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Introduction

1.1 Dynamics

In order to investigate some physical phenomenon usually one constructs its mathematical model. The model is a system of equations which describe a process under study in mathematical terms. Equations involved in a system may be of different nature. The dependence between quantities involved in equations may be linear, i.e. this dependence is represented by a linear function, or nonlinear. Parameters may be included in equations, and in this case we have the equations with parameters. Equations may contain both functions sought for and their derivatives – differential equations. Such models are commonly known, e.g. a model of the pendulum motion, a model of the fluid motion, a model of the heat diffusion, a model of the bacteria reproduction, and other. By the process we mean the observed parameters variables which depend on the time t. Parameter values at a time t determine the state of a process. The set of process states constitutes the phase space of a system. Thus, a system of equations describing a given process is determined on the phase space.

For an example, the law of radioactive decay can be stated as: the rate of the decay at a given moment is proportional to an amount of a substance remaining at this moment. In this case the state of a process is determined by the amount of a substance. The process of bacteria reproduction under wide enough amount of a nutritive material can be stated as: the rate of population reproduction is proportional to the population size. In this case the state of a process is determined by the bacteria quantity. In the cases just discussed above, the phase space is one-dimensional and constitutes the set of positive real numbers.

Let us consider a mechanical system that describes the motion of a mass point. The state of the mass point is specified by two quantities: coordinates and velocity. In order to determine uniquely the state of the mass point one needs different number of characteristics depending on where the movement occurs. If the mass point moves along the straight line, one needs two quantities: line coordinate and velocity. Thus, the phase space is the plane $\mathbb{R}^2$ or its part. If the mass point moves in the plane, the point position is determined by its two coordinates and by two components of the velocity vector. Hence, the phase space is four-dimensional Euclidean space $\mathbb{R}^4$. Similarly, to describe the motion of a mass point in the three-dimensional space one needs six quantities that determine the point state at a given time, and the phase space is $\mathbb{R}^6$.

A system of equations governs changes in the object state that occurs with time via some law. If this law is expressed by a system of differential equations then one says that a continuous-time system is given. If equations that govern a system determine changes of the object state through a fixed time interval then the system is called a discrete-time system. A length of the time interval is determined by a problem at hand. Thus, we can became aware of the behavior of an object at hand by treating the movement of points in a phase space at given instants of time with the law of this movement governed by the system of equations.

One of the mostly known classes of systems is that describing so-called determinate processes. This means that there exists a rule in terms of a system of equations that uniquely determines the future and the past of the process on the basis of knowledge of its state at present. The systems describing radioactive decay and bacteria reproduction as well as mechanical systems of a mass point motion outlined above are determinate, i.e. the process progress is uniquely determined by initial conditions and equations. Needless to say that there exist also indeterminate systems, e.g. the process of heat propagation in a medium is semi-determinate as the future is determined by the present whereas the past is not. It is well known that the motion of particles in quantum mechanics is an indeterminate process.

It should be noted that whether or not a process is determinate can be established only experimentally, hence with a certain degree of accuracy. In the subsequent discussion we will return to this subject, but now we suppose that a mathematical model reflects closely a given physical process, i.e. the model is sufficiently accurate. In what follows we will treat both discrete and continuous dynamical systems.

A discrete system is given by a mapping (a difference equation) of the form

$$x_{n+1} = f(x_n),$$

where each subsequent system state x_{n+1} is uniquely determined by its previous state x_n and the mapping f, n can be viewed as the discrete time. Thus, the evolution of the system is governed by the sequence $\{x_n, n \in \mathbb{Z}\}$ in the phase space. A continuous dynamical system is generally given by an equation of the form

$$\frac{dx}{dt} = F(x)$$

or by a system of such equations. Let $\Phi(t, x_0)$ be a solution of the equation, where x_0 is an initial state at $t = 0$, t is viewed as the time. In this case, the system evolution is governed by the curve $\{x = \Phi(t, x_0),\ t \in \mathbb{R}\}$ in the phase space. Fundamental theorems of the differential equations theory ensure the existence of the solution Φ under some reasonable conditions posed on the mapping F, however, its explicit finding (integration of a system) is a sufficiently challenging task. Moreover, solutions of the most part of differential equations cannot be expressed in elementary functions. In practice, when solving an actual problem, Φ is often constructed numerically.

At this point of view, discrete dynamical systems are more favored for the study as the mapping f is similar to the solution Φ and the integration of a system does not complicate understanding of the system evolution. Computer modeling allows to construct easily a trajectory of the system on each finite-time interval that gives a possibility to solve many problems. If we simulate an orbit of a dynamical system for a given initial condition we reach to an attractor of this system and in general, we are not be able to locate any other objects existing in the state space. Although several coexisting attractors might be detected by variation of initial conditions, it is not possible to find unstable objects like, for instance, unstable limit cycles. In this context we need methods that studies the global structure of dynamical system rather than tracing single orbits in the state space.

The method presented approaches this task. It provides a unified framework for the acquisition of information about the system flow without any restrictions concerning the stability of specific invariant sets.

1.2 Order and Disorder

Since the behavior of the process described by a determinate system is uniquely determined by a given initial state, it is reasonable to assume that the behavior of such a system is sufficiently regular, i.e. it obeys a certain law. This mode of thought prevailed in the 19th century. However, with the advance of science our concepts on outward things have been changed. In the 20th century, theory of relativity, quantum mechanics, and theory of chaos have been created.

The theory of relativity dispelled Newton's ideas about the absolute nature of time and space. The quantum mechanics showed that many physical phenomena cannot be considered determinate. The theory of chaos proved that many determinate systems can exhibit irregularity, i.e. they obey solutions that depend on the time in an unpredictable way. One example of chaotic dependence is the decimal representation of an irrational number, where each subsequent digit may be arbitrary independently of preceding digits, i.e. being aware of the first n digits one cannot predict the next one.

The term "chaos" was likely introduced by J. Yorke in 60th. However, H. Poincaré is recognized a pioneer in the study of chaotic behavior of trajectories [117]. In 1888, H. Poincaré [116] revealed strongly unstable trajectories in the three-body problem. For this work, in 1889 he was awarded a prize of Swedish King Oscar II. More precisely, H. Poincaré proved the existence of so-called doubly asymptotic orbits in the three-body problem. Now these orbits are called homoclinic. The main property of such an orbit is that it starts and ends near the same periodic orbit. It should be noted that in this case chaotic trajectories appear in a fully determinate mechanical system that obeys Newton's laws.

In 1935, G. Birkhoff [13] applied symbolic dynamics for coding trajectories near a homoclinic orbit. The same technique was used by S. Smale [136] in construction of the so-called "horseshoe" – a simple model of the chaotic dynamics. Smale's "horseshoe" influenced very much on the theory of chaos as this example is typical and the symbolic dynamics methods turned out to be just an instrument that allows to describe the nature of chaos.

The systematic study of chaos begins in 1960, when researches perceived that even very simple nonlinear models can provide as much disorder as the most violent waterfall. Minor distinctions between initial conditions produce considerable difference in results that is called a "sensitive dependence on initial conditions". One of the pioneer investigators of chaos, E. Lorenz, called this phenomenon a "butterfly effect": trembling of the butterfly wings may cause a tornado in New York within a month. However, the majority of researches continue to hold the viewpoint of Laplace, a philosopher and mathematician of the 18th century, who reasoned that there exists formulas that describe the motion of all physical bodies and hence there is nothing indeterminate neither in the future nor in the past. They believe that by adding complexity to a mathematical model and by increasing accuracy of calculations on can achieve an absolute determinate description of a system, the chaos in a model is viewed as a weakness of the model and the work of investigator is negatively appreciated. If in the course of investigation or in the performance of experiment it emerges that instability or chaos are inherent characteristics of an object of study then this is explained by extraneous "noise", unaccounted perturbations, or bad quality of the experiment performance. It is reasonable that biologists, physiologists, economists and others desire to decompose systems investigated into "elements" and then to construct their determinate models. However, it should be remembered the following:

1) the absolute accuracy of calculations cannot be achieved;
2) the more complicated mathematical models, the greater is the dependence on initial conditions.

In addition, many of system parameters are known with a certain degree of accuracy, e.g. the acceleration of gravity. Moreover, every model describes a real system only approximately and an initial state is also not known precisely. An attempt to achieve a closer description of a system implies a complication of a mathematical model which generally becomes nonlinear. This inevitably leads to systems admitting indeterminate or chaotic solutions (trajectories). Hence, we cannot circumvent chaotic behavior of systems and must foresee the chaos and control it. A practical implementation of such an approach is a solution of the problem of transmitting information. It is known that the transmission of information (in computers, telephone nets, etc.) is attended with interference or noise: intervals of pure transmission alternate with intervals with noise. The unexpected appearance of noise was believed to be associated with a “human element”. Costly attempts to improve characteristics of nets or to increase signal power did not lead to solution of the problem of noise. Intervals of pure transmission and intervals of noise are arranged highly chaotic both in duration and in order. However, it turned out that in the chaos of noise and pure intervals there is a certain regularity: the mean ratio of the summarized time of pure transmission and the summarized time of noise is kept constant and, in addition, this ratio is independent of the scale, i.e. it is the same both for an hour and for a second. This means that the problem of noise is not a local problem and is associated not only with a “human element”. The way out from this seemingly hopeless situation is very simple: it is reasonable to use a rather weak and inexpensive communication network but duplicate it for correcting errors. This strategy of communicating information is applied now in computer networks.

Economics also provides examples of the chaotic behavior. Studying the variation diagram of prices of cotton within eight years, Hautxacker, a professor of economics at the Harvard university, revealed that there were too many big jumps and that the frequency curve did not correlate with the normal distribution curve. He consulted B. Mandelbrot who worked in the IBM research center. A computer analysis of the variation of prices showed that the points which do not fall on the normal distribution curve form a strange symmetry. Each individual jump of the price is random, but the sequence of such jumps is independent of the scale: day’s and month’s jumps correspond well to each other under appropriate scaling of the time. Such a regularity persists during the last sixty years with two world wars and many crises. Thus, a striking regularity appears within chaotic dynamics.

Chaotic behavior can be viewed not only in statistic processes but in determinate ones. Let us consider a pendulum built up from two or more rigid components. The first component is secured at a fixed point, to the end of the first component is secured the second component, and so forth. This mechanical system is entirely determinate and described by a collection of differential equations. If one actuates the pendulum in such away that it highly rotates then a chaotic motion can be observed: The pendulum will change the direction of rotation in a chaotic manner. In addition, it is impossible to repeat exactly the motion in subsequent experiments. Thus, we can observe chaos in fully determinate mechanical systems. An explanation is very simple: the system offers the property of sensitive dependence on initial conditions [136], [21].

1.3 Orbit Coding

The modern theory and practice of dynamical systems require the necessity of studying structures that fall outside the scope of traditional subjects of mathematical analysis — analytic formulas, integrals, series, etc. An important tool that allows to investigate such complicate phenomena as chaos and strange attractors is the method of symbolic dynamics. The name reflects the main idea of the method — the description of system dynamics by admissible sequences (admissible words) of symbols from a finite symbol collection (alphabet). We explain this idea by the following hypothetic sample.

Assume that a "device" (realizable or hypothetic) note a system state (a position of the phase point) by some values. These values are obtained with certain accuracy. For example, an electronic clock displays the value t_i, when the exact time t lies in the interval $[t_i, t_i + h)$, where $h > 0$ depends on clock's design. It is convenient to suppose that the phase space M of the system studied is covered by a finite number of cells $\{M_i\}$ and the "device" marks the cell number (index) i when the point x is in the cell M_i. The cells M_i and M_j can intersect when the device indicator is exactly on the boundary between M_i and M_j. In the last case any of i and j are accepted as correct. For simplicity we suppose that the device marks indices of cells through equal time intervals and the trajectory (the sequence of phase points under the action of a system) is coded by the sequence of indices of the system $\{z(k),\ k \in \mathbb{Z}\}$. As indices, we can use symbols of different nature: numbers, letters, coordinates etc. If symbols are letters of some alphabet then the number of letters coincides with the number of cells and trajectories are coded by sequences of letters named admissible words. For transmission of communications by telegraph, as an example, an alphabet with two symbols ("dot" and "dash") is usually used.

Thus, the set of potential system states (phase space) is divided into a finite number of cells. Each cell is coded by a symbol and the "device" in every unit of time "displays" a symbol which corresponds to that cell where the system occurs. Notice that given a sequence of symbols, we can uniquely restore the sequence of cells a trajectory passes through. Clearly, the smaller are cells, the closer is the description of dynamics. The transition from an infinite phase space to a finite collection of symbols can be viewed as a discretization of the phase space.

Thus, the behavior of a system is "coded" with a specially constructed language; in so doing there is a certain correspondence between sequences of symbols and the system dynamics. For example, to a periodic orbit there corresponds a sequence formed by repeated blocks of symbols. The property of orbit recurrence is expressed in repetition of a symbol in an admissible word. Thus, the system dynamics is determined not by values of symbols but by their order in the sequence. Notice that the system dynamics specifies the permissibility of transition from one cell to another and, hence, from one symbol to other symbol; the transition from one symbol to several ones is not excluded. In this case the set of all admissible words is infinite. As an illustration, if the alphabet is formed by the symbols $\{0, 1\}$ and transitions from each symbol to an each one are allowed then we obtain the set of infinite binary sequences with continuum cardinality. If the transition from 1 to 0 is forbidden, we obtain sequences that differ where the transition from 0 to 1 occurs; such sequences form a denumerable set. The first system has the infinite number of periodic orbits, whereas the second one has only two periodic orbits: $\{\dots 0 \dots\}$ and $\{\dots 1 \dots\}$.

G. Hadamard was the first who used coding of trajectories. In 1898 he applied coding of trajectories by sequences of symbols to obtain the global behavior of geodesics on surfaces of negative curvature [50]. M. Morse [89] is recognized as a founder of symbolic dynamics methods. The term "symbolic dynamics" was introduced by M. Morse and Hedlund [90] who laid the foundations of its methods. They described the main subject as follows.

"The methods used in the study of recurrence and transitivity frequently combine classical differential analysis with a more abstract symbolic analysis. This involves a characterization of the ordinary dynamical trajectory by an unending sequence of symbols termed symbolic trajectory such that the properties of recurrence and transitivity of the dynamical trajectory are reflected in analogous properties of its symbolic trajectory."

These ideas led in the 1960's an 1970's to the development of powerful mathematical tools to investigate a class of extremely non-trivial dynamical systems. R. Bowen [14, 15] made an essential contribution to their development. Smale's "horseshoe" mentioned above influenced very much the advancement of the theory. In 1972 V.M. Alekseev [3] applied symbolic dynamics to investigate some problems of celestial mechanics. He put into use the term "symbolic image" to name the space of admissible sequences in coding trajectories of a system. For theoretical background and applications of symbolic dynamics we refer the reader to the lectures by V.M. Alekseev [4].

In an attempt to find an approach to computer modeling of dynamical systems, C. Hsu [57] elaborated the "cell-to-cell mapping" method. This method performs well in studying the global structure of dynamical systems with chaotic behavior of trajectories. The idea of the method is to approximate a given mapping by a mapping of "cells"; the image of the cell M_i is considered to coincide with the cell M_j provided the center of M_i is mapped by f to some point of M_j. The method suggested by C. Hsu is computer-oriented and admits a straightforward computer implementation. One of the weaknesses of the method is its insufficient theoretical justification. That is why results and conclusions of simulation require detailed analysis and verification. It is also known a generalized version of the method when the image $f(M_i)$ of M_i may consists of several cells $\{M_j\}$ with probability proportional to the volume (the measure) of the intersection $f(M_i) \cap M_j$. Such approach leads to finite Markov's chains which theory is well developed. In this case the computer implementation is rather complicate and presents certain difficulties. A detailed description of these methods can be found in [57].

In 1983 G.S. Osipenko [95] introduced the notion of symbolic image of a dynamical system with respect to a finite covering. A symbolic image is an oriented graph with vertices i corresponding to the cells M_i and edges $i \to j$; the edge $i \to j$ exists if and only if there is a point $x \in M_i$ whose image $f(x)$ lies in M_j. By transforming the system flow into graph we are able to formulate investigation methods as graph algorithms. The following relations between an initial system and its symbolic image hold:

trajectories of a system agree with admissible paths on the graph;
symbolic image reflects the global structure of a dynamical system;
symbolic image can be considered as a finite approximation of a system;
the maximal diameter of cells control an accuracy of approximation.

We notice that there exist several other approaches which use concepts similar to the construction of the symbolic image graph. In Mischaikow [84], a symbolic image-like graph, called a *multivalued mapping*, is constructed in order to compute isolated blocks in the context of the Conley Index Theory [28]. The *set-oriented* methods of Dellnitz, Hohmann and Junge [7, 31, 33, 36] use a scheme similar to our graph and apply a subdivision technique which is also used slightly modified in our implementation. Hruska [56] makes a *box chain construction* to get a directed graph with the aim to compute an *expanding* metric for dynamical systems. An analogous tool for discretization of dynamical systems was applied by F.S. Hunt [58] and Diamond et al [38]. Furthermore, there are many other constructive and computer-oriented methods, of this kind [29, 30, 46, 48, 78, 134, 135].

M. Dellnitz et al [32, 33, 36] elaborated a subdivision technique for the numerical study of dynamical systems. The main point of this method is as follows: a studied domain is covered by boxes or cells, according to certain rules, a part of cells is excluded from consideration while the remainder part is subdivided, then this procedure is repeated. This approach was used in construction of algorithms localizing various invariant sets, in particular, a numerical method for construction of stable and unstable invariant manifolds was obtained [32]. Algorithms for calculating approximations of the invariant measure and the Lyapunov exponent were also created [35, 36]. Based on the algorithms just mentioned, the package GAIO (available at http://math-www.uni-paderborn.de/agdellnotz/gaio/) was elaborated.

A general scheme of the symbolic analysis proposed is as follows. By a finite covering of the phase space of a dynamical system we construct a directed graph (symbolic image) with vertices corresponding to cells of the covering and edges corresponding to admissible transitions. A symbolic image can be viewed as a finite discrete approximation of a dynamical system; the fine is the covering, the closer is the approximation. A process of adaptive subdivision of cells allows to construct a sequence of symbolic images and in so doing to refine qualitative characteristics of a system. The method described above can be used to solve the following problems:

1. Localization of periodic orbits with a given period,
2. Construction of periodic orbit,
3. Localization of the chain recurrent set,
4. Construction of positive (negative) invariant sets,
5. Construction of attractors and domains of attraction,
6. Construction of filtrations and fine sequence of filtrations,
7. Construction of the structural graph,
8. Estimation of the topological entropy,
9. Estimation of Lyapunov exponents,
10. Estimation of the Morse spectrum,
11. Verification of hyperbolicity,
12. Verification of structural stability,
13. Verification of controllability,
14. Construction of isolating neighborhoods of invariant sets.
15. Calculation of the Conley index.

We remark that the symbolic image construction opens the door to applications of several new methods for the investigation of dynamical systems. Quite a lot of information can be gathered by this, and there might be even some more techniques, yet undiscovered, which could be built around symbolic image in the future.

1.4 Dynamical Systems

Let M be a subset in the q-dimensional Euclidean space $\mathbb{R}^q$. In what follows we assume that M is a closed bounded set (a compact) or a smooth manifold in $\mathbb{R}^q$. Let $\mathbb{Z}$ and $\mathbb{R}$ stand for the sets of integers and real numbers, respectively. By a dynamical system we mean a continuous mapping $\Phi(x,t)$, where $x \in M$, $t \in \mathbb{Z}$ ($t \in \mathbb{R}$), such that $\Phi : M \times \mathbb{Z} \to M$ ($\Phi : M \times \mathbb{R} \to M$) and

$$\Phi(x,0) = x,$$
$$\Phi(\Phi(x,t),s) = \Phi(x,t+s),$$

for all $t, s \in \mathbb{Z}$ ($t, s \in \mathbb{R}$). The variable t is thought of as the time and M is named the phase space. If $t \in \mathbb{Z}$ then we have a discrete time system called, for brevity, discrete system (cascade). Discrete dynamical systems result generally from iterative processes or difference equations $x_{n+1} = f(x_n)$. In the case when $t \in \mathbb{R}$ we deal with a continuous time system called, for brevity, continuous system (flow). Continuous dynamical systems result generally from autonomous systems of ordinary differential equations $\dot{x} = f(x)$, i.e. from systems with right hand sides independent of time.

Example 1. Linear equation.

Consider the linear differential equation $\dot{x} = ax$ on the straight line R. The solution with initial conditions (x_0, t_0) is of the form $F(x_0, t - t_0)$

$= x_0 \exp a(t - t_0)$. In this case the continuous dynamical system is given by the mapping $F(x, t)$, i.e.

$$\Phi(x, t) = x \exp at.$$

If $a < 0$ then $x \exp at \to 0$ as $t \to +\infty$. If $a > 0$ and $x \neq 0$ then $x \exp at \to \pm\infty$ as $t \to +\infty$. By fixing the time t of the shift along trajectories, e.g. $t = 1$, we reach to the discrete dynamical system

$$x_{n+1} = bx_n$$

where $b = \exp a$ is a positive constant. The discrete system $x_{n+1} = bx_n$ can be considered independently of the differential equation and, as this holds, the constant b may be negative. In the last case the mapping $\Phi(x) = bx$ is said to reverse orientation.

Example 2. The Lotka-Volterra equations.

The Lotka-Volterra equations are a system of differential equations of the form

$$\begin{aligned} \dot{x_1} &= (a - bx_2)x_1 \\ \dot{x_2} &= (-c + dx_1)x_2, \end{aligned} \tag{1.1}$$

where a, b, c, and d are positive parameters. The Lotka-Volterra equations are one of the mostly known examples that present dynamics of two interacting biological populations. In (1.1) x_1 and x_2 stand for quantities of preys and predators, respectively, a is the reproduction rate of predators in the absence of preys, the term $-bx_2$ means losses via preys. Thus, for predators the population growth per one predator $\dot{x_1}/x_1$ equals $a - bx_2$. In the absence of predators the population of preys decreases, so that $\dot{x_2}/x_2 = -c, c > 0$ provided $x_1 = 0$. The term dx_1 compensates this decrease in the case of "lucky hunting".

1.4.1 Discrete Dynamical Systems

Assume that a continuous mapping $f : M \to M$ has the continuous inverse f^{-1}, i.e. f is a homeomorphism. Then f generates a discrete dynamical system of the form $\Phi(x, n) = f^n(x),\ n \in \mathbb{Z}$. The mapping $f^m(x)$ is an m-times composition of the function f for $m > 0$ and an m-times composition of the function f^{-1} for $m < 0$; if $m = 0$ then f is the identity mapping.

Thus, we study the dynamics of the cascade

$$x_{k+1} = f(x_k),\ x_k \in M \subset \mathbb{R}^q,\ k \in \mathbb{Z}.$$

Sometimes we will require a homeomorphism f to be a diffeomorphism. This means that there exists continuous partial derivatives of f and f^{-1}.

The trajectory (or the orbit) of the point x_0 is an infinite two-sided sequence

$$T(x_0) = \{x_k = f^k(x_0),\ k \in \mathbb{Z}\}.$$

A point x_0 is called fixed point if $f(x_0) = x_0$. The trajectory of a fixed point consists of a single point $T(x_0) = \{x_0\}$. A point x_0 is called p-periodic point if $f^p(x_0) = x_0$; a least positive integer p with this property is called the least period. For example, a fixed point is a p-periodic point for each positive integer p but its least period is 1. The trajectory of a periodic point x_0 with the least period p consists of p distinct points $T(x_0) = \{x_0, x_1, ..., x_{p-1}\}$.

Example 3. Consider the mapping of the plane $\mathbb{R}^2$ into itself:

$$f : (x, y) \to (ay + bx^2, -ax).$$

Since $f(0,0) = (0,0)$ the origin $(0,0)$ is a fixed point with trajectory $T(0,0) = \{(0,0)\}$. If $b \neq 0$ there exists one more fixed point (x_0, y_0), where $x_0 = (1 + a^2)/b$, $y_0 = -a(1 + a^2)/b$, with trajectory $T(x_0, y_0) = \{(x_0, y_0)\}$. If $b = 0$ then the mapping f is a composition of two linear mappings: $f = L_1 \circ L_2$, where L_1 is a multiplication by a and $L_2 = (y, -x)$ is a rotation through the angle $\alpha = -90°$. When $a = 1$, f is reduced to a rotation; each point $(x, y) \neq (0,0)$ generates the periodic trajectory with least period $p = 4$, i.e. $f^4(x, y) = (x, y)$. As an example, the trajectory of the point $(1,1)$ is of the form $T(1,1) = \{(1,1), (1,-1), (-1,-1), (-1,1)\}$. It turns out that under certain values of a and b the dynamical system posses infinitely many periodic trajectories with unbounded least periods (see [57]).

1.4.2 Continuous Dynamical Systems

To describe a continuous dynamical system given by ordinary differential equations we use the shift operator along its trajectories defined as follows. Consider the system of differential equations

$$\dot{x} = F(t, x),$$

where $x \in M$, $F(t, x)$ is a C^1 vector field periodic in t with period ω. Let $\Phi(t, t_0, x_0)$ be the solution of the system with initial conditions $\Phi(t_0, t_0, x_0) = x_0$. The investigation of the global dynamics of the system can be performed by studying the Poincaré mapping $f(x) = \Phi(\omega, 0, x)$ of the system which is nothing that the shift operator along trajectories through the period ω.

Example 4. Duffing equation with forcing.

Consider the damped Duffing equation with forcing

$$\ddot{x} + k\dot{x} + \alpha x + \beta x^3 = B\cos(ht),$$

where t is an independent variable, k, α, β, B, and $h \neq 0$ are parameters, x is a function sought for. Setting $y = \dot{x}$ we get an equivalent system of the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ky - \alpha x - \beta x^3 + B\cos(ht). \end{aligned}$$

If $B \neq 0$ then the system is periodic in t with least period $\omega = \frac{2\pi}{h}$. Let $(X(t,x,y), Y(t,x,y))$ be its solution with initial conditions (x,y) at $t=0$. If we put, say, $h=2$ then the Poincaré mapping takes the form

$$f : (x,y) \to (X(\pi,x,y), Y(\pi,x,y)).$$

If the system is autonomous (i.e. the vector field F is independent of t), an arbitrary $\omega \neq 0$ can be reasoned as a period. For example, without loss of generality we may take 1. The shift operator takes the form $f(x) = \Phi(\omega, x)$, where $\Phi(t,x)$ is the solution of autonomous system such that $\Phi(0,x) = x$. When differential equations are solved numerically, for instance, by the Runge-Kutta or the Adams methods, we get the shift operator approximately.

Example 5. Duffing equation without forcing.

Consider the damped Duffing equation without forcing

$$\ddot{x} + k\dot{x} + \alpha x + \beta x^3 = 0.$$

The corresponding system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ky - \alpha x - \beta x^3, \end{aligned}$$

is autonomous and the shift operator may be written as

$$f : (x,y) \to (X(1,x,y), Y(1,x,y)).$$

To study the systems listed above methods of computer modeling are widely applied. For example, the use of the MAPLE yields good results. Obtained with the Runge-Kutta method, the phase portrait of the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - 0.27x^3 - 0.48y, \end{aligned}$$

is depicted in Fig. 1.1.

The system has three equilibriums O, A, and B. There are two trajectories that approach O as $t \to +\infty$. These trajectories are called stable separatrices and denoted by $W^s(O)$. Thus, for each $x \in W^s(O)$ the omega limit set (ω-limit set) of x coincides with O. There are also two trajectories called unstable separatrices and denoted by $W^u(O)$ that approach O as $t \to -\infty$. Similarly, for each $x \in W^u(O)$ the alpha limit set (α-limit set) of x is O. Other trajectories, except for $W^s(O)$ approach equilibriums A and B as $t \to +\infty$.

Relationship between discrete and continuous dynamical systems. Historically, in the dynamical systems theory continuous dynamical systems governed by ordinary differential equations have been the main object of investigation. However, recent trends are to give much attention to

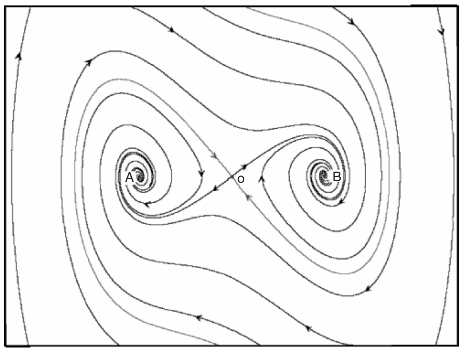

Fig. 1.1. The phase portrait of Duffing's equation

discrete systems governed by diffeomorphisms. Let us show that there is a connection between continuous and discrete systems. We will convince that each continuous system generates a discrete system and vice versa, moreover there is a natural correspondence between trajectories of the systems. The most simple way to obtain a discrete system from a continuous one is to consider the shift mapping (shift operator) at a fixed time along trajectories. The method for constructing the shift mapping was discussed above. By the theorems of existence of ODE solutions and differentiability of solutions with respect to initial data, the shift mapping is a diffeomorphism provided the original system is smooth. In connection with this an inverse problem of including a diffeomorphism in a flow arises: for a given diffeomorphism one needs to find a vector field whose shift operator coincides with the diffeomorphism. However, as M.I. Brin [16] showed, most of diffeomorphisms cannot be included in flows as shift operators. For example, if a diffeomorphism is orientation revising, i.e. its Jacobian is negative, it cannot be included in a flow since the shift operator is always continuously transformed into the identity mapping with positive Jacobian. Thus, diffeomorphisms constitute essentially wide class than flows generated by differential equations on the same manifold. However, using the notion of a section mapping introduced by Poincaré one can construct the correspondence where the opposite situation appears. As an example, consider the section of a torus. A torus can be viewed as the product of two circles $T = S \times S$ with the coordinates $(x, y), 0 \leq x, y \leq 1$. Let a vector field F on T be such that its trajectories intersect transversally the circle $S \times 0$, which called a section of the flow on a torus. Suppose that the trajectory which starts from the point $(x, 0), x \in S$ returns back to S in a unit time at the point $(f(x), 0)$. In this manner the diffeomorphism $f : S \mapsto S$ called a first return mapping arises. Poincaré was the first who applied this construction to study the system dynamics near a periodic trajectory. In this case, the section is a surface transverse to a periodic trajectory and the return time depends on an initial point. Consider now the inverse passage from a diffeomorphism to a vector field. Let $f : M \mapsto M$ be a diffeomorphism of a manifold M. First of all we define the new manifold M^* by identifying the points $(x, 1)$ and $(f(x), 0)$ in the product $M \times [0, 1]$. Clearly, for the unit vector field $F = (0, 1)$ on $M \times [0, 1]$, the manifold $M \times 0 \cong M$ is a section. The field F generates the vector field F^* on M^* such that its trajectories intersect transversally M and take the point x to $f(x)$ in a unit time. Thus, the diffeomorphism f on M generates the vector field F^* on M^* for which the shift mapping on the zero section M coincides with f, $\dim M^* = \dim M + 1$. Both of the methods discussed for correlation of flows and diffeomorphisms indicate that the qualitative theory of smooth flows (differential equations) and the theory of discrete systems develop in parallel though can differ in details.