9

Topological Aspects

Singular covering maps take historical precedence (in the work of Riemann in analysis) over the more recent concept of covering map that occurs in algebraic topology. This chapter emphasizes the point of view that topos complete spreads may be regarded as a class of generalized singular coverings. We deal with aspects of singular covering toposes of interest in topology, and in particular, we focus on a special class of complete spreads that we call branched coverings.

We call the geometric morphism $\mathscr{E}/X \longrightarrow \mathscr{E}$ associated with any object X of a topos $\mathscr{E}$ a local homeomorphism. We shall see that if X is a locally constant object (or locally trivial, or locally split) in a locally connected topos $\mathscr{E}$, then $\mathscr{E}/X \longrightarrow \mathscr{E}$ is a complete spread geometric morphism. Let us call a local homeomorphism that is also a complete spread an unramified covering, or an unramified covering topos. If X is a locally constant object of a locally connected topos $\mathscr{E}$, then the geometric morphism $\mathscr{E}/X \longrightarrow \mathscr{E}$ is thus unramified in this sense.

Our definition (or intrinsic characterization) of branched coverings in the context of toposes employs the notions of complete spread, pure subobject, locally trivial covering, and a newly isolated concept of purely skeletal geometric morphism. We establish the equivalence of this (axiomatic) definition with an alternative notion of branched covering that is almost directly motivated by Fox's topological concept of branched covering, and which was given independently by M. Bunge and S. Niefield [BN00], and by J. Funk [Fun00].

We show that a van Kampen theorem, obtained in joint work by M. Bunge and S. Lack [BL03], holds for what we call fibrations of regular coverings. These include both the locally constant as well as the larger class of unramified coverings, which are shown to enjoy similar topological properties yet do not agree in general.

Finally, we introduce a notion of index of a complete spread, which is related to branched coverings.

9.1 Locally Constant versus Unramified Coverings

Let $\mathscr{E}$ be a topos bounded over $\mathscr{S}$. For the moment, no assumptions on $\mathscr{E}$ will be made. Occasionally, we will need to suppose that $\mathscr{E}$ is locally connected.

We say that an object has *global support* if its unique morphism to the terminal is an epimorphism.

Definition 9.1.1 *An object X of $\mathscr{E}$ is said to be U-*split *by an object U of $\mathscr{E}$ if there is a morphism $\alpha : S \longrightarrow I$ in $\mathscr{E}$, and a morphism $\eta : U \longrightarrow e^*I$, such that there is a morphism $X \times U \longrightarrow e^*S$ for which the square*

$$\begin{array}{ccc} X \times U & \longrightarrow & e^*S \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle e^*\alpha} \\ U & \xrightarrow[\eta]{} & e^*I \end{array}$$

is a pullback. An object X of $\mathscr{E}$ is said to be locally constant*, or* locally trivial*, if X is U-split by an object U of $\mathscr{E}$ with global support. We call the geometric morphism $\mathscr{E}/X \longrightarrow \mathscr{E}$ associated with a locally constant object X* a locally constant covering.

Remark 9.1.2 *A notion of* constant object *(as U-split where $U = 1$) is implicit in the notion of locally constant object. In view of the central role which definable morphisms play in this book, it is worthwhile to note that an object X of an $\mathscr{S}$-topos $\mathscr{E}$ is 1-split iff it is a definable object, in the sense that $X \longrightarrow 1$ is a definable morphism. Then implicitly and automatically an object is locally constant iff it is locally definable.*

Lemma 9.1.3 *Assume that X and U are objects of a locally connected topos $\mathscr{E}$. Then the following are equivalent:*

1. *X is U-split.*
2. *there exists a morphism $\alpha : S \longrightarrow e_!U$ in $\mathscr{S}$, and a pullback*

$$\begin{array}{ccc} X \times U & \longrightarrow & e^*S \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle e^*\alpha} \\ U & \xrightarrow[\eta_U]{} & e^*e_!U \end{array}$$

   where η is the unit of the adjunction $e_! \dashv e^*$.
3. *the adjunction square*

$$\begin{array}{ccc} X \times U & \longrightarrow & e^*e_!(X \times U) \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle e^*e_!\pi_2} \\ U & \xrightarrow[\eta_U]{} & e^*e_!U \end{array}$$

   is a pullback, where η is the unit of $e_! \dashv e^*$.

It follows from the preservation properties of inverse image functors of geometric morphisms that they preserve locally constant objects. We now wonder about the question of inverse image functors reflecting locally constant objects. We are led to consider restrictions on the geometric morphism.

Lemma 9.1.4 *Pullback along a locally connected surjection reflects locally constant objects.*

Proof. Let $\mathscr{G} \xrightarrow{\varphi} \mathscr{E}$ be a locally connected surjection. Assume that $\varphi^* X$ is split by some object U of global support, with the help of morphisms $\alpha : S \longrightarrow I$ in $\mathscr{S}$, and $\eta : U \longrightarrow \varphi^* e^* I$ in $\mathscr{G}$, so that there is a pullback diagram

$$\begin{array}{ccc} \varphi^* X \times U & \xrightarrow{\zeta} & \varphi^* e^* S \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \varphi^* e^* \alpha} \\ U & \xrightarrow[\eta]{} & \varphi^* e^* I \end{array}$$

in $\mathscr{G}$. Since φ is locally connected, it preserves definable morphisms in the sense that the diagram

$$\begin{array}{ccc} \varphi_!(\varphi^* X \times U) & \xrightarrow{\zeta'} & e^* S \\ {\scriptstyle \varphi_! \pi_2}\downarrow & & \downarrow{\scriptstyle e^* \alpha} \\ \varphi_! U & \xrightarrow[\eta']{} & e^* I \end{array}$$

is a pullback in $\mathscr{E}$. By the Frobenius condition for a locally connected geometric morphism over $\mathscr{S}$, which in this case says that the canonical map

$$\varphi_!(\varphi^* X \times U) \longrightarrow X \times \varphi_! U$$

is invertible, the above diagram gives a pullback

$$\begin{array}{ccc} X \times \varphi_! U & \xrightarrow{\zeta'} & e^* S \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle e^* \alpha} \\ \varphi_! U & \xrightarrow[\eta']{} & e^* I \end{array}$$

which shows that X is $\varphi_! U$-split.

It remains to prove that if U has global support, then so does $\varphi_! U$. Since $U \longrightarrow 1$ is an epimorphism, also $\varphi_! U \longrightarrow \varphi_! 1$ is an epimorphism as $\varphi_!$ is a left adjoint. If φ is a surjection (φ^* is faithful), then $\varphi_! 1$ has global support since every component of the counit of $\varphi_! \dashv \varphi^*$ is an epimorphism. In particular, $\varphi_! 1 \cong \varphi_! \varphi^* 1 \longrightarrow 1$ is an epimorphism. This completes the proof. □

We now turn to a consideration of the ‘analytic’ notion of an unramified covering, which we take to mean a local homeomorphism that is also a complete spread (Def. 9.1.7). We first analyse some further relevant properties of complete spreads, beginning with the following analogue of Lemma 9.1.4 for complete spreads.

Lemmas 9.1.4 and 9.1.5 will prepare us for Theorem 9.1.6, and for the van Kampen theorem for locally constant coverings and unramified coverings.

Lemma 9.1.5 *Pullback along a locally connected surjection reflects complete spreads.*

Proof. Suppose in a pullback

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{\ \xi\ } & \mathscr{G} \\ \downarrow & & \downarrow \varphi \\ \mathscr{F} & \xrightarrow[\ \psi\ ]{} & \mathscr{E} \end{array}$$

that ξ is a complete spread, where φ is a locally connected surjection. Then ξ is a spread, so that by Lemma 3.1.10, ψ is a spread. Form the spread completion of ψ and its pullback along φ.

$$\begin{array}{ccccc} \mathscr{P} & \rightarrowtail & \mathscr{X} & \longrightarrow & \mathscr{G} \\ \downarrow & & \downarrow & & \downarrow \varphi \\ \mathscr{F} & \underset{\eta}{\rightarrowtail} & \mathscr{Y} & \longrightarrow & \mathscr{E} \end{array}$$

η is a pure spread, hence an inclusion (Lemma 3.1.12). The pullback of η is an equivalence because ξ is a complete spread. Therefore, η is a surjection, hence an equivalence. This proves that ψ is a complete spread. ▫

Theorem 9.1.6 *If X is a locally constant object of a locally connected topos $\mathscr{E}$, then $\mathscr{E}/X \longrightarrow \mathscr{E}$ is a complete spread.*

Proof. Let X be locally constant. By Exercise 2.4.12, 3, for any morphism $S \xrightarrow{m} I$ of $\mathscr{S}$, $\mathscr{S}/S \longrightarrow \mathscr{S}/I$ is a complete spread. Complete spreads are pullback stable along locally connected (or even essential) geometric morphisms, so

$$\mathscr{E}/e^*S \longrightarrow \mathscr{E}/e^*I \ ,$$

and hence $\mathscr{E}/X \times U \longrightarrow \mathscr{E}/U$, is a complete spread. By Lemma 9.1.5 we are done. ▫

Definition 9.1.7 *Assume that $\mathscr{E}$ is locally connected over $\mathscr{S}$. We shall refer to an object X of a topos $\mathscr{E}$ over $\mathscr{S}$ for which $\mathscr{E}/X \longrightarrow \mathscr{E}$ is a complete spread as a* complete spread object. *In this case, we call the geometric morphism $\mathscr{E}/X \longrightarrow \mathscr{E}$* an unramified covering.

Theorem 9.1.6 says, in our terminology, that a locally constant covering is an unramified covering. Example 9.1.8 describes an unramified cover that is not locally constant; the class of unramified coverings is strictly larger than the class of locally constant coverings, even over a locally connected space. The domain space in this example is connected, so this map cannot even be a coproduct of locally constant coverings.

Example 9.1.8 *This example describes a local homeomorphism* $Y \xrightarrow{\psi} X$ *(for which* Y *is connected) into a locally path-connected and connected space* X *that is a complete spread (an unramified covering), but is not locally constant. The space* X *is the 'Hawaiian earring:'* X *is the pencil of tangent circles* C_n *of radius* $\frac{1}{n}$, $n = 1, 2, 3 \ldots$, *topologized as a subspace of the Euclidean plane. We have*

$$X = \bigcup_{n=1}^{\infty} C_n$$

with a single tangent point a *such that* $\forall\, m \neq n$, $C_m \cap C_n = \{a\}$. *The Hawaiian earring is not semi-locally simply connected (defined below). The domain space* Y *consists of countably many copies of the real line* $\mathbb{R}$ *and of* X, *topologized as a subset of Euclidean* 3*-space. To be precise, let*

$$Y = (\bigcup_{n=1}^{\infty} \mathbb{R}_n) \cup (\bigcup_{|z|=1}^{\infty} X_z)\ ,$$

where n *is a natural number and* z *is an integer. Let* $Y \xrightarrow{\psi} X$ *be the map such that:*

1. *each* $\mathbb{R}_n$ *is a homeomorphic copy of the real line, and* ψ *restricted to* $\mathbb{R}_n$ *is a universal covering map* $\mathbb{R}_n \longrightarrow C_n$,
2. $\psi^{-1}(a) = \{\ldots, -2, -1, 1, 2, \ldots\}$ *ordered consecutively on* R_1, *and* $\psi^{-1}(a) \cap \mathbb{R}_n = \{\ldots, -n-1, -n, n, n+1, \ldots\}$,
3. ψ *carries* X_z *homeomorphically onto* $\bigcup_{j=|z|+1}^{\infty} C_j$, $|z| = 1, 2, \ldots$,
4. *each* $y \in Y - \psi^{-1}(a)$ *has an open neighbourhood that is homeomorphic to the real line,*
5. $X_z \cap (\bigcup_{n=1}^{\infty} \mathbb{R}_n) = \{z\}$, $|z| = 1, 2, \ldots$.

The space Y *is connected and locally path-connected. We readily see that the map* ψ *is a local homeomorphism, even at the points of the fiber* $\psi^{-1}(a)$. *Furthermore,* ψ *is a spread, and it also holds that the fiber of any point of* X *is in bijection with its cogerms, so that* ψ *is a complete spread. On the other hand,* ψ *is not locally constant because the point* $a \in X$ *does not have an evenly covered neighbourhood. Indeed, any neighbourhood* B *of* a *contains a circle* C_n, *for some* n. *For this* n, *the point* n *of* $\psi^{-1}(a)$ *(according to our naming convention) is a member of* R_n. *The connected component of* $\psi^{-1}(B)$ *that contains this point must contain all of* $\mathbb{R}_n$, *so that* ψ *cannot restrict to a homeomorphism of this component onto* B.

Remark 9.1.9 *The following result provides further evidence beyond Exercise 6 that unramified coverings are locally constant under hypotheses of the locally simply connected kind. A space is said to be* semi-locally simply connected *if it has a cover* $\{U_\alpha\}$ *of open neighbourhoods such that each* U_α *has the property that any two paths in* U_α *with common endpoints are homotopic in the whole space by a homotopy that fixes the endpoints.*

> *Theorem [FT01]:* A non-0 local homeomorphism over a connected, locally path-connected, semi-locally simply connected space that is also a complete spread is a surjective covering space.

This theorem may be proved using a path-lifting argument, but we omit this proof as we shall not use the result and the proof would take us beyond the scope of this book.

Remark 9.1.10 *In some ways, unramified coverings are better behaved than locally constant coverings. Locally constant coverings do not generally compose: it can happen that* $\mathscr{E}/X \longrightarrow \mathscr{E}/Y$ *and* $\mathscr{E}/Y \longrightarrow \mathscr{E}$ *are locally constant, but* $\mathscr{E}/X \longrightarrow \mathscr{E}$ *is not (even in a locally connected topos* $\mathscr{E}$*). However, unramified coverings do compose. We shall see also that unramified coverings, just like locally constant coverings, satisfy a (coverings) van Kampen theorem with respect to the same class of geometric morphisms of effective descent, namely locally connected surjections. For this purpose, we shall define a notion of regular covering morphism. The locally constant coverings and the unramified coverings are two examples of such regular classes. Of course, we also think of complete spreads as coverings, but of a singular (or ramified), not regular kind. We shall develop this point of view in § 9.3.*

Let us denote by

$$\mathrm{L} : \mathbf{Top}_{\mathscr{S}}{}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$$

the pseudofunctor that assigns to a topos $\mathscr{E}$ the slice category $\mathbf{Top}_{\mathscr{S}}/\mathscr{E}$ and to a geometric morphism $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}$ the functor given by pulling back along φ.

A geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is said *to be of effective descent* if any object X of $\mathscr{F}$ equipped with descent data already comes from $\mathscr{E}$ under ψ^*. We are concerned with classes Φ of geometric morphisms of effective descent that are closed under composition and pullbacks. For instance, the class of locally connected surjections is such a class Φ, which we shall meet again below.

Definition 9.1.11 *Let* Φ *be a class of geometric morphisms of effective descent in* $\mathbf{Top}_{\mathscr{S}}$ *that is closed under composition and pullbacks. We shall say that a subpseudofunctor* Γ *of* L *is a* Φ*-stack, if for any pullback diagram*

$$\begin{array}{ccc} \mathscr{P} & \longrightarrow & \mathscr{G} \\ {\scriptstyle \varphi^*(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathscr{F} & \xrightarrow[\varphi]{} & \mathscr{E} \end{array}$$

of geometric morphisms for which φ *is a member of* Φ*, we have*

$$\varphi^*(\alpha) \in \Gamma(\mathscr{F}) \text{ implies } \alpha \in \Gamma(\mathscr{E}) .$$

Let us return to a pseudofunctor we have already encountered in § 1.3:

$$\mathbf{A} : \mathbf{Top}_{\mathscr{S}}{}^{\mathrm{op}} \longrightarrow \mathbf{CAT} ,$$

such that $\mathbf{A}(\mathscr{E})$ is the topos-frame $\mathscr{E}$ itself. $\mathbf{A}$ is a subpseudofunctor of L. This is a basic example of an intensive quantity that Lawvere has emphasized. By the very definition of effective descent, $\mathbf{A}$ is trivially a Φ-stack for any class Φ of effective descent geometric morphisms, and we have already seen in Proposition 1.3.2 that $\mathbf{A}$ preserves binary products. We make the following definition.

Definition 9.1.12 *Let* $\mathscr{K}$ *be an extensive sub-2-category of* $\mathbf{Top}_{\mathscr{S}}$*, and* Φ *a class of geometric morphisms of effective descent, closed under composition and pullbacks. A subpseudofunctor* $\mathscr{K}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$ *of* L *is said to be* a fibration of regular covering morphisms with respect to Φ *if it is a* Φ*-stack, and if it preserves binary products.*

Denote by

$$\mathscr{C} : \mathbf{Top}_{\mathscr{S}}{}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$$

the subpseudofunctor of L such that $\mathscr{C}(\mathscr{E})$ is the full subcategory of $\mathrm{L}(\mathscr{E})$ consisting of the local homeomorphisms determined by its locally constant objects. The objects of $\mathscr{C}(\mathscr{E})$ are called *locally constant coverings of* $\mathscr{E}$. We have already observed that this assignment is pseudofunctorial, as inverse images of geometric morphisms preserve locally constant objects in general. In particular, we may consider $\mathscr{C}$ as a contravariant pseudofunctor defined on $\mathbf{LTop}_{\mathscr{S}}$, the full sub 2-category of $\mathbf{Top}_{\mathscr{S}}$ whose objects are locally connected toposes.

For a locally connected topos $\mathscr{E}$, let $\mathscr{U}(\mathscr{E})$ denote the full subcategory of $\mathrm{L}(\mathscr{E})$ determined by the complete spread objects of $\mathscr{E}$, or unramified coverings of $\mathscr{E}$ (Definition 9.1.7). This assignment extends to a pseudofunctor

$$\mathscr{U} : \mathbf{LTop}_{\mathscr{S}}{}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$$

since unramified geometric morphisms are stable under pullback along geometric morphisms with locally connected domain (Exercise 9.1, 7).

Lemma 9.1.13 *The subpseudofunctors* $\mathscr{C}$ *and* $\mathscr{U}$ *of* L *are both fibrations of regular coverings with respect to the class* Φ *of locally connected surjections.*

Proof. It is not difficult to show that $\mathscr{C}$ and $\mathscr{U}$ preserve binary products (Exercise 9.1, 8). The fact that $\mathscr{C}$ is a Φ-stack depends on Lemma 9.1.4. The same fact for $\mathscr{U}$ depends on Lemma 9.1.5. □

Definition 9.1.14 *Let $\mathscr{K}$ be an extensive sub-2-category of $\boldsymbol{Top}_{\mathscr{S}}$, and Φ a class of geometric morphisms of effective descent, closed under composition and pullbacks. Let Γ be a pseudofunctor on $\mathscr{K}$. We shall say that* the van Kampen theorem holds *for Γ with respect to Φ if whenever*

$$\begin{array}{ccc} \mathscr{E}_0 & \xrightarrow{\beta_1} & \mathscr{E}_1 \\ {\scriptstyle \beta_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ \mathscr{E}_2 & \xrightarrow[\alpha_2]{} & \mathscr{E} \end{array}$$

is a bipushout (in $\boldsymbol{Top}_{\mathscr{S}}$) of objects in $\mathscr{K}$ in which the induced map $\mathscr{E}_1 + \mathscr{E}_2 \longrightarrow \mathscr{E}$ is a member of Φ, the diagram

$$\begin{array}{ccc} \Gamma(\mathscr{E}_0) & \xleftarrow{\beta_1^*} & \Gamma(\mathscr{E}_1) \\ {\scriptstyle \beta_2^*}\uparrow & & \uparrow{\scriptstyle \alpha_1^*} \\ \Gamma(\mathscr{E}_2) & \xleftarrow[\alpha_2^*]{} & \Gamma(\mathscr{E}) \end{array}$$

is a bipullback in **CAT**.

The conditions of Definition 9.1.12 imply the condition of Definition 9.1.14, which we state as the next theorem.

Theorem 9.1.15 *Let $\mathscr{K}$ be an extensive sub-2-category of $\boldsymbol{Top}_{\mathscr{S}}$, and let Φ be a class of effective descent morphisms in $\mathscr{K}$, closed under pullback and composition. Let Γ be a fibration of regular covering morphisms in $\mathscr{K}$ with respect to Φ. Then Γ satisfies the van Kampen theorem with respect to Φ.*

Proof. The proof reduces, using the given pushout (testing it with geometric morphisms whose codomain is the object classifier), and since Γ is a sub-pseudofunctor of L, to showing that an object X of $\mathscr{E}$ is in $\Gamma(\mathscr{E})$ if $X_1 = \alpha_1^*(X)$ and $X_2 = \alpha_2^*(X)$ are in $\Gamma(\mathscr{E}_1)$ and $\Gamma(\mathscr{E}_2)$ respectively. Since Γ preserves binary products, $\alpha^*(X) = (\alpha_1^*(X), \alpha_2^*(X))$ is in $\Gamma(\mathscr{E}_1) \times \Gamma(\mathscr{E}_2) \simeq \Gamma(\mathscr{E}_1 + \mathscr{E}_2)$. Since α is of effective descent for Γ, and by our assumption that Γ is a Φ-stack, X is indeed in $\Gamma(\mathscr{E})$. □

Corollary 9.1.16 *The van Kampen theorem holds for both $\mathscr{C}$ and $\mathscr{U}$ regarded as pseudofunctors $\boldsymbol{LTop}_{\mathscr{S}}^{\mathrm{op}} \longrightarrow$ **CAT**, with respect to the class Φ of locally connected surjections.*

Proof. This follows directly from Theorem 9.1.15 and Lemma 9.1.13. □

Remark 9.1.17 *Assume that the toposes in Theorem 9.1.15 are all locally connected and* locally simply connected, *in the sense that there is a single $U \twoheadrightarrow 1_{\mathscr{E}}$ that splits all locally constant objects. If $\mathscr{E}$ is locally connected and* *locally simply connected, then $\mathscr{C}(\mathscr{E})$ is a (Galois) topos of the form $\mathscr{B}(\pi_1(\mathscr{E}))$, the classifying topos of the fundamental group of $\mathscr{E}$. In this case, the van Kampen theorem takes the form of "a pushout-to-pushout" result, as follows. Let*

$$\begin{array}{ccc} \mathscr{E}_0 & \xrightarrow{\beta_1} & \mathscr{E}_1 \\ {\scriptstyle \beta_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ \mathscr{E}_2 & \xrightarrow[\alpha_2]{} & \mathscr{E} \end{array}$$

be a pushout diagram in $\boldsymbol{LTop}_{\mathscr{S}}$ in which all four toposes are locally simply connected, and where the induced map $\mathscr{E}_1 + \mathscr{E}_2 \longrightarrow \mathscr{E}$ is a locally connected surjection. Then the diagram

$$\begin{array}{ccc} \mathscr{C}(\mathscr{E}_0) & \xrightarrow{\beta_1} & \mathscr{C}(\mathscr{E}_1) \\ {\scriptstyle \beta_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ \mathscr{C}(\mathscr{E}_2) & \xrightarrow[\alpha_2]{} & \mathscr{C}(\mathscr{E}) \end{array}$$

is a pushout diagram in $\boldsymbol{Top}_{\mathscr{S}}$. The conclusion uses the fact that a bipushout in $\boldsymbol{Top}_{\mathscr{S}}$ is calculated as a bipullback in **Cat** *via the inverse images of the geometric morphisms. We warn the reader that trying to deduce this result from the possible existence of a reflection of the inclusion of Galois toposes into locally simply connected toposes meets with some difficulties.*

Remark 9.1.18 *A pseudofunctor*

$$\Gamma : \mathscr{K}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$$

provides a notion of homotopy. I.e., by definition, a Γ-homotopy $\psi \Rightarrow \varphi$ between two geometric morphisms with the same domain and codomain toposes is a pseudonatural transformation (or isomorphism)

$$\Gamma(\psi) \Rightarrow \Gamma(\varphi) .$$

For example, **A**, *$\mathscr{C}$, and $\mathscr{U}$ are all fibrations of regular coverings with respect to locally connected surjections, but $\mathscr{C}$ and $\mathscr{U}$ are distinguished from* **A** *by their homotopies. Indeed, ordinary homotopies (for locally path-connected topological spaces) induce $\mathscr{C}$ and $\mathscr{U}$-homotopies, but they do not induce* **A***-homotopies.*

Exercises 9.1.19

1. *Show that any constant object e^*A (including 0) is a locally constant object.*
2. *Prove Lemma 9.1.3.*

3. *If $\mathscr{E}$ is connected and locally connected, and the base topos is Set, show that an object X of $\mathscr{E}$ is locally constant (as in Definition 9.1.1) iff there is a $U \twoheadrightarrow 1_{\mathscr{E}}$ and an object S of $\mathscr{S}$ such that $X \times U \cong e^*S \times U$ over U. (This condition was taken by Barr and Diaconescu as the definition of locally constant; however, it is only suitable in the connected case.)*
4. *An open set $V \subseteq X$ is* evenly covered by a map $F \xrightarrow{\pi} X$ *if $\pi^{-1}(V)$ has an open partition such that the restriction of π to each member of the partition is a homeomorphism with V. Then π is a (necessarily surjective)* covering space *if X has a cover of open sets each of which is evenly covered by π. Let X be locally connected. Show that a local homeomorphism $F \longrightarrow X$ is a covering space in this sense iff it is a non-0 locally constant object in $Sh(X)$.*
5. *Show that presheaf on a connected small category is locally constant iff its transition maps are isomorphisms.*
6. *A presheaf (= discrete fibration) is a complete spread object iff it is also a discrete opfibration. Show that a presheaf on a connected category is locally constant iff it is a complete spread object.*
7. *Prove that the unramified geometric morphisms with locally connected codomain (hence also locally connected domain) are stable under pullback along geometric morphisms with locally connected domain.*
8. *Prove that the pseudofunctors $\mathscr{C}$ and $\mathscr{U}$ preserve binary products.*
9. *Let $\mathscr{F} \xrightarrow{\rho} \mathscr{E}$ be a pure geometric morphism between locally connected toposes. Show that the induced functor*

$$\rho^* : \mathscr{C}(\mathscr{E}) \longrightarrow \mathscr{C}(\mathscr{F})$$

is full and faithful.

9.2 Purely Skeletal Geometric Morphisms

In order to prepare for branched coverings in §9.3, we first investigate the class of geometric morphisms that respect pure (mono)morphisms. Throughout this section, $\mathscr{E}$ denotes a locally connected topos.

Definition 9.2.1 *A* pure morphism *of $\mathscr{E}$ is a morphism $X \longrightarrow Y$ for which $\mathscr{E}/X \longrightarrow \mathscr{E}/Y$ is a pure geometric morphism. We say an object X is pure if $X \longrightarrow 1$ is a pure morphism.*

Lemma 9.2.2 *A morphism $X \xrightarrow{a} Y$ of $\mathscr{E}$ is pure iff the induced morphism*

$$Y \times e^*(\Omega_{\mathscr{S}}) \longrightarrow (Y \times e^*(\Omega_{\mathscr{S}}))^a$$

is an isomorphism. In particular, the lattice of definable subobjects of a pure object is isomorphic to the lattice of definable subobjects of $1_{\mathscr{E}}$.

Proof. This follows directly from the definitions. □

Example 9.2.3 *The balloon's shadow map $S^2 \longrightarrow S^2$ (§ 2.1) has the property that every pure open subset remains pure under inverse image. On the other hand, the image part of this map $S^2 \twoheadrightarrow D$, where D is a closed disk, does not have this pure-respecting property (eg., consider the interior of D). Both these maps are complete spreads.*

A map $D \twoheadrightarrow S^2$ that envelopes the sphere by collapsing the boundary of D to a point of S^2 also has the pure-respecting property.

Definition 9.2.4 *We shall say that* a geometric morphism respects (reflects) pure morphisms *if its inverse image functor preserves (reflects) pure morphisms. We shall call* purely skeletal *any geometric morphism that respects* pure monomorphisms *(meaning simply a monomorphism that is a pure).*

Geometric morphisms that respect pure (mono) morphisms are analogous to geometric morphisms that respect double-negation dense monomorphisms, the so-called skeletal geometric morphisms.

Remark 9.2.5 *In the following diagram if p reflects pure morphisms and φ respects pure morphisms, then ψ respects pure morphisms.*

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{p} & \mathscr{Y} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathscr{E} & \end{array}$$

Proposition 9.2.6 *A locally connected geometric morphism respects pure morphisms. A locally connected surjection reflects pure morphisms.*

Proof. These statements follow from Lemma 2.2.11. □

Proposition 9.2.6 has the following dual statement for pure geometric morphisms.

Proposition 9.2.7 *A pure geometric morphism reflects pure morphisms. A pure inclusion respects pure morphisms.*

Proof. Consider the following diagram for pure p, and object X.

$$\begin{array}{ccc} \mathscr{F}/p^*X & \longrightarrow & \mathscr{E}/X \\ \downarrow & & \downarrow \\ \mathscr{F} & \xrightarrow[p]{} & \mathscr{E} \end{array}$$

The top horizontal is pure. If p^*X is pure, then X is pure by Lemma 2.2.8. We also use Lemma 2.2.8 for the second statement. □

We may always consider the largest topology in a topos for which a given object of the topos is a sheaf. If X denotes an object, then a monomorphism $m : A \rightarrowtail B$ is dense for this largest topology iff $B^*X \longrightarrow B^*X^m$ (transpose of the projection) is an isomorphism. We have the following.

Proposition 9.2.8 *Let $\mathscr{E}$ be a topos bounded over $\mathscr{S}$. Then a monomorphism is dense for the largest topology for which $e^*(\Omega_{\mathscr{S}})$ is a sheaf iff it is a pure morphism. Thus, the pure monomorphisms in $\mathscr{E}$ are the dense monomorphisms for a topology in $\mathscr{E}$.*

Definition 9.2.9 *We refer to the topology of pure monomorphisms as* the pure topology in $\mathscr{E}$*, and to its sheaves as* pure-sheaves*. We refer to monomorphisms that are closed for the pure topology as* pure-closed*. We denote the subtopos of pure-sheaves by $\mathscr{E}_p \rightarrowtail \mathscr{E}$.*

Every topos has a smallest dense subtopos: the subtopos of double-negation sheaves, which is a Boolean topos. Similarly every locally connected topos has a smallest pure subtopos.

Proposition 9.2.10 *If $\mathscr{E}$ is locally connected, then $\mathscr{E}_p$ is locally connected. $\mathscr{E}_p$ is the smallest pure subtopos of $\mathscr{E}$.*

Proof. The unit $e^*(\Omega_{\mathscr{S}}) \longrightarrow i_*i^*e^*(\Omega_{\mathscr{S}})$ for the inclusion $i : \mathscr{E}_p \rightarrowtail \mathscr{E}$ is an isomorphism because $e^*(\Omega_{\mathscr{S}})$ is a pure-sheaf. But this says that i is pure. $\mathscr{E}_p$ is locally connected because a pure subtopos of a locally connected topos is locally connected (2.2.16). Let $Sh_j(\mathscr{E}) \rightarrowtail \mathscr{E}$ be a pure inclusion. By the definition of pure, $e^*(\Omega_{\mathscr{S}})$ is a j-sheaf. Hence, every j-dense monomorphism is pure. Therefore, $\mathscr{E}_p \rightarrowtail Sh_j(\mathscr{E})$. □

Example 9.2.11 *Let R denote the real numbers. Then $Sh(R)_p = Sh(R)$ because a pure inclusion of open intervals must be an equality. However, $Sh(R^2)_p$ is a proper subtopos of $Sh(R^2)$. For instance, the complement in R^2 of a curve is a pure-closed open subset of R^2. On the other hand, a punctured plane is a pure subset of R^2. The complement of a surface in R^3 is a pure-closed open subset of R^3.*

Example 9.2.12 *The dense topology in a presheaf topos $P(\mathbb{C})$ is given by the sieves $R \rightarrowtail h_c$ such that for every morphism $d \xrightarrow{f} c$ in $\mathbb{C}$, $f^*(R) \rightarrowtail h_d$ is non-empty. The subtopos of sheaves for this topology is precisely the smallest dense subtopos of $P(\mathbb{C})$. A sieve $R \rightarrowtail h_c$ is a member of the pure topology in $P(\mathbb{C})$ if for every f, $f^*(R)$ is connected. Implicitly, a connected sieve is non-empty, so the pure topology is contained in the dense (or double-negation) topology. The smallest pure subtopos $P(\mathbb{C})_p$ is locally connected.*

Definition 9.2.13 A density object *of an locally connected topos is an object of the form $\boldsymbol{d}(\mu)$, for some distribution μ, where $\boldsymbol{d}$ is the density monad, introduced in § 6.2.*

Proposition 9.2.14 *Density objects are pure-sheaves. If $\mathscr{E}$ is locally connected, then $\mathscr{E}_p$ is the smallest subtopos of $\mathscr{E}$ containing the density objects.*

Proof. Let $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ denote the complete spread associated with a distribution μ. Let $A \rightarrowtail B$ be a pure monomorphism in a locally connected topos $\mathscr{E}$. We know that morphisms $A \longrightarrow \mathbf{d}(\mu)$ are in bijection with geometric morphisms between the spread completion of $\mathscr{E}/A$ and $\mathscr{Y}$ over $\mathscr{E}$. However, the spread completions of $\mathscr{E}/A$ and $\mathscr{E}/B$ are equivalent because $\mathscr{E}/A \rightarrowtail \mathscr{E}/B$ is a pure geometric morphism. This shows that $\mathbf{d}(\mu)$ is a sheaf for the pure topology. The second statement holds because any constant object is a density object. In particular, the constant object $e^*(\Omega_{\mathscr{S}})$ is a density object. ◻

Proposition 9.2.15 *For any geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ over $\mathscr{S}$ with locally connected domain, the following are equivalent:*

1. *ψ is purely skeletal;*
2. *ψ restricts to smallest pure subtoposes;*

$$\begin{array}{ccc} \mathscr{F}_p & \rightarrowtail & \mathscr{F} \\ \downarrow & & \downarrow \psi \\ \mathscr{E}_p & \rightarrowtail & \mathscr{E} \end{array}$$

3. *the distribution algebra $H = \psi_*(f^*\Omega_{\mathscr{S}})$ in $\mathscr{E}$ is a pure-sheaf.*

Proof. These conditions are clearly equivalent once we recall the definitions of pure, and pure-sheaf. ◻

The spread completion of an object Y of a locally connected topos $\mathscr{E}$ is the complete spread geometric morphism ψ in the diagram

$$\begin{array}{ccc} \mathscr{E}/Y & \xrightarrow{p} & \mathscr{Y} \\ & \searrow \quad \swarrow \psi & \\ & \mathscr{E} & \end{array} \tag{9.1}$$

in which p is pure. These complete spreads correspond to Lawvere's *absolutely continuous distributions*, meaning a distribution of the kind $Y.e_!$.

Proposition 9.2.16 *The spread completion of an object of a locally connected topos is purely skeletal.*

Proof. Use Proposition 9.2.6 and Remark 9.2.5 applied to diagram 9.1. ◻

Exercises 9.2.17

1. *How are preservation of pure sets under inverse image and change in codimension of singular sets related?*

2. *Show that the inclusion of a single point into the real line is purely skeletal, but not skeletal. Find an example showing that skeletal does not imply purely skeletal.*
3. *Let $\mathbf{E}_p(\mathscr{E})$ be the category of distributions on a locally connected topos $\mathscr{E}$ that carry pure monomorphisms to isomorphisms. Show that an absolutely continuous distribution $Y.e_!$ is a member of $\mathbf{E}_p(\mathscr{E})$. Show that $\mathbf{E}_p(\mathscr{E}) \simeq \mathbf{E}(\mathscr{E}_p)$.*
4. *Provide a detailed proof of Proposition 9.2.15.*

9.3 Branched Covering Toposes

Our development of branched coverings stems in part from the ideas of R. H. Fox. We shall define a branched covering of a topos as a special kind of complete spread: we shall define a branched covering as a complete spread that is purely skeletal (Def. 9.2.4), and which is, as we shall say, a purely locally constant covering.

We shall say that an object *has pure support* if its support (which is a subobject of $1_{\mathscr{E}}$) is a pure monomorphism.

Definition 9.3.1 *A geometric morphism $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ is* a purely locally constant covering *if there is $U \longrightarrow 1_{\mathscr{E}}$ with pure support such that $U^*(\psi)$ in the pullback*

$$\begin{array}{ccc} \mathscr{Y}/\psi^*(U) & \longrightarrow & \mathscr{Y} \\ {\scriptstyle U^*(\psi)}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathscr{E}/U & \longrightarrow & \mathscr{E} \end{array}$$

in $\boldsymbol{Top}_{\mathscr{S}}$ is a local homeomorphism determined by a definable object in $\mathscr{E}/U$, in the sense of Remark 9.1.2.

Note that any locally constant covering of $\mathscr{E}$ is a purely locally constant covering of $\mathscr{E}$.

Definition 9.3.2 *A geometric morphism $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ is said to be* a branched covering *if:*

1. *it is a complete spread,*
2. *it is a purely locally constant covering, and*
3. *it is purely skeletal.*

We regard branched coverings of $\mathscr{E}$ as a full subcategory of the category of complete spreads over $\mathscr{E}$. Denote by $\mathscr{B}(\mathscr{E})$ the corresponding category of branched coverings of $\mathscr{E}$.

Example 9.3.3 *The balloon's shadow map $S^2 \longrightarrow S^2$ (Example 9.2.3) is a complete spread that satisfies the third requirement of branched covering, but* *not the second. The image part $S^2 \twoheadrightarrow D$ of this map is a complete spread that satisfies the second requirement, but not the third. Both of these maps exhibit folding, but this folding is detected differently in each case. Intuitively, the last two conditions of a branched covering together rule out folding in a complete spread.*

Any geometric morphism satisfying the last two conditions of Definition 9.3.2, but possibly not the first, can be 'normalized' in the following sense.

Proposition 9.3.4 *The spread completion of a geometric morphism with locally connected domain satisfying the last two conditions of a branched covering is a branched covering.*

Proof. We must verify that the spread completion $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ of such a geometric morphism $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}$ satisfies the last two conditions of a branched covering as in Definition 9.3.2. ψ has the third property because pure geometric morphisms reflect pure monomorphisms. The second condition holds because the pure

$$\mathscr{F}/\varphi^*U \longrightarrow \mathscr{Y}/\psi^*U$$

is a complete spread, hence an equivalence, since both $\mathscr{F}/\varphi^*U \longrightarrow \mathscr{E}/U$ and $\mathscr{Y}/\psi^*U \longrightarrow \mathscr{E}/U$ are complete spreads. □

Remark 9.3.5 *The normalization process may have trivial results. The bag map $D \twoheadrightarrow S^2$ (collapse the boundary of D to a point forming a sphere) satisfies the last two requirements of a branched geometric morphism, but it is not a complete spread. In fact, this map is pure, so its spread completion is the identity on S^2. It may be interesting to note that the zipper map $D \twoheadrightarrow S^2$ (collapse the boundary of D to a closed line segment forming a sphere) is not branched because it is not purely locally constant (but it is purely skeletal). However, it is already normalized, as it is a complete spread.*

The following is immediately clear.

Proposition 9.3.6 *A locally constant covering of a locally connected topos $\mathscr{E}$ is a branched covering. For a locally connected topos $\mathscr{E}$, the category $\mathscr{C}(\mathscr{E})$ of locally constant coverings of $\mathscr{E}$ is a full subcategory of the category $\mathscr{B}(\mathscr{E})$ of branched coverings of $\mathscr{E}$.*

Remark 9.3.7 *If V denotes the support of a splitting object U for a branched covering $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$, then the support of $\psi^*(U)$ is $\psi^*(V)$. We refer to the support of $\psi^*(U)$ associated with ψ as the* non-singular part of $\mathscr{Y}$*. By definition, this is a pure subobject of $1_{\mathscr{Y}}$. Caution: the non-singular part of $\mathscr{Y}$ is not a well-defined subobject of $1_{\mathscr{Y}}$. In some cases a largest non-singular part is available, but we have not made this a requirement in Definition 9.3.2. However, it is reasonable to expect that in applications the notion of a branched covering be further specified by including its non-singular part, or its pure splitting object, as part of the data.*

Lemma 9.3.8 *Consider a commutative triangle*

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\;\rho\;} & \mathscr{Y} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathscr{E} & \end{array}$$

of localic geometric morphisms. If ρ and φ are local homeomorphisms, and ρ is a surjection, then ψ is a local homeomorphism.

Proof. This fact may be established using the well-known fact that a localic geometric morphism is a local homeomorphism iff it is open and its diagonal is open. □

Proposition 9.3.9 *A branched covering is a locally constant covering iff it has a splitting object with global support.*

Proof. If a branched covering $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ has a splitting object with global support, then we have a pullback diagram

$$\begin{array}{ccc} \mathscr{Y}/\psi^*(U) & \twoheadrightarrow & \mathscr{Y} \\ {\scriptstyle U^*(\psi)}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathscr{E}/U & \twoheadrightarrow & \mathscr{E} \end{array}$$

in which all geometric morphisms are local homeomorphisms except ostensibly ψ. But then by Lemma 9.3.8, ψ must be a local homeomorphism, whence locally constant. □

Lemma 9.3.10 is analogous to the simple fact from topology that in a square

$$\begin{array}{ccc} C & \rightarrowtail & \overline{C} \\ \downarrow & & \downarrow \\ S & \rightarrowtail & X \end{array}$$

where S is a subspace of X, C is a closed subset of S, and $\overline{C}$ is the closure of C in X, we have $C = S \cap \overline{C}$. In other words, the square is a pullback.

Lemma 9.3.10 *Let $\mathscr{F} \rightarrowtail \mathscr{E}$ be a subtopos. Let $\mathscr{X} \xrightarrow{\psi} \mathscr{F}$ be a complete spread. Then the spread completion of ψ over $\mathscr{E}$ forms a topos pullback square.*

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\;\rho\;} & \mathscr{Y} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ \mathscr{F} & \underset{i}{\rightarrowtail} & \mathscr{E} \end{array}$$

Consequently, the pure factor ρ is an inclusion.

Proof. We may construct the spread completion ψ' as follows. Let λ ($= x_! \cdot \psi^*$) denote the distribution on $\mathscr{F}$ associated with ψ, so that the distribution on $\mathscr{E}$ associated with ψ' is $\lambda \cdot i^*$. We choose any site for $\mathscr{E}$, say with underlying category $\mathbb{C}$. Then ψ' fits in the following pullback.

$$\begin{array}{ccc} \mathscr{Y} & \rightarrowtail & P(\mathbb{Y}) \\ {\scriptstyle \psi'}\downarrow & & \downarrow \\ \mathscr{E} & \rightarrowtail & P(\mathbb{C}) \end{array}$$

$\mathbb{Y}$ denotes the amalgamation site for $\lambda \cdot i^*$. A typical object of $\mathbb{Y}$ is a pair (c, α), where $\alpha \in \lambda i^*(\epsilon_c)$, and where $\mathbb{C} \xrightarrow{\epsilon} \mathscr{E}$ denotes the canonical functor. But we may also regard $\mathbb{C}$ as (the underlying category of) a site for $\mathscr{F}$, where now we have $\mathbb{C} \xrightarrow{\epsilon} \mathscr{F}$ such that $\epsilon' \cong i^* \cdot \epsilon$. Then $\mathbb{Y}$ is isomorphic to the amalgamation site for λ, since $\lambda \cdot \epsilon' \cong \lambda \cdot i^* \cdot \epsilon$. Hence, we have a commutative diagram

$$\begin{array}{ccccc} \mathscr{X} & \longrightarrow & \mathscr{Y} & \rightarrowtail & P(\mathbb{Y}) \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi'} & & \downarrow \\ \mathscr{F} & \underset{i}{\rightarrowtail} & \mathscr{E} & \rightarrowtail & P(\mathbb{C}) \end{array}$$

in which the outer and right squares are pullbacks. The essentially unique factoring morphism $\mathscr{X} \longrightarrow \mathscr{Y}$ must of course be the pure ρ. But then the left square must be a pullback. $\square$

Proposition 9.3.11 *The category of purely skeletal complete spreads over a topos $\mathscr{E}$ is equivalent to the category of complete spreads over $\mathscr{E}_p$. The equivalence is given on the one hand by pullback along $\mathscr{E}_p \rightarrowtail \mathscr{E}$, and on the other by spread completion.*

Proof. Assume first that $\mathscr{X} \xrightarrow{\varphi} \mathscr{E}_p$ is a complete spread. By Lemma 9.3.10, the spread completion $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ of the composite of φ with the pure inclusion $\mathscr{E}_p \rightarrowtail \mathscr{E}$ forms a topos pullback with φ. In particular, $\mathscr{X}$ is a pure subtopos of $\mathscr{Y}$. Therefore, $\mathscr{Y}_p$ factors through $\mathscr{X}$, so ψ restricts to smallest pure subtoposes, i.e., ψ respects pure monomorphisms.

On the other hand, assume that a complete spread $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ respects pure monomorphisms. We must show that the pullback

$$\begin{array}{ccc} \mathscr{X} & \rightarrowtail & \mathscr{Y} \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathscr{E}_p & \rightarrowtail & \mathscr{E} \end{array}$$

of ψ is a complete spread, and that the inclusion $\mathscr{X} \rightarrowtail \mathscr{Y}$ is pure. By assumption, ψ restricts to $\mathscr{Y}_p \xrightarrow{\psi_p} \mathscr{E}_p$. Since the domain $\mathscr{Y}$ of the complete spread ψ is locally connected, so is $\mathscr{Y}_p$. Form the spread completion of ψ_p, say $\mathscr{X} \xrightarrow{\varphi} \mathscr{E}_p$. But then the given ψ must be the spread completion of $\mathscr{X} \longrightarrow \mathscr{E}$, so again by Lemma 9.3.10, φ is the pullback of ψ. □

Proposition 9.3.11 has the following refinement. The proof technique is essentially the same.

Theorem 9.3.12 *The category $\mathscr{B}(\mathscr{E})$ of branched coverings over $\mathscr{E}$ is canonically equivalent to the category $\mathscr{C}(\mathscr{E}_p)$ of locally constant coverings of its smallest pure subtopos $\mathscr{E}_p$.*

Proof. We pass from a branched covering ψ of $\mathscr{E}$ to a locally constant covering of $\mathscr{E}_p$ by pullback.

$$\begin{array}{ccc} \mathscr{X} & \overset{\rho}{\rightarrowtail} & \mathscr{Y} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathscr{E}_p & \underset{i}{\rightarrowtail} & \mathscr{E} \end{array} \tag{9.2}$$

The pullback φ is i^*U-split if ψ is U-split, where U has pure support in $\mathscr{E}$. But then i^*U has global support in $\mathscr{E}_p$. By Lemma 9.3.8, φ must be a local homeomorphism, hence a locally constant covering. We shall show that ρ is pure. We may regard (9.2) as a composite of two pullback squares.

$$\begin{array}{ccccc} \mathscr{X} & \rightarrowtail & \mathscr{Y}/\psi^*V & \rightarrowtail & \mathscr{Y} \\ {\scriptstyle\varphi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\psi} \\ \mathscr{E}_p & \rightarrowtail & \mathscr{E}/V & \rightarrowtail & \mathscr{E} \end{array}$$

Here V denotes the (pure) support of U, which must include the smallest pure subtopos $\mathscr{E}_p$ by a pure inclusion. It follows again by Lemma 9.3.8 that the middle vertical is a local homeomorphism, and in fact it is a locally constant covering. The left top horizontal factor is therefore pure, since it is the pullback of a pure along a locally connected (in fact, a local homeomorphism). The right top horizontal factor is pure by the assumption that ψ is purely skeletal, hence, since $V \rightarrowtail 1$ is pure, so is $\psi^*V \rightarrowtail 1$. This shows that ρ in (9.2) is a composite of pures, hence pure itself. ψ is therefore the spread completion of $i \cdot \varphi$.

If we begin with a locally constant object X of $\mathscr{E}_p$, then we pass to a branched covering of $\mathscr{E}$ by spread completion.

$$\begin{array}{ccc} \mathscr{E}_p/X & \longrightarrow & \mathscr{Y} \\ \downarrow & & \downarrow{\scriptstyle\psi} \\ \mathscr{E}_p & \underset{i}{\rightarrowtail} & \mathscr{E} \end{array}$$

By Lemma 9.3.10, this square is a pullback. We must show that ψ is indeed a branched covering. From Proposition 9.2.16, we get that ψ is purely skeletal. We have a pullback

$$\begin{array}{ccc} \mathscr{E}_p/X & \rightarrowtail & \mathscr{E}/i_*X \\ \downarrow & & \downarrow \\ \mathscr{E}_p & \underset{i}{\rightarrowtail} & \mathscr{E} \end{array}$$

so that the top inclusion is pure. Therefore, ψ is also the spread completion of $\mathscr{E}/i_*X$.

$$\begin{array}{ccc} \mathscr{E}/i_*X & \longrightarrow & \mathscr{Y} \\ & \searrow \quad \swarrow {\scriptstyle \psi} & \\ & \mathscr{E} & \end{array}$$

By composing with the projection $i_*(X \times U) \longrightarrow i_*(X)$ we obtain the following commutative diagram of geometric morphisms.

$$\begin{array}{ccc} \mathscr{E}/i_*(X \times U) & \longrightarrow & \mathscr{Y} \\ \downarrow & & \downarrow {\scriptstyle \psi} \\ \mathscr{E}/i_*U & \longrightarrow & \mathscr{E} \end{array} \tag{9.3}$$

Since the direct image functor of a pure geometric morphism preserves definable objects (Exercise 1.5.7, 2), the left vertical is a definable object. The support $V \rightarrowtail 1_{\mathscr{E}}$ of i_*U is pure because it must be dense for the pure topology. It remains to show that (9.3) is a pullback and that $\psi^*(V)$ is a pure subobject of $1_{\mathscr{Y}}$. Consider the pullbacks

$$\begin{array}{ccccc} i_*(X \times U) & \twoheadrightarrow & Y & \rightarrowtail & i_*X \\ \downarrow & & \downarrow & & \downarrow \\ i_*U & \twoheadrightarrow & V & \rightarrowtail & 1_{\mathscr{E}} \end{array}$$

in $\mathscr{E}$. The top monomorphism is pure so that the top horizontal geometric morphism in the following diagram is pure.

$$\begin{array}{ccc} \mathscr{E}/Y & \longrightarrow & \mathscr{Y} \\ \downarrow & & \downarrow {\scriptstyle \psi} \\ \mathscr{E}/V & \rightarrowtail & \mathscr{E} \end{array}$$

Hence, this is a spread completion diagram. But then $\mathscr{E}/Y \longrightarrow \mathscr{E}/V$ is locally constant, whence a complete spread. By Lemma 9.3.10, this square is a pullback, which shows that (9.3) is a pullback. It also shows that $\psi^*(V) \cong Y$ is a subobject of $1_{\mathscr{Y}}$, which we already know is pure. This completes our argument that ψ is a branched covering, and the proof of the theorem. □

The following result, which relates the definition of a branched covering that we have given with one that is closer in spirit to the topological notion given by R. H. Fox, is implicit in the above proof. Fox does mention in passing that by defining branched cover as a completion of a locally constant (or unramified) map, folds are excluded, at least intuitively. We have seen by means of examples that in our definition of branched covering, folds are also intuitively excluded. A topos-theoretic (or for that matter, a topological) definition of a folded covering has not been given.

Corollary 9.3.13 *A geometric morphism over a topos $\mathscr{E}$ is a branched covering iff it is the spread completion of a locally constant covering of $\mathscr{E}/V$, for some pure subobject $V \rightarrowtail 1_{\mathscr{E}}$. Moreover, if a branched covering $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ is the spread completion of a locally constant covering $\mathscr{E}/Y \longrightarrow \mathscr{E}/V$, with pure $V \rightarrowtail 1_{\mathscr{E}}$, then the spread completion diagram*

$$\begin{array}{ccc} \mathscr{E}/Y & \longrightarrow & \mathscr{Y} \\ \downarrow & & \downarrow \psi \\ \mathscr{E}/V & \rightarrowtail & \mathscr{E} \end{array}$$

is a topos pullback.

Remark 9.3.14 *Sometimes we may wish to focus on a particular pure subobject $V \rightarrowtail 1_{\mathscr{E}}$. For instance, in topology V may be the complement of a knot. Given such a V, consider two functors given by completion and by pullback.*

$$\begin{array}{ccccc} & & \mathscr{C}(\mathscr{E}/V) & & \\ & \text{completion} \swarrow & & \searrow \text{pullback} & \\ \mathscr{B}(\mathscr{E}) & & \longrightarrow & & \mathscr{C}(\mathscr{E}_p) \end{array}$$

The bottom arrow is an equivalence, and the other two functors are full and faithful. The pullback functor may be equivalently described just as i^* for $i : \mathscr{E}_p \rightarrowtail \mathscr{E}/V$. By Exercise 9.1.19, 9,

$$i^* : \mathscr{C}(\mathscr{E}/V) \longrightarrow \mathscr{C}(\mathscr{E}_p)$$

is full and faithful. Denote by $\mathscr{B}_V(\mathscr{E})$ the full image category of $\mathscr{B}(\mathscr{E})$ under the completion functor: an object of $\mathscr{B}_V(\mathscr{E})$ is thus a branched covering of $\mathscr{E}$ that is the completion of a locally constant covering of $\mathscr{E}/V$. There is an equivalence $\mathscr{B}_V(\mathscr{E}) \simeq \mathscr{C}(\mathscr{E}/V)$. In particular, for a locally connected and locally simply connected topos $\mathscr{E}/V$, $\mathscr{B}_V(\mathscr{E})$ is equivalent to the atomic topos $\mathscr{C}(\mathscr{E}/V)$, also denoted ${\Pi_1}^c(\mathscr{E}/V)$. This is a version of the (coverings) fundamental group of a "knot" in $\mathscr{E}$ with "complement" V.

Remark 9.3.15 *Continuing with Remark 9.3.14, if* $V \rightarrowtail 1_{\mathscr{E}}$ *is pure, then we may regard* $\mathscr{C}(\mathscr{E}/V)$ *as a full subcategory of* $\mathscr{C}(\mathscr{E}_p)$ *under the full and faithful* i^**. For instance, every knot group* $\mathscr{C}(\mathrm{Sh}(S^3)/V)$ *can be regarded as a full subcategory of* $\mathscr{C}(\mathrm{Sh}(S^3)_p)$*. Hence, any two knot groups have an intersection, by which we mean their overlap in* $\mathscr{C}(\mathrm{Sh}(S^3)_p)$*.*

Proposition 9.3.16 *Branched coverings are stable under pullback along locally connected geometric morphisms.*

Proof. One way to prove this is to work with the formulation of branched cover that Corollary 9.3.13 provides. We invite the reader to complete the details. □

Remark 9.3.17 *One can argue that historically singular coverings precede locally constant coverings in the theory of Riemann surfaces, and that only on account of a desired connection with the fundamental group had additional assumptions been made on the maps, assumptions that in practice have the effect of reducing singular coverings to locally constant coverings. In fact, the familiar concept of locally constant covering is a topological concept formed from the analytical concept of a Riemann surface, or rather, that part of the Riemann surface remaining after the branch points have been deleted.*

Exercises 9.3.18

1. *Establish a version of Theorem 9.3.12 with unramified coverings in place of locally constant ones. Of course the notion of branched covering must be appropriately changed.*
2. *Prove Corollary 9.3.13.*
3. *Prove the following variation of Corollary 9.3.13: a complete spread over* $\mathscr{E}$ *is a branched covering iff it is the spread completion of a locally constant object of* $\mathscr{E}/W$*, for some pure object* $W \longrightarrow 1_{\mathscr{E}}$*.*
4. *Prove Proposition 9.3.16.*

9.4 The Index of a Complete Spread

In this section we study the category $\mathbf{Dist}_{\mathscr{E}}(\mathscr{E}, \mathscr{Y})$ associated with an $\mathscr{S}$-complete spread $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$. Note that the base topos is $\mathscr{E}$, not $\mathscr{S}$. We sometimes informally refer to this category as the 'index category' of the $\mathscr{S}$-complete spread ψ for the following reason. Let $W \subseteq Y$ denote the non-singular part of a branched covering $Y \xrightarrow{\psi} X$. The familiar index of branching of ψ is defined in terms of a functor

$$\varsigma : \mathscr{O}(Y) \longrightarrow Sh(X)$$

that associates with an open set $U \subseteq Y$ the locally constant map

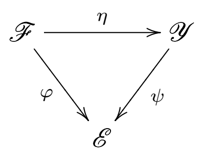
$$\psi : U \cap W \longrightarrow \psi(U \cap W) ,$$

by which we mean the restriction of ψ to $U \cap W$. Indeed, a number $\mathrm{b}(y)$ is then the index of branching of ψ at a point $y \in Y$ if there is a base $\{U_i\}$ of the neighbourhood system of y such that each locally constant map $U_i \cap W \longrightarrow \psi(U_i \cap W)$ has fiber $\mathrm{b}(y)$. Now suppose that $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ is a branched covering in our sense. In particular, ψ is the spread completion $\mathscr{E}/Y \xrightarrow{\eta} \mathscr{Y}$ of some $\mathscr{E}/Y \longrightarrow \mathscr{E}$. We may thus interpret the 'index-functor' simply as the functor

$$\varsigma_Y = \Sigma_Y \cdot \eta^* : \mathscr{Y} \longrightarrow \mathscr{E} .$$

This functor is an $\mathscr{E}$-distribution that we call *the index-distribution associated with the branched covering*, motivated by the usual notion.

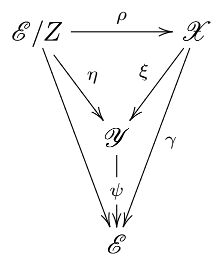
To begin our investigation, we know that for any geometric morphism $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$, an $\mathscr{E}$-distribution on $\mathscr{Y}$ is isomorphic to $\varphi_! \cdot \eta^*$, for some locally connected φ.



(9.4)

Proposition 9.4.1 *In diagram* (9.4)*, assume that ψ is an $\mathscr{S}$-complete spread. Then η is an $\mathscr{E}$-complete spread iff φ is a local homeomorphism. In particular, any $\mathscr{E}$-distribution $\mathscr{Y} \longrightarrow \mathscr{E}$ preserves pullbacks.*

Proof. Assume that φ is a local homeomorphism: $\mathscr{E}/Z \longrightarrow \mathscr{E}$. Consider the $\mathscr{E}$-pure, complete spread factorization of η.

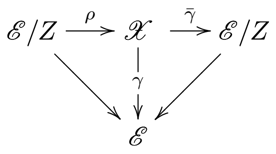


(9.5)

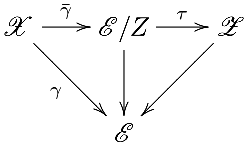
The geometric morphism γ is locally connected, ρ is $\mathscr{E}$-pure so that $\Sigma_Z \cdot \rho^* \cong \gamma_!$, and we have

$$Z \cong \Sigma_Z(\rho^*(\xi^*(1))) \cong \gamma_!(\xi^*(1)) \cong \gamma_!(1) .$$

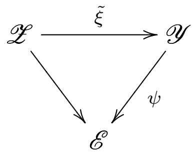
We shall show that ρ is an equivalence. Let $\mathscr{X} \xrightarrow{\bar{\gamma}} \mathscr{E}/Z$ denote the connected part of γ. Then the composite $\bar{\gamma} \cdot \rho$ is uniquely isomorphic to the identity geometric morphism on $\mathscr{E}/Z$.



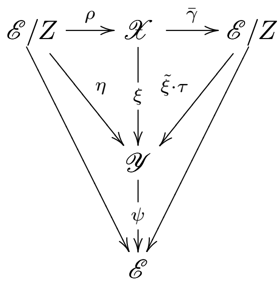
It remains to show that $\rho \cdot \bar{\gamma}$ is isomorphic to the identity on $\mathscr{X}$. Let $\mathscr{Z} \longrightarrow \mathscr{E}$ be the spread completion of $\mathscr{E}/Z \longrightarrow \mathscr{E}$ over $\mathscr{S}$. The $\mathscr{S}$-pure, complete spread factorization of γ is as follows.



Since the codomain topos of ξ is an $\mathscr{S}$-complete spread, there must be a factorization $\xi \cong \tilde{\xi} \cdot \tau \cdot \bar{\gamma}$, where



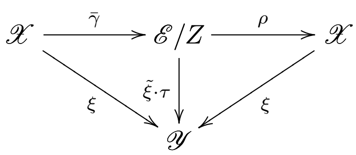
commutes. Thus, $\bar{\gamma}$ is defined over $\mathscr{Y}$ as in the following diagram.



But we also wish to define an isomorphism $\tilde{\xi} \cdot \tau \cdot 1_Z \cong \eta$ such that the isomorphism $\bar{\gamma} \cdot \rho \cong 1_Z$ is over $\mathscr{Y}$. We simply take the composite isomorphism

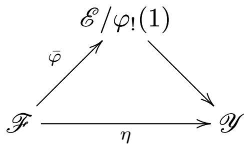
$$\tilde{\xi} \cdot \tau \cdot 1_Z \cong \tilde{\xi} \cdot \tau \cdot \bar{\gamma} \cdot \rho \cong \xi \cdot \rho \cong \eta \ .$$

The identity geometric morphism on $\mathscr{X}$ and $\rho \cdot \bar{\gamma}$ are two geometric morphisms from the $\mathscr{E}$-complete spread ξ to itself.



We have $\rho \cdot \bar{\gamma} \cdot \rho \cong \rho$ over $\mathscr{Y}$. Since ρ is $\mathscr{E}$-pure, $\rho \cdot \bar{\gamma}$ must be isomorphic to the identity on $\mathscr{X}$ over $\mathscr{Y}$. This proves that η is an $\mathscr{E}$-complete spread.

Conversely, assume that η is an $\mathscr{E}$-complete spread. We have seen in the previous paragraph that because ψ is an $\mathscr{S}$-complete spread there is a factorization



over $\mathscr{E}$. But the connected $\bar{\varphi}$ is $\mathscr{E}$-pure, and a complete spread cannot have a non-trivial first factor that is pure. Thus $\bar{\varphi}$ is an equivalence, so that φ is a local homeomorphism. □

Corollary 9.4.2 *Let Y be any object of a locally connected topos $\mathscr{E}$, with associated complete spread $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$. Then the $\mathscr{S}$-pure factor $\mathscr{E}/Y \longrightarrow \mathscr{Y}$ is an $\mathscr{E}$-complete spread.*

Intuitively, Corollary 9.4.2 says that an object of a topos $\mathscr{E}$ is $\mathscr{E}$-closed in its $\mathscr{S}$-closure.

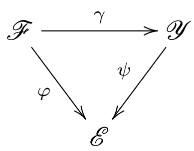
Corollary 9.4.3 *A locally connected $\mathscr{S}$-complete spread is a local homeomorphism. Thus, an $\mathscr{S}$-complete spread is an unramified cover iff it is locally connected.*

Proof. If an $\mathscr{S}$-complete spread $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ is locally connected, then the identity geometric morphism $\mathscr{Y} \longrightarrow \mathscr{Y}$ is an $\mathscr{E}$-complete spread with locally connected domain. By Proposition 9.4.1, ψ is a local homeomorphism. □

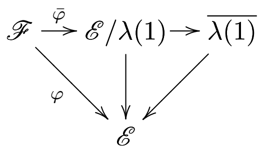
Let $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ denote an arbitrary $\mathscr{S}$-complete spread, with interior $\mathscr{E}/X \xrightarrow{\tau} \mathscr{Y}$: $X = \mathbf{d}(\psi)$. ($\mathbf{d}(\psi)$ coincides with the density of the distribution associated with ψ, but in the case of a geometric morphism we sometimes use the term interior.) There is a functor

$$\Phi : \mathscr{E}/X \longrightarrow \mathbf{Dist}_{\mathscr{E}}(\mathscr{E}, \mathscr{Y})$$

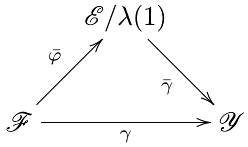
that associates with $Z \xrightarrow{m} X$ the distribution $\Sigma_X(m \times \tau^*(\ \ ))$. Now let λ be an arbitrary $\mathscr{E}$-distribution $\mathscr{Y} \longrightarrow \mathscr{E}$. We know there is a diagram



such that φ is locally connected and $\lambda \cong \varphi_! \cdot \gamma^*$. We have $\varphi_!(1) = \lambda(1)$. We know that the $\mathscr{S}$-pure, complete spread factorization of φ is



where $\overline{\lambda(1)}$ is the $\mathscr{S}$-spread completion of $\lambda(1)$. Since ψ is an $\mathscr{S}$-complete spread, γ must factor through $\overline{\lambda(1)}$, hence through $\lambda(1)$ over $\mathscr{E}$:



Since $\bar{\varphi}$ is $\mathscr{E}$-pure, we have $\lambda \cong \Sigma_{\lambda(1)} \cdot \bar{\gamma}^*$. Thus, the object $\lambda(1)$ factors through the interior X, say by $\lambda(1) \xrightarrow{m} X$. This gives us a functor

$$\Psi : \mathbf{Dist}_{\mathscr{E}}(\mathscr{E}, \mathscr{Y}) \longrightarrow \mathscr{E}/X$$

that associates with a λ the object $\lambda(1) \xrightarrow{m} X$ just constructed. Moreover, for any λ, we have $\lambda \cong \Sigma_X(\, m \times \tau^*(\;)\,) = \Phi(\Psi(\lambda))$.

Proposition 9.4.4 *Let $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ denote an $\mathscr{S}$-complete spread with interior $\mathscr{E}/X \xrightarrow{\tau} \mathscr{Y}$: $X = \boldsymbol{d}(\psi)$. Then the functors Φ and Ψ establish an equivalence of categories:*

$$\mathscr{E}/X \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathbf{Dist}_{\mathscr{E}}(\mathscr{E}, \mathscr{Y}) \;.$$

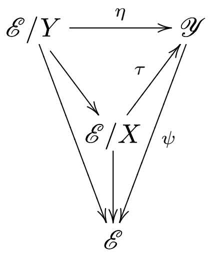
Moreover, for any object $Z \xrightarrow{m} X$ of $\mathscr{E}/X$, the $\mathscr{E}$-complete spread associated with the distribution $\Phi(m)$ is $\mathscr{E}/Z \xrightarrow{m} \mathscr{E}/X \xrightarrow{\tau} \mathscr{Y}$.

Proof. For the first statement, we have only to show that for any $Z \xrightarrow{m} X$, we have $\Psi(\Phi(m)) \cong m$. But this is clear because

$$\Phi(m)(1) = \Sigma_X(\, m \times \tau^*(1)\,) = \Sigma_X(m) = Z \;,$$

and the morphism $\Phi(m)(1) \longrightarrow X$ is easily seen to be m. $\square$

Corollary 9.4.5 *Let Y be any object of a locally connected topos $\mathscr{E}$ with $\mathscr{S}$-spread completion $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ and interior $\mathscr{E}/X \xrightarrow{\tau} \mathscr{Y}$, as in the following diagram.*

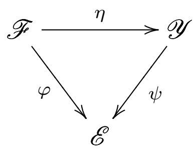


Then the functor $Z \xrightarrow{m} X \mapsto \Sigma_X(\, m \times \tau^*(\;\;)\,)$ *is an equivalence*

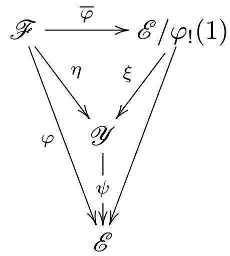
$$\mathscr{E}/X \;\simeq\; \mathbf{Dist}_{\mathscr{E}}(\mathscr{E}, \mathscr{Y})\;.$$

Remark 9.4.6 *Let* $\mathscr{Y} \xrightarrow{\psi} \mathscr{E}$ *be an* $\mathscr{S}$*-complete spread, with interior* $\mathscr{E}/X \xrightarrow{\tau} \mathscr{Y}$*. Then the terminal object of* $\mathbf{Dist}_{\mathscr{E}}(\mathscr{E}, \mathscr{Y})$ *is the* $\mathscr{E}$*-distribution* $\Sigma_X \cdot \tau^*$*, and* τ *is the terminal* $\mathscr{E}$*-complete spread.*

The previous discussion examines $\mathscr{E}$-distributions on a given arbitrary $\mathscr{S}$-complete spread to $\mathscr{E}$. We now begin with a locally connected geometric morphism $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}$ over a topos $\mathscr{S}$, and then consider its $\mathscr{S}$-spread completion.



We may also consider the $\mathscr{E}$-pure, complete spread factorization of φ, which is the perimeter of the following diagram.



(9.6)

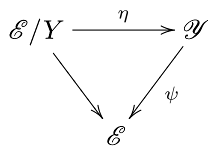
Remark 9.4.7 *By Proposition 9.4.1, the* $\mathscr{S}$*-pure* ξ *in diagram (9.6) is an* $\mathscr{E}$*-complete spread, so that* $\overline{\varphi}$ *and* ξ *give the* $\mathscr{E}$*-pure, complete spread factorization of* η*.*

We conclude with the following result describing the nature of the index-distribution that we had introduced at the beginning of this section.

Corollary 9.4.8 *For any* Y *in a locally connected topos* $\mathscr{E}$*, the functor*

$$\mathscr{E}/Y \longrightarrow \mathbf{Dist}_{\mathscr{E}}(\mathscr{E}, \mathscr{Y})/\varsigma_Y\;; \quad E \xrightarrow{m} Y \mapsto \varsigma_m = \Sigma_Y(m \times \eta^*(\;\;))$$

is an equivalence, where



is the $\mathscr{S}$*-spread completion of* Y*, and* ς_Y *denotes the index-distribution* $\Sigma_Y \cdot \eta^*$*. For any* Y *in* $\mathscr{E}$*, if* η *is an inclusion* (as in the case of a branched cover)*, then* ς_Y *is a weak terminal object in* $\mathbf{Dist}_{\mathscr{E}}(\mathscr{E}, \mathscr{Y})$*.*

Proof. We have

$$\mathscr{E}/Y \simeq \mathbf{Dist}_{\mathscr{E}}(\mathscr{E}, \mathscr{E}/Y) \simeq \mathbf{Dist}_{\mathscr{E}}(\mathscr{E}, \mathscr{Y})/\varsigma_Y \, .$$

Indeed, the first equivalence is by Exercise 1.3.8, 3. The second equivalence is by Proposition 2.5.4, since $\mathscr{E}/Y \xrightarrow{\eta} \mathscr{Y}$ is an $\mathscr{E}$-complete spread (Remark 9.4.7). The second statement of the proposition holds because for any object $E \xrightarrow{m} Y$ there is exactly one natural transformation $m \times Y^* \Rightarrow Y^*$, hence exactly one $Y^* \Rightarrow Y^*(\ )^m$. If η is an inclusion, then there is exactly one natural transformation $\eta_* Y^* \Rightarrow \eta_*(Y^*(\ )^m)$. Now consider left adjoints to see that there is exactly one natural transformation $\varsigma_m \Rightarrow \varsigma_Y$. This shows that ς_Y is a weak terminal because if there is a morphism $\mu \Rightarrow \varsigma_Y$, then $\mu \cong \varsigma_m$ for some m. □

Further reading: Barr & Diaconescu [BD81, BD80], Brown [Bro88], Bunge [Bun04], Bunge & Funk [BF98], Bunge & Lack [BL03], Bunge & Moerdijk [BM97], Bunge & Niefield [BN00], Bunge & Paré [BP79], Fox [Fox57], Funk [Fun00], Funk & Tymchatyn [FT01], Janelidze [Jan90], Johnstone [Joh02], Joyal & Tierney [JT84], Kock & Reyes [KR99], Mulero [Mul98], Springer [Spr57].