (sufficiently rich) perturbations, and other resonant tori disintegrate as well. In two degrees of freedom surviving 2-tori form barriers on the 3-dimensional energy shells, from which one can infer that all motions are bounded, cf. [222]. Where the system has three or more degrees of freedom there is no such obstruction to orbits connecting distant points of the phase space. The existence of this kind of orbits has been termed *Arnol'd diffusion*, for an up-to-date discussion see [91] and references therein.

While KAM theory concerns the fate of "most" trajectories and for all times, a complementary theorem has been obtained in [220, 221, 226]. It concerns all trajectories and states that they stay close to the unperturbed tori for *long* times that are exponential in the inverse of the perturbation strength. Here a form of smoothness exceeding the mere existence of ∞ many derivatives of the Hamiltonian is a necessary ingredient, for finitely differentiable Hamiltonians one only obtains polynomial times. Most results in this direction are formulated for analytic Hamiltonians, in [190] the neccessary regularity assumptions have been lowered to Gevrey Hamiltonians. For trajectories starting close to surviving tori the diffusion is even superexponentially slow, cf. [213, 214].

A new type of invariant sets, not present in integrable systems, is constructed for generic Hamiltonian systems in [192, 203], using a construction from [25]. Starting point is an elliptic periodic orbit around which another elliptic periodic orbit encircling the former is shown to exist. Iterating this procedure yields a whole sequence of elliptic periodic orbits which converges to a *solenoid*. The construction in [25, 192] not only yields the existence of one solenoid near a given elliptic periodic orbit, but the simultaneous existence of representatives of all homeomorphy-classes of solenoids.

Hyperbolic tori form the core of a construction proposed in [7] of trajectories that venture off to distant points of the phase space. The key ingredience are resonant tori that disintegrate under perturbation leading to lower dimensional hyperbolic tori, cf. [275, 276]. In the unperturbed system the union of a family of hyperbolic tori, parametrised by the actions conjugate to the toral angles, forms a *normally hyperbolic manifold*. The latter is persistent under perturbations, cf. [151, 211], and carries again a Hamiltonian flow, with fewer degrees of freedom.

Perturbed resonant lower dimensional tori that bifurcate according to a quasi-periodic *Hamiltonian pitchfork bifurcation* are studied in [180, 178, 181, 182]. Such parabolic resonances (PR) exhibit large dynamical instabilities. This effect can be significantly amplified by increasing the number of degrees of freedom. This is not only due to multiple resonances (m-PR), but can also be induced by an additional vanishing derivative of the unperturbed Hamiltonian at the parabolic torus for so-called tangent (or 1-flat) parabolic resonances. This latter condition makes a larger part of the energy shell accessible in the perturbed system. In high degrees of freedom, combinations like l-flat m-PR become a common phenomenon as well.

1.1.1 Symmetry reduction

To fix thoughts, let the phase space $\mathcal{P}$ be a symplectic manifold of dimension $2(n+1)$, on which a locally free symplectic n-torus action

$$\tau \;:\; \mathbb{T}^n \times \mathcal{P} \;\longrightarrow\; \mathcal{P}$$

is given. Reduction then leads to a one-degree-of-freedom problem. If the action τ is free then the symmetry reduction is regular, cf. [206, 194, 3], and the reduced phase space is a (2-dimensional) symplectic manifold.

Singularities of the reduced phase space are related to points with non-trivial isotropy group $\mathbb{T}^n_z$, cf. [4, 265, 230]. Note that all points in the orbit

$$\mathbb{T}^n(z) \;=\; \left\{ \tau_\xi(z) \in \mathcal{P} \;\middle|\; \xi \in \mathbb{T}^n \right\}$$

have, up to conjugation, that same isotropy group, which can be given the form

$$\mathbb{T}^n_z \;\cong\; \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_n}$$

with $k \in \mathbb{N}^n$. Thus, if we pass to a $(k_1, \ldots, k_n)$-fold covering of $\mathcal{P}$, the action τ becomes a free[8] action and regular reduction can again be applied.

[8] Strictly speaking this is only true locally around the lift of the torus $\mathbb{T}^n(z)$.

On the covering space the isotropy group $\mathbb{T}^n_z$ acts as the group of deck transformations, fixing the lift of $\mathbb{T}^n(z)$. This induces a symplectic $\mathbb{T}^n_z$-action on the reduced phase space, which we locally identify with $\mathbb{R}^2$. Here the origin is the image of the lift of $\mathbb{T}^n(z)$ under the reduction mapping and, by Bochner's theorem we may assume that $\mathbb{T}^n_z$ acts linearly on $\mathbb{R}^2$. Recall that the only finite subgroups of $SL_2(\mathbb{R})$ are the cyclic groups $\mathbb{Z}_\ell$. This yields an epimorphism from the deck group onto $\mathbb{Z}_\ell$, the kernel of which we denote by N.

Identifying all points on the $(k_1, \ldots, k_n)$-fold covering space of $\mathcal{P}$ that are mapped to each other by elements of N we pass to an ℓ-fold covering of $\mathcal{P}$. This has no influence on the reduced phase space $\mathbb{R}^2$, in particular the image of the lift of $\mathbb{T}^n(z)$ remains a regular point. In this way the action of the deck group $\mathbb{Z}_\ell = \mathbb{T}^n_z/_N$ on $\mathbb{R}^2$ becomes faithful.

Only if we go further and also identify points within the $\mathbb{Z}_\ell$-orbit on the ℓ-fold covering space do we introduce a singularity on the reduced phase space. In particular, if we reduce the n-torus action τ directly on $\mathcal{P}$ we are led to a singularity of type $\mathbb{R}^2/_{\mathbb{Z}_\ell}$ of the reduced phase space. This has been used in [49] to study n-tori with a normal-internal resonance; the necessary action τ was introduced by means of normalization.

1.1.2 Distinguished Parameters

Torus bifurcations occur in families of invariant tori, and the necessary parameters enter Hamiltonian systems in various fashions. This leads to a hierarchical structure where some parameters are *distinguished* with respect to others. To explain the basic mechanism let us start with a family of Hamiltonian systems that depends on an external parameter α. Then co-ordinate transformations $z \mapsto \tilde{z}$ on the phase space[9] $\mathcal{P}$ may clearly depend on the parameter α, while re-parametrisations $\alpha \mapsto \tilde{\alpha}$ are not allowed to depend on the phase space variable z. This ensures that after re-parametrisation and co-ordinate transformation the distinction between phase space variables $\tilde{z}$ and external parameters $\tilde{\alpha}$ remains valid.

Let the Hamiltonian system now be symmetric with respect to a symplectic action of a compact Lie group G. According to Noether's theorem every (continuous) symmetry induces a conserved quantity. If we divide out the group action, then the latter become Casimirs. Hence, we can treat their value μ as a parameter the reduced system depends upon. In the hierarchy the place of μ is "between" the external parameter α and the variable ζ on the reduced phase space. Indeed, while co-ordinate transformations $\zeta \mapsto \tilde{\zeta}$ now may depend on both α and μ, re-parametrisations $\alpha \mapsto \tilde{\alpha}$ are not allowed to depend on either ζ or μ – recall that (μ, ζ) constitute together with the reduced variable along the orbit of the Lie group G the "original" variable z on the phase space $\mathcal{P}$. While a re-parametrisation $\mu \mapsto \tilde{\mu}$ may (still) depend on the external parameter α, there should be no dependence on ζ. We say

[9] For simplicity we let the phase space be the same for all parameter values α.

that the (internal) parameter μ is *distinguished* with respect to the (external) parameter α, cf. [288]. If the reduction of the G-action is not regular, but a singular reduction, then the re-parametrisation $\mu \mapsto \tilde{\mu}(\alpha, \mu)$ has to be restricted to preserve the singular values, cf. [43]. A typical example is that μ is the value of angular momenta and the restriction $\tilde{\mu}(\alpha, 0) = 0 \; \forall_\alpha$ imposes that the zero level be preserved.

In applications the existing symmetries often do not suffice to render the system integrable. A possible approach is then to introduce additional symmetries by means of a normal form. After a co-ordinate transformation the Hamiltonian is split into an integrable part and a small perturbation. The first step then is to understand the dynamics defined by the integrable part of the Hamiltonian.

Typically the additional symmetry introduced by normalization is a torus symmetry. Dividing out the group action turns the actions conjugate to the toral angles into Casimirs, the value I of which again plays the rôle of parameter. Clearly I is distinguished with respect to α and a re-parametrisation $I \mapsto \tilde{I}$ should not depend on the variable of the twice reduced phase space. But we also want I to be distinguished with respect to μ, i.e. our parameter changes should be of the form

$$(\alpha, \mu, I) \quad \mapsto \quad \Big(\tilde{\alpha}(\alpha), \tilde{\mu}(\alpha, \mu), \tilde{I}(\alpha, \mu, I)\Big) \quad . \tag{1.3}$$

In this way the new $\tilde{I}$ is still the value of the momentum mapping of the approximate symmetry, and when adding the small perturbation to the integrable part of the normal form the perturbation analysis may be performed for fixed $\tilde{\alpha}$ and $\tilde{\mu}$. Where the symmetry reduction is singular the re-parametrisation (1.3) should preserve the singular values.

Our aim is to understand what happens to the ramified torus bundle defined by a single integrable Hamiltonian system under generic pertrubations. In that setting there are no external parameters, and the perturbation does not leave part of the symmetry of the unperturbed system intact. However, in applications one easily encounters simultaneously two or even all three hierarchical levels of parameters. This leads to changes in the unfolding properties, cf. [288, 43, 188, 53]. Nevertheless, the starting point for such modifications would be a theory with a single class of parameters.

1.2 Outline

Bifurcations of invariant tori are to a large extent governed by their normal dynamics. In the following *Chapter 2* we therefore study bifurcations of equilibria in their own right. To this end we let the system depend on external parameters.

We first concentrate on bifurcating equilibria in Hamiltonian systems with one degree of freedom. This is indeed the situation one is led to when studying bifurcations of invariant n-tori in $n + 1$ degrees of freedom. In one degree of freedom the symplectic form becomes an area form, the Hamiltonian is a planar function and the equilibria correspond to planar *singularities*. Morse singularities lead to *centres* and *saddles*. Local bifurcations are in turn governed by unstable singularities and their *universal unfoldings*.

Next to the simple planar singularities, which form two infinite series $(A_k)_{k\geq 1}$, $(D_k)_{k\geq 4}$ and a finite series E_6 , E_7 , E_8, there are various series of planar singularities with *moduli*. In *Chapter 2* we address the latter only sporadically and leave a more systematic approach to *Appendices A and B*. It turns out that the moduli of planar singularities do not lead to moduli of bifurcations of equilibria in Hamiltonian systems with one degree of freedom.

Motivated by the reduction of the toral symmetry τ in Section 1.1.1 we also study bifurcations of equilibria at singular points of 2-dimensional Poisson spaces. There are two possibilities. Similar to bifurcations of regular equilibria the Hamiltonian may change under parameter variation. Alternatively, the bifurcation may be triggered by local changes of the phase space, e.g. leading to a singular point when the parameter attains the bifurcation value. In multi-parameter systems there may also be combinations of these two mechanisms.

Next to the cyclic symmetry groups $\mathbb{Z}_\ell$ which lead to singular phase spaces there are other (discrete) symmetries of one-degree-of-freedom systems, sometimes reversing. The main example for the latter is the reflection

$$(q, p) \quad \mapsto \quad (q, -p) \enspace .$$

Such symmetries strongly influence the bifurcations that degenerate equilibria can undergo. The ensuing possibilities are detailed in *Chapter 2* as well.

The local bifurcations of one-degree-of-freedom systems can occur in more degrees of freedom as well. Indeed, for an equilibrium in d degrees of freedom that has a linearization with $2d-2$ eigenvalues off the imaginary axis this hyperbolic part can be dealt with by means of a centre manifold. The flow on the latter is that of a one-degree-of-freedom Hamiltonian system, and the equilibrium undergoes one-degree-of-freedom bifurcations where the remaining 2 eigenvalues vanish. Where a zero eigenvalue with (algebraic) multiplicity 2 coexists with further purely imaginary pairs of eigenvalues the situation is much more complicated, cf. [43, 122].

We focus on two degrees of freedom and also content ourselves with bifurcations of regular equilibria, leaving aside a systematic study of local bifurcations of singular points in two (or more) degrees of freedom. In fact, already a complete understanding of co-dimension 2 bifurcations of regular equilibria in two degrees of freedom is beyond our present possibilities.

A new phenomenon in two degrees of freedom is that one may have two pairs of purely imaginary eigenvalues in resonance. The most important of these is the 1:-1 resonance. In generic 1-parameter families this resonance triggers a Hamiltonian Hopf bifurcation. The double pair of imaginary eigenvalues leads to an S^1-symmetry, and reduction yields a one-degree-of-freedom problem where the bifurcating equilibrium is a singular point of the phase space.

Here and also for other resonant equilibria normalization is an important tool. This procedure allows to "push a toral symmetry through the Taylor series" whence the system can be approximated by the integrable part of a normal form. For the convenience of the reader this well-known method is recalled in *Appendix C*.

In families of two-degree-of-freedom systems with at least 2 parameters one may encounter equilibria with nilpotent linearization. In case the system has an S^1-symmetry one can again reduce to one degree of freedom and can proceed as for the 1:−1 resonance. However, in the absence of symmetry the phenomena become much more complicated. For instance, all forms of resonant equilibria occur in an unfolding of nilpotent equilibria.

In *Chapter 3* we consider bifurcations of periodic orbits. Here the *Floquet multipliers* play a rôle similar to that of the eigenvalues of the linearization of an equilibrium. One Floquet multiplier is always equal to 1 as it corresponds to the direction tangential to the periodic orbit. All other multipliers are in 1-1 correspondence with the eigenvalues of (the linearization of) the Poincaré mapping. In the present case of Hamiltonian systems one of these eigenvalues is equal to 1. The (generalized) eigenvector of this Floquet multiplier spans the direction conjugate to that of the "first" multiplier 1. Correspondingly, periodic orbits of Hamiltonian systems form 1-parameter families. Occurring bifurcations are determined by the distribution of the remaining Floquet multipliers.

In contrast to our treatment of equilibria we concentrate on a single Hamiltonian system, without dependence on external parameters. Therefore, the bifurcations of periodic orbits we encounter are of co-dimension 1. In this way we recover the well-known three types of bifurcations triggered by an additional double Floquet multiplier 1, by a double Floquet multiplier −1 and by a double pair of Floquet multpliers on the unit[10] circle. These are the periodic centre-saddle bifurcation, the *period-doubling bifurcation* and the periodic Hamiltonian Hopf bifurcation, respectively.

For all these bifurcations the key information is already contained in the behaviour of the corresponding bifurcation of equilibria. For the period-doubling bifurcation this is the *Hamiltonian flip bifurcation* treated in Section 2.1.2 in which a singular equilibrium loses its stability. Since we use a similar strategy for bifurcations of invariant tori the reasons that allow to carry the bifurcations of equilibria over to bifurcations of periodic orbits are presented in detail, although the results on periodic orbits themselves are well documented in the existing literature, cf. [208, 38] and references therein. Specifically, in [38] also multiparameter bifurcations with one distinguished parameter are considered; this allows to understand bifurcations of periodic orbits in families of Hamiltonian systems.

[10] This double pair is different from 1 or −1.

Invariant tori and their bifurcations are then studied in *Chapter 4*. Since the n actions $y_1, \ldots, y_n$ conjugate to the toral angles of an invariant n-torus serve as (internal) parameters, we may encounter bifurcations of arbitrary co-dimension already in a single Hamiltonian system, provided the number $d > n$ of degrees of freedom is sufficiently large. We therefore abstain again from including external parameters into this setting.

An important assumption we make, which is automatically fulfilled for lower dimensional invariant tori of integrable systems, is that the torus $y = y_0$ be reducible to Floquet form

$$\dot{x} = \omega(y_0) + \mathcal{O}(y - y_0, z^2) \tag{1.4a}$$

$$\dot{y} = \mathcal{O}(y - y_0, z^3) \tag{1.4b}$$

$$\dot{z} = \Omega(y_0)\, z + \mathcal{O}(y - y_0, z^2) \tag{1.4c}$$

where the matrix $\Omega(y_0) \in \mathfrak{sp}(2m, \mathbb{R})$, $m = d - n$ is independent of the toral angles $x_1, \ldots, x_n$. The eigenvalues of this matrix are called Floquet exponents. Their distribution determines occurring bifurcations.

In the integrable case where there is no dependence at all on x we can reduce (1.4) to m degrees of freedom and end up with the Hamiltonian system defined by (1.4c). Here the origin $z = 0$ is an equilibrium, which undergoes a bifurcation as the parameter y passes through y_0. This puts us in the framework of Chapter 2 – and the main purpose of that chapter is indeed to address this problem independent of where it originates from. In this way the results obtained there carry over to bifurcations of invariant tori in integrable Hamiltonian systems.

We therefore concentrate on those bifurcations that could be satisfactorily treated in Chapter 2. This means we mainly restrict to $m = 1$ normal degree of freedom and consider $m = 2$ only insofar as there is an S^1-symmetry that again allows reduction to one normal degree of freedom. In this way we clarify the structure of the ramified d-torus bundle around invariant n-tori for integrable systems with $d = n + 1$ degrees of freedom and also for some cases with $d = n + 2$ degrees of freedom.

The remaining question then is what happens to this integrable picture under small Hamiltonian perturbations. Inevitably, where perturbations of quasi-periodic motions are concerned, small denominators enter the scene. Correspondingly, Diophantine conditions are needed to obtain the necessary estimates. The persistence of the bifurcation scenario is obtained by a combination of KAM theory and singularity theory.

To prove persistence of invariant tori one often uses a Kolmogorov-like condition

$$\det D\omega(y) \neq 0 \tag{1.5}$$

to let the actions $y_1, \ldots, y_n$ control the frequencies $\omega_1, \ldots, \omega_n$. In the present bifurcational setting we already need the actions to control the unfolding parameters $\lambda_1, \ldots, \lambda_k$. In particular, if the co-dimension k of the bifurcation is equal to the dimension n of the invariant torus, then the most degenerate torus is isolated and may disappear in a resonance gap. We therefore restrict to co-dimensions $k \leq n-1$ where even the most degenerate tori still form continuous families in the unperturbed integrable system. Replacing (1.5) by a Rüssmann-like condition that involves also higher derivatives of the frequency mapping then yields a *Cantor family* of invariant tori in the perturbed system. In this way one can decouple the frequencies from the Hamiltonian and obtain persistence of invariant tori of the latter by treating the former as independent parameters. This strategy was already very successful in the study of normally elliptic lower dimensional tori, cf. [55] and references therein.

When proving a persistence result for a whole bifurcation scenario, the difficult part is to keep track of the most degenerate "object" in the perturbed system. To this end a KAM iteration scheme is used, performing two operations at each iteration step. First the lower[11] order terms are made x-independent. Here one has to deal with small denominators to solve a (linear) homological equation. Then these lower order terms are transformed into the universal unfolding of the central singularity. This is achieved by explicit co-ordinate changes known from singularity theory. The technical details of this procedure are deferred to *Appendices D and E*, where we also discuss in how far this proof is still open to generalizations.

In the final *Chapter 5* we put the results obtained into context to describe the dynamics in integrable and nearly integrable Hamiltonian systems. A completely integrable system with d degrees of freedom has d commuting integrals $G_1 = H, G_2, \ldots, G_d$ and according to Liouville's theorem [3, 13, 16] bounded motions starting at regular points of $G : \mathcal{P} \longrightarrow \mathbb{R}^d$ are conditionally periodic. Singular values of G give rise to lower dimensional invariant subsets and yield the whole hierarchy of singular fibres of the ramified torus bundle defined by G. Excitation of normal modes of non-hyperbolic equilibria generates periodic orbits (this is Lyapunov's theorem, see [3, 16, 208]) and the same mechanism explains how families of n-tori shrink down to k-tori, $k < n$. In Chapter 2 we encounter many more mechanisms how the various families of invariant tori fit together.

Under small non-integrable perturbations the ramified torus bundle is "Cantorised" as the smooth action manifolds parametrising invariant tori are replaced by Cantor sets of large relative measure. In the non-degenerate case this implies that most motions of the perturbed system are quasi-periodic, and the question arises how the various Cantor families of tori fit together. For the excitation of normal modes it has been shown in [164, 261] that the persistent k-tori consist of Lebesgue density points of persistent n-tori. Similar results are obtained in Chapter 4 for all the cases treated in Chapter 2. The destruction of maximal tori with a single resonance exemplifies that "Cantorised"

[11] This notion is defined by means of the singularity at hand.

bifurcations of lower dimensional tori ocur in virtually every nearly integrable Hamiltonian system.

In case there are more integrals than degrees of freedom the system is superintegrable. The G_i no longer commute, but the compact connected components of their level sets are still invariant tori carrying a conditionally periodic flow. In important cases it is possible to construct an "intermediate system" that is still integrable, but non-degenerately so. It is the ramified torus bundle defined by this "intermediate system" that gets "Cantorised" when passing to the original perturbed dynamics.