8

Localic and Algebraic Aspects

In this chapter we consider distributions on locales, and the lower power locale from a constructive point of view. We also consider factorizations other than the comprehensive one (or pure, complete spread), and compare them. The lower bagdomain B_L, and the probability distribution classifier T are two variants of the symmetric KZ-monad M; the equation $\mathrm{M} = \mathrm{B}_L \cdot \mathrm{T}$ offers a new perspective on distributions and complete spread geometric morphisms. Our notion of discrete complete spread structure provides yet another single universe for both local homeomorphisms and complete spreads. We illustrate some of the ideas discussed in this book with an example from algebraic geometry involving coschemes. We make a special analysis of distributions on the Jonsson-Tarski topos.

8.1 Distributions on Locales

Power domains (lower, upper, mixed) had been introduced in the 1970's in order to analyse the semantics of non-deterministic and parallel computation. Some computer scientists now believe that this is not an ideal solution to the problem, since infinite communicating processes are hardly ever determined by the finite (or partial) observations one can make about them. On the other hand, the lower bagdomains had emerged from efforts to make more accurate the model provided by the lower power domain, in which the 'partial information' about a database should not only be specified by individual partial records, but by an indexed family of such partial records (a 'bag'). Even if the domain from which one starts has only one point, the points of the bagdomain should correspond to the 'space' of all sets, and the refinement ordering on them, to arbitrary functions. The result is not a topological space, or even a locale. However, the space of all sets can easily be handled by passing to toposes by means of the object classifier $\mathscr{S}[U]$. The lower bagdomain has been constructed by S. Vickers, and put on a categorical foundation by P. T. Johnstone. We will return to bagdomains in § 8.2; however, power domains have other aspects that we shall address here.

We shall refer to a complete upper semilattice in an elementary topos $\mathscr{S}$ as *a suplattice*. Let **sl** denote the 2-category of of suplattices and sup-preserving maps, so $\mathbf{sl}(M, N)$ denotes the poset of sup-preserving maps from a suplattice M to another one N. We may consider distributions on a locale (or more generally, on a suplattice), in the following sense.

Definition 8.1.1 A distribution on a locale X *in* $\mathscr{S}$ *is a sup-preserving morphism* $\mathscr{O}(X) \longrightarrow \Omega_{\mathscr{S}}$*, where as always* $\mathscr{O}(X)$ *denotes the frame associated with the locale* X*.*

We denote by $\Sigma(M)$ *the symmetric frame of* M , defined to be the frame of the classifying locale for the theory of distributions on M , i.e., of sup-preserving maps $M \longrightarrow \Omega_{\mathscr{S}}$. Equivalently, the following universal property defines $\Sigma(M)$: for any frame $\mathscr{O}(X)$ in $\mathscr{S}$, there is an isomorphism

$$\mathbf{Fr}(\Sigma(M), \mathscr{O}(X)) \cong \mathbf{sl}(M, \mathscr{O}(X)) ,$$

of posets natural in $\mathscr{O}(X)$, where **Fr** denotes the 2-category of frames, and frame homomorphisms. In other words, Σ is left adjoint to the forgetful functor

$$U : \mathbf{Fr} \longrightarrow \mathbf{sl} \; ; \; \Sigma \dashv U \; .$$

We call $\Sigma(M)$ the *symmetric frame* of a suplattice M.

Just like our treatment of the symmetric topos, it is appropriate to take a 'geometric' point of view: we may regard $\Sigma \cdot U$ as an endofunctor of locales. If X is any locale, we define $\mathrm{P}_L(X)$ as the locale whose frame is

$$\mathscr{O}(\mathrm{P}_L(X)) = \Sigma(\mathscr{O}(X)) \; .$$

Of course we mean $\Sigma(U(\mathscr{O}(X))$ on the right, but we do not need to write U. Thus, $\Sigma(\mathscr{O}(X))$ is none other than the frame of opens of the *Hoare locale*, or *lower power locale* $\mathrm{P}_L(X)$ of X, as it is called in the literature.

Classically, the frame of the lower power locale $\mathrm{P}_L(X)$ is freely generated by symbols $\Diamond U$, $U \in \mathscr{O}(X)$, so that $U \mapsto \Diamond U$ preserves arbitrary joins. If the topos $\mathscr{S}$ is Boolean, it is known that a point of $\mathrm{P}_L(X)$, i.e., a sup-preserving map $\mathscr{O}(X) \longrightarrow \Omega_{\mathscr{S}}$, is completely determined by a closed sublocale of X. Before examining the extent of the validity of this assertion for an arbitrary topos $\mathscr{S}$, we give a construction of $\mathrm{P}_L(X)$ that parallels the construction of the symmetric topos.

The finite inf-completion $Q^\bullet$ of a poset Q can be given as the collection of equivalence classes $[S]$, where S is a (Kuratowski) finite subset of Q, and where $[S] = [S']$ iff S and S' generate the same upper set in Q. As a poset, $Q^\bullet$ has the partial order given by $[S] \leq [T]$ iff T is contained in the upper set generated by S.

Any frame $\mathscr{O}(X)$ is canonically presented as a coinverter

$$D(Q) \underset{d_1}{\overset{d_0}{\underset{\Downarrow}{\rightrightarrows}}} D(\mathscr{O}(X)) \longrightarrow\!\!\!\!\!\rightarrow \mathscr{O}(X)$$

in **sl**, where Q is the poset whose elements are pairs (R, U) such that $R \subseteq \downarrow U$, $U \in \mathscr{O}(X)$, and $\bigvee R = U$. The maps d_0 and d_1 are induced by the assignments $(R, U) \mapsto R$ and $(R, U) \mapsto \downarrow U$, respectively, where $\Rightarrow$ is the unique 2-cell from d_0 to d_1, i.e., $d_0 \leq d_1$. It is well-known that the free frame on an inf-semilattice Z is given by the frame $D(Z)$ of down-closed subsets of Z.

We have left the proof of the following as a exercise, since this proof proceeds by analogy with the construction of (topos-frame of) the symmetric topos. Note that the forgetful functor from frames to suplattices creates coinverters.

Proposition 8.1.2 *The symmetric frame $\Sigma(\mathscr{O}(X))$ is defined by the coinverter*

$$D(Q^\bullet) \underset{d_1^\bullet}{\overset{d_0^\bullet}{\rightrightarrows}} D(\mathscr{O}(X)^\bullet) \xrightarrow{i^*} \Sigma(\mathscr{O}(X)) \, .$$

in **Fr** (created in **sl**)*, where the parallel arrows $d_0^\bullet$, $d_1^\bullet$ are induced from the canonical suplattice presentation of $\mathscr{O}(X)$ via finite inf-completions at the level of the posets.*

We now turn to an identification of the points of $\mathrm{P}_L(X)$ where $\mathscr{S}$ is now an arbitrary topos. A locale morphism $f : Y \longrightarrow X$ in a topos $\mathscr{S}$ is said to be *strongly dense* if the canonical inequality $\omega \leq f_* f^* \omega$ is an equality, for every $\omega \in \Omega_{\mathscr{S}}$. (Notice the parallel with what we call a pure geometric morphism.) In particular, a strongly dense locale morphism is *dense* in the sense that $0 = f_* 0$. It turns out that f is strongly dense iff f is dense under pullback along every closed sublocale of the terminal locale (whose frame is $\Omega_{\mathscr{S}}$). Every locale inclusion may be factored uniquely into a strongly dense inclusion followed by *a weakly closed* sublocale. Tautologically speaking, we may say that an inclusion of locales $B \rightarrowtail X$ is weakly closed iff any strongly dense inclusion $B \rightarrowtail B'$ is an isomorphism, where $B' \rightarrowtail X$ is any sublocale.

Let $\mathrm{Sub}(X)$ denote the coframe of sublocales of a locale X. We denote by $\mathrm{W}(X)$ the poset of weakly closed sublocales of X. $\mathrm{W}(X)$ is a subcoframe of $\mathrm{Sub}(X)$ (Jibladze and Johnstone), and it contains $\mathrm{C}(X) = \mathscr{O}(X)^{\mathrm{op}}$ as a subcoframe. For an open $U \in \mathscr{O}(X)$, we use the same symbol U to denote the sublocale of X corresponding to the nucleus $U \Rightarrow (\_)$. This association $\mathscr{O}(X) \longrightarrow \mathrm{Sub}(X)$ (of a frame into a coframe) preserves arbitrary suprema and finite infima.

For any locale morphism $X \xrightarrow{f} Y$ and $B \in \mathrm{Sub}(X)$, we shall use $\|B\|_f$ to denote the image of B in Y under f . When no subscript is supplied, then the unique map to the terminal locale is intended. Consider the functor

$$\chi : \mathrm{Sub}(X) \longrightarrow \mathbf{sl}(\mathscr{O}(X), \mathrm{Sub}(1))$$

that carries a sublocale $B \rightarrowtail X$ to the suplattice map

$$\chi_B : U \mapsto \|B \wedge U\| \, .$$

Observe that χ has as right adjoint the functor that associates to a sup-preserving map $\mathscr{O}(X) \xrightarrow{f} \mathrm{Sub}(1)$ the sublocale

$$A_f = \bigwedge \left\{ (X - U) \vee \gamma^\sharp(fU) \mid U \in \mathscr{O}(X) \right\} , \tag{8.1}$$

where $\gamma^\sharp$ denotes locale pullback along the unique locale morphism $\gamma : X \longrightarrow 1$. Moreover, observe that the sublocale A_f is weakly closed. This follows from the fact that every sublocale of 1 is weakly closed, so that $\gamma^\sharp(fU)$ is weakly closed by pullback stability, and from the fact that closed sublocales are weakly closed, using then the fact that $\mathrm{W}(X)$ is a subcoframe of $\mathrm{Sub}(X)$.

Let $\delta : 1' \longrightarrow 1$ denote the *splitting locale* of 1 , i.e., $\mathscr{O}(1')$ is the frame of nuclei on $\Omega_{\mathscr{S}}$, and δ^* is the frame morphism that associates to $\omega \in \Omega_{\mathscr{S}}$ the nucleus $\omega \vee (\_)$. Pullback along δ yields an isomorphism $\mathrm{Sub}(1) \longrightarrow \mathrm{C}(1')$. Our explanation of Theorem 8.1.4 below relies on the following result, for which we do not include a proof.

Proposition 8.1.3 *Let X be an arbitrary locale, and let*

$$\begin{array}{ccc} Z & \xrightarrow{\psi} & 1' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \delta} \\ X & \xrightarrow{\gamma} & 1 \end{array}$$

be a pullback. Then pullback along p gives an isomorphism $\mathrm{W}(X) \cong \mathrm{C}(Z)$ *.*

Theorem 8.1.4 *For any locale X, the restriction of χ to $\mathrm{W}(X)$ yields an isomorphism*

$$\mathrm{W}(X) \cong \mathbf{sl}(\mathscr{O}(X), \mathrm{Sub}(1)) .$$

Proof. We employ the 'module' framework for frames and suplattices from the work of Joyal and Tierney. If M and N are $\mathscr{O}(Y)$-modules (suplattices that carry an $\mathscr{O}(Y)$-action in a suitable sense), then $\mathbf{sl}_{\mathscr{O}(Y)}(M, N)$ shall denote the poset of suplattice maps that preserve the $\mathscr{O}(Y)$-action. We start with the fact that for any locale morphism $X \xrightarrow{f} Y$, there are canonical isomorphisms

$$\mathrm{C}(X) \cong \mathbf{sl}_{\mathscr{O}(Y)}(\mathscr{O}(Y), \mathrm{C}(X)) \cong \mathbf{sl}_{\mathscr{O}(Y)}(\mathscr{O}(X), \mathrm{C}(Y)) . \tag{8.2}$$

This composite isomorphism sends a closed sublocale $B = X - W$ to the suplattice map

$$\begin{aligned} U \mapsto & \bigvee \{ V \in \mathscr{O}(Y) \mid U \leq f^* V \Rightarrow W \} \\ = & \bigvee \{ V \in \mathscr{O}(Y) \mid f^* V \leq U \Rightarrow W \} . \end{aligned}$$

When written in terms of closed parts, this suplattice map is

$$U \mapsto \overline{\| B \wedge U \|_f} ,$$

on account of the identities

$$\bigwedge\{D \in C(Y) \mid B \wedge U \leq f^* D\} = \bigwedge\{D \in C(Y) \mid \|B \wedge U\|_f \leq D\}$$

and

$$\bigwedge\{D \in C(Y) \mid \|B \wedge U\|_f \leq D\} = \overline{\|B \wedge U\|_f} \ .$$

On combining Proposition 8.1.3 with (8.2) applied to the morphism $Z \xrightarrow{\psi} 1'$ of Proposition 8.1.3, we obtain

$$\mathrm{W}(X) \cong \mathrm{C}(Z) \cong \mathbf{sl}_{\mathscr{O}(1')}(\mathscr{O}(Z), C(1')) \ . \tag{8.3}$$

Since $\mathscr{O}(Z) = \mathscr{O}(1)' \otimes \mathscr{O}(X)$, by adjointness this is isomorphic to

$$\mathbf{sl}(\mathscr{O}(X), \mathrm{C}(1')) \cong \mathbf{sl}(\mathscr{O}(X), \mathrm{Sub}(1)) \ . \tag{8.4}$$

The isomorphism in (8.4) is composition with the isomorphism

$$\mathrm{C}(1') \longrightarrow \mathrm{Sub}(1) \ ,$$

which carries a closed sublocale $E \rightarrowtail 1'$ to $\|E\|$, and furthermore, satisfies $\|\overline{I}\| = \|I\|$, for any sublocale $I \rightarrowtail 1'$, where $\overline{I}$ denotes the closure of I in $1'$. It remains to verify that the composite of (8.3) and (8.4) is indeed equal to χ restricted to $\mathrm{W}(X)$. By Proposition 8.1.3, and since B is weakly closed, we have

$$\|p^*(B \wedge U)\|_p = B \wedge U \ .$$

Then the composite of (8.3) and (8.4) sends $B \in \mathrm{W}(X)$ to the suplattice map

$$U \mapsto \| \overline{\|p^* B \wedge p^* U\|_\psi} \| = \| \|p^*(B \wedge U)\|_\psi \| \ ,$$

which is equal to

$$U \mapsto \| \|p^*(B \wedge U)\|_p \| = \|B \wedge U\| = \chi_B(U) \ .$$

□

Remark 8.1.5 *In effect, χ is weak closure. Let χ^{-1} denote the right adjoint of χ (8.1). The notation is justified since by Theorem 8.1.4, this right adjoint is full and faithful. Thus, for any sublocale $S \rightarrowtail X$, $\chi^{-1}(\chi_S)$ is its weak closure, and S is weakly closed iff*

$$S = \bigwedge \big\{(X - U) \vee \gamma^\sharp \|S \wedge U\| \mid U \in \mathscr{O}(X)\big\} \ .$$

Let $\mathrm{Sub}_o(X)$, respectively $\mathrm{W}_o(X)$, denote the poset of sublocales, respectively weakly closed sublocales, of X *with open domain*, i.e., those $B \rightarrowtail X$ for which the unique locale morphism $B \longrightarrow 1$ to the terminal locale is open. By definition, $B \longrightarrow 1$ is open if the unique frame morphism $\Omega_{\mathscr{S}} \longrightarrow \mathscr{O}(B)$ has a left adjoint $\exists$. This is equivalent to the condition that for all $U \in \mathscr{O}(X)$,

$\|B \wedge U\|$ is an open sublocale of 1 . It follows that the restriction of χ to sublocales with open domain yields the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Sub}_o(X) & \xrightarrow{\chi_o} & \mathbf{sl}(\mathscr{O}(X), \Omega_{\mathscr{S}}) \\ \downarrow & & \downarrow \\ \mathrm{Sub}(X) & \xrightarrow{\chi} & \mathbf{sl}(\mathscr{O}(X), \mathrm{Sub}(1)) \end{array}$$

By definition, if a sublocale $B \rightarrowtail X$ has open domain, then $\chi_o(B)$ is the suplattice map

$$\mathscr{O}(X) \xrightarrow{b^*} \mathscr{O}(B) \xrightarrow{\exists} \Omega_{\mathscr{S}} \ .$$

The vertical arrow on the right is composition with the canonical lattice inclusion $\Omega_{\mathscr{S}} \longrightarrow \mathrm{Sub}(1)$. It is full and faithful. We now have the main result of this section.

Theorem 8.1.6 *For any locale X, the restriction of χ_o to $\mathrm{W}_o(X)$ yields an isomorphism*

$$\mathrm{W}_o(X) \cong \mathbf{sl}(\mathscr{O}(X), \Omega_{\mathscr{S}}) \ .$$

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{W}_o(X) & \xrightarrow{\chi_o} & \mathbf{sl}(\mathscr{O}(X), \Omega_{\mathscr{S}}) \\ \downarrow & & \downarrow \\ \mathrm{W}(X) & \xrightarrow{\chi} & \mathbf{sl}(\mathscr{O}(X), \mathrm{Sub}(1)) \end{array}$$

where the vertical arrow on the right is full and faithful. But then we see that χ_o is an isomorphism. Indeed, χ_o is clearly full and faithful, and if $g \in \mathbf{sl}(\mathscr{O}(X), \Omega_{\mathscr{S}})$, then by Theorem 8.1.4, there is a weakly closed $B \rightarrowtail X$ such that

$$\forall U \in \mathscr{O}(X),\ g(U) = \|B \wedge U\| \ .$$

This says that $B \longrightarrow 1$ is open, i.e., that $B \in \mathrm{W}_o(X)$. □

Exercises 8.1.7

1. *Prove Proposition 8.1.2 by analogy with the proof of the corresponding theorem for the (frame of the) symmetric topos.*
2. *Prove that the following are equivalent, for any elementary topos $\mathscr{S}$.*
   (a) $\mathscr{S}$ *is Boolean.*
   (b) $\mathrm{C}(X) = \mathrm{W}(X)$ *for every locale X in $\mathscr{S}$.*
   (c) $\mathrm{C}(I) = \mathrm{W}(I)$ *for every object I of $\mathscr{S}$.*
   (d) $\mathrm{W}_o(X) = \mathrm{W}(X)$ *for every locale X in $\mathscr{S}$.*
   (e) $\mathrm{W}_o(1) = \mathrm{W}(1)$.

Hint: The equivalence of the first three conditions, and that they imply the last two is reasonably easy to establish. In order to prove that the last implies the first, observe that we always have $\mathrm{Sub}(1) = \mathrm{W}(1)$ *and* $\Omega_{\mathscr{S}} = \mathrm{Sub}_o(1) = \mathrm{W}_o(1)$ *. Thus, if* $\mathrm{W}_o(1) = \mathrm{W}(1)$ *, then* $\Omega_{\mathscr{S}} = \mathrm{Sub}(1)$*, i.e., then every sublocale of* 1 *is open. It is well-known that this implies* $\Omega_{\mathscr{S}}$ *is Boolean.*

3. *Show that for any locale* X *in* $\mathscr{S}$*, the topos* $Sh(\mathrm{P}_L(X))$ *of sheaves on the lower power locale of* X *is equivalent to the localic reflection of the symmetric topos* $\mathrm{M}(Sh(X))$*.*
4. *The hyperconnected geometric morphism*

$$h : \mathrm{M}(Sh(X)) \longrightarrow Sh(\mathrm{P}_L(X))$$

mediates support in a sense that can be described in terms of its action on geometric points. For any topos $\mathscr{F}$*, composite with* h *may be equivalently described as a functor*

$$\boldsymbol{Dist}_{\mathscr{S}}(\mathscr{F}, \mathrm{Sh}(X)) \longrightarrow \mathbf{sl}(\mathscr{O}(X), f_*(\Omega_{\mathscr{F}}))$$

given by composition with the 'support' functor $\sigma : \mathscr{F} \longrightarrow \Omega_{\mathscr{F}}$*, which assigns to an* I*-indexed family* $F \longrightarrow f^*I$ *the characteristic map of its image, and with Yoneda* $\mathscr{O}(X) \longrightarrow Sh(X)$*.*
5. *Show that the lower power monad* P_L *falls within the theory of completion KZ-monads.*
6. *Directly establish the correspondence of Theorem 8.1.6 in terms of bicomma objects: if* $1 \xrightarrow{p} \mathrm{P}_L(X)$ *is a localic point corresponding to a weakly closed sublocale* $B \rightarrowtail X$ *with open domain, then there is a bicomma object*

$$\begin{array}{ccc} B & \longrightarrow & 1 \\ \big\downarrow & \leq & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[\delta]{} & \mathrm{P}_L(X) \end{array}$$

of locales.

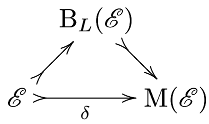
8.2 Symmetric versus Lower Bagdomain

Given a topos $\mathscr{E}$ over $\mathscr{S}$, $\mathrm{B}_L(\mathscr{E})$ is a topos whose points are bags of points of $\mathscr{E}$. We remarked in the previous section that the lower bagdomain generalizes the lower power locale. The symmetric topos $\mathrm{M}(\mathscr{E})$ is also a sort of generalization of $\mathrm{P}_L(X)$ (Exercises 8.1.7). Also, we have

$$\mathrm{M}(\mathscr{S}) = \mathrm{B}_L(\mathscr{S}) = \mathscr{S}[U] \ .$$

We begin by examining $\mathrm{B}_L(\mathscr{E})$ from a symmetric viewpoint.

The essential inclusion $\delta : \mathscr{E} \rightarrowtail \mathrm{M}(\mathscr{E})$ factors through the lower bagdomain topos $\mathrm{B}_L(\mathscr{E})$ by essential inclusions as in the following diagram.



These morphisms are induced by corresponding site inclusions

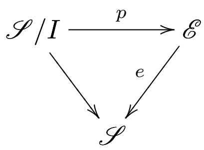
$$\langle \mathbb{C}, J \rangle \longrightarrow \langle \mathbb{C}_{\mathrm{fp}}, J_{\mathrm{fp}} \rangle \longrightarrow \langle \mathbb{C}^{\star}, J^{\star} \rangle \ ,$$

where $\langle \mathbb{C}, J \rangle$ is a site definition of $\mathscr{E}$, $\mathbb{C}_{\mathrm{fp}}$ is the finite products completion of $\mathbb{C}$ is a site for $\mathrm{B}_L(\mathscr{E})$, and $\mathbb{C}^{\star}$ is the lex completion of $\mathbb{C}$, which we have seen is a site for $\mathrm{M}(\mathscr{E})$ in § 4.2. (We use the notation $J^{\star}$ for the topology in $\mathbb{C}^{\star}$, but it is not the lex completion of the total poset of J.)

In terms of the models of the theories that these toposes classify, the geometric morphism $\mathscr{E} \longrightarrow \mathrm{B}_L(\mathscr{E})$ corresponds to forgetting that a lex distribution $\mathscr{E} \longrightarrow \mathscr{X}$ preserves the terminal object, whereas $\mathrm{B}_L(\mathscr{E}) \longrightarrow \mathrm{M}(\mathscr{E})$ corresponds to forgetting that a pullback preserving distribution preserves pullbacks. $\mathrm{M}(\mathscr{E})$ classifies distributions, whereas $\mathrm{B}_L(\mathscr{E})$ classifies *bags of points* of $\mathscr{E}$, meaning a geometric morphism $\mathscr{S}/I \longrightarrow \mathscr{E}$. Equivalently, a bag of points is a pullback preserving distribution on $\mathscr{E}$.

Proposition 8.2.1 *The following conditions on a locally connected topos $\mathscr{E}$ are equivalent:*

1. *the connected components functor $e_!$ preserves pullbacks,*
2. *$\mathscr{E}$ has a pure 'bag' of points, in the sense that there is a diagram*



in which p is pure.

The topos is also connected iff it has a pure point ($I = 1$ in this case).

Proof. 1 implies 2 because we may take $I = e_!(1)$. It follows that there is a geometric morphism p as above such that $e_! \cong \Sigma_I \cdot p^*$, which says that p is pure. 2 implies 1 because if p is pure, then $e_! \cong \Sigma_I \cdot p^*$. Thus, $e_!$ preserves pullbacks. □

The comparison between M and B_L can also be phrased in terms of complete spreads: M classifies complete spreads with locally connected domain, and B_L classifies complete spreads whose domain has totally connected components, in the following sense.

Definition 8.2.2 *We shall say that a locally connected topos* has totally connected components *if either of the conditions in 8.2.1 holds. If a topos is connected, locally connected, and has totally connected components, then we say it is* totally connected.

Proposition 8.2.3 $\boldsymbol{Top}_{\mathscr{S}}(\mathscr{X}, \mathrm{B}_L(\mathscr{E}))$ *is equivalent to the category of* $\mathscr{X}$*-complete spreads* $\mathscr{Y} \longrightarrow \mathscr{X} \times_{\mathscr{S}} \mathscr{E}$*, whose* $\mathscr{X}$*-domain has totally connected components. Moreover, a geometric morphism*

$$\mathscr{X} \xrightarrow{\rho} \mathrm{M}(\mathscr{E})$$

factors through $\mathrm{B}_L(\mathscr{E})$ *iff in the bicomma object*

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\gamma} & \mathscr{X} \\ {\scriptstyle\psi}\downarrow & \Rightarrow & \downarrow{\scriptstyle\rho} \\ \mathscr{E} & \underset{\delta}{\rightarrowtail} & \mathrm{M}(\mathscr{E}) \end{array}$$

the locally connected γ *has totally connected components, in which case the resulting inside square with* $\mathrm{B}_L(\mathscr{E})$ *is a bicomma object.*

Proof. The topos $\mathrm{B}_L(\mathscr{E})$ is the partial product of $\mathscr{S}[U]/U \longrightarrow \mathscr{S}[U]$ with $\mathscr{E}$, where U denotes the generic object of the object classifier $\mathscr{S}[U]$. It follows immediately that a geometric morphism $\mathscr{X} \longrightarrow \mathrm{B}_L(\mathscr{E})$ amounts to a pair consisting of an object F of $\mathscr{X}$ and a geometric morphism $\mathscr{X}/F \longrightarrow \mathscr{E}$ over $\mathscr{S}$, equivalently, to a pair consisting of an object F of $\mathscr{X}$ and a geometric morphism

$$\mathscr{X}/F \longrightarrow \mathscr{X} \times_{\mathscr{S}} \mathscr{E}$$

over $\mathscr{X}$. Now form the $\mathscr{X}$-comprehensive factorization of this geometric morphism.

We leave the second assertion of the proposition as an exercise. □

Remark 8.2.4 *Proposition 8.2.3 gives a characterization of those toposes* $\mathscr{E}$ *for which all distributions on it are* Riemman sums *by which we mean, in this context, bags of points. They are precisely those toposes* $\mathscr{E}$ *such that* $\mathrm{M}(\mathscr{E})$ *and* $\mathrm{B}_L(\mathscr{E})$ *coincide. Equivalently, they are the toposes* $\mathscr{E}$ *for which the domain topos of every complete spread over* $\mathscr{E}$ *is locally connected and has totally connected components (equivalently, locally connected and the connected components functor preserves pullbacks).*

We now turn to a third KZ-monad in $\mathbf{Top}_{\mathscr{S}}$: the classifier of probability distributions.

Definition 8.2.5 A probability distribution *on a topos is a distribution that preserves the terminal object. We denote the topos classifier of probability distributions on a topos* $\mathscr{E}$ *by* $\mathrm{T}(\mathscr{E})$*; the category of geometric morphisms* $\mathscr{X} \longrightarrow \mathrm{T}(\mathscr{E})$ *is naturally equivalent to the category of probability distributions* $\mathscr{E} \longrightarrow \mathscr{X}$.

Proposition 8.2.6 *For any topos $\mathscr{E}$ over $\mathscr{S}$, there is a subtopos $\mathrm{T}(\mathscr{E})$ of $\mathrm{M}(\mathscr{E})$ that classifies probability distributions. Furthermore, there is a factorization*

$$\begin{array}{ccc} \mathscr{E} & \overset{\delta}{\rightarrowtail} & \mathrm{M}(\mathscr{E}) \\ & {\scriptstyle\overline{\delta}}\searrow \quad \nearrow{\scriptstyle i} & \\ & \mathrm{T}(\mathscr{E}) & \end{array}$$

where $\overline{\delta}$ is essential and satisfies $\overline{\delta}_!(1) \cong 1$. A geometric morphism $\mathscr{X} \xrightarrow{\rho} \mathrm{M}(\mathscr{E})$ factors through $\mathrm{T}(\mathscr{E})$ iff in the bicomma object

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\gamma} & \mathscr{X} \\ {\scriptstyle\psi}\downarrow & \Rightarrow & \downarrow{\scriptstyle\rho} \\ \mathscr{E} & \underset{\delta}{\rightarrowtail} & \mathrm{M}(\mathscr{E}) \end{array}$$

the locally connected γ is connected, in which case the resulting inside square with $\mathrm{T}(\mathscr{E})$ is a bicomma object.

Proof. Let $\mathrm{T}(\mathscr{E})$ denote the subtopos of $\mathrm{M}(\mathscr{E})$ given by the least topology forcing the morphism $\delta_! 1 \longrightarrow 1$ to be an isomorphism. Then a geometric morphism $\mathscr{X} \xrightarrow{\rho} \mathrm{M}(\mathscr{E})$ factors (uniquely) through $\mathrm{T}(\mathscr{E})$ iff $\rho^* \delta_! 1 \cong 1$ iff $\gamma_! 1 \cong \gamma_! \psi^* 1 \cong 1$ iff the locally connected γ is connected. Note that δ factors through $\mathrm{T}(\mathscr{E})$ since $\delta^* \delta_! 1 \cong 1$. We have $\overline{\delta}^* = \delta^* i_*$, and since $i^* \delta_! \dashv \delta^* i_*$, $\overline{\delta}$ is essential with $\overline{\delta}_! 1 = i^* \delta_! 1 \cong 1$. □

In terms of models of the theories classified by these toposes, $\overline{\delta}$ corresponds to forgetting that a lex distribution $\mathscr{E} \longrightarrow \mathscr{X}$, i.e., a geometric morphism $\mathscr{X} \longrightarrow \mathscr{E}$, preserves pullbacks, and the second factor corresponds to forgetting that a probability distribution $\mathscr{E} \longrightarrow \mathscr{X}$, which corresponds to a geometric morphism $\mathscr{X} \longrightarrow \mathrm{T}(\mathscr{E})$, preserves 1 .

Theorem 8.2.7 *Let $\mathscr{E}$ denote an arbitrary topos over $\mathscr{S}$. Then $\mathrm{M}(\mathscr{E}) \simeq \mathrm{B}_L(\mathrm{T}(\mathscr{E}))$, naturally in $\mathscr{E}$.*

Proof. This follows easily from universal properties. The category

$$\mathbf{Top}_{\mathscr{S}}(\mathscr{X}, \mathrm{B}_L(\mathrm{T}(\mathscr{E})))$$

is equivalent to the category of pairs $F \in \mathscr{X}$ and $\mathscr{X}/F \longrightarrow \mathrm{T}(\mathscr{E})$ over $\mathscr{S}$, as we had mentioned in Proposition 8.2.3. This data is equivalently given by an object $F \in \mathscr{X}$ and a probability distribution $\mathscr{E} \longrightarrow \mathscr{X}/F$. The category of such pairs is clearly equivalent to $\mathbf{Dist}_{\mathscr{S}}(\mathscr{E}, \mathscr{X}) \simeq \mathbf{Top}_{\mathscr{S}}(\mathscr{X}, \mathrm{M}(\mathscr{E}))$. □

The following results gives an alternative construction of $\mathrm{T}(\mathscr{E})$ in terms of sites. A finite connected limit is one whose diagram is finite, non-empty, and connected. Finite connected limits can be freely adjoined to an arbitrary small category. Let

$$\kappa : \mathbb{C} \longrightarrow \mathbb{C}^{\oplus}$$

denote the *finite connected limit completion* of a small category $\mathbb{C}$.

Lemma 8.2.8 *κ is a final functor.*

Proof. Let $\mathbb{D}_{\oplus}$ denote the finite connected *colimit* completion of a small category $\mathbb{D}$. $\mathbb{D}_{\oplus}$ can be constructed as the full subcategory of $P(\mathbb{D})$ determined by those presheaves that are finite connected colimits of representables. Then the canonical functor $\mathbb{D} \longrightarrow \mathbb{D}_{\oplus}$ is an initial functor, so that, since $\mathbb{C}^{\oplus} = (\mathbb{C}^{\mathrm{op}}{}_{\oplus})^{\mathrm{op}}$, the functor κ is final. □

Proposition 8.2.9 *Let $\mathscr{E} = Sh(\mathbb{C}, J)$, so that $\mathrm{M}(\mathscr{E}) \simeq Sh(\mathbb{C}^{\star}, J^{\star})$, where $\mathbb{C}^{\star}$ is the free lex completion of $\mathbb{C}$. Then $\mathrm{T}(\mathscr{E})$ can be constructed as the pullback*

$$\begin{array}{ccc} \mathrm{T}(\mathscr{E}) & \rightarrowtail & \mathrm{M}(\mathscr{E}) \\ \downarrow & & \downarrow \\ P(\mathbb{C}^{\oplus}) & \rightarrowtail & P(\mathbb{C}^{\star}) \end{array}$$

in $\boldsymbol{Top}_{\mathscr{S}}$, where the bottom geometric morphism is induced by the unique factorization of δ through κ.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\kappa} & \mathbb{C}^{\oplus} \\ & {}_{\delta}\searrow & \downarrow \\ & & \mathbb{C}^{\star} \end{array}$$

Proof. We must show that for an arbitrary topos $\mathscr{X}$, the category of $\mathscr{X}$-valued probability distributions on $\mathscr{E}$ is equivalent to the category of cones

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{h} & \mathrm{M}(\mathscr{E}) \\ {\scriptstyle k}\downarrow & & \downarrow \\ P(\mathbb{C}^{\oplus}) & \rightarrowtail & P(\mathbb{C}^{\star}) \end{array}$$

by an equivalence that is natural in $\mathscr{X}$. Intuitively, this is clear because such a cone is simply an h for which $\mathscr{X} \xrightarrow{h} \mathrm{M}(\mathscr{E}) \rightarrowtail P(\mathbb{C}^{\star})$ factors through $P(\mathbb{C}^{\oplus})$. In any case, some explanation is necessary. Suppose we are given a $\mathscr{X}$-valued probability distribution with corresponding cosheaf $G : \mathbb{C} \longrightarrow \mathscr{X}$, and geometric morphism $\mathscr{X} \xrightarrow{h} \mathrm{M}(\mathscr{E})$. G satisfies $\varinjlim (G) \cong 1$. Then G lifts to a functor $G^{\oplus} : \mathbb{C}^{\oplus} \longrightarrow \mathscr{X}$ that preserves finite connected limits, and which furthermore, since κ is final (Lemma 8.2.8), satisfies $\varinjlim (G^{\oplus}) \cong 1$. It follows that the left extension $k^* : P(\mathbb{C}^{\oplus}) \longrightarrow \mathscr{X}$ of $G^{\oplus}$ preserves finite connected limits and also 1. Therefore, k^* is left exact, so that we have a geometric morphism k and a cone as above.

Conversely, a cone such as above gives a cosheaf $G : \mathbb{C} \longrightarrow \mathscr{X}$ corresponding to h, and at the same time a flat functor $K : \mathbb{C}^{\oplus} \longrightarrow \mathscr{X}$ corresponding to k. K satisfies $\varinjlim K \cong 1$, so that, again since κ is final, we have $\varinjlim (K \cdot \kappa) \cong 1$ also. But since the cone commutes, we have $G \cong K \cdot \kappa$, so that the corresponding distribution is a probability distribution. □

The following diagram depicts the two canonical factorizations of the unit $\delta : \mathscr{E} \rightarrowtail \mathrm{M}(\mathscr{E})$, one through the bag-domain $\mathrm{B}_L(\mathscr{E})$, and the other through the probability distribution classifier $\mathrm{T}(\mathscr{E})$.

$$\begin{array}{ccc} \mathscr{E} & \rightarrowtail & \mathrm{B}_L(\mathscr{E}) \\ \downarrow & & \downarrow \\ \mathrm{T}(\mathscr{E}) & \rightarrowtail & \mathrm{M}(\mathscr{E}) \end{array}$$

In terms of freely adjoining finite limits to a site for the topos, the above corresponds to the diagram below.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathrm{fp}} & \mathbb{C}_{\mathrm{fp}} \\ {\scriptstyle \mathrm{fc}}\downarrow & & \downarrow{\scriptstyle \mathrm{eq}} \\ \mathbb{C}^{\oplus} & \xrightarrow[\mathrm{fp}]{} & \mathbb{C}^{\star} \end{array}$$

For example, the functor labeled 'eq' is the unit for freely adjoining equalizers.

8.3 Discrete Complete Spread Structure

Our discussion in this section focuses on a certain notion of discreteness that implies that the lower bagdomain and symmetric topos agree (Remark 8.2.4). It turns out that this notion also provides a suitable 'single universe' for local homeomorphisms and complete spreads.

Definition 8.3.1 *We shall say that a topos $\mathscr{X}$ over a topos $\mathscr{S}$ has* discrete $\mathscr{S}$-complete spread structure *if in any commutative diagram*

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\eta} & \mathscr{X} \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \mathscr{S} & \end{array} \qquad (8.5)$$

with f locally connected, η is an $\mathscr{S}$-complete spread iff f is discrete. We say a locale has discrete complete spread structure just when its topos of sheaves has. Sometimes we omit the prefix $\mathscr{S}$ when it is clear what is the base topos.

Proposition 8.3.2 *If a topos has discrete complete spread structure, then its bagdomain and symmetric toposes are equivalent.*

Proof. A distribution on a topos with discrete complete spread structure must preserves pullbacks, because the domain of the corresponding complete spread has the form $\mathscr{S}/I \longrightarrow \mathscr{S}$. ◻

For locales, discrete complete structure may be described with other equivalent conditions.

Proposition 8.3.3 *The following are equivalent for any locale X in a topos $\mathscr{S}$:*

1. *X has discrete complete spread structure;*
2. *any locale morphism $Y \longrightarrow X$ with locally connected domain is a complete spread iff Y is discrete;*
3. *The counit $|X| \longrightarrow X$ is the Gleason core* (locally connected coreflection) *of X, where $|X|$ denotes the object of points of X;*
4. *the canonical functor*

$$\mathscr{S}/|X| \longrightarrow \mathbf{E}(X) = \mathbf{Dist}_{\mathscr{S}}(\mathscr{S}, Sh(X))$$

is an equivalence.

Proof. The first two conditions are equivalent because the localic reflection of a locally connected topos is locally connected. The second and third conditions are obviously equivalent. The third and fourth are equivalent because the complete spread corresponding to the terminal distribution is precisely the Gleason core of the locale. ◻

A proof of the following may be found in the literature.

Lemma 8.3.4 *The quasi-components of an open set of a topological space are open in its Gleason core.*

By definition, a zero-dimensional space is one in which the clopen sets generate the topology.

Proposition 8.3.5 *A zero-dimensional T_0 topological space has discrete Set-complete spread structure.*

Proof. The Gleason core of a space has the same underlying set as the given space. The Gleason core of a zero-dimensional T_0 space is discrete. Indeed, the quasi-components of a zero-dimensional T_0 space are its singletons. By Lemma 8.3.4, singletons are open in the Gleason core. ◻

Proposition 8.3.6 *The product of two locales with discrete complete spread structure has complete spread structure.*

Proof. This follows because $|X \times Y| \cong |X| \times |Y|$. □

Proposition 8.3.7 *Let $\mathscr{E}$ denote a locally connected topos over $\mathscr{S}$. Then an $\mathscr{S}$-complete spread has discrete $\mathscr{E}$-complete spread structure.*

Proof. This is Proposition 9.4.1, which we prove in that section. □

Remark 8.3.8 *A local homeomorphism $\mathscr{E}/X \longrightarrow \mathscr{E}$ also has discrete $\mathscr{E}$-complete spread structure (Exercise 5). Thus, the locales in $\mathscr{E}$ with discrete $\mathscr{E}$-complete spread structure contain as full subcategories both the local homeomorphisms and the $\mathscr{S}$-complete spreads. This notion of single universe is largely unexplored. For example, consider Exercise 9.*

Exercises 8.3.9

1. *Show that $\mathrm{M}(\mathscr{E}) \simeq \mathrm{B}_L(\mathscr{E})$ iff every complete spread over $\mathscr{E}$ has totally connected components.*
2. *Show that a zero-dimensional sober topological space has discrete complete spread structure.*
3. *Construct the probability distribution classifier $\mathrm{T}(\mathscr{E})$ by a coinverter argument using $\mathbb{C}^{\oplus}$ instead of $\mathbb{C}^{\star}$.*
4. *It is well known that for any small category $\mathbb{C}$, the (finite) product completion $\mathbb{C}_{\mathrm{fp}}$ is finitely complete iff $\mathbb{C}$ has all (finite) small connected limits (Diers). Moreover, the universal functor $\mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{fp}}$ preserves any connected limits that might exist in $\mathbb{C}$. Deduce from this that*
$$\mathbb{C}^{\star} \simeq (\mathbb{C}^{\oplus})_{\mathrm{fp}}.$$
In turn, conclude that $\mathrm{M}(\mathscr{E}) \simeq \mathrm{B}_L(\mathrm{T}(\mathscr{E}))$. This gives an alternative proof, in terms of sites, of Theorem 8.2.7.
5. *Show that a locale is locally connected and has discrete complete spread structure iff it is discrete.*
6. *Show that the intersection of a (non-empty) family of locally connected topologies is locally connected.*
7. *Show directly that a topological space has a Gleason core: it has the same underlying set, but retopologized with the smallest locally connected topology larger than the given one.*
8. *Prove Lemma 8.3.4.*
9. *Prove or refute: a discrete Conduché fibration over a small category $\mathbb{C}$ has discrete $\mathrm{P}(\mathbb{C})$-complete spread structure, regarded as a locale in $\mathrm{P}(\mathbb{C})$.*
10. *Give a direct description of the algebras ('convex toposes') for the probability distributions classifier KZ-monad T in $\boldsymbol{Top}_{\mathscr{S}}$.*

8.4 Algebraic Geometry: Coschemes

In this section we discuss an example for the purpose of fixing ideas. We provide more details than an informed reader may need; however, we feel this is worthwhile as it illustrates in detail, and in a special case - the one associated with the Zariski topos (classifier of local rings) - the notions of distribution, distribution algebra, and complete spread geometric morphism. In the process we answer two questions. The first is how is the topos classifier of local rings with a given residue field constructed? Second, what is the nature of a distribution on the Zariski topos?

Throughout, the term 'ring' means a commutative ring with unit. Let *Ring* denote the category of such rings. Let *FP* denote the category of finitely presented rings $A = Z[x_1, \ldots, x_n]/I$. The ring $Z[x]$ is a coring object in *FP* for the tensor product $Z[x] \otimes Z[x] = Z[x, y]$. The comultiplication and coaddition $Z[x] \longrightarrow Z[x, y]$ are given by $x \mapsto xy$ and $x \mapsto x + y$.

We denote the topos of set-valued functors on *FP* by Set^{FP}. As always, we have the Yoneda functor

$$h : FP^{\mathrm{op}} \longrightarrow Set^{FP} .$$

Let $U = h(Z[x])$ denote the covariant representable

$$U(A) = Ring(Z[x], A) = A .$$

U is a universal ring object, or 'affine line,' making Set^{FP} a ringed topos. Set^{FP} classifies rings in the sense that there is a canonical equivalence of categories

$$\mathbf{Top}(Set, Set^{FP}) \;\simeq\; \mathbf{Lex}(FP^{\mathrm{op}}, Set) \;\simeq\; Ring . \tag{8.6}$$

The equivalence is given on the one hand by associating with a left exact functor F the ring $F(Z[x])$, and on the other hand by associating with a ring R the contravariant representable functor

$$\overline{R}(A) = Ring(A, R) . \tag{8.7}$$

Notice that $F(Z[x])$ is a ring because F is left exact and $Z[x]$ is a coring. The functor $\overline{R}$ is left exact because it is representable. The above equivalence can also be regarded in terms of points of Set^{FP}: $p \mapsto p^*(U)$.

The so-called (gros) Zariski topos denoted $\mathscr{Z}$, may be defined as the topos of sheaves on the site FP^{op} for the Grothendieck topology whose cocovers in *FP* are finite families of localizations $\{A \longrightarrow A[a_i^{-1}]\}$, such that $a_1 + \cdots + a_n = 1$. We call this topology the Zariski topology. The covering sieves are connected, so that $\mathscr{Z}$ is a locally connected topos.

Let *LRing* denote the category of commutative local rings with unit. (The trivial ring in which $0 = 1$ is not considered to be a local ring.) The equivalence (8.6) restricts to one

$$\mathbf{Top}(Set, \mathscr{Z}) \;\simeq\; LRing ,$$

so that the Zariski topos classifies local rings. The representables $h(A)$ are sheaves for the Zariski topology. In particular, the universal ring $U = h(Z[x])$ is a sheaf. Moreover, U is a local ring object in $\mathscr{Z}$. Local rings are preserved under inverse image of a geometric morphism. In terms of points of $\mathscr{Z}$, the above equivalence is given by $p \mapsto p^*(U)$. More generally, for any Grothendieck topos $\mathscr{E}$, there is a natural equivalence of categories

$$\mathbf{Top}(\mathscr{E}, \mathscr{Z}) \;\simeq\; \mathit{LRing}(\mathscr{E}) \; .$$

If R is a (local) ring in a topos $\mathscr{E}$, let $\mathit{Ring}_{\mathscr{E}}(R^{\Delta A})$ denote the object of ring homomorphisms from the constant ring ΔA to R in $\mathscr{E}$. If c is an object of a site $\mathbb{C}$ for $\mathscr{E}$, then

$$\mathit{Ring}_{\mathscr{E}}(R^{\Delta A})(c) = \mathit{Ring}(A, R(c)) \; .$$

This describes the geometric morphism $p : \mathscr{E} \longrightarrow \mathscr{Z}$ corresponding to a local ring object R:

$$p^*(A) = \mathit{Ring}_{\mathscr{E}}(R^{\Delta A}) \rightarrowtail R^n \; .$$

We sometimes refer to $p^*(A)$ as the R-variety defined by A in $\mathscr{E}$. We have $p^*(Z[x]) = p^*(U) = R$.

Example 8.4.1 *The frame of the Zariski spectrum* $\mathit{Spec}(R)$ *of a ring* R *is the lattice of radical ideals of* R*, ordered by inclusion:*

$$\mathscr{O}(\mathit{Spec}(R)) = \{\textit{radical ideals of } R\} \; .$$

This frame is generated by the basic radical ideals:

$$D(r) = \{a \in R \mid \exists n, t \; a^n = tr\} \; .$$

We have

$$\mathscr{O}(\mathit{Spec}(R[r^{-1}]) \cong \{\textit{radical } I \mid I \subseteq D(r)\} \; .$$

Thus, it makes sense to denote the locale $\mathit{Spec}(R[r^{-1}])$ *by* $D(r)$*. This is an open sublocale of* $\mathit{Spec}(R)$*.*

We define the structure sheaf on $\mathit{Spec}(R)$*:*

$$\mathscr{O}_R(D(r)) = R[r^{-1}] \; .$$

For any $r, s \in R$, $D(r) \subseteq D(s)$ *iff* $r \in D(s)$ *iff* $r^n = ts$*, for some natural number* n *and some* $t \in R$*. Therefore, if* $D(r) \subseteq D(s)$*, then the ring homomorphism* $R \longrightarrow R[r^{-1}]$ *inverts* s*, so that it factors through* $R \longrightarrow R[s^{-1}]$*.*

$$\begin{array}{ccc} R & \longrightarrow & R[s^{-1}] \\ & \searrow & \downarrow \\ & & R[r^{-1}] \end{array}$$

The structure sheaf $\mathcal{O}_R$ is a local ring object in $\mathcal{E} = Sh(Spec(R))$. As above, we may explicitly describe the geometric morphism $\mathcal{E} \xrightarrow{p} \mathcal{Z}$ that corresponds to $\mathcal{O}_R$ as a left exact cosheaf $FP^{\mathrm{op}} \xrightarrow{p^*} \mathcal{E}$:

$$p^*(A)(D(r)) = Ring_{\mathcal{E}}({\mathcal{O}_R}^{\Delta A})(D(r)) = Ring(A, R[r^{-1}]) \ .$$

In particular, we have $p^*(U) \cong \mathcal{O}_R$.

Definition 8.4.2 *A ring homomorphism* with local domain *is a ring homomorphism $L \longrightarrow R$ for which L is a local ring.*

Example 8.4.3 *Let k denote any field, and let R denote the local ring $k[x]_{(x)}$ with maximal ideal (x). The quotient ring $R/(x)$ is isomorphic to k. A ring homomorphism $L \xrightarrow{\varphi} R$ with local domain amounts to a commutative square*

$$\begin{array}{ccc} L/P & \rightarrowtail & R \\ \downarrow & & \downarrow \pi \\ L/Q & \rightarrowtail & k \end{array}$$

where $P = \ker(\varphi)$ and $Q = \ker(\pi\varphi) =$ maximal ideal of L.

Our goal is to produce the topos classifier of ring homomorphisms with local domain to a given ring R:

$$\mathbf{Top}(\mathcal{E}, \mathcal{Z}_R) \ \simeq \ LRing(\mathcal{E})/\Delta_{\mathcal{E}}(R) \ .$$

Theorem 8.4.4 *Let R be a commutative ring with 1. Then there is a topos $\mathcal{Z}_R$ that classifies ring homomorphisms to R with local domain. Furthermore, there is a geometric morphism*

$$\psi_R : \mathcal{Z}_R \longrightarrow \mathcal{Z} \ .$$

If R is a local ring, then the left exact $\overline{R}$ (8.7) *is a cosheaf* (for the Zariski topology)*, and ψ_R is a complete spread geometric morphism: $\mathcal{Z}_R$ occurs as the middle topos in the comprehensive factorization of the point $Set \longrightarrow \mathcal{Z}$ corresponding to R. $\mathcal{Z}_R$ is totally connected, and has a pure point $Set \longrightarrow \mathcal{Z}_R$.*

Proof. We first establish the following equivalences.

$$\mathbf{Top}(Set, Set^{FP/R}) \ \simeq \ \mathbf{Lex}(FP/R^{\mathrm{op}}, Set) \ \simeq \ Ring/R \ .$$

FP/*R* has finite colimits: they are created in *FP*. Associated with a ring homomorphism $f : T \longrightarrow R$ we have the representable functor

$$\overline{f} : FP/R^{\mathrm{op}} \longrightarrow Set \ ,$$

which is left exact.

Conversely, suppose we have a left exact functor

$$F : \mathit{FP}/R^{\mathrm{op}} \longrightarrow \mathit{Set}\ .$$

We regard elements $r \in R$ as homomorphisms $Z[x] \xrightarrow{r} R$. Let

$$f_F : T = \coprod_{r \in R} F(r) \longrightarrow R$$

be the evident projection. We denote elements of T as pairs (t, r), where $t \in F(r)$. We have $f_F(t, r) = r$.

The set T has a commutative ring structure such that:

1. f_F is a ring homomorphism,
2. $\overline{f_F} = F$,
3. $f_{\overline{f}} = f$.

Let (t, r) and (t', r') be elements of T. The object $Z[x, y] \xrightarrow{r,r'} R$ is the coproduct in FP/R of $Z[x] \xrightarrow{r} R$ and $Z[x] \xrightarrow{r'} R$. We use the map

$$F(x + y) : F(r, r') = F(r) \times F(r') \longrightarrow F(r + r')$$

to add elements of T:

$$(t, r) + (t', r') = (F(x + y)(t, t'), r + r')\ .$$

To see what is the 0-element of T observe that we have

$$\begin{array}{ccc} Z[x] & \xrightarrow{\ 0\ } & Z \\ & {\scriptstyle 0_R}\searrow \quad \swarrow{\scriptstyle !} & \\ & R & \end{array}$$

in FP, where $Z \xrightarrow{!} R$ is the initial object of FP/R. Therefore $F(!) \cong 1$ and we have

$$F(0) : 1 \longrightarrow F(0_R)\ .$$

The 0-element of T is then $(F(0), 0_R)$. The remaining details are routinely verified.

We next turn to a construction of $\mathscr{Z}_R$. Consider the topos pullback of the essential geometric morphism associated with the discrete fibration $\mathit{FP}/R \longrightarrow \mathit{FP}$ for $\overline{R}$ (8.7).

$$\begin{array}{ccc} \mathscr{Z}_R & \rightarrowtail & \mathit{Set}^{\mathit{FP}/R} \\ {\scriptstyle \psi_R}\downarrow & & \downarrow \\ \mathscr{Z} & \rightarrowtail & \mathit{Set}^{\mathit{FP}} \end{array}$$

Then a point $\mathit{Set} \longrightarrow \mathscr{Z}_R$ corresponds to a commutative square of geometric morphisms (up to isomorphism) as follows.

$$\begin{array}{ccc} Set & \longrightarrow & Set^{FP/R} \\ \big\downarrow & & \big\downarrow \\ \mathscr{Z} & \rightarrowtail & Set^{FP} \end{array}$$

This says that a point of $\mathscr{Z}_R$ amounts to a local ring L that as a 'ring' $Set \longrightarrow Set^{FP}$ factors through $Set^{FP/R}$. Thus, a point of $\mathscr{Z}_R$ is equivalently given by a local ring L that is equipped with a ring homomorphism $L \longrightarrow R$.

Finally, if R is a local ring, then $\overline{R}$ preserves Zariski-covers. I.e., the left exact $\overline{R}$ is a cosheaf, and the above construction of ψ_R is just the usual construction of the complete spread factor of the comprehensive factorization of the point $Set \longrightarrow \mathscr{Z}$ corresponding to R. In this case, $\mathscr{Z}_R$ is locally connected, in fact totally connected, and the pure factor is a point $Set \longrightarrow \mathscr{Z}_R$. □

Remark 8.4.5 *The universal ring homomorphism in* $Set^{FP/R}$ *is* $U \xrightarrow{\pi} V$, *where*

$$U(A \xrightarrow{f} R) = A \ ,$$

and V *is the constant presheaf* $\Delta(R)$. *The component* $\pi_f : U(f) \longrightarrow V(f)$ *of the natural transformation* π *is given by* $\pi_f = f$. *Also, we have*

$$U = \coprod_{r \in R} h_R(r) \ ,$$

where

$$h_R : FP/R^{\mathrm{op}} \longrightarrow Set^{FP/R}$$

denotes Yoneda. As always, we regard an element $r \in R$ *as the object* $Z[x] \xrightarrow{r} R$ *of* FP/R. *We have* $h_R(r)(A \xrightarrow{f} R) = f^{-1}(r)$.

Example 8.4.6 *Consider the finite field* $R = Z/pZ = F_p$, *which is a local ring. If* L *is any local ring, then a ring homomorphism* $L \longrightarrow F_p$ *amounts to an isomorphism of the residue field* L/M *with* F_p. *Thus, the totally connected topos* $\mathscr{Z}_{F_p}$ *classifies local rings paired with an isomorphism of its residue field with* F_p. *The geometric morphism* $\psi_{F_p} : \mathscr{Z}_{F_p} \longrightarrow \mathscr{Z}$ *is a complete spread geometric morphism.*

Example 8.4.6 generalizes as follows.

Corollary 8.4.7 *For any field* k, *there is a topos classifier of local rings with residue field* k. *This topos is a subtopos of* $\mathscr{Z}_k$.

Proof. Let j denote the least topology in $\mathscr{Z}_k$ that forces the universal morphism $U \xrightarrow{\pi} V$ in $\mathscr{Z}_k$ to be an epimorphism. Then $Sh_j(\mathscr{Z}_k)$ classifies local rings whose residue field is isomorphic to k. Indeed, the kernel of a ring epimorphism $L \longrightarrow k$ with local domain must be equal to the single maximal ideal of L, so that the residue field of L is k. $\square$

Corollary 8.4.7 answers our first question. For the second question, we introduce the following terminology.

Definition 8.4.8 *A* coscheme *is a distribution on $\mathscr{Z}$. Equivalently, a coscheme is a cosheaf on the Zariski site FP^{op}.*

Left exact coschemes correspond precisely to local rings: if R is a local ring, then $\overline{R} = Ring(\_\,, R)$ is a left exact coscheme. More generally, we may associate with a local ring object R in a locally connected topos $\mathscr{E}$ a coscheme (possibly not left exact):

$$G_{(\mathscr{E},R)}(A) = \pi_0(Ring_{\mathscr{E}}(R^{\Delta A})) \ , \tag{8.8}$$

where $\pi_0 \dashv \Delta$ is the connected components functor for $\mathscr{E}$.

The canonical equivalence of topos distributions with complete spread geometric morphisms implies the following.

Theorem 8.4.9 *The functor defined in* (8.8) *is a coscheme, and every coscheme has this form for some ringed topos $(\mathscr{E}, R)$, where $\mathscr{E}$ is locally connected, and R is local.*

Proof. Let R be a local ring object in a locally connected topos $\mathscr{E}$, corresponding to geometric morphism $\mathscr{E} \xrightarrow{p} \mathscr{Z}$. Then $G_{(\mathscr{E},R)}$ is the functor

$$FP^{\mathrm{op}} \xrightarrow{h} \mathscr{Z} \xrightarrow{p^*} \mathscr{E} \xrightarrow{\pi_0} Set \ ,$$

which is a Zariski cosheaf. Conversely, for any coscheme G there is a complete spread geometric morphism

$$\psi : \mathscr{E} \longrightarrow \mathscr{Z}$$

for which $\mathscr{E}$ is locally connected, satisfying $G \cong \pi_0 \cdot \psi^* \cdot h$. But $R = \psi^* U$ is a local ring and $\psi^*(h(A)) \cong Ring_{\mathscr{E}}(R^{\Delta A})$. $\square$

The function $D : R \longrightarrow \mathscr{O}(Spec(R))$ provides a bijection between the set $E(R)$ of idempotents of R and the definable subobjects of 1 in $Sh(Spec(R))$.

Definition 8.4.10 *A ring R is* idempotent finite *if every localization $R[r^{-1}]$ has only finitely many idempotents. A* minimal idempotent *is an idempotent e for which $D(e)$ is connected. We write $E_m(R)$ for the set of minimal idempotents of a ring R.*

For example, an integral domain is idempotent finite because every localization has no idempotents other than $0, 1$.

We have already mentioned the spatial version of the following result (Exercise 1.1.2).

Lemma 8.4.11 *A ring is idempotent finite iff its Zariski spectrum is a locally connected locale. In this case the terminal cosheaf on Spec(R) is*

$$\pi_0(D(r)) = \{\text{connected components of } D(r)\} \cong E_m(R[r^{-1}]) \; .$$

Proof. Any locale with only finitely many definable subobjects (= clopens) may be covered with finitely many connected clopens. Thus, if a ring has only finitely many idempotents, then its spectrum may covered with finitely many connected clopens. We may repeat this argument for every $D(r) = Spec(R[r^{-1}]$. The upshot is that the connected opens generate the topology of the spectrum.

If $Spec(R)$ is locally connected, then so is every locale $Spec(R[r^{-1}])$. The connected components of the latter form an open cover. However, $Spec(R[r^{-1}])$ is compact so these components must be finite in number. Hence the definable subobjects of any $D(r)$ are finite in number, which coincides with the number of idempotents in $R[r^{-1}]$. □

Corollary 8.4.12 *If a ring R is idempotent finite, then*

$$G_R(A) = G_{(\mathscr{E},\mathscr{O}_R)}(A) = \pi_0(Ring_{\mathscr{E}}({\mathscr{O}_R}^{\Delta A}))$$

is a coscheme, where $\mathscr{E} = Sh(Spec(R))$.

Example 8.4.13 *(Example 8.4.1 continued.) Consider the sheaf of idempotents of* $\mathscr{O}_R$*:*

$$E_R(D(r)) = E(D(r)) = \{\text{idempotents of } R[r^{-1}]\} \; .$$

E is a subsheaf of $\mathscr{O}_R$. *E is also a Boolean algebra in Sh(Spec(R)):* $e \vee f = e + f - ef$, $e \wedge f = ef$. *E is isomorphic to the Boolean algebra* $\Delta(2)$. *If R is idempotent finite, then* $E \cong \Delta(2)$ *is isomorphic to the distribution algebra* $\mathbf{U}(\pi_0)$ *for the underlying sheaf functor (Def. 7.1.6)*

$$\mathbf{U} : \mathbf{E}(\mathscr{E})^{\mathrm{op}} \longrightarrow \mathscr{E} \; ,$$

for $\mathscr{E} = Sh(Spec(R))$. *I.e., E is the initial distribution algebra in* $\mathscr{E}$, *in the idempotent finite case.*

Example 8.4.14 *To sum up, we may associate with any idempotent finite ring R the following:*

1. *the coscheme* G_R, *such that* $G_R(A)$ *equals the set of connected components of the* $\mathscr{O}_R$*-variety defined by A in Sh(Spec(R));*
2. *the complete spread* $\mathscr{V}_R \longrightarrow \mathscr{X}$ *associated with* G_R, *which is the complete spread factor of the comprehensive factorization*

$$\begin{array}{ccc} Sh(Spec(R)) & \xrightarrow{\eta} & \mathscr{V}_R \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \psi} \\ & & \mathscr{X} \end{array}$$

where p corresponds to the local ring $\mathscr{O}_R$ ($\mathscr{V}_R$ is a kind of completion of Spec(R));

3. *the distribution algebra $B_R = p_*(E_R) \cong p_*(\Delta 2)$ in $\mathscr{X}$. B_R is a sheaf on FP^{op}. $B_R(A)$ is equal to set of the definable subobjects of the $\mathscr{O}_R$-variety defined by A in $\mathscr{E} = \mathrm{Sh}(\mathrm{Spec}(R))$. We may equivalently describe B_R as the sheaf*

$$B_R(A) = \mathscr{E}(p^*(A), E_R)$$

where

$$p^*(A)(D(r)) = Ring(A, R[r^{-1}]) \ ,$$

and

$$E_R(D(r)) = E(R[r^{-1}]) \ .$$

Note: B_R is covariant in A because Yoneda: $FP^{\mathrm{op}} \longrightarrow \mathscr{X}$ is contravariant.

Exercises 8.4.15

1. *Explicitly describe the Grothendieck topology in $\mathrm{FP}/R^{\mathrm{op}}$ that defines $\mathscr{X}_R$.*
2. *Show that an idempotent e is minimal iff $R[e^{-1}]$ has no idempotents other than $0, 1$ iff e is non-0 and for all idempotents f, either $ef = e$ or $ef = 0$. Show that the sum of two distinct minimal idempotents is an idempotent. Show that 1 is minimal iff 0 and 1 are the only idempotents.*
3. *The Zariski spectrum of a Noetherian ring is a Noetherian space. Show that a Noetherian ring is idempotent finite by showing that a Noetherian space is locally connected, and then use Lemma 8.4.11.*
4. *Let R be idempotent finite with coscheme G_R (Cor. 8.4.12). Show that*

$$G_R(Z[x]) = \pi_0(\mathscr{O}_R) \cong \left(\coprod_{r \in R} R[r^{-1}] \times E_m(R[r^{-1}]) \right) / \sim \ ,$$

where the equivalence relation $\sim$ is defined by a colimit:

$$\varinjlim \left(El(\mathscr{O}_R) \longrightarrow \mathscr{O}(\mathrm{Spec}(R)) \xrightarrow{\pi_0} Set \right) \ .$$

$El(\mathscr{O}_R)$ denotes the category of elements of the sheaf $\mathscr{O}_R$.

5. *What sort of idempotent finite ring R is recovered from its coscheme G_R? Technically, the question can be phrased in various and related ways. When is the corresponding geometric morphism $\mathrm{Sh}(\mathrm{Spec}(R)) \xrightarrow{p} \mathscr{X}$ is a complete spread, i.e., when is the pure factor η in Example 8.4.14 an equivalence? For what R is the unit*

$$\psi^*(U) \longrightarrow \eta_*\eta^*(\psi^*(U)) \cong \eta_*(\mathscr{O}_R)$$

a ring isomorphism in $\mathscr{V}_R$, where U is the generic ring in $\mathscr{X}$? For what R is the canonical ring homomorphism $\Gamma(\psi^* U) \longrightarrow R$ an isomorphism?

8.5 Distributions on the Jonsson-Tarski Topos

In this section we shall consider a Grothendieck topos $\mathscr{C}$ for which $\mathbf{E}(\mathscr{C})$, as it happens, is a Grothendieck topos (Prop. 1.4.13). $\mathscr{C}$ is an étendue, meaning a topos $\mathscr{E}$ that has an object F with full support such that $\mathscr{E}/F$ is a localic topos. $\mathscr{C}$ is not locally connected, but we are able to explicitly describe its Gleason core. The topos $\mathscr{C}$ in question is the topos analogue of Cantor space: it has also been called the Jonsson-Tarski topos.

Definition 8.5.1 A Jonsson-Tarski algebra *is a set X equipped with an isomorphism $X \xrightarrow{(f,g)} X \times X$. We refer to such an object as a* JT-algebra. *A morphism of JT-algebras is a morphism $X \longrightarrow Y$ that commutes with the structure isomorphisms. Let $\mathscr{C}$ denote this category.*

We first exhibit a site presentation of $\mathscr{C}$, showing that $\mathscr{C}$ is a Grothendieck topos. Let M_2 denote the free monoid on two generators a, b. We refer to the elements of M_2 as words. We have the topos $P(M_2)$ of right M_2-sets (= presheaves on M_2). We use $\hom(X, Y)$ to denote the set of right M_2-set maps between two right M_2-sets X and Y. Usually we write $\hat{a} : X \longrightarrow X$ for the action of a in X: $\hat{a}(x) = xa$.

Let $\star$ denote the single object of the monoid M_2. We denote the single representative right M_2-set by $h(\star) = M_2$. It carries an action of M_2 by right multiplication. A word V provides a right M_2-set map $h_V : M_2 \longrightarrow M_2$ given by left multiplication: $h_V(W) = VW$.

The slice category $M_2/\star$ is the partially ordered set of all words, such that $V \le W$ if W is a prefix of V. We denote the empty word by $\top$ since it is the top word in this ordering. We have

$$P(M_2/\star) \simeq P(M_2)/M_2 \, .$$

We may make a JT-algebra $X \xrightarrow{(f,g)} X \times X$ into a right M_2-set by defining $\hat{a} = f$ and $\hat{b} = g$. On the other hand, a right M_2-set X is a JT-algebra if $X \xrightarrow{(\hat{a},\hat{b})} X \times X$ is an isomorphism.

Put another way, a JT-algebra is a right M_2-set that perceives a certain right M_2-set map as an isomorphism. To be precise, an M_2-set X is a JT-algebra if the following unique extension property holds.

$$\begin{array}{ccc} M_2 + M_2 & \xrightarrow{(h_a, h_b)} & M_2 \\ & \searrow^{\forall} & \downarrow{\exists!} \\ & & X \end{array}$$

This condition is of course a sheaf condition. The morphism (h_a, h_b) is a monomorphism. Its image is the set of non-empty words.

Thus, $\mathscr{C}$ is a subtopos of $P(M_2)$. A sieve is a subset of M_2 closed under right multiplication, which we typically denote R. The Grothendieck topology

J in M_2 for JT-algebras is thus a collection of certain covering sieves, which we wish to identify. If $\{W_i\}$ is any set of words, let

$$\langle W_i \rangle = \{ W \in M_2 \mid \exists\, i\; W \leq W_i \}$$

denote the sieve generated by the W_i. For instance, the sieve

$$\langle a, b \rangle = \{ W \in M_2 \mid W \text{ begins with } a \text{ or } b \}$$

consists of all non-empty words. This sieve generates J. The top sieve is $M_2 = \langle \top \rangle$. A covering sieve is then one of the form $\langle a, b \rangle$, $\langle aa, ab, ba, bb \rangle$, $\langle aa, ab, b \rangle$, and so on.

J is itself a right M_2-set: the action of M_2 is given by pullback along h_a:

$$R \cdot a = \hat{a}(R) = \{ W \mid aW \in R \} \;,$$

and similarly for b. We have $\hat{a}(R) \rightarrowtail R : W \mapsto aW$.

We may identify the covering sieves with rooted binary trees of finite depth. For instance, we identify $\langle aa, ab, b \rangle$ with the tree

$$\begin{array}{ccccc} & & & \top & \\ & a & & & b \\ a & & b & & \end{array}$$

We may describe the action of M_2 in J in terms of these trees. For example, $\langle aa, ab, b \rangle \cdot a = \langle a, b \rangle$, and $\langle aa, ab, b \rangle \cdot b = \top$.

If R is a covering sieve, and X is any right M_2-set, then we may identify a member of $\hom(R, X)$ with the tree R whose leaves are paired with elements of X. For instance, if $x \in X$, then the tree

$$\top, x \tag{8.9}$$

depicts the morphism $M_2 \xrightarrow{x} X$. For another example,

$$\begin{array}{ccccc} & & & \top & \\ & a & & & b, z \\ a, x & & b, y & & \end{array}$$

depicts the right M_2-map $\langle aa, ab, b \rangle \longrightarrow X$ that sends aa to x, ab to y, and b to z.

The (covering) sieves are ordered by containment. As binary trees, this ordering appears as reverse inclusion. For instance, we have

$$\langle aa, ab, b \rangle \;\leq\; \langle a, b \rangle \;,$$

which we depict as follows.

$$\begin{array}{ccccc} & & \top & & \\ & a & & & b \\ a & & b & & \end{array} \;\leq\; \begin{array}{ccc} & \top & \\ a & & b \end{array}$$

The induced map $\hom(\langle a, b\rangle, X) \longrightarrow \hom(\langle aa, ab, b\rangle, X)$ appears as

$$\begin{array}{ccc} & \top & \\ a, x & & b, y \end{array} \quad \mapsto \quad \begin{array}{ccccc} & & & \top & \\ & a & & & b, y \\ a, xa & & b, xb & & \end{array}$$

The colimit of this system is a right M_2-set that we denote $L(X)$:

$$L(X) = \varinjlim_{J^{\mathrm{op}}} \hom(R, X) \ .$$

If we denote members of $L(X)$ by $[\xi]$, where $\xi : R \longrightarrow X$, then the action of M_2 in $L(X)$ is given by

$$[\xi] \cdot a = [R \cdot a \rightarrowtail R \xrightarrow{\xi} X] \ ,$$

and similarly for b. For example, the action of a on

$$\begin{array}{ccccc} & & & \top & \\ & a & & & b, z \\ a, x & & b, y & & \end{array}$$

is equal to

$$\begin{array}{ccc} & \top & \\ a, x & & b, y \end{array}$$

and the action of b on this same tree is equal to $\top, z$.

From sheaf theory, if a presheaf has the uniqueness property for every covering sieve, then it is said to be separated. It turns out that if a right M_2-set X has this property for $\langle a, b\rangle \rightarrowtail M_2$, then it is separated. Moreover, if X is separated, then $L(X)$ is a JT-algebra: $L(X)$ is the best approximation of a separated X by a JT-algebra.

The single representable M_2-set M_2 is not a JT-algebra, but it is separated. An element of the JT-algebra $L(M_2)$ is an equivalence class of binary trees whose leaves are paired with elements of M_2. The canonical morphism of JT-algebras

$$(\, L(h_a), L(h_b)\,) : L(M_2) + L(M_2) \longrightarrow L(M_2) \tag{8.10}$$

is an isomorphism.

Now consider Cantor space $C = 2^N$. This is a product space, where 2 carries the discrete topology. We may regard C as the set of infinite strings $\{abb\cdots\}$. A typical basic open set U_V of Cantor space is then the collection of infinite strings that have the finite word V as a prefix. We have $V \leq W$ iff $U_V \leq U_W$. Thus, the partial order $M_2/\star$ is isomorphic to this base of open sets $\{U_V\}$. This brings us to the following result due to P. Freyd.

Proposition 8.5.2 *$\mathscr{C}$ is an étendue: $L(M_2)$ has full support in $\mathscr{C}$, and we have $\mathscr{C}/L(M_2) \simeq Sh(C)$. The sheaf on C associated with a JT-algebra morphism $X \xrightarrow{f} L(M_2)$ is*

$$F(U_V) = \{x \in X \mid f(x) = [\top, V]\} ,$$

where $\top, V$ *is defined in (8.9).*

Proof. The pullback of the JT-topology J in $P(M_2)$ to $P(M_2/\star)$ coincides with the canonical topology in $M_2/\star$. The category of sheaves for this topology is precisely the sheaf topos $Sh(C)$. On the other hand, since $P(M_2/\star) \simeq P(M_2)/M_2$, this pullback coincides with $\mathscr{C}/L(M_2)$. □

Definition 8.5.3 A Kennison algebra, or K-algebra*, in a topos is an object* A *of the topos equipped with an isomorphism* $A + A \longrightarrow A$*. Let* $\mathscr{K}$ *denote the category of K-algebras in Set.*

Example 8.5.4 *The set of natural numbers* N *is a K-algebra in Set such that* $n \mapsto 2n$ *or* $n \mapsto 2n+1$*.*

Example 8.5.5 *By (8.10),* $L(M_2)$ *is a K-algebra in* $\mathscr{C}$*. In fact, we shall see in Proposition 8.5.6 that* $L(M_2)$ *is the generic K-algebra.*

A few remarks about the structure of $\mathscr{K}$ follow. In particular, we show that $\mathscr{K} \simeq \mathbf{E}(\mathscr{C})$.

Proposition 8.5.6 *The functor* $\mu \mapsto \mu(L(M_2))$ *is an equivalence* $\mathbf{E}(\mathscr{C}) \simeq \mathscr{K}$*. More generally, for any Grothendieck topos* $\mathscr{F}$*,* $\mathbf{Dist}(\mathscr{F}, \mathscr{C})$ *is equivalent to the category of K-algebras in* $\mathscr{F}$*. The symmetric topos* $\mathrm{M}(\mathscr{C})$ *classifies Kennison algebras.*

Proof. The category of topos distributions $\mathbf{E}(P(M_2))$ is equivalent to the topos of left (= covariant) M_2-sets. If Y is a left M_2-set, then we denote the left action of a by $\bar{a} : Y \longrightarrow Y$. A left M_2-set Y carries the generating dense monomorphism $(h_a, h_b) : M_2 + M_2 \rightarrowtail M_2$ (of right M_2-sets) to the function $(\bar{a}, \bar{b}) : Y + Y \longrightarrow Y$. A K-algebra is thus a left M_2-set that carries the generating dense monomorphism to an isomorphism. This is precisely a cosheaf on M_2 for the JT-topology. □

Definition 8.5.7 *Let* T_C *denote the left* M_2*-set* 2^N*, such that the left action* $\bar{a}$ *prefixes an infinite string with the generator symbol* a*, and likewise for* $\bar{b}$*.*

Proposition 8.5.8 T_C *is the terminal K-algebra in Set. Equivalently,* T_C *is the terminal distribution on* $\mathscr{C}$*.*

Proof. T_C is a cosheaf for the JT-topology because T_C carries the generating dense monomorphism $(h_a, h_b) : M_2 + M_2 \rightarrowtail M_2$ to $2^N + 2^N \xrightarrow{(\bar{a}, \bar{b})} 2^N$, which is an isomorphism.

Let A be any K-algebra. We define the unique left M_2-set map $f : A \longrightarrow 2^N$ as follows. Let $x = x_0 \in A$. Then there is a unique $x_1 \in A$ such that $x_0 = c_0 x_1$, where $c_0 \in \{a, b\}$. Similarly, $x_1 = c_1 x_2$ for a unique x_2. We define $f(x) = c_0 c_1 \ldots$. Thus, T_C is terminal. □

Remark 8.5.9 *J. Kennison has shown that the category $\mathscr{K}$ of K-algebras in Set is a topos, and M. Barr has remarked that although $\mathscr{K}$ is Boolean, it is not well-pointed. In fact,*

$$\mathscr{K} \;\simeq\; \mathrm{Sh}_{\neg\neg}(Set^{M_2}/T_C).$$

We mention an application of the Monadicity Theorem (Thm. 7.1.15) and Proposition 8.5.6. Every K-algebra A has an underlying JT-algebra $\mathbf{U}(A) = 2^A$, which is the distribution algebra in $\mathscr{C}$ associated with the K-algebra A. The right action of M_2 in 2^A is as follows: for any subset $S \subseteq A$,

$$S \cdot a = \{x \in A \mid a \cdot x \in S\} \;,$$

and similarly for b. The underlying JT-algebra functor $\mathbf{U}$ is contravariant.

Corollary 8.5.10 *The underlying JT-algebra functor*

$$\mathbf{U} : \mathscr{K}^{\mathrm{op}} \longrightarrow \mathscr{C}$$

is monadic.

We conclude with an instance of the construction of the Gleason core of a topos: the Gleason core of $\mathscr{C}$.

$$\gamma : \widehat{\mathscr{C}} \longrightarrow \mathscr{C}$$

Any geometric morphism $\mathscr{F} \longrightarrow \mathscr{C}$ for which $\mathscr{F}$ is locally connected has an essentially unique factorization through the above γ.

Let $\mathbb{A}$ denote the category whose objects are infinite strings $x = aaba\ldots$ in the two generators a, b. A morphism $x \longrightarrow y$ is a word $V \in M_2$ such that $Vx = y$. $\mathbb{A}$ is not a poset: let $x = \overline{ab} = abab\ldots$, and let $V = ab$. Then $Vx = x$, so V and V^2 are distinct endomorphisms of x. There is a functor $\mathbb{A} \longrightarrow M_2$ that carries a morphism W to itself.

We regard $\mathbb{A}$ as a site as follows. We define a single generating covering sieve of a string y as the set of all non-empty words $W \in M_2$ that appear as a prefix of y: so we have a morphism $W : \frac{y}{W} \xrightarrow{W} y$, where $\frac{y}{W}$ denotes the string y after deleting the prefix W. Let $\widehat{\mathscr{C}}$ denote the topos of sheaves for this site. It happens that $\widehat{\mathscr{C}}$ is part of a topos pullback.

$$\begin{array}{ccc} \widehat{\mathscr{C}} & \rightarrowtail & P(\mathbb{A}) \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ \mathscr{C} & \rightarrowtail & P(M_2) \end{array}$$

The fact that $\widehat{\mathscr{C}}$ is locally connected is a special case of a fact shown already in a more general context (Proposition 2.4.1). The terminal distribution on $\mathscr{C}$ is $\pi_0 \cdot \gamma^*$, and we recover the terminal Kennison algebra as

$$T_C \cong (\pi_0 \cdot \gamma^*)(L(M_2)) \ .$$

Exercises 8.5.11

1. *What is the underlying JT-algebra of the natural numbers as a K-algebra (Example 8.5.4)?*
2. *Show that $\mathscr{C}$ is not locally connected. Show directly (without appealing to Proposition 2.4.1) that $\widehat{\mathscr{C}}$ is locally connected, and that γ is the Gleason core of $\mathscr{C}$.*
3. *Show that $\widehat{\mathscr{C}}$ is an étendue.*
4. *A geometric morphism is said to be a surjection if its inverse image functor is faithful. Show that $\gamma : \widehat{\mathscr{C}} \longrightarrow \mathscr{C}$ is a surjection.*

Further reading: Barr & Kennison [BK02], Bunge & Carboni [BC95], Bunge & Funk [BF98], Jibladze & Johnstone [JJ91], Johnstone [Joh82b, Joh02, Joh92, Joh89], Joyal & Tierney [JT84], Kock & Reyes [KR99], Vickers [Vic89, Vic92, Vic95].