6

Closed and Linear KZ-Monads

In this chapter we introduce additional axioms one may impose on a completion Kock-Zöberlein monad in order to further develop the theory of complete spreads in a more general context.

If a KZ-monad M is *closed* as we shall call it, we prove that any 1-cell whose domain admits an M-adjoint can be factored in an essentially unique way into a final 1-cell followed by a discrete M-fibration whose domain admits an M-adjoint. The final 1-cells for the symmetric monad M in $\mathbf{Top}_{\mathscr{S}}$ are precisely the pure geometric morphisms (relative to $\mathscr{S}$), and the discrete fibrations are precisely complete spread geometric morphisms with locally connected domain (§ 5.2).

In the context of a closed completion KZ-monad on $\mathscr{K}$ we discuss the existence of a *density functor* in connection with the validity of a *Gleason core* axiom. We consider a 'single universe' containing both the discrete M-fibrations and the discrete M-opfibrations, for any closed completion KZ-monad M in a 2-category $\mathscr{K}$. It has the required properties when the Gleason core axiom holds.

We also investigate what we call *additive and $\mathscr{K}$-equivariant KZ-monads*, and the nature of their M-algebras and M-homomorphisms. $\mathscr{K}$-equivariant KZ-monads may be used to establish Pitts' theorem on bicomma objects (§ 6.5).

6.1 Closed KZ-Monads and Comprehension

We shall prove two results about what we shall call a closed KZ-monad. The first result says that every closed, completion KZ-monad has an associated comprehensive factorization. The second shows that discrete fibrations compose for such a KZ-monad. Throughout, M is a KZ-monad in a 2-category with terminal object T.

Definition 6.1.1 *Let $G \xrightarrow{p} D$ be a 1-cell, and assume that both $G \xrightarrow{g} T$ and $D \xrightarrow{d} T$ admit an* M*-adjoint. We shall say that p is* a final 1-cell for M *if the canonical 2-cell $\mathrm{M}(p) \cdot r_g \Rightarrow r_d$ is an isomorphism.*

A 1-cell p is final iff $\mathrm{M}(p) \cdot r_g \cdot \delta_T \Rightarrow r_d \cdot \delta_T$ is an isomorphism because both 1-cells are homomorphisms.

Lemma 6.1.2 *Let* M *be a locally full and faithful KZ-monad. Let $G \xrightarrow{p} D$ be a 1-cell, and assume that $G \xrightarrow{g} T$ and $D \xrightarrow{d} T$ admit an* M*-adjoint. Then p is final iff for every 1-cell $D \xrightarrow{b} X$ and every 'constant' $T \xrightarrow{a} X$, composition with p gives a bijection*

$$\frac{b \Rightarrow a \cdot d}{b \cdot p \Rightarrow a \cdot d \cdot p}$$

of 2-cells, where $D \xrightarrow{d} T$ is the unique 1-cell.

Proof. If p is final, then because M is locally full and faithful we have the following natural bijections.

$$\begin{array}{c} b \Rightarrow a \cdot d \\ \hline \mathrm{M}b \Rightarrow \mathrm{M}a \cdot \mathrm{M}d \\ \hline \mathrm{M}b \cdot r_d \Rightarrow \mathrm{M}a \\ \hline \mathrm{M}b \cdot \mathrm{M}p \cdot r_g \Rightarrow \mathrm{M}a \\ \hline \mathrm{M}b \cdot \mathrm{M}p \Rightarrow \mathrm{M}a \cdot \mathrm{M}g \\ \hline b \cdot p \Rightarrow a \cdot g \cong a \cdot d \cdot p \end{array}$$

If the stated condition holds, then there are the following natural bijections. Here, h and k denote arbitrary M-homomorphisms with the appropriate domain and codomain.

$$\begin{array}{c} h \cdot \mathrm{M}p \cdot r_g \Rightarrow k \\ \hline h \cdot \mathrm{M}p \Rightarrow k \cdot \mathrm{M}g \cong k \cdot \mathrm{M}d \cdot \mathrm{M}p \\ \hline h \cdot \mathrm{M}p \cdot \delta_G \Rightarrow k \cdot \mathrm{M}d \cdot \mathrm{M}p \cdot \delta_G \\ \hline h \cdot \delta_D \cdot p \Rightarrow k \cdot \delta_T \cdot d \cdot p \\ \hline h \cdot \delta_D \Rightarrow k \cdot \delta_T \cdot d \\ \hline h \cdot \delta_D \Rightarrow k \cdot \mathrm{M}d \cdot \delta_D \\ \hline h \Rightarrow k \cdot \mathrm{M}d \end{array}$$

This shows that $\mathrm{M}(d) \dashv \mathrm{M}(p) \cdot r_g$ as M-homomorphisms, and hence as ordinary 1-cells. I.e., this shows that $\mathrm{M}(p) \cdot r_g \cong r_d$. Note: r_g is an M-homomorphism (4.3.4). □

Proposition 6.1.3 *A geometric morphism between locally connected toposes is final for the symmetric monad iff it is pure.*

Proof. A geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is final for the symmetric monad iff

$$\mathrm{M}(\psi) \cdot r_f \cdot \delta_{\mathscr{S}} \Rightarrow r_e \cdot \delta_{\mathscr{S}}$$

is an isomorphism. This holds iff $f_! \cdot \psi^* \cong e_!$ iff ψ is pure. ◻

Proposition 6.1.4 *A geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ satisfies the condition in Lemma 6.1.2 iff ψ_* preserves $\mathscr{S}$-coproducts.*

Proof. Take for X in Lemma 6.1.2 the object classifier $\mathrm{M}(\mathscr{S})$. It follows that for every I in $\mathscr{S}$, the unit $e^*I \longrightarrow \psi_*\psi^*(e^*I)$ is an isomorphism. This is precisely the property that ψ_* preserves $\mathscr{S}$-coproducts. ◻

A 1-cell f in a 2-category is said to reflect isomorphisms if for every composable 2-cell t, ft invertible implies t invertible. Consider the following condition on a KZ-monad.

Definition 6.1.5 *We shall say that a KZ-monad* M *is* closed *if for every discrete* M*-fibration ψ the 1-cell* $\mathrm{M}(\psi)$ *reflects isomorphisms.*

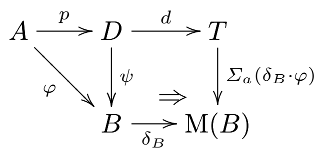
As always, T denotes the terminal object in the 2-category.

Theorem 6.1.6 (Comprehensive factorization for KZ-monads)
Suppose that M *is a closed completion KZ-monad. Then every 1-cell whose domain admits an* M*-adjoint has an essentially unique factorization as a final 1-cell followed by a discrete* M*-fibration whose domain admits an* M*-adjoint.*

Proof. Let $A \xrightarrow{\varphi} B$ be an arbitrary 1-cell such that the essentially unique 1-cell $A \xrightarrow{a} T$ admits an M-adjoint. Consider the universal 1-cell $A \xrightarrow{p} D$, where D denotes

$$\delta_B \Downarrow \Sigma_a(\delta_B \cdot \varphi)$$

as in the following diagram.



The 1-cell ψ is a discrete M-fibration, witnessed by $\Sigma_a(\delta_B \cdot \varphi)$. In order to show that p is final, let $i : \mathrm{M}(p) \cdot r_a \Rightarrow r_d$ denote the canonical 2-cell. We must show that i is an isomorphism. We know that

$$\Sigma_a(\delta_B \cdot \varphi) \cong \Sigma_d(\delta_B \cdot \psi) ,$$

and hence that

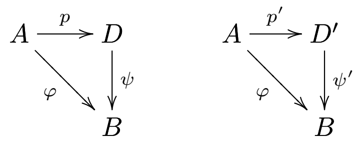
$$\mathrm{M}(\varphi) \cdot r_a \cdot \delta_T \cong \mathrm{M}(\psi) \cdot r_d \cdot \delta_T .$$

Therefore, the 2-cell

$$\mathrm{M}(\psi)\, i\, \delta_T : \mathrm{M}(\psi) \cdot \mathrm{M}(p) \cdot r_a \cdot \delta_T \Rightarrow \mathrm{M}(\psi) \cdot r_d \cdot \delta_T$$

is an isomorphism. Since $\mathrm{M}(T)$ is free, $\mathrm{M}(\psi)i$ is an isomorphism. Since M is closed, i is an isomorphism. This establishes the existence of the factorization.

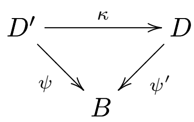
Now suppose we have two final, discrete fibration factorizations of φ.



Let q and q' denote the 1-cells $T \longrightarrow \mathrm{M}(B)$ corresponding to ψ and ψ', respectively. Since p and p' are final, by reversing the steps of the previous paragraph we have

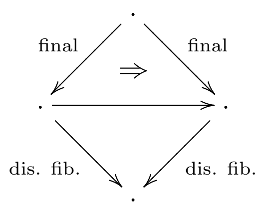
$$q \cong \Sigma_a(\delta_B \cdot \varphi) \cong q' \ .$$

Therefore, there is an equivalence of discrete fibrations over B as follows.



We obtain $p' \cong \kappa \cdot p$ from the uniqueness of the universal property of the bicomma object defining ψ'. □

Remark 6.1.7 *The comprehensive factorization associated with a closed completion KZ-monad is 2-dimensional. Indeed, the construction in 6.1.6 shows that a 2-cell between 1-cells whose domains admit an* M*-adjoint has a unique decomposition of the following kind.*



Theorem 6.1.8 *Let* M *be a locally full and faithful, closed, completion KZ-monad. Then the composite of two discrete* M*-fibrations is a discrete fibration. If a composite and its second factor are discrete* M*-fibrations, then its first factor is also.*

Proof. Throughout, we drop the prefix M from M-fibration. Let $G \xrightarrow{\psi} D$ and $D \xrightarrow{\varphi} E$ be two discrete fibrations. First, we write the bicomma object for φ as the following composite diagram.

$$\begin{array}{ccccc} D & \xrightarrow{\delta_D} & \mathrm{M}(D) & \xrightarrow{z} & T \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{M}(\varphi)} & \Rightarrow\ \swarrow{\scriptstyle p} & \\ E & \xrightarrow[\delta_E]{} & \mathrm{M}(E) & & \end{array}$$

This can be done because there is a unique 2-cell $\mathrm{M}(\varphi) \Rightarrow p \cdot z$ corresponding to the 2-cell

$$\mathrm{M}(\varphi) \cdot \delta_D \cong \delta_E \cdot \varphi \Rightarrow p \cdot d \cong p \cdot z \cdot \delta_D \ .$$

This correspondence is by 4.3.7 applied to the M-homomorphism $\mathrm{M}(\varphi)$. As in Exercise 2.5.9, 1, if the composite $\varphi \cdot \psi$ is a discrete fibration, then its corresponding point $T \longrightarrow \mathrm{M}(E)$ must be $\mathrm{M}(\varphi) \cdot q$, where $T \xrightarrow{q} \mathrm{M}(D)$ is the point of ψ. Thus we should consider the bicomma object

$$K = \delta_E \Downarrow \mathrm{M}(\varphi) \cdot q \ ,$$

and the intervening final 1-cell $G \xrightarrow{h} K$.

$$\begin{array}{ccc} K & \xrightarrow{k} & T \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle \mathrm{M}(\varphi)\cdot q} \\ & \Rightarrow & \\ E & \xrightarrow[\delta_E]{} & \mathrm{M}(E) \end{array}$$

This is the comprehensive factorization of $\varphi \cdot \psi$. Our assumption that M is closed allows us to conclude that h is final. Now regard the bicomma object for φ. Since there is a 2-cell

$$\delta_E \cdot \kappa \Rightarrow \mathrm{M}\varphi \cdot q \cdot k \Rightarrow p \cdot z \cdot q \cdot k \cong p \cdot k \ ,$$

we can factor κ through φ by a 1-cell $K \xrightarrow{\gamma} D$. By the universal property of this bicomma object we also have $\gamma \cdot h \cong \psi$. Using this isomorphism we obtain a 2-cell

$$\delta_D \cdot \gamma \cdot h \cong \delta_D \cdot \psi \Rightarrow q \cdot g \cong q \cdot k \cdot h \ ,$$

which by Lemma 6.1.2 corresponds to a 2-cell

$$\delta_D \cdot \gamma \Rightarrow q \cdot k \ .$$

The bicomma object for ψ now gives a 1-cell $K \xrightarrow{j} G$ such that $\psi \cdot j \cong \gamma$, and it is a routine matter to show that h is an equivalence with pseudo-inverse j. This shows that $\varphi \cdot \psi$ is a discrete fibration, witnessed by $\mathrm{M}(\varphi) \cdot q$.

The second assertion of the theorem is a formal consequence of the first and the comprehensive factorization. □

Proposition 6.1.9 *The symmetric monad in* $\boldsymbol{Top}_{\mathscr{S}}$ *is closed.*

Proof. Let $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ be a spread with locally connected domain. We shall show that $\mathrm{M}(\psi)$ reflects isomorphisms. Let

$$t : p \Rightarrow q : \mathscr{X} \longrightarrow \mathrm{M}(\mathscr{F})$$

be a 2-cell such that $\mathrm{M}(\psi) \cdot t$ is an isomorphism. Equivalently, t is a natural transformation between ($\mathscr{X}$-valued) distributions

$$t : p^* \cdot \delta_! \Rightarrow q^* \cdot \delta_! \ ,$$

such that $t \cdot \psi^*$ is a natural isomorphism. Let $\{X_i\}$ be a generating family for $\mathscr{E}$ over $\mathscr{S}$. Then every component morphism $t_{\psi^* X_i}$ is an isomorphism. Since distributions preserve coproducts, it follows that for any component α of any $\psi^* X_i$, t_α is an isomorphism. We know from § 3.3 that the α generate $\mathscr{F}$ over $\mathscr{S}$, so that t must be an isomorphism. ▫

Remark 6.1.10 *Thus, the symmetric monad is closed (Proposition 6.1.9). An example of a different nature of a closed completion KZ-monad is the identity monad* Id *in a 2-category with bicomma objects. Consider the* lift monad $\mathbf{L}$ *associated with a domain structure* $(\mathbb{C}, \mathbb{D})$ *in the sense of M. Fiore. (The functor* $\mathbf{L}$ *is the right adjoint of the inclusion of the category* $\mathbb{C}$ *of "total maps" into the category* $\mathrm{p}(\mathbb{C}, \mathbb{D})$ *of "partial maps.") Algebras for* $\mathbf{L}$ *are "pointed objects," and their homomorphisms are "strict maps." Fiore's axiom states that in* $\mathbb{C}$ *every morphism with pointed domain factors as a strict map followed by an upper-closed monomorphism with pointed domain. We claim that this axiom can in fact be derived in the context of KZ-monads in view of the following remarks. An object of* $\mathbb{C}$ *has an* Id*-adjoint in* $\mathbb{C}^{\mathrm{op}}$ *iff its unique map to the terminal has a left adjoint iff it has a bottom, thus, it is pointed iff it is an algebra for the associated lift monad. The discrete* Id*-fibrations in* $\mathbb{C}^{\mathrm{op}}$ *are the principal upper-closed monomorphisms. A map between pointed objects is final for* Id *iff it preserves the bottom, in other words, iff it is strict. We can now apply the comprehension factorization in the context of closed KZ-monads to prove our claim.*

6.2 The Gleason Core and Density Axioms

The existence of a locally connected coreflection (or to use Lawvere's term 'Gleason core') was proved by Gleason (1963) for spaces, and by Funk (1999) for Grothendieck toposes. Although we know of no geometric morphism $\mathscr{E} \longrightarrow \mathscr{S}$ for which the Gleason core $\hat{\mathscr{E}} \longrightarrow \mathscr{E}$ does not exist, we do not know either whether the Gleason core always exists. Therefore, the assumption that every bounded topos over an arbitrary base topos $\mathscr{S}$ have a Gleason core may restrict the generality of our results. On the other hand, we do wish to retain the generality of $\mathscr{S}$ for other reasons. For instance, in Chapter 9 we consider $\mathscr{E}$-valued distributions over $\mathscr{E}$ in order to introduce a notion of index of a complete spread.

Throughout, M denotes a completion KZ-monad in a 2-category $\mathscr{K}$.

Definition 6.2.1 *An object X of $\mathscr{K}$ is said to admit a* Gleason core *relative to* M *if there exists a 1-cell $\hat{X} \xrightarrow{\varepsilon_X} X$ in $\mathscr{K}$, such that $\hat{X} \longrightarrow T$ has an* M*-adjoint, universal with this property. In other words, any 1-cell $Y \longrightarrow X$ in $\mathscr{K}$, such that $Y \longrightarrow T$ admits an* M*-adjoint, factors uniquely (up to 2-isomorphism) through $\hat{X} \xrightarrow{\varepsilon_X} X$.*

Proposition 6.2.2 *For an object X of $\mathscr{K}$, the Gleason core axiom relative to* M *holds for X iff the category $\mathscr{K}(T, \mathrm{M}(X))$ has a terminal object, denoted $\top$. The Gleason core is always a discrete* M*-fibration. Explicitly, the Gleason core of such an object X may be constructed by means of the bicomma object*

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{x}} & T \\ {\scriptstyle \varepsilon_X}\downarrow & \Rightarrow & \downarrow{\scriptstyle \top} \\ X & \xrightarrow[\delta_X]{} & \mathrm{M}(X) \end{array}$$

in $\mathscr{K}$.

Proof. The category $\mathscr{K}(T, \mathrm{M}(X)) = \mathscr{K}_M(T, X)$ is equivalent to the category of discrete M-fibrations over X, where the equivalence is given by the above bicomma object construction (Cor. 5.1.5). If $\mathscr{K}(T, \mathrm{M}(X))$ has a terminal, then $\hat{x}$ in the bicomma object admits an M-adjoint and ε_X is the Gleason core. On the other hand, if X has a Gleason core, then it is necessarily the bicomma of the left extension of $\delta_X \cdot \varepsilon_X$ along $\hat{x}$, denoted $\top$. In other words, the Gleason core must be a discrete M-fibration, and $\top$ is indeed the terminal object of $\mathscr{K}(T, \mathrm{M}(X))$. □

Proposition 6.2.3 *Let* M *be a locally full and faithful, closed, completion KZ-monad on a 2-category $\mathscr{K}$. Then for each object X of $\mathscr{K}$ for which the Gleason core axiom holds, composition with the discrete* M*-fibration $\hat{X} \xrightarrow{\varepsilon_X} X$ induces an equivalence between the categories of discrete* M*-fibrations over X and over $\hat{X}$.*

Proof. This result is a consequence of Theorem 6.1.8, the universal property of the bicomma objects defining discrete M-fibrations over X, and of the equivalence between $\mathscr{K}(T, \mathrm{M}(X))$ and discrete M-fibrations over X. □

Remark 6.2.4

1. *By Proposition 6.2.2 for the symmetric monad* M *in* $\boldsymbol{Top}_{\mathscr{S}}$*, the Gleason core axiom for an object $\mathscr{E}$ of* $\boldsymbol{Top}_{\mathscr{S}}$ *is equivalent to the existence of a terminal distribution on $\mathscr{E}$. We have* $\mathrm{M}(\mathscr{E}) \simeq \mathrm{M}(\hat{\mathscr{E}})$.

2. *We have* $\mathbf{E}(\mathscr{E}) \simeq \mathbf{E}(\hat{\mathscr{E}})$*, when* $\mathscr{E}$ *has a Gleason core. Thus, distributions on such an* $\mathscr{E}$ *are supported on its Gleason core.*
3. *There exist non-trivial Grothendieck toposes whose Gleason cores are trivial. For such a Grothendieck topos, no non-zero distributions exist - in particular, the terminal distribution agrees with the zero distribution.*

Proposition 6.2.5 *Assume that* M *is a completion KZ-monad. Then for any object* B*, the identity* 1_B *is a discrete* M*-fibration iff* $B \longrightarrow T$ *admits an* M*-adjoint. In this case, the 1-cell* $T \longrightarrow \mathrm{M}(B)$ *that corresponds to* 1_B *is* $r_b \cdot \delta_T$*. This 1-cell is the terminal object in* $\mathscr{K}(T, \mathrm{M}(B))$*.*

Proof. If 1_B is a discrete fibration, then by definition $B \longrightarrow T$ admits an M-adjoint. If $B \longrightarrow T$ admits an M-adjoint, then

$$\begin{array}{ccc} B & \xrightarrow{\ b\ } & T \\ {\scriptstyle 1}\downarrow & \Rightarrow & \downarrow{\scriptstyle r_b \cdot \delta_T} \\ B & \xrightarrow[\ \delta_B\ ]{} & \mathrm{M}(B) \end{array}$$

is a bicomma object. □

Assume that B has the property described in Proposition 6.2.5. Let us denote the left extension $\Sigma_b(\delta_B) = r_b \cdot \delta_T$ by $\top$. If $X \xrightarrow{f} B$ is any discrete opfibration, corresponding to $\Sigma_f(\delta_T \cdot x)$, then taking $p = \top$ in (5.1) gives

$$f.\top = \mathrm{M}(f) \cdot r_f \cdot \top \cong \mathrm{M}(f) \cdot r_f \cdot r_b \cdot \delta_T \cong \mathrm{M}(f) \cdot r_x \cdot \delta_T = \Sigma_x(\delta_B \cdot f) \ .$$

Thus, this special case of the action provides a functor

$$\Phi_B : \mathscr{K}_{\mathrm{M}}(B, T) \longrightarrow \mathscr{K}_{\mathrm{M}}(T, B) \ ; \ f \mapsto f.\top \qquad (6.1)$$

given by 'inversion':

$$\Sigma_f(\delta_T \cdot x) \mapsto \Sigma_x(\delta_B \cdot f) \ .$$

Example 6.2.6 *For any topos* $\mathscr{E}$*,* $\mathrm{id}_{\mathscr{E}}$ *is a discrete fibration for the symmetric monad iff* $\mathscr{E}$ *is a locally connected topos. The inversion formula is another way of describing the Lawvere action* $F \mapsto F.\pi_0$*, such that* $F.\pi_0(X) = \pi_0(F \times X)$*. Distributions of the form* $F.\pi_0$ *are what Lawvere has termed* absolutely continuous (relative to π_0)*.*

If B has a Gleason core, corresponding to a terminal 1-cell $\top : T \longrightarrow \mathrm{M}(B)$, then the functor $\Phi_B(f) = f.\top$ (6.1) still makes sense.

Definition 6.2.7 *Suppose that* B *has a Gleason core. We say that* B admits a density *(for* M*) if the inversion functor* Φ_B *has a right adjoint. We call this right adjoint the* density functor *for* B*, denoted* $\boldsymbol{d}_B$*. This defines a monad in* $\mathscr{K}_{\mathrm{M}}(B, T)$ *that we call* the density monad associated with M*.*

Proposition 6.2.8 *A locally connected topos admits a density for the symmetric monad.*

Proof. Let $\langle \mathbb{C}, J \rangle$ denote a locally connected site, with sheaves $Sh(\mathbb{C}, J)$. We have $\pi_0 \dashv \Delta$. Define

$$\mathbf{d}(\mu)(c) = \{\text{nat. trans. } h_c.\pi_0 \longrightarrow \mu\} .$$

Then $\mathbf{d}(\mu)$ is a sheaf, and $\mathbf{d}$ is right adjoint to inversion (Example 6.2.6). □

Remark 6.2.9

1. *Consider again the specific case of the symmetric monad in the 2-category* $\boldsymbol{Top}_{\mathscr{S}}$*. Then* $\boldsymbol{Top}_{\mathscr{S}}(\mathscr{E}, \mathrm{M}(\mathscr{S}))$ *is equivalent to the topos frame* $\mathscr{E}$*, and* $\boldsymbol{Top}_{\mathscr{S}}(\mathscr{S}, \mathrm{M}(\mathscr{E}))$ *is equivalent to the distribution category* $\mathbf{E}(\mathscr{E})$*. Thus, for a locally connected topos* $\mathscr{E}$*, the adjoint pair* $\Phi_{\mathscr{E}} \dashv \boldsymbol{d}_{\mathscr{E}} = \boldsymbol{d}$ *connects* $\mathscr{E}$ *with* $\mathbf{E}(\mathscr{E})$*, where*
$$\Phi_{\mathscr{E}} : \mathscr{E} \longrightarrow \mathbf{E}(\mathscr{E}) ,$$
associates with an object X *of* $\mathscr{E}$ *the distribution* $X.e_!$*, such that*
$$(X.e_!)(Y) = e_!(X \times Y) .$$
2. *The density of a distribution on a locally connected topos* $\mathscr{E}$ *coincides with the object of* $\mathscr{E}$*-points of the corresponding (localic) complete spread. Thus, we may generalize the density to any topos:*
$$\boldsymbol{d}(\mu) = \text{object of } \mathscr{E}\text{-points of the complete spread of } \mu .$$
If $\mathscr{E}$ *has a Gleason core, so that* $\mathbf{E}(\mathscr{E})$ *has a terminal object* $\mathbf{1}$*, then the above* $\boldsymbol{d}$ *is given by*
$$\boldsymbol{d}(\mu) = \mathrm{Hom}(\mathbf{1}, \mu) ,$$
and moreover $\boldsymbol{d}$ *has a left adjoint in this case:*
$$(\_).\mathbf{1} : \mathscr{E} \longrightarrow \mathbf{E}(\mathscr{E}) .$$
3. *Intuitively, the density monad is in some ways similar to the* regularization *monad in the lattice of open sets of a topological space, which associates with an open set the interior of its closure. But consider the example of the real numbers* R*. Regularization in* $\mathscr{O}(R)$ *is non-trivial, but the density monad in* $\mathrm{Sh}(R)$ *is the identity monad.*

Exercises 6.2.10

1. *Prove Proposition 5.1.3.*

2. *Suppose that $\mathscr{K}$ has bipullbacks, and that* M *is a locally full and faithful completion KZ-monad (as always). Show that the action $f.\psi$ of discrete opfibrations on discrete fibrations can be equivalently described:*
$$f.\psi = \Sigma_k(\delta_B \cdot f \cdot \pi_1)\ ,$$
where
$$\begin{array}{ccc} K & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow[f]{} & B \end{array}$$
is a bipullback.
3. *Investigate what are the algebras for the density monad in a locally connected topos.*
4. *Fill in the details of the proof of Proposition 6.2.3.*

6.3 The Twisted Single Universe

We define a category, *the twist category*, whose objects are twisted maps between M-bifibrations.

Definition 6.3.1 *For objects A and B in $\mathscr{K}$, denote by* $\mathrm{Tw}(A,B)$ *the following category. An* object *is a 3-tuple $(yE\psi, t, xD\varphi)$ where the spans $xD\varphi : A \longrightarrow B$ and $yE\psi : B \longrightarrow A$ are* M*-bifibrations, and $E \xrightarrow{t} D$ is a 1-cell for which the following diagram commutes (up to 2-isomorphism).*
$$\begin{array}{ccc} B & \xleftarrow{y} & E \\ {\scriptstyle \varphi}\uparrow & \swarrow{\scriptstyle t} & \downarrow{\scriptstyle \psi} \\ D & \xrightarrow[x]{} & \top \end{array}$$
We shall denote such an object more simply by (E,t,D). A morphism
$$(E,t,D) \longrightarrow (E',t',D')$$
is a pair (α,β) such that $\alpha : E \longrightarrow E'$ is a 1-cell of bifibrations $B \longrightarrow A$, $\beta : D \longrightarrow D'$ is a 1-cell of bifibrations $A \longrightarrow B$, such that there is a commutative diagram
$$\begin{array}{ccc} E & \xrightarrow{\alpha} & E' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ D & \xrightarrow[\beta]{} & D' \end{array}$$
up to 2-isomorphism.

Under the assumption that $\mathscr{K}$ has a terminal object $\top$, we denote the category $\mathrm{Tw}(\top,B)$ *just by* $\mathrm{Tw}(B)$, *and call it* the twist category of B.

Assume now that M is a closed completion KZ-monad in a 2-category $\mathscr{K}$, and that B is an object of $\mathscr{K}$ that has a Gleason core relative to M. Using the existence of the density functor, we may now explicitate $\mathrm{Tw}(B)$ as a *single universe* for discrete M-fibrations and discrete M-opfibrations over B.

Theorem 6.3.2 *Let* M *be a closed completion KZ-monad in* $\mathscr{K}$ *and let* B *be any object such that* $B \xrightarrow{b} T$ *admits an* M*-adjoint. Assume also that* B *admits a density* $\boldsymbol{d}$*. Then* $\mathrm{Tw}(B)$ *is equivalent to the category* $\mathscr{K}_{\mathrm{M}}(B, \top) \downarrow \boldsymbol{d}$*, obtained by Artin glueing along* $\boldsymbol{d}$*. The glueing category* $\mathscr{K}_{\mathrm{M}}(B, \top) \downarrow \boldsymbol{d}$ *contains as full subcategories both* $\mathscr{K}_{\mathrm{M}}(B, \top)$ *and* $\mathscr{K}_{\mathrm{M}}(\top, B)$*.*

Proof. An object of $\mathrm{Tw}(B)$ is a 3-tuple (E, t, D) (Definition 6.3.1); such a t may be regarded as a morphism

$$t : \Sigma_\psi(\delta_B \cdot y) \longrightarrow \Sigma_x(\delta_B \cdot \varphi) \ .$$

Equivalently, for $q = \Sigma_y(\delta_\top \cdot \psi)$ and $r = \Sigma_x(\delta_B \cdot \varphi)$, we have $t : \Phi_B(q) \longrightarrow r$, which is given, suggestively, as

$$\Sigma_{x \cdot t}(\delta_B \cdot \varphi \cdot t) \longrightarrow \Sigma_x(\delta_B \cdot \varphi).$$

By the adjointness $\Phi_B \dashv \mathbf{d}_B$, this t corresponds uniquely to a morphism $\hat{t} : q \longrightarrow \mathbf{d}(r)$, i.e., to an object of the glueing category $\mathscr{K}_{\mathrm{M}}(B, \top) \downarrow \mathbf{d}$. This process is functorial and reversible, giving the desired equivalence. It follows from that the glueing category contains both full subcategories as claimed. □

Exercises 6.3.3

1. *Show that if* $X \xrightarrow{f} B$ *is a discrete opfibration, and* B *admits a Gleason core, then so does* X*.*
2. *Generalize Theorem 6.3.2, by assuming only that* B *has a Gleason core and a density.*
3. *The work of Carboni and Johnstone [CJ95] shows that the twist category* $\mathrm{Tw}(P(\mathbb{C}))$ *is again a presheaf topos, say* $P(\mathbb{K})$*. Give an explicit description of* $\mathbb{K}$ *as a full subcategory of the category* $\mathrm{Tw}(P(\mathbb{C}))$*, using the definition of the latter. In particular, identify* $\mathbb{K}$ *with the* collage *of* $\mathbb{C}$*.*
4. *The category* $\mathrm{ULF}/\mathbb{C}$ *of* unique lifting of factorizations*, also known as the category of discrete Giraud-Conduché fibrations over* $\mathbb{C}$*, is not a topos in general, but it is a topos for any* $\mathbb{C}$ *that is "paths linearizable" [BN00, BF00]. This too is a single universe for local homeomorphisms and complete spreads, as it can be generalized to toposes by the familiar amalgamation construction. Produce a comparison map between the two "single universes"* $\mathrm{Tw}(P(\mathbb{C}))$ *and* $\mathrm{ULF}/\mathbb{C}$ *and study its properties. Notice that, unlike* $\mathrm{ULF}/\mathbb{C}$*, the category* $\mathrm{Tw}(P(\mathbb{C}))$ *is a topos for any* $\mathbb{C}$*.*
5. *Show how the action of discrete fibrations on discrete opfibrations may be naturally described in* $\mathrm{ULF}/\mathbb{C}$ *using the comprehensive factorization of functors.*

6.4 Linear KZ-Monads

Distributions have a basic additive property: two distributions

$$\mathscr{E} \xrightarrow{\mu} \mathscr{X} \ , \ \mathscr{F} \xrightarrow{\lambda} \mathscr{X}$$

may be paired into a single distribution $\langle \mu, \lambda \rangle$ on the coproduct $\mathscr{E} +_{\mathscr{S}} \mathscr{F}$, such that

$$\langle \mu, \lambda \rangle (E, F) = \mu(E) + \lambda(F) \ .$$

We remind the reader that the topos frame of $\mathscr{E} +_{\mathscr{S}} \mathscr{F}$ is the underlying category product $\mathscr{E} \times \mathscr{F}$ (of course not to be confused with the topos product $\mathscr{E} \times_{\mathscr{S}} \mathscr{F}$). The functor $\langle \mu, \lambda \rangle$ just defined is indeed a distribution because $\langle \mu, \lambda \rangle \dashv \langle \mu_*, \lambda_* \rangle$. It should be noted that a coproduct inclusion $\mathscr{E} \xrightarrow{\iota} \mathscr{E} +_{\mathscr{S}} \mathscr{F}$ is locally connected such that $\iota_!(E) = (E, 0)$. We recover μ by composing with $\iota_!$ (and similarly λ) thereby establishing an equivalence between such pairs of distributions and distributions on $\mathscr{E} +_{\mathscr{S}} \mathscr{F}$. Furthermore, a distribution on the coproduct is isomorphic to the pairing of its restriction to the summands, because the two distributions have the same right adjoint.

Since the symmetric monad classifies distributions we immediately conclude that there is a canonical equivalence

$$\langle r_{\mathscr{E}}, r_{\mathscr{F}} \rangle : \mathrm{M}(\mathscr{E} +_{\mathscr{S}} \mathscr{F}) \ \simeq \ \mathrm{M}(\mathscr{E}) \times_{\mathscr{S}} \mathrm{M}(\mathscr{F})$$

of toposes.

We are indicating here that the equivalence is given explicitly by pairing right adjoints $r_{\mathscr{E}}$ and $r_{\mathscr{F}}$, where $\mathrm{M}(\iota_{\mathscr{E}}) \dashv r_{\mathscr{E}}$ (geometric morphisms). Finally, we remark that since the coproduct inclusion ι is indeed an inclusion in the sense of geometric morphisms, we conclude that $r_{\mathscr{E}} \cdot \mathrm{M}(\iota) \cong id_{M(\mathscr{E})}$. We say that ι has *a coreflection* M-*adjoint*.

The additivity of the symmetric monad is reflected in the fact that its discrete fibrations, the complete spreads, may be summed. To be more precise, the '$\mathbf{Top}_{\mathscr{S}}$ is extensive' equivalence

$$\mathbf{Top}_{\mathscr{S}}/(\mathscr{E} +_{\mathscr{S}} \mathscr{F}) \ \simeq \ \mathbf{Top}_{\mathscr{S}}/\mathscr{E} \times \mathbf{Top}_{\mathscr{S}}/\mathscr{F}$$

restricts to complete spreads. This is intuitively plausible, but to prove it we must invoke the comprehensive factorization. This suggests and we prove that a similar result holds in the generic context (Theorem 6.4.6).

We may now posit what we mean by an additive KZ-monad in the generic context. It makes sense to study additivity in an extensive 2-category $\mathscr{K}$. Throughout, f^* denotes pullback along a 1-cell f in $\mathscr{K}$.

Definition 6.4.1 *A KZ-monad* M *in an extensive 2-category* $\mathscr{K}$ *is said to be* additive*, or to satisfy the* exponential law*, if*

1. *for any two objects* X *and* Y *of* $\mathscr{K}$*, the coproduct injections* $X \xrightarrow{\iota_X} X + Y$ *and* $Y \xrightarrow{\iota_Y} X + Y$ *have coreflection* M*-adjoints* r_X *and* r_Y*, and*

2. *'$\mathrm{M}(X+Y) \simeq \mathrm{M}(X) \times \mathrm{M}(Y)$': for any coproduct diagram in $\mathscr{K}$, below left,*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle \iota_X} \\ Y & \xrightarrow{\iota_Y} & Z \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{M}(Z) & \xrightarrow{r_X} & \mathrm{M}(X) \\ {\scriptstyle r_Y}\downarrow & & \\ \mathrm{M}(Y) & & \end{array}$$

the right-hand diagram is a product diagram.

Remark 6.4.2 *If* M *is additive, then for any two objects X and Y of $\mathscr{K}$, the functor*

$$\mathrm{M}(\iota_X)^* : \mathscr{K}/\mathrm{M}(X+Y) \longrightarrow \mathscr{K}/\mathrm{M}(X)$$

is just composition with r_X. This is so because we are assuming that r_X is a coreflection: $r_X \cdot \mathrm{M}(\iota_X) \cong \mathrm{id}_{M(X)}$.

Lemma 6.4.3 *Let $\mathscr{K}$ be an extensive 2-category. Let $G_0 \xrightarrow{p_0} D_0$ and $G_1 \xrightarrow{p_1} D_1$ be two 1-cells in $\mathscr{K}$. Then the sum*

$$G_0 + G_1 \xrightarrow{p_0+p_1} D_0 + D_1$$

is M*-final iff each of the 1-cells p_0, p_1 is* M*-final.*

Proof. We may establish this using the characterization of M-final 1-cells given in Lemma 6.1.2. □

It is our aim in this section to introduce a notion of linear KZ-monad in an extensive 2-category $\mathscr{K}$ with pullbacks. Let K be an object of a 2-category $\mathscr{K}$. Let $K_! : \mathscr{K}/K \longrightarrow \mathscr{K}$ denote composition with $K \longrightarrow T$, where T denotes the pseudo-terminal object in $\mathscr{K}$. We have $K_! \dashv K^*$.

Definition 6.4.4 *A KZ-monad* M *in $\mathscr{K}$ is said to be $\mathscr{K}$-*equivariant*, or just* equivariant*, if it is given by the following data and conditions:*

1. *For every object K, a KZ-monad*

$$(\mathrm{M}^K, \delta^K, \mu^K)$$

in $\mathscr{K}/K$. The case $K = T$ gives a KZ-monad $(\mathrm{M}, \delta, \mu)$ in $\mathscr{K}$.

2. *We have '$\mathrm{M}^K(K \times X) \simeq K \times \mathrm{M}(K)$'. Precisely, we require a connecting pseudo-natural transformation*

$$\rho : K_! \circ \mathrm{M}^K \longrightarrow \mathrm{M} \circ K_!$$

such that for any object X of $\mathscr{K}$ the right hand square below is a pullback and the left hand one commutes. The left hand square is therefore a pullback.

$$\begin{array}{ccccc} K \times X & \xrightarrow{\delta^K_{K^*X}} & \mathrm{M}^K(K \times X) & \xrightarrow{\pi_1} & K \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_X} & & \downarrow \\ X & \xrightarrow[\delta_X]{} & \mathrm{M}(X) & \longrightarrow & T \end{array}$$

We systematically abuse the notation slightly. For instance, $\mathrm{M}^K(K \times X) \xrightarrow{\pi_1} K$ *means*

$$\mathrm{M}^K \left(\begin{array}{c} K \times X \\ {\scriptstyle \pi_1}\downarrow \\ K \end{array} \right)$$

as an object of $\mathscr{K}/K$. π_X *denotes the composite 1-cell*

$$\mathrm{M}^K(K \times X) \xrightarrow{\rho_{K^*X}} \mathrm{M}(K \times X) \xrightarrow{\mathrm{M}(\pi_2)} \mathrm{M}(X) \; .$$

3. *The* ρ*'s commute with the multiplications* μ^K *and* μ*: for any object* $X \longrightarrow K$ *of* $\mathscr{K}/K$*, the diagram*

$$\begin{array}{ccc} (\mathrm{M}^K)^2(X) & \xrightarrow{\mathrm{M}(\rho_X)\cdot\rho_{\mathrm{M}^K X}} & \mathrm{M}^2(X) \\ {\scriptstyle \mu^K_X}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ \mathrm{M}^K(X) & \xrightarrow[\rho_X]{} & \mathrm{M}(X) \end{array}$$

   commutes in $\mathscr{K}$.
4. *Finally, we require that if a 1-cell* $A \xrightarrow{q} Y$ *over* K *admits an* M^K*-adjoint* $\mathrm{M}^K(q) \dashv r^K_q$*, then it admits an* M*-adjoint* $\mathrm{M}(q) \dashv r_q$ *in* $\mathscr{K}$*, and the canonical 2-cell*

$$\rho_A \cdot r^K_q \Rightarrow r_q \cdot \rho_Y$$

   is an isomorphism.

Definition 6.4.5 *A* linear *KZ-monad in an extensive 2-category* $\mathscr{K}$ *with pullbacks is one that is both additive (Definition 6.4.1) and* $\mathscr{K}$*-equivariant (Definition 6.4.4).*

Theorem 6.4.6 *Let* M *be a KZ-monad in an extensive 2-category with pullbacks, and assume that coproduct injections have coreflection* M*-adjoints. If* M *is closed, completion, and equivariant, then* M *is additive.*

Proof. Since $\mathscr{K}$ is extensive, the functor

$$\Phi : \mathscr{K}/X \times \mathscr{K}/Y \longrightarrow \mathscr{K}/(X+Y) \; ,$$

given by coproduct, is an equivalence with pseudoinverse

$$\Psi : \mathscr{K}/(X+Y) \longrightarrow \mathscr{K}/X \times \mathscr{K}/Y$$

given by bipullback along the coproduct injections. For a completion KZ-monad M, there is for each object X a full and faithful functor

$$\mathscr{K}(T, \mathrm{M}(X)) \longrightarrow \mathscr{K}/X \ , \tag{6.2}$$

via a bicomma object constuction equating $\mathscr{K}(T, \mathrm{M}(X))$ with the full subcategory of $\mathscr{K}/X$ whose objects are the discrete M-fibrations (whose domain admits an M-adjoint). Furthermore, Ψ restricts to

$$\langle r_X, r_Y \rangle : \mathrm{M}(X+Y) \longrightarrow \mathrm{M}(X) \times \mathrm{M}(Y) \ ,$$

because discrete fibrations are pullback stable along 1-cells with M-adjoints.

We now claim that Φ restricts to a functor

$$\mathrm{M}(X) \times \mathrm{M}(Y) \longrightarrow \mathrm{M}(X+Y) \ .$$

This is the case iff the coproduct $A + B \xrightarrow{\varphi+\psi} X + Y$ of two discrete M-fibrations $A \xrightarrow{\varphi} X$ and $B \xrightarrow{\psi} Y$ is again a discrete M-fibration. We shall prove that this is so when M is closed.

Note that if $A \longrightarrow T$ and $B \longrightarrow T$ have M-adjoints, then $(A+B) \longrightarrow T$ has an M-adjoint. Consider the comprehensive factorization of $A + B \xrightarrow{\varphi+\psi} X + Y$. Observe that the M-final factor ρ is the sum of two 1-cells ρ_0 and ρ_1, each of which must be M-final, by Lemma 6.4.3. Our assumption that the given 1-cells φ and ψ are discrete M-fibrations implies that ρ_0 and ρ_1 are both isomorphisms, hence so is their sum ρ.

Finally, when M is equivariant (6.2) holds in any slice $\mathscr{K}/K$, so that we may essentially repeat the above argument for 'generalized points' $K \longrightarrow \mathrm{M}(X)$. □

Example 6.4.7

1. *We saw at the beginning of this section that the symmetric monad is additive in* $\boldsymbol{Top}_{\mathscr{S}}$. *It is also equivariant. For instance, the forward preservation of* M*-adjoints (requirement 4) amounts to the observation that for any geometric morphism* $\mathscr{T} \longrightarrow \mathscr{S}$*, a* $\mathscr{T}$*-essential geometric morphism over* $\mathscr{T}$ *is* $\mathscr{S}$*-essential as a geometric morphism over* $\mathscr{S}$. *The pullback equivalence* $\mathrm{M}^K(K \times X) \simeq K \times \mathrm{M}(X)$ *for toposes is in a new guise another important fact about distributions and change of base discovered by A. M. Pitts. This fact says that the topos pullback is universal for distributions, not just left exact distributions.*
2. *The lower bagdomain* B_L *and probability distributions classifier* T *are both closed, completion and equivariant KZ-monads in* $\boldsymbol{Top}_{\mathscr{S}}$. B_L *is additive since* $\langle \mu, \lambda \rangle$ *preserves pullbacks if both* μ *and* λ *do; however,* T *is not additive because* $\langle \mu, \lambda \rangle$ *will not preserve* 1 *if* μ *and* λ *do. This is consistent with Theorem 6.4.6, because coproduct inclusions in* $\boldsymbol{Top}_{\mathscr{S}}$ *have (coreflection)* B_L*-adjoints, but they do not have* T*-adjoints.*

Remark 6.4.8 *The present exposition of additive and equivariant KZ-monads is only the beginning of an interesting subject. For instance, a simpler characterization of completion KZ-monads is available in the $\mathscr{K}$-equivariant case. This leads to a slightly different characterization of the algebras, which holds for the symmetric monad, resulting in another Waelbroeck theorem with locally connected geometric morphisms in place of essential ones, and pullbacks in place of bicomma objects.*

Exercises 6.4.9

1. *Prove that the homomorphisms for the algebras in a linear KZ-monad* M *in an extensive 2-category* $\mathscr{K}$ *are linear in the usual sense, meaning that they preserve addition and scalar multiplication.*

6.5 Pitts' Theorem Revisited

We shall now explain Pitts' theorem (herein Theorem 4.3.1) in terms of a linear KZ-monad. Although this is not a direct explanation, we feel that it is worthwhile. For instance, we gain new information about the geometric morphism opposite the upper one in a topos bicomma object in which the lower geometric morphism is essential.

The following lemma is at the heart of the bicomma object construction in the presence of an equivariant KZ-monad: it shows how to construct a bicomma object by localizing.

Lemma 6.5.1 *Under the above notation, let* M *be a locally full and faithful, 'closed, equivariant KZ-monad in* $\mathscr{K}$*. Suppose that a 1-cell* $X \xrightarrow{\varphi} K$ *admits an* M*-adjoint, and let* n *denote the pairing* $(1_K, r_\varphi \cdot \delta_K)$*, where* $\mathrm{M}(\varphi) \dashv r_\varphi$*. Suppose that*

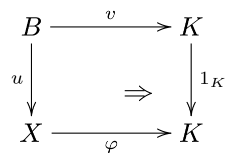
$$\begin{array}{ccc} A & \xrightarrow{\quad q \quad} & K \\ {\scriptstyle p}\downarrow & \Rightarrow & \downarrow{\scriptstyle n=(1_K, r_\varphi \cdot \delta_K)} \\ K \times X & \xrightarrow[\delta^K_{K^*X}]{} & \mathrm{M}^K(K \times X) \end{array}$$

is a bicomma object in $\mathscr{K}/K$ *such that* q *admits an* M^K*-adjoint and the BCC holds for* M^K*:* $\mathrm{M}^K(p) \cdot r^K_q \cong \mu^K_{K^*X} \cdot \mathrm{M}^K(n)$*. Then*

$$\begin{array}{ccc} A & \xrightarrow{\quad q \quad} & K \\ {\scriptstyle \pi_2 \cdot p}\downarrow & \Rightarrow & \downarrow{\scriptstyle 1_K} \\ X & \xrightarrow[\varphi]{\quad\quad} & K \end{array}$$

is a bicomma object in $\mathscr{K}$ *and the BCC holds for* M *in* $\mathscr{K}$ (by equivariance, q admits an M-adjoint)*:* $r_\varphi \cong \mathrm{M}(\pi_2 \cdot p) \cdot r_q$*.*

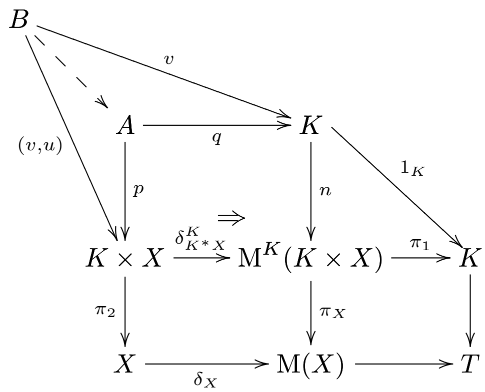
Proof. Since M is locally full and faithful, 2-cells in a diagram



are in bijection with 2-cells

$$\frac{\mathrm{M}(\varphi)\cdot\delta_X\cdot u \cong \delta_K\cdot\varphi\cdot u \Rightarrow \delta_K\cdot v}{\delta_X\cdot u \Rightarrow r_\varphi\cdot\delta_K\cdot v\ .}$$

Now chase through the following diagram:



where $\pi_X\cdot n = r_\varphi\cdot\delta_K$.

As for the BCC for M in $\mathscr{K}$, it suffices to show

$$r_\varphi\cdot\delta_K \cong \mathrm{M}(\pi_2\cdot p)\cdot r_q\cdot\delta_K\ ,$$

since both 1-cells in the BCC equation are M-homomorphisms. We have

$$\begin{aligned}\mathrm{M}(\pi_2\cdot p)\cdot r_q\cdot\delta_K &\cong \pi_X\cdot\mathrm{M}^K(p)\cdot r^K_q\cdot\delta^K_K\\ &\cong \pi_X\cdot\mu^K_{K*X}\cdot\mathrm{M}^K(n)\cdot\delta^K_K\\ &\cong \pi_X\cdot\mu^K_{K*X}\cdot\delta^K_{\mathrm{M}^K(K*X)}\cdot n\\ &\cong \pi_X\cdot n\\ &\cong r_\varphi\cdot\delta_K\ .\end{aligned}$$

□

Remark 6.5.2 *The 1-cell p in the hypothesis of Lemma 6.5.1 is a discrete* M^K*-fibration.*

We turn to the existence of topos bicomma objects, and a proof of Pitts' bicomma object theorem.

By Lemma 6.5.1, any topos bicomma object

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\;q\;} & \mathscr{T} \\ {\scriptstyle p}\Big\downarrow & \Rightarrow & \Big\downarrow{\scriptstyle \mathrm{id}_{\mathscr{T}}} \\ \mathscr{X} & \xrightarrow[\;\varphi\;]{} & \mathscr{T} \end{array} \tag{6.3}$$

in $\mathbf{Top}_{\mathscr{S}}$ in which φ is $\mathscr{S}$-essential can be obtained by localizing over $\mathscr{T}$. Indeed, the above bicomma object reduces to the bicomma object

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\;q\;} & \mathscr{T} \\ \Big\downarrow & \Rightarrow & \Big\downarrow{\scriptstyle f} \\ \mathscr{X} \times_{\mathscr{S}} \mathscr{T} & \xrightarrow[\;\delta^{\mathscr{T}}\;]{} & \mathrm{M}^{\mathscr{T}}(\mathscr{X} \times_{\mathscr{S}} \mathscr{T}) \end{array} \tag{6.4}$$

in $\mathbf{Top}_{\mathscr{T}}$ for some point f. Thus, it suffices to construct bicomma objects $\delta \Downarrow f$ of diagrams

$$\begin{array}{ccc} & & \mathscr{S} \\ & & \Big\downarrow{\scriptstyle f} \\ \mathscr{E} & \xrightarrow[\;\delta\;]{} & \mathrm{M}(\mathscr{E}) \end{array} ,$$

where now the usual $\mathscr{S}$ denotes the base topos. The geometric morphism opposite f in such a bicomma object is what we call a discrete M-fibration. We have seen in § 5.2 that such bicomma objects exist, and that by their very construction a discrete M-fibration is precisely a complete spread geometric morphism.

Remark 6.5.3

1. *The combination of bicomma objects (6.3) and the pullback stability of locally connected geometric morphisms (Exercise 2) gives Pitts' Theorem.*
2. *We gain the extra information that in (6.4)* $\mathscr{A} \xrightarrow{(p,q)} \mathscr{X} \times_{\mathscr{S}} \mathscr{T}$ *is a* $\mathscr{T}$*-complete spread.*
3. *For toposes* $\mathscr{E}$ *and* $\mathscr{G}$ *over* $\mathscr{S}$*, the functor which associates to a geometric morphism* $\mathscr{G} \xrightarrow{\rho} \mathrm{M}(\mathscr{E})$ *the complete spread* $(\gamma, \lambda) : \mathscr{Y} \longrightarrow \mathscr{G} \times_{\mathscr{S}} \mathscr{E}$ *for the bicomma object*

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\;\gamma\;} & \mathscr{G} \\ {\scriptstyle \lambda}\Big\downarrow & \Rightarrow & \Big\downarrow{\scriptstyle \rho} \\ \mathscr{E} & \xrightarrow[\;\delta\;]{} & \mathrm{M}(\mathscr{E}) \end{array}$$

is an equivalence of $\boldsymbol{Top}_{\mathscr{S}}(\mathscr{G}, \mathrm{M}(\mathscr{E}))$ *with the category of complete spreads over* $\mathscr{G} \times_{\mathscr{S}} \mathscr{E}$ *with locally connected* $\mathscr{G}$*-domain. It is reasonable to call the bifibration* (γ, λ) *a generalized complete spread over* $\mathscr{S}$*. Such a* λ *may not be localic, and* $\mathscr{Y} \longrightarrow \mathscr{S}$ *may not be locally connected, although* γ *is.*

Exercises 6.5.4

1. *Show that "admits an* M*-adjoint" is stable under bipullback.*
2. *Let* M *be* $\mathscr{K}$*-equivariant. Show that if* $X \longrightarrow T$ *admits an* M*-adjoint, then for any* K*,* $K \times X \longrightarrow K$ *admits an* M^K*-adjoint (and hence admits an* M*-adjoint) and the BCC holds for* M *and the bipullback*

$$\begin{array}{ccc} K \times X & \longrightarrow & K \\ \downarrow & & \downarrow \\ X & \longrightarrow & T \end{array}$$

   in $\mathscr{K}$*.*
3. *What is the point* f *in the bicomma object (6.3)?*

Further reading: Bunge [Bun95, Bun74, Bun04], Bunge & Funk [BF99], Bunge & Niefield [BN00], Bunge & Fiore [BF00], Carboni & Johnstone [CJ95], Johnstone [Joh99], Kock [Koc75], Street [Str, Str74], Waelbroeck [Wae67], Zöeberlein [Zoe76].