3

The Newton polygon method

Almost all techniques for solving asymptotic systems of equations are explicitly or implicitly based on the Newton polygon method. In this section we explain this technique in the elementary case of algebraic equations over grid-based algebras $C[[\mathfrak{M}]]$, where C is a constant field of characteristic zero and $\mathfrak{M}$ a totally ordered monomial group with $\mathbb{Q}$-powers. In later chapters of this book, the method will be generalized to linear and non-linear differential equations.

In section 3.1, we first illustrate the Newton polygon method by some examples. One important feature of our exposition is that we systematically work with "asymptotic algebraic equations", which are polynomial equations $P(f) = 0$ over $C[[\mathfrak{M}]]$ together with asymptotic side-conditions, like $f \prec \mathfrak{v}$. Asymptotic algebraic equations admit natural invariants, like the "Newton degree", which are useful in the termination proof of the method. Another important ingredient is the consideration of equations $P'(f) = 0$, $P''(f) = 0$, etc. in the case when $P(f) = 0$ admits almost multiple roots.

In section 3.2, we prove a version of the implicit function theorem for grid-based series. Our proof uses a syntactic technique which will be further generalized in chapter 6. The implicit function theorem corresponds to the resolution of asymptotic algebraic equations of Newton degree one. In section 3.3, we show how to compute the solutions to an asymptotic algebraic equation using the Newton polygon method. We also prove that $C[[\mathfrak{M}]]$ is algebraically closed or real closed, if this is the case for C.

The end of this chapter contains a digression on "Cartesian representations", which allow for a finer calculus on grid-based series. This calculus is based on the observation that any grid-based series can be represented by a multivariate Laurent series. By restricting these Laurent series to be of a special form, it is possible to define special types of grid-based series, such as convergent, algebraic or effective grid-based series. In section 3.5, we will show that the Newton polygon method can again be applied to these more special types of grid-based series.

Cartesian representations are essential for the development of effective asymptotics [vdH97], but they will only rarely occur later in this book (the main exceptions being section 4.5 and some of the exercises). Therefore, sections 3.4 and 3.5 may be skipped in a first reading.

3.1 The method illustrated by examples

3.1.1 The Newton polygon and its slopes

Consider the equation

$$P(f) = \sum_{i \geqslant 0} P_i f^i = z^3 f^6 + z^4 f^5 + f^4 - 2 f^3 + f^2 + \frac{z}{1 - z^2} f + \frac{z^3}{1 - z} = 0 \qquad (3.1)$$

and a Puiseux series $f = c\, z^\mu + \cdots \in \mathbb{C}[c][[z^{\mathbb{Q}}]]$, where $c \neq 0$ is a formal parameter. We call $\mu = \operatorname{val} f$ the *dominant exponent* or *valuation* of f. Then

$$\alpha = \min_i \operatorname{val}(P_i z^{i\mu}) = \min\{3, \mu + 1, 2\mu, 3\mu, 4\mu, 5\mu + 4, 6\mu + 3\}$$

is the dominant exponent of $P(f) \in \mathbb{C}[c][[z^{\mathbb{Q}}]]$ and

$$N_{P, z^\mu}(c) := P(f)_{z^\alpha} = 0 \qquad (3.2)$$

is a non-trivial polynomial equation in c. We call N_{P,z^μ} and (3.2) the *Newton polynomial* resp. *Newton equation* associated to z^μ.

Let us now replace c by a non-zero value in C, so that $f = c\, z^\mu + \cdots \in \mathbb{C}[[z^{\mathbb{Q}}]]$. If f is a solution to (3.1), then we have in particular $N_{P,z^\mu}(c) = 0$. Consequently, N_{P,z^μ} must contain at least two terms, so that α occurs at least twice among the numbers $3, \mu + 1, 2\mu, 3\mu, 4\mu, 5\mu + 4, 6\mu + 3$. It follows that

$$\mu \in \{2, 1, 0, -\tfrac{3}{2}\}.$$

We call 2, 1, 0 and $-\frac{3}{2}$ the *starting exponents* for (3.1). The corresponding monomials z^2, z, 1 and $z^{-3/2}$ are called *starting monomials* for (3.1).

The starting exponents may be determined graphically from the *Newton polygon* associated to (3.1), which is defined to be the convex hull of all points (i, ν) with $\nu \geqslant \operatorname{val} P_i$. Here points $(i, \nu) \in \mathbb{N} \times \mathbb{Q}$ really encode points $(f^i, z^\nu) \in f^{\mathbb{N}} \times z^{\mathbb{Q}}$ (recall the explanations below figure 2.1). The Newton polygon associated to (3.1) is drawn at the left hand side of figure 3.1. The diagonal slopes

$$\begin{array}{rcll} (1, z^3) & \rightarrow & (f, z) & (\mu = 2) \\ (f, z) & \rightarrow & (f^2, 1) & (\mu = 1) \\ (f^2, 1) & \rightarrow & (f^4, 1) & (\mu = 0) \\ (f^4, 1) & \rightarrow & (f^6, z^3) & (\mu = -\frac{3}{2}) \end{array}$$

correspond to the starting exponents for (3.1).

Given a starting exponent $\mu \in \mathbb{Q}$ for (3.1), a non-zero solution c of the corresponding Newton equation is called a *starting coefficient* and $c\, z^\mu$ a *starting term*. Below, we listed the starting coefficients c as a function of μ in the case of equation (3.2):

μ	$N_{P,\mu}$	c	multiplicity
2	$c+1$	-1	1
1	c^2+c	-1	1
0	$c^4-2\,c^3+c^2$	1	2
$\frac{3}{2}$	c^6+c^4	$-\mathrm{i}, \mathrm{i}$	1

Notice that the Newton polynomials can again be read off from the Newton polygon. Indeed, when labeling each point (f^i, z^μ) by the coefficient of z^μ in P_i, the coefficients of N_{P,z^μ} are precisely the coefficients on the edge with slope μ.

Given a starting term $c\, z^\mu \in \mathbb{C}\, z^{\mathbb{Q}}$, we can now consider the equation $\tilde{P}(\tilde{f}) = 0$ which is obtained from (3.1), by substituting $c\, z^\mu + \tilde{f}$ for f, and where $\tilde{f}$ satisfies the asymptotic constraint $\tilde{f} \prec z^\mu$. For instance, if $c\, z^\mu = 1\, z^0$, then we obtain:

$$\begin{aligned} \tilde{P}(\tilde{f}) \;=\; & z^3\, \tilde{f}^6 + (6z^3)\, \tilde{f}^5 + (15\, z^3 + 5\, z^4 + 1)\, \tilde{f}^4 + \\ & (20\, z^3 + 10\, z^4 + 2)\, \tilde{f}^3 + (15\, z^3 + 10\, z^4 + 1)\, \tilde{f}^2 + \\ & \left(6\, z^3 + 5\, z^4 + \frac{z}{1-z^2}\right) \tilde{f} + z^4 + z^3 + \frac{z^4+z^3+z}{1-z^2} = 0 \quad (\tilde{f} \prec 1) \end{aligned} \tag{3.3}$$

The Newton polygon associated to (3.3) is illustrated at the right hand side of figure 3.1. It remains to be shown that we may solve (3.3) by using the same method in a recursive way.

3.1.2 Equations with asymptotic constraints and refinements

First of all, since the new equation (3.3) comes with the asymptotic side-condition $\tilde{f} \prec 1$, it is convenient to study polynomial equations with asymptotic side-conditions

$$P(f) = 0 \quad (f \prec z^\nu) \tag{3.4}$$

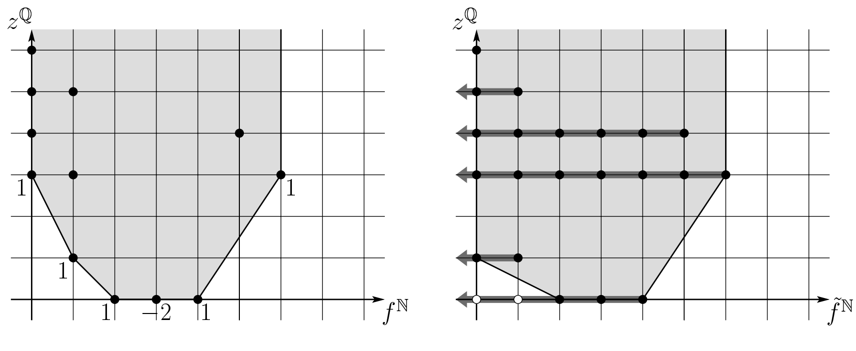

Fig. 3.1. The left-hand side shows the Newton polygon associated to the equation (3.1). The slopes of the four edges correspond to the starting exponents 2, 1, 0 and $-\frac{3}{2}$ (from left to right). After the substitution

$$f \to 1 + \tilde{f} \ \ (\tilde{f} \prec 1),$$

we obtain the equation (3.3), whose Newton polygon is shown at the right-hand side. Each non-zero coefficient P_{i,z^α} in the equation (3.1) for f induces a "row" of (potentially) non-zero coefficients $\tilde{P}_{\tilde{\imath},z^\alpha}$ in the equation for $\tilde{f}$, in the direction of the arrows. The horizontal direction of the arrows corresponds to the slope of the starting exponent 0. Moreover, the fact that 1 is a starting term corresponds to the fact that the coefficient of the lowest leftmost induced point vanishes.

in a systematic way. The case of usual polynomial equations is recovered by allowing $\nu = -\infty$. In order to solve (3.4), we now only keep those starting monomials z^μ for $P(f) = 0$ which satisfy the asymptotic side condition $z^\mu \prec z^\nu$, i.e. $\mu > \nu$.

The highest degree of N_{P,z^μ} for a monomial $z^\mu \prec z^\nu$ is called the Newton degree of (3.4). If $d > 0$, then P is either divisible by f (and $f = 0$ is a solution to (3.4)), or (3.4) admits a starting monomial (and we can carry out one step of the above resolution procedure). If $d = 0$, then (3.4) admits no solutions.

Remark 3.1. Graphically speaking, the starting exponents for (3.4) correspond to sufficiently steep slopes in the Newton polygon (see figure 3.2). Using a substitution $f = z^\nu \tilde{f}$, the equation (3.4) may always be transformed into an equation

$$\tilde{P}(\tilde{f}) = 0 \quad (\tilde{f} \prec 1)$$

with a normalized asymptotic side-condition (the case $\nu = -\infty$ has to be handled with some additional care). Such transformations, called *multiplicative conjugations*, will be useful in chapter 8, and their effect on the Newton polygon is illustrated in figure 3.2.

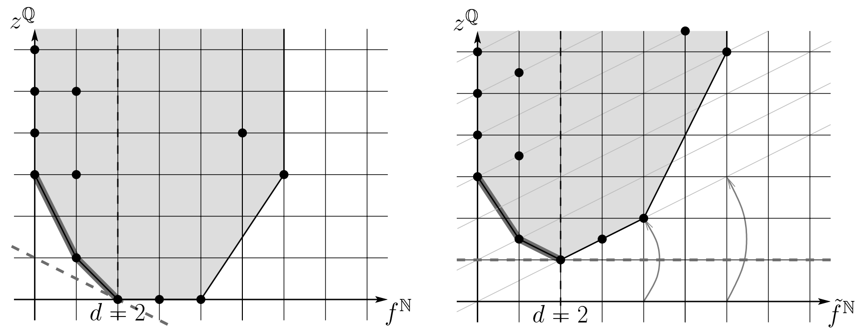

Fig. 3.2. At the left-hand side, we have illustrated the Newton polygon for the asymptotic equation $P(f)=0$ $(f \prec z^{1/2})$. The dashed line corresponds to the slope $1/2$ and the edges of the Newton polygon with slope $>1/2$ have been highlighted. Notice that the Newton degree $d=2$ corresponds to the first coordinate of the rightmost point on an edge with slope $>1/2$. At the right-hand side, we have shown the "pivoting" effect around the origin of the substitution $f=z^{1/2}\,\tilde{f}$ on the Newton polygon.

Given a starting term $\varphi=\tau=c\,z^{\mu}$ or a more general series $\varphi=c\,z^{\mu}+\cdots\in\mathbb{C}[[z^{\mathbb{Q}}]]$, we next consider the transformation

$$f=\varphi+\tilde{f}\quad(\tilde{f}\prec z^{\tilde{\nu}}),\tag{3.5}$$

with $z^{\tilde{\nu}}\preccurlyeq z^{\mu}$, which transforms (3.4) into a new asymptotic polynomial equation

$$\tilde{P}(\tilde{f})=0\quad(\tilde{f}\prec z^{\tilde{\nu}}).\tag{3.6}$$

Transformations like (3.5) are called *refinements*. A refinement is said to be *admissible*, if the Newton degree of (3.6) does not vanish.

Now the process of computing starting terms and their corresponding refinements is generally infinite and even transfinite. *A priori*, the process therefore only generates an infinite number of necessary conditions for Puiseux series f to satisfy (3.4). In order to really *solve* (3.4), we have to prove that, after a finite number of steps of the Newton polygon method, and whatever starting terms we chose (when we have a choice), we obtain an asymptotic polynomial equation with a unique solution. In the next section, we will prove an implicit function theorem which guarantees the existence of such a unique solution for equations of Newton degree one. Such equations will be said to be *quasi-linear*.

Returning to our example equation (3.1), it can be checked that each of the refinements

$$\begin{array}{rcll} f &=& -z^2+\tilde{f} & (\tilde{f}\prec z^2);\\ f &=& -z+\tilde{f} & (\tilde{f}\prec z);\\ f &=& -\mathrm{i}\,z^{-3/2}+\tilde{f} & (\tilde{f}\prec z^{-3/2});\\ f &=& \mathrm{i}\,z^{-3/2}+\tilde{f} & (\tilde{f}\prec z^{-3/2}) \end{array}$$

leads to a quasi-linear equation in $\tilde{f}$. The case

$$f = 1 + \tilde{f} \quad (\tilde{f} \prec 1)$$

leads to an equation of Newton degree 2 (it will be shown later that the Newton degree of (3.6) coincides with the multiplicity of c as a root of N_{P,z^μ}). Therefore, the last case necessitates one more step of the Newton polygon method:

$$\begin{aligned}\tilde{f} &= -\mathrm{i}\sqrt{z} + \tilde{\tilde{f}} \quad (\tilde{\tilde{f}} \prec z^{1/2});\\ \tilde{f} &= \mathrm{i}\sqrt{z} + \tilde{\tilde{f}} \quad (\tilde{\tilde{f}} \prec z^{1/2}).\end{aligned}$$

For both refinements, it can be checked that the asymptotic equation in $\tilde{\tilde{f}}$ is quasi-linear. Hence, after a finite number of steps, we have obtained a complete description of the set of solutions to (3.1). The first terms of these solutions are as follows:

$$\begin{aligned}f_I &= -z^2 - 2\,z^3 - 4\,z^4 - 13\,z^5 - 50\,z^6 + O(z^7);\\ f_{II} &= -z + 3\,z^2 - 8\,z^3 + 46\,z^4 - 200\,z^5 + O(z^6);\\ f_{III} &= 1 - \mathrm{i}\,z^{1/2} + \tfrac{1}{2}\,z + \tfrac{5\,\mathrm{i}}{8}\,z^{3/2} - z^2 + O(z^{5/2});\\ f_{IV} &= 1 + \mathrm{i}\,z^{1/2} + \tfrac{1}{2}\,z - \tfrac{5\,\mathrm{i}}{8}\,z^{3/2} - z^2 + O(z^{5/2});\\ f_V &= -\mathrm{i}\,z^{-3/2} - 1 - \tfrac{1}{2}\,z - \mathrm{i}\,z^{3/2} - \tfrac{\mathrm{i}}{2}\,z^{5/2} + O(z^3);\\ f_{VI} &= \mathrm{i}\,z^{-3/2} - 1 - \tfrac{1}{2}\,z + \mathrm{i}\,z^{3/2} + \tfrac{\mathrm{i}}{2}\,z^{5/2} + O(z^3).\end{aligned}$$

3.1.3 Almost double roots

Usually the Newton degrees rapidly decreases during refinements and we are quickly left with only quasi-linear equations. However, in the presence of almost multiple roots, the Newton degree may remain bigger than two for quite a while. Consider for instance the equation

$$\left(f - \frac{1}{1-z}\right)^2 = \varepsilon^2 \tag{3.7}$$

over $\mathbb{C}[[z;\varepsilon]]$, with $z \prec 1$ and $\varepsilon \prec 1$. This equation has Newton degree 2, and after n steps of the ordinary Newton polygon method, we obtain the equation

$$\left(\tilde{f} - \frac{z^n}{1-z}\right)^2 = \varepsilon^2 \quad (\tilde{f} \prec z^{n-1}),$$

which still has Newton degree 2. In order to enforce termination, an additional trick is applied: consider the first derivative

$$2\,f - \frac{2}{1-z} = 0$$

of the equation (3.7) w.r.t. f. This derived equation is quasi-linear, so it admits a unique solution

$$\varphi = \frac{1}{1-z}.$$

Now, instead of performing the usual refinement $f = 1 + \tilde{f}$ $(\tilde{f} \prec 1)$ in the original equation (3.7), we perform refinement

$$f = \varphi + \tilde{f} \quad (\tilde{f} \prec 1).$$

This yields the equation

$$\tilde{f}^2 = \varepsilon^2 \quad (\tilde{f} \prec 1).$$

Applying one more step of the Newton polygon method yields the admissible refinements

$$\begin{aligned} \tilde{f} &= -\varepsilon + \tilde{\tilde{f}} && (\tilde{\tilde{f}} \prec \varepsilon); \\ \tilde{f} &= \varepsilon + \tilde{\tilde{f}} && (\tilde{\tilde{f}} \prec \varepsilon). \end{aligned}$$

In both cases, we obtain a quasi-linear equation in $\tilde{\tilde{f}}$:

$$\begin{aligned} -2\,\varepsilon\,\tilde{\tilde{f}} + \tilde{\tilde{f}}^2 &= 0 && (\tilde{\tilde{f}} \prec \varepsilon); \\ 2\,\varepsilon\,\tilde{\tilde{f}} + \tilde{\tilde{f}}^2 &= 0 && (\tilde{\tilde{f}} \prec \varepsilon). \end{aligned}$$

In section 3.3.2, we will show that this trick applies in general, and that the resulting method always yields a complete description of the solution set after a finite number of steps.

Remark 3.2. The idea of using repeated differentiation in order to handle almost multiple solutions is old [Smi75] and has been used in computer algebra before [Chi86, Gri91]. Our contribution has been to incorporate it directly into the Newton polygon process, as will be shown in more detail in section 3.3.2.

3.2 The implicit series theorem

In the previous section, we have stressed the particular importance of quasi-linear equations when solving asymptotic polynomial equations. In this section, we will prove an implicit series theorem for polynomial equations. In the next section, we will apply this theorem to show that quasi-linear equations admit unique solutions. The implicit series theorem admits several proofs (see the exercises). The proof we present here uses a powerful syntactic technique, which will be generalized in chapter 6.

Theorem 3.3. *Let C be a ring and $\mathfrak{M}$ a monomial monoid. Consider the polynomial equation*

$$P_n\,f^n + \cdots + P_0 = 0 \tag{3.8}$$

with coefficients P_0, ..., $P_n \in C[[\mathfrak{M}_{\preccurlyeq}]]$, such that $P_{0,1} = 0$ and $P_{1,1} \in C^*$. Then (3.8) *admits a unique solution in $C[[\mathfrak{M}_{\prec}]]$.*

Proof. Since $P_{1,1} \in C^*$, the series P_1 is invertible in $C[[\mathfrak{M}_{\preccurlyeq}]]$. Modulo division of (3.8) by P_1, we may therefore assume without loss of generality that $P_1 = 1$. Setting $Q_i = -P_i$ for all $i \neq 1$, we may then rewrite (3.8) as

$$f = Q_0 + Q_2 f^2 + \cdots + Q_n f^n. \tag{3.9}$$

Now consider the set $\mathscr{T}$ of trees with nodes of arities in $\{0, 2, \ldots, n\}$ and such that each node of arity i is labeled by a monomial in $\operatorname{supp} Q_i$. To each such tree

$$t = \begin{array}{c} \mathfrak{v} \\ \swarrow \ \downarrow \ \searrow \\ t_1 \ \cdots \ t_i \end{array} \in \mathscr{T}$$

we recursively associate a coefficient $c_t \in C$ and a monomial $\mathfrak{m}_t \in \mathfrak{M}$ by

$$\begin{aligned} c_t &= Q_{i,\mathfrak{v}}\, c_{t_1} \cdots c_{t_i}; \\ \mathfrak{m}_t &= \mathfrak{v}\, \mathfrak{m}_{t_1} \cdots \mathfrak{m}_{t_i}. \end{aligned}$$

Now we observe that each of these monomials $\mathfrak{m}_t$ is infinitesimal, with

$$\mathfrak{m}_t \in (\operatorname{supp} Q_0) \cdot (\operatorname{supp} Q_0 \cup \operatorname{supp} Q_2 \cup \cdots \cup \operatorname{supp} Q_n)^*. \tag{3.10}$$

Hence the mapping $t \mapsto \mathfrak{m}_t$ is strictly increasing, when $\mathscr{T}$ is given the embeddability ordering from section 1.4. From Kruskal's theorem, it follows that the family $(c_t \mathfrak{m}_t)_{t \in \mathscr{T}}$ is well-based and even grid-based, because of (3.10). We claim that $f = \sum_{t \in \mathscr{T}} c_t \mathfrak{m}_t$ is the unique solution to (3.9).

First of all, f is indeed a solution to (3.9), since

$$\begin{aligned} f &= \sum_{i \in \{0,2,\ldots,n\}} \sum_{\mathfrak{v} \in \operatorname{supp} Q_i} \sum_{t_1, \ldots, t_i \in \mathscr{T}} c_{\begin{array}{c}\scriptstyle\mathfrak{v}\\ \swarrow\downarrow\searrow \\ \scriptstyle t_1 \cdots t_i\end{array}} \mathfrak{m}_{\begin{array}{c}\scriptstyle\mathfrak{v}\\ \swarrow\downarrow\searrow \\ \scriptstyle t_1 \cdots t_i\end{array}} \\ &= \sum_{i \in \{0,2,\ldots,n\}} \sum_{\mathfrak{v} \in \operatorname{supp} Q_i} \sum_{t_1, \ldots, t_i \in \mathscr{T}} (Q_{i,\mathfrak{v}}\, \mathfrak{v})\, (c_{t_1} \mathfrak{m}_{t_1}) \cdots (c_{t_i} \mathfrak{m}_{t_i}) \\ &= \sum_{i \in \{0,2,\ldots,n\}} \left(\sum_{\mathfrak{v} \in \operatorname{supp} Q_i} Q_{i,\mathfrak{v}}\, \mathfrak{v} \right) \left(\prod_{j=1}^{i} \sum_{t_j \in \mathscr{T}} c_{t_j} \mathfrak{m}_{t_j} \right) \\ &= \sum_{i \in \{0,2,\ldots,n\}} Q_i f^i = Q_0 + Q_2 f^2 + \cdots + Q_n f^n. \end{aligned}$$

In order to see that f is the unique solution to (3.8), consider the polynomial $R(\delta) = P(f + \delta)$. Since $f \prec 1$, we have $R_i = P_i + o(1)$ for all i, whence in particular $R_1 = 1 + o(1)$. Furthermore, $P(f) = 0$ implies $R_0 = 0$. Now assume that $g \prec 1$ were another root of P. Then $\delta = g - f \prec 1$ would be a root of R, so that

$$\delta = (R_1 + R_2\, \delta + \cdots + R_{n-1}\, \delta^{n-1})^{-1} R(\delta) = 0, \tag{3.11}$$

since $R_1 + R_2\, \delta + \cdots + R_{n-1}\, \delta^{n-1} = 1 + o(1)$ is invertible. □

Exercise 3.1. Generalize theorem 3.3 to the case when (3.8) is replaced by

$$P_0 + P_1 f + P_2 f^2 + \cdots = 0,$$

where $(P_i)_{i \in \mathbb{N}} \in C[[\mathfrak{M}_{\preccurlyeq}]]$ is a grid-based family with $P_{0,1} = 0$ and $P_{1,1} \in C^*$.

Exercise 3.2. Give an alternative proof of theorem 3.3, using the fact that (3.9) admits a unique power series solution in $\mathbb{Z}[[Q_2\, Q_0, \ldots, Q_n\, Q_0^{n-1}]]\, Q_0$, when considered as an equation with coefficients in $\mathbb{Z}[[Q_0, Q_2, \ldots, Q_n]]$.

Exercise 3.3. Assuming that $\mathfrak{M}$ is totally ordered, give yet another alternative proof of theorem 3.3, by computing the terms of the unique solution by transfinite induction.

Exercise 3.4. Let $C\langle\langle z_1, \ldots, z_n\rangle\rangle$ denote the ring of non-commutative power series in $z_1, \ldots, z_n$ over C. Consider the equation

$$f(g(z_1, \ldots, z_n), z_1, \ldots, z_n) = 0 \tag{3.12}$$

with $f \in C\langle\langle y, z_1, \ldots, z_n\rangle\rangle$, $f_1 = 0$ and invertible f_y. Prove that (3.12) admits a unique infinitesimal solution $g \in C\langle\langle z_1, \ldots, z_n\rangle\rangle$.

3.3 The Newton polygon method

3.3.1 Newton polynomials and Newton degree

Let C be a constant field of characteristic zero and $\mathfrak{M}$ a totally ordered monomial group with $\mathbb{Q}$-powers. Consider the *asymptotic polynomial equation*

$$P_n f^n + \cdots + P_0 = 0 \quad (f \prec \mathfrak{v}), \tag{3.13}$$

with coefficients in $C[[\mathfrak{M}]]$ and $\mathfrak{v} \in \mathfrak{M}$. In order to capture ordinary polynomial equations, we will also allow $\mathfrak{v} = \top_{\mathfrak{M}}$, where $\top_{\mathfrak{M}}$ is a formal monomial with $\top_{\mathfrak{M}} \succ \mathfrak{M}$. A *starting monomial* of f relative to (3.13) is a monomial $\mathfrak{m} \prec \mathfrak{v}$ in $\mathfrak{M}$, such that there exist $0 \leqslant i < j \leqslant n$ and $\mathfrak{n} \in \mathfrak{M}$ with $P_i\, \mathfrak{m}^i \asymp P_j\, \mathfrak{m}^j \asymp \mathfrak{n}$ and $P_k\, \mathfrak{m}^k \preccurlyeq \mathfrak{n}$ for all other k. To such a starting monomial $\mathfrak{m}$ we associate the equation

$$N_{P,\mathfrak{m}}(c) = P_{n,\mathfrak{n}/\mathfrak{m}^d}\, c^n + \cdots + P_{0,\mathfrak{n}} = 0, \tag{3.14}$$

and $N_{P,\mathfrak{m}}$ is called the *Newton polynomial* associated to $\mathfrak{m}$. A *starting term* of f relative to (3.13) is a term $\tau = c\, \mathfrak{m}$, where $\mathfrak{m}$ is a starting monomial of f relative to (3.13) and $c \in C^{\neq}$ a non-zero root of $N_{P,\mathfrak{m}}$. In that case, the *multiplicity* of τ is defined to be the multiplicity of c as a root of $N_{P,\mathfrak{m}}$. Notice that there are only a finite number of starting terms relative to (3.13).

Proposition 3.4. *Let f be a non-zero solution to* (3.13). *Then τ_f is a starting term for* (3.13). □

The *Newton degree* d of (3.13) is defined to be the largest degree of the Newton polynomial associated to a monomial $\mathfrak{m} \prec \mathfrak{v}$. In particular, if there exists no starting monomial relative to (3.13), then the Newton degree equals the valuation of P in f. If $d = 1$, then we say that (3.13) is *quasi-linear*. The previous proposition implies that (3.13) does not admit any solution, if $d = 0$.

Lemma 3.5. *If* (3.13) *is quasi-linear, then it admits a unique solution in* $C[[\mathfrak{M}]]$.

Proof. If $P_0 = 0$, then our statement follows from proposition 3.4, since there are no starting monomials. Otherwise, our statement follows from theorem 3.3, after substitution of $f\,\mathfrak{n}$ for f in (3.13), where $\mathfrak{n}$ is chosen $\preccurlyeq$-maximal such that $\mathfrak{d}_{P_1} \succcurlyeq \mathfrak{d}_{P_i}\,\mathfrak{n}^{i-1}$ for all i, and after division of (3.13) by $\mathfrak{d}_{P_1}$. □

3.3.2 Decrease of the Newton degree during refinements

A *refinement* is a change of variables together with the imposition of an asymptotic constraint:

$$f = \varphi + \tilde{f} \qquad (\tilde{f} \prec \tilde{\mathfrak{v}}), \tag{3.15}$$

where $\varphi \prec \mathfrak{v}$ and $\tilde{\mathfrak{v}} \preccurlyeq \mathfrak{v}$. Such a refinement transforms (3.13) into an asymptotic polynomial equation in $\tilde{f}$:

$$\tilde{P}_n \tilde{f}^n + \cdots + \tilde{P}_0 = 0 \quad (\tilde{f} \prec \tilde{\mathfrak{v}}), \tag{3.16}$$

where

$$\tilde{P}_i = \frac{1}{i!} P^{(i)}(\varphi) = \sum_{k=i}^{n} \binom{k}{i} P_k \varphi^{k-i}, \tag{3.17}$$

for each i. We say that the refinement (3.15) is *admissible* if the Newton degree of (3.16) is strictly positive.

Lemma 3.6. *Consider the refinement* (3.15) *with* $\tilde{\mathfrak{v}} = \mathfrak{d}_\varphi$. *Then*

a) The Newton degree of (3.16) *coincides with the multiplicity of* c *as a root of* $N_{P,\mathfrak{m}}$. *In particular,* (3.15) *is admissible if and only if* $c\,\mathfrak{m}$ *is a starting term for* (3.13).
b) The Newton degree of (3.16) *is bounded by the Newton degree of* (3.13).

Proof. Let d be maximal such that $P_d\ \mathfrak{m}^d \succcurlyeq P_i\ \mathfrak{m}^i$ for all i, and denote $\mathfrak{n} = \mathfrak{d}(P_d)\,\mathfrak{m}^d$. Then d is bounded by the Newton degree of (3.13) and

$$\begin{aligned} \tilde{P}_i &= \frac{1}{i!} \sum_{k=i}^{n} \binom{k}{i} P_k \varphi^{k-i} \\ &= \frac{1}{i!} \sum_{k=i}^{n} \binom{k}{i} (P_{k,\mathfrak{n}\mathfrak{m}^{-k}} + o(1))\, \mathfrak{n}\,\mathfrak{m}^{-k} (c + o(1))^{k-i}\, \mathfrak{m}^{k-i} \\ &= \frac{1}{i!} N_{P,\mathfrak{m}}^{(i)}(c)\, \mathfrak{n}\,\mathfrak{m}^i + o(\mathfrak{n}\,\mathfrak{m}^i), \end{aligned}$$

for all i. In particular, denoting the multiplicity of c as a root of $N_{P,\mathfrak{m}}$ by $\tilde{d}$, we have $\tilde{P}_{\tilde{d}} \asymp \mathfrak{n}\,\mathfrak{m}^{-\tilde{d}}$. Moreover, for all $i \geqslant \tilde{d}$, we have $\tilde{P}_i \preccurlyeq \mathfrak{n}\,\mathfrak{m}^{-i}$. Hence, for any $i > \tilde{d}$ and $\tilde{\mathfrak{m}} \prec \mathfrak{m}$, we have $\tilde{P}_i\,\tilde{\mathfrak{m}}^i \prec \tilde{P}_{\tilde{d}}\,\tilde{\mathfrak{m}}^{\tilde{d}}$. This shows that the Newton degree of (3.16) is at most $\tilde{d}$.

Let us now show that the Newton degree of (3.16) is precisely $\tilde{d}$. Choose $\tilde{\mathfrak{m}} \prec \mathfrak{m}$ large enough, so that

$$\tilde{\mathfrak{m}} \succcurlyeq \sqrt[\tilde{d}-i]{\frac{\mathfrak{d}_{\tilde{P}_i(\tilde{f})}}{\mathfrak{d}_{\tilde{P}_{\tilde{d}}(\tilde{f})}}}$$

for all $i < \tilde{d}$. Then $\deg N_{\tilde{P},\tilde{\mathfrak{m}}} = \tilde{d}$. □

If one step of the Newton polygon method does not suffice to decrease the Newton degree, then two steps do, when applying the trick from the next lemma:

Lemma 3.7. *Let d be the Newton degree of* (3.13)*. If f admits a unique starting monomial $\mathfrak{m}$ and $N_{P,\mathfrak{m}}$ a unique root c of multiplicity d, then*

a) *The equation*

$$P^{(d-1)}(\varphi) = 0 \quad (\varphi \prec \mathfrak{v}) \tag{3.18}$$

is quasi-linear and its unique solution satisfies $\varphi = c\,\mathfrak{m} + o(\mathfrak{m})$.

b) *The Newton degree of any refinement*

$$\tilde{f} = \tilde{\varphi} + \tilde{\tilde{f}} \quad (\tilde{\tilde{f}} \prec \tilde{\tilde{\mathfrak{v}}})$$

relative to (3.16) *with $\tilde{\tilde{\mathfrak{v}}} = \mathfrak{d}_{\tilde{\tilde{\varphi}}}$ is strictly inferior to d.*

Proof. Notice first that $N_{P',\mathfrak{m}} = N_{P,\mathfrak{m}}'$ for all polynomials P and monomials $\mathfrak{m}$. Consequently, (3.18) is quasi-linear and c is a single root of $N_{P^{(d-1)},\mathfrak{m}}$. This proves ($a$).

As to (b), we first observe that $\tilde{P}_{d-1} = P^{(d-1)}(\varphi) = 0$. Given $\tilde{\mathfrak{m}} \prec \tilde{\mathfrak{v}}$, it follows that $N_{\tilde{P},\mathfrak{m},d-1} = 0$. In particular, there do not exist $\alpha \neq 0$, $\beta \neq 0$ with $N_{\tilde{P},\tilde{\mathfrak{m}}}(\tilde{c}) = \alpha\,(\tilde{c} - \beta)^d$. In other words, $N_{\tilde{P},\tilde{\mathfrak{m}}}$ does not admit roots of multiplicity d. We conclude by lemma 3.6. □

3.3.3 Resolution of asymptotic polynomial equations

Theorem 3.8. *Let C be an algebraically closed field of characteristic zero and $\mathfrak{M}$ a totally ordered monomial group with $\mathbb{Q}$-powers. Then $C[[\mathfrak{M}]]$ is algebraically closed.*

Proof. Consider the following algorithm:

Algorithm `polynomial_solve`
Input: An asymptotic polynomial equation (3.13).
Output: The set of solutions to (3.13).

1. Compute the starting terms $c_1\,\mathfrak{m}_1, \ldots, c_\nu\,\mathfrak{m}_\nu$ of f relative to (3.13).
2. If $\nu = 1$ and c_1 is a root of multiplicity d of $N_{P,\mathfrak{m}_1}$, then let φ be the unique solution to (3.18). Refine (3.15) and apply `polynomial_solve` to (3.16). Return the so obtained solutions to (3.13).

3. For each $1 \leqslant i \leqslant \nu$, refine

$$f = c_i\, \mathfrak{m}_i + \tilde{f} \quad (\tilde{f} \prec \mathfrak{m}_i)$$

and apply `polynomial_solve` to the new equation in $\tilde{f}$. Collect and return the so obtained solutions to (3.13), together with 0, if P is divisible by f.

The correctness of `polynomial_solve` is clear; its termination follows from lemmas 3.6(b) and 3.7(b). Since C is algebraically closed, all Newton polynomials considered in the algorithm split over C. Hence, `polynomial_solve` returns d solutions to (3.13) in $C[[\mathfrak{M}]]$, when counting with multiplicities. In particular, when taking $\mathfrak{v} = \top_{\mathfrak{M}} \succ \mathfrak{M}$, we obtain n solutions, so $C[[\mathfrak{M}]]$ is algebraically closed. □

Corollary 3.9. *Let C be a real closed field and $\mathfrak{M}$ a totally ordered monomial group with $\mathbb{Q}$-powers. Then $C[[\mathfrak{M}]]$ is real closed.*

Proof. By the theorem, a polynomial equation $P(n) = 0$ of degree n over $C[[\mathfrak{M}]]$ admits n solutions in $C[\mathrm{i}][[\mathfrak{M}]]$, when counting with multiplicities. Moreover, each root $\varphi \in C[\mathrm{i}][[\mathfrak{M}]] \setminus C[[\mathfrak{M}]]$ is imaginary, because

$$\mathrm{i} = \frac{\varphi - \operatorname{Re}\varphi}{\operatorname{Im}\varphi} \in C[[\mathfrak{M}]][\varphi]$$

for such φ. Therefore all real roots of P are in $C[[\mathfrak{M}]]$. □

Corollary 3.10. *The field $C[[z^{\mathbb{Q}}]]$ of Puiseux series over an algebraically resp. real closed field C is algebraically resp. real closed.* □

Exercise 3.5. Consider an asymptotic algebraic equation (3.13) of Newton degree d. Let $\tau_1, \ldots, \tau_k$ be the starting terms of (3.13), with multiplicities $\mu_1, \ldots, \mu_k$. Prove that

$$\mu_1 + \cdots + \mu_k \leqslant d.$$

Also show that $\mu_1 + \cdots + \mu_k = d$ if C is algebraically closed.

Exercise 3.6.

a) Show that the computation of all solutions to (3.13) can be represented by a finite tree, whose non-root nodes are labeled by refinements. Applied to (3.1), this would yields the following tree:

$$f = \tilde{f}$$

$$\begin{array}{ccccc} f = -z^2 + \tilde{f} & f = -z + \tilde{f} & f = 1 + \tilde{f} & f = -\mathrm{i}\, z^{-3/2} + \tilde{f} & f = \mathrm{i}\, z^{-3/2} + \tilde{f} \\ \tilde{f} \prec z^2 & \tilde{f} \prec z & \tilde{f} \prec 1 & \tilde{f} \prec z^{-3/2} & \tilde{f} \prec z^{-3/2} \end{array}$$

$$\begin{array}{cc} f = 1 - \mathrm{i}\, z^{1/2} + \tilde{f} & f = 1 + \mathrm{i}\, z^{1/2} + \tilde{f} \\ \tilde{f} \prec z^{1/2} & \tilde{f} \prec z^{1/2} \end{array}$$

b) Show that the successors of each node may be ordered in a natural way, if C is a real field, and if we restrict our attention to real algebraic solutions. Prove that the natural ordering on the leaves, which is induced by this ordering, corresponds to the usual ordering of the solutions.

Exercise 3.7.

a) Generalize the results of this chapter to asymptotic equations of infinite degree in f, but of finite Newton degree.
b) Give an example of an asymptotic equation of infinite degree in f, with infinitely many solutions.

Exercise 3.8. Consider an asymptotic polynomial equation

$$P(f)=0 \ (f \prec \mathfrak{v})$$

of Newton degree d, with $P \in C[[\mathfrak{M}]][F]$ and $\mathfrak{v} \in \mathfrak{M}$. Consider the monomial monoid $\mathfrak{U}=\mathfrak{M} \times F^{\mathbb{N}}$ with

$$\mathfrak{m}\, F^i \prec 1 \Leftrightarrow \mathfrak{m}\, \mathfrak{v}^i \prec 1 \vee (\mathfrak{m}\, \mathfrak{v}^i = 1 \wedge i > 0).$$

a) Show that there exists a unique invertible series $u \in C[[\mathfrak{U}]]$ such that $\tilde{P} = u\, P$ is a monoic polynomial in $C[[\mathfrak{M}]][F]$.
b) Show that $\deg \tilde{P} = d$.

3.4 Cartesian representations

In this section, we show that grid-based series may be represented by (finite sums of) multivariate Laurent series in which we substitute an infinitesimal monomial for each variable. Such representations are very useful for finer computations with grid-based series.

3.4.1 Cartesian representations

Let $C[[\mathfrak{M}]]$ be a grid-based algebra. A *Cartesian representation* for a series $f \in C[[\mathfrak{M}]]$ is a multivariate Laurent series $\check{f} \in C((\check{\mathfrak{z}}_1, \ldots, \check{\mathfrak{z}}_k))$, such that $f = \hat{\varphi}(\check{f})$ for some morphism of monomial monoids $\varphi\colon \check{\mathfrak{z}}_1^{\mathbb{Z}} \cdots \check{\mathfrak{z}}_k^{\mathbb{Z}} \rightarrow \mathfrak{M}$. Writing $\check{f} = \check{g}\, \check{\mathfrak{z}}_1^{\alpha_1} \cdots \check{\mathfrak{z}}_k^{\alpha_k}$, with $\check{g} \in C[[\check{\mathfrak{z}}_1, \ldots, \check{\mathfrak{z}}_k]]$, we may also interpret f as the product of a "series" $\hat{\varphi}(\check{g})$ in $\varphi(\check{\mathfrak{z}}_1), \ldots, \varphi(\check{\mathfrak{z}}_k)$ and the monomial $\mathfrak{m} = \varphi(\check{\mathfrak{z}}_1^{\alpha_1} \cdots \check{\mathfrak{z}}_k^{\alpha_k})$.

More generally, a *semi-Cartesian representation* for $f \in C[[\mathfrak{M}]]$ is an expression of the form

$$f = \hat{\varphi}(\check{g}_1)\, \mathfrak{m}_1 + \cdots + \hat{\varphi}(\check{g}_l)\, \mathfrak{m}_l\,,$$

where $g_1, \ldots, g_l \in C[[\check{\mathfrak{z}}_1, \ldots, \check{\mathfrak{z}}_k]]$, $\mathfrak{m}_1, \ldots, \mathfrak{m}_l \in \mathfrak{M}$ and $\varphi\colon \check{\mathfrak{z}}_1^{\mathbb{N}} \cdots \check{\mathfrak{z}}_k^{\mathbb{N}} \rightarrow \mathfrak{M}$ is a morphism of monomial monoids.

Proposition 3.11.

a) Any grid-based series $f \in C[[\mathfrak{M}]]$ admits a semi-Cartesian representation.
b) If $\mathfrak{M}$ is a monomial group, which is generated by its infinitesimal elements, then each grid-based series $f \in C[[\mathfrak{M}]]$ admits a Cartesian representation.

Proof.

a) Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \in \mathfrak{M}^{\prec}$ and $\mathfrak{n}_1, \ldots, \mathfrak{n}_l \in \mathfrak{M}$ be such that

$$\operatorname{supp} f \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}^* \{\mathfrak{n}_1, \ldots, \mathfrak{n}_l\}.$$

For each $\mathfrak{v} \in \operatorname{supp} f$, let

$$n_{\mathfrak{v}} = \operatorname{card} \{(\alpha_1, \ldots, \alpha_k, i) \in \mathbb{N}^k \times \{1, \ldots, l\}: \mathfrak{v} = \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_k^{\alpha_k} \mathfrak{n}_i\}.$$

Let

$$\breve{g}_i = \sum_{\alpha_1, \ldots, \alpha_k \in \mathbb{N}^k} \frac{f_{\mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_k^{\alpha_k} \mathfrak{n}_i}}{n_{\mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_k^{\alpha_k} \mathfrak{n}_i}} \breve{\mathfrak{z}}_1^{\alpha_1} \cdots \breve{\mathfrak{z}}_k^{\alpha_k}$$

for all $1 \leqslant i \leqslant l$ and let $\varphi: \breve{\mathfrak{z}}_1^{\mathbb{N}} \cdots \breve{\mathfrak{z}}_k^{\mathbb{N}} \to \mathfrak{M}, \breve{\mathfrak{z}}_1^{\alpha_1} \cdots \breve{\mathfrak{z}}_k^{\alpha_k} \mapsto \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_k^{\alpha_k}$. Then

$$f = \hat{\varphi}(\breve{g}_1)\, \mathfrak{n}_1 + \cdots + \hat{\varphi}(\breve{g}_l)\, \mathfrak{n}_l.$$

b) For certain $\mathfrak{m}_{k+1}, \ldots, \mathfrak{m}_p \in \mathfrak{M}^{\prec}$ and $\beta_{i,j} \in \mathbb{Z}$, we may write

$$\mathfrak{n}_i = \mathfrak{m}_{k+1}^{\beta_{i,k+1}} \cdots \mathfrak{m}_p^{\beta_{i,p}},$$

for all $1 \leqslant i \leqslant l$. Let $\psi: \breve{\mathfrak{z}}_1^{\mathbb{Z}} \cdots \breve{\mathfrak{z}}_p^{\mathbb{Z}} \to \mathfrak{M}, \breve{\mathfrak{z}}_1^{\alpha_1} \cdots \breve{\mathfrak{z}}_p^{\alpha_p} \mapsto \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_p^{\alpha_p}$ and

$$\breve{f} = \sum_{i=1}^{l} \breve{g}_i\, \breve{\mathfrak{z}}_{k+1}^{\beta_{i,k+1}} \cdots \breve{\mathfrak{z}}_p^{\beta_{i,p}}.$$

Then $f = \hat{\psi}(\breve{f})$. □

Cartesian or semi-Cartesian representations $f_1 = \hat{\varphi}_1(\breve{f}_1)$ and $f_2 = \hat{\varphi}_2(\breve{f}_2)$ are said to be *compatible*, if $\breve{f}_1$ and $\breve{f}_2$ belong to the same algebra $C((\breve{\mathfrak{z}}_1, \ldots, \breve{\mathfrak{z}}_k))$ of Laurent series, and if $\varphi_1 = \varphi_2$.

Proposition 3.12.

a) *Any $f_1, \ldots, f_n \in C[[\mathfrak{M}]]$ admit compatible semi-Cartesian representations.*
b) *If $\mathfrak{M}$ is a monomial group, which is generated by its infinitesimal elements, then any $f_1, \ldots, f_n \in C[[\mathfrak{M}]]$ admit compatible Cartesian representations.*

Proof. By the previous proposition, $f_1, \ldots, f_n$ admit semi-Cartesian representations $f_i = \hat{\varphi}_i(\breve{f}_i)$, where $\breve{f}_i \in C((\breve{\mathfrak{z}}_{i,1}, \ldots, \breve{\mathfrak{z}}_{i,k_i}))$ and $\varphi_i: \breve{\mathfrak{z}}_{i,1}^{\mathbb{N}} \cdots \breve{\mathfrak{z}}_{i,k_i}^{\mathbb{N}} \to \mathfrak{M}$ for each i. Now consider

$$\begin{aligned} \psi: \prod_{i=1}^{n} \prod_{j=1}^{k_i} \breve{\mathfrak{z}}_{i,j}^{\mathbb{N}} &\longrightarrow \mathfrak{M} \\ \prod_{i=1}^{n} \prod_{j=1}^{k_i} \breve{\mathfrak{z}}_{i,j}^{\alpha_{i,j}} &\longmapsto \prod_{i=1}^{n} \hat{\varphi}_i\left(\prod_{j=1}^{k_i} \breve{\mathfrak{z}}_{i,j}^{\alpha_{i,j}} \right). \end{aligned}$$

Then $f_i = \hat{\psi}(\breve{F}_i)$ for each i, where $\breve{F}_i$ is the image of $\breve{f}_i$ under the natural inclusion of $C((\breve{\mathfrak{z}}_{i,1}, \ldots, \breve{\mathfrak{z}}_{i,k_i}))$ into $C((\breve{\mathfrak{z}}_{1,1}, \ldots, \breve{\mathfrak{z}}_{1,k_1}, \ldots, \breve{\mathfrak{z}}_{n,1}, \ldots, \breve{\mathfrak{z}}_{n,k_n}))$. This proves (*a*); part (*b*) is proved in a similar way. □

3.4.2 Inserting new infinitesimal monomials

In proposition 3.12 we drastically increased the size of the Cartesian basis in order to obtain compatible Cartesian representations. The following lemma is often useful, if one wants to keep this size as low as possible.

Lemma 3.13. *Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_k, \mathfrak{m}_1, \ldots, \mathfrak{m}_l$ be infinitesimal elements of a totally ordered monomial group $\mathfrak{M}$ with $\mathbb{Q}$-powers, such that $\mathfrak{m}_1, \ldots, \mathfrak{m}_l \in \mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_k^{\mathbb{Z}}$. Then there exist infinitesimal $\mathfrak{z}_1', \ldots, \mathfrak{z}_k' \in \mathfrak{z}_1^{\mathbb{Q}} \cdots \mathfrak{z}_k^{\mathbb{Q}}$ with $\mathfrak{z}_1, \ldots, \mathfrak{z}_k, \mathfrak{m}_1, \ldots, \mathfrak{m}_l \in (\mathfrak{z}_1')^{\mathbb{N}} \cdots (\mathfrak{z}_k')^{\mathbb{N}}$.*

Proof. It suffices to prove the lemma for $l = 1$; the general case follows by induction over l. The case $l = 1$ is proved by induction over k. For $k = 0$, there is nothing to prove. So assume that $k \geqslant 1$ and let $\mathfrak{m}_1 = \mathfrak{z}_1^{\alpha_1} \cdots \mathfrak{z}_k^{\alpha_k}$ with $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}$. Without loss of generality, we may assume that $\alpha_k > 0$, modulo a permutation of variables. Putting $\mathfrak{n} = \mathfrak{z}_1^{\alpha_1} \cdots \mathfrak{z}_{k-1}^{\alpha_{k-1}}$, we now distinguish the following three cases:

1. If $\mathfrak{n} \prec 1$, then there exist infinitesimal $\mathfrak{z}_1' \cdots \mathfrak{z}_{k-1}' \in \mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_{k-1}^{\mathbb{Z}}$, such that $\mathfrak{z}_1, \ldots, \mathfrak{z}_{k-1}, \mathfrak{n} \in (\mathfrak{z}_1')^{\mathbb{N}} \cdots (\mathfrak{z}_{k-1}')^{\mathbb{N}}$, by the induction hypothesis. Taking $\mathfrak{z}_k' = \mathfrak{z}_k$, we now have $\mathfrak{z}_k, \mathfrak{m}_1 = \mathfrak{n}\, \mathfrak{z}_k^{\alpha_k} \in (\mathfrak{z}_1')^{\mathbb{N}} \cdots (\mathfrak{z}_k')^{\mathbb{N}}$, since $\alpha_k > 0$.
2. If $\mathfrak{n} = 1$, then $\mathfrak{m}_1 = \mathfrak{z}_k^{\alpha_k}$, and we may take $\mathfrak{z}_1' = \mathfrak{z}_1, \ldots, \mathfrak{z}_k' = \mathfrak{z}_k$.
3. If $\mathfrak{n} \succ 1$, then there exists infinitesimal $\mathfrak{z}_1' \cdots \mathfrak{z}_{k-1}' \in \mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_{k-1}^{\mathbb{Z}}$, such that $\mathfrak{z}_1^{1/\alpha_k}, \ldots, \mathfrak{z}_{k-1}^{1/\alpha_k}, \mathfrak{n}^{-1/\alpha_k} \in (\mathfrak{z}_1')^{\mathbb{N}} \cdots (\mathfrak{z}_{k-1}')^{\mathbb{N}}$. Taking $\mathfrak{z}_k' = \mathfrak{z}_1^{\alpha_1/\alpha_k} \cdots \mathfrak{z}_{k-1}^{\alpha_{k-1}/\alpha_k} \mathfrak{z}_k$, we again have $\mathfrak{z}_k = \mathfrak{z}_k' \, \mathfrak{n}^{-1/\alpha_k}, \mathfrak{m}_1 = (\mathfrak{z}_k')^{\alpha_k} \in (\mathfrak{z}_1')^{\mathbb{N}} \cdots (\mathfrak{z}_k')^{\mathbb{N}}$. □

When doing computations on grid-based series in $C[[\mathfrak{M}]]$, one often works with respect to a *Cartesian basis* $\mathfrak{Z} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_k)$ of infinitesimal elements in $\mathfrak{M}$. Each time one encounters a series $f \in C[[\mathfrak{M}]]$ which cannot be represented by a series in $C((\check{\mathfrak{z}}_1, \ldots, \check{\mathfrak{z}}_k))$, one has to replace $\mathfrak{Z}$ by a *wider* Cartesian basis $\mathfrak{Z}' = (\mathfrak{z}_1', \ldots, \mathfrak{z}_{k'}')$ with $\mathfrak{z}_1, \ldots, \mathfrak{z}_k \in (\mathfrak{z}_1')^{\mathbb{N}} \cdots (\mathfrak{z}_{k'}')^{\mathbb{N}}$. The corresponding mapping $C((\check{\mathfrak{z}}_1, \ldots, \check{\mathfrak{z}}_k)) \to C((\check{\mathfrak{z}}_1', \ldots, \check{\mathfrak{z}}_{k'}'))$ is called a *widening*. Lemma 3.13 enables us to keep the Cartesian basis reasonably small during the computation.

3.5 Local communities

Let C be a ring and $\mathfrak{M}$ a monomial group which is generated by its infinitesimal elements. Given a set $A_k \subseteq C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]$ for each $k \in \mathbb{N}$, we denote by $C[[\mathfrak{M}]]_A$ the set of all grid-based series $f \in C[[\mathfrak{M}]]$, which admit a Cartesian representation $\check{f} \in A_k \, \mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_k^{\mathbb{Z}}$ for some $k \in \mathbb{N}$. In this section, we will show that if the A_k satisfy appropriate conditions, then many types of computations which can be carried out in $C[[\mathfrak{M}]]$ can also be carried out in $C[[\mathfrak{M}]]_A$.

3.5.1 Cartesian communities

Let C be a ring. A sequence $(A_k)_{k\in\mathbb{N}}$ with $A_k \subseteq C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]$ is said to be a *Cartesian community* over C, if the following conditions are satisfied:

CC1. $\mathfrak{z}_1 \in A_1$.
CC2. A_k is a C-algebra for each $k \in \mathbb{N}$.
CC3. The A_k are stable under strong monomial morphisms.

In **CC3**, a *strong monomial morphism* is strong C-algebra morphism which maps monomials to monomials. In our case, a monomial preserving strong morphism from $C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]$ into $C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_{k'}]]$ is always of the form

$$\begin{aligned}\sigma: C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]] &\longrightarrow C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_{k'}]];\\ f(\mathfrak{z}_1, \ldots, \mathfrak{z}_k) &\longmapsto f(\mathfrak{z}_1^{\alpha_{1,1}} \cdots \mathfrak{z}_{k'}^{\alpha_{1,k'}}, \ldots, \mathfrak{z}_1^{\alpha_{k,1}} \cdots \mathfrak{z}_{k'}^{\alpha_{k,k'}}),\end{aligned}$$

where $\alpha_{i,j} \in \mathbb{N}$ and $\sum_j \alpha_{i,j} \neq 0$ for all i. In particular, **CA3** implies that the A_k are stable under widenings.

Proposition 3.14. *Let $(A_k)_{k\in\mathbb{N}}$ be a Cartesian community over C and let $\mathfrak{M}$ be a monomial group. Then $C[[\mathfrak{M}]]_A$ is a C-algebra.*

Proof. We clearly have $C \subseteq C[[\mathfrak{M}]]_A$. Let $\hat{f}, \hat{g} \in C[[\mathfrak{M}]]_A$. Mimicking the proof of proposition 3.12, we observe that f and g admit compatible Cartesian representations $f, g \in A_k\, \mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_k^{\mathbb{Z}}$. Then $f + g$, $f - g$ and $f g$ are Cartesian representations of $\hat{f} + \hat{g}$, $\hat{f} - \hat{g}$ resp. $\hat{f} \hat{g}$. □

3.5.2 Local communities

A *local community* is a Cartesian community $(A_k)_{k\in\mathbb{N}}$, which satisfies the following additional conditions:

LC1. For each $f \in A_k$ with $[\mathfrak{z}_k^0] f = 0$, we have $f/\mathfrak{z}_k \in A_k$.
LC2. Given $g \in A_k$ and $f_1, \ldots, f_k \in A_l^{\prec}$, we have $g \circ (f_1, \ldots, f_k) \in A_l$.
LC3. Given $f \in A_{k+1}$ with $[\mathfrak{z}_1^0 \cdots \mathfrak{z}_{k+1}^0] f = 0$ and $[\mathfrak{z}_1^0 \cdots \mathfrak{z}_k^0\, \mathfrak{z}_{k+1}^1] f \in C^*$, the unique series $\varphi \in C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]$ with $f \circ (\mathfrak{z}_1, \ldots, \mathfrak{z}_k, \varphi) = 0$ belongs to A_k.

In **LC1** and **LC3**, the notation $[\mathfrak{z}_1^{\alpha_1} \cdots \mathfrak{z}_p^{\alpha_p}] f$ stands for the coefficient of $\mathfrak{z}_1^{\alpha_1} \cdots \mathfrak{z}_p^{\alpha_p}$ in f. The condition **LC3** should be considered as an implicit function theorem for the local community. Notice that A_k is stable under $\partial/\partial \mathfrak{z}_i$ for all $\{i \in 1, \ldots, k\}$, since

$$\frac{\partial f}{\partial \mathfrak{z}_i} = \frac{f \circ (\mathfrak{z}_1, \ldots, \mathfrak{z}_i + \mathfrak{z}_{k+1}, \ldots, \mathfrak{z}_k) - f}{\mathfrak{z}_{k+1}} \circ (\mathfrak{z}_1, \ldots, \mathfrak{z}_k, 0). \tag{3.19}$$

Remark 3.15. In [vdH97], the conditions **LC2** and **LC3** were replaced by a single, equivalent condition: given $f \in A_{k+1}$ as in **LC3**, we required that $\operatorname{im} \varphi \subseteq A_k$, for the unique strong C-algebra morphism φ: $C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_{k+1}]] \to C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]$, such that $\varphi_{|C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]} = \mathrm{Id}_{C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]}$ and $\varphi(f) = 0$. We also explicitly requested the stability under differentiation, although (3.19) shows that this is superfluous.

Example 3.16. Let C be a subfield of $\mathbb{C}$ and let $A_k = C\{\{\mathfrak{z}_1, \ldots, \mathfrak{z}_k\}\}$ be the set of convergent power series in k variables over C, for each $k \in \mathbb{N}$. Then the A_k form a local community. If $\mathfrak{M}$ is a monomial group, then $C\{\!\{\mathfrak{M}\}\!\} = C[\![\mathfrak{M}]\!]_A$ will also be called the set of *convergent grid-based series* in $\mathfrak{M}$ over C.

Example 3.17. For each $k \in \mathbb{N}$, let A_k be the set of power series in $C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]$, which satisfy an algebraic equation over $C[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]$. Then the A_k form a local community.

3.5.3 Faithful Cartesian representations

In this and the next section, $A = (A_k)_{k \in \mathbb{N}}$ is a local community. A Cartesian representation $f \in C((\mathfrak{z}_1, \ldots, \mathfrak{z}_k))$ is said to be *faithful*, if for each dominant monomial $\mathfrak{d}$ of f, there exists a dominant monomial $\hat{\mathfrak{d}}'$ of $\hat{f}$, with $\hat{\mathfrak{d}} \preccurlyeq \hat{\mathfrak{d}}'$.

Proposition 3.18. *Let $(A_i)_{i \in \mathbb{N}}$ be a local community and $f \in A_k$. Then*

a) For each $1 \leqslant i \leqslant k$ and $\alpha \in \mathbb{Z}$, we have $[\mathfrak{z}_k^\alpha] f \in A_{k-1}$.
b) For each initial segment $\mathfrak{I} \subseteq \mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_k^{\mathbb{Z}}$, we have

$$f_{\mathfrak{I}} = \sum_{\mathfrak{m} \in \mathfrak{I}} f_{\mathfrak{m}} \, \mathfrak{m} \in A_k.$$

Proof. For each α, let $f_\alpha = [\mathfrak{z}_k^\alpha] f$. We will prove (a) by a weak induction over α. If $\alpha = 0$, then $[\mathfrak{z}_k^0] f = f \circ (\mathfrak{z}_1, \ldots, \mathfrak{z}_{k-1}, 0) \in A_{k-1}$. If $\alpha > 0$, then

$$[\mathfrak{z}_k^\alpha] f = \frac{f - ([\mathfrak{z}_k^0] f) \, \mathfrak{z}_k^0 - \cdots - ([\mathfrak{z}_k^{\alpha-1}] f) \, \mathfrak{z}_k^{\alpha-1}}{\mathfrak{z}_k^\alpha}.$$

By the weak induction hypothesis and **LC1**, we thus have $[\mathfrak{z}_k^\alpha] f \in A_k$.

In order to prove (b), let $\mathfrak{D} = \{\mathfrak{d}_1, \ldots, \mathfrak{d}_l\}$ be the finite anti-chain of maximal elements of $\mathfrak{I}$, so that $\mathfrak{I} = \operatorname{in}(\mathfrak{d}_1, \ldots, \mathfrak{d}_l)$. Let n be the number of variables which effectively occur in $\mathfrak{D}$, i.e. the number of $i \in \{1, \ldots, k\}$, such that $\mathfrak{d}_j = \mathfrak{z}_1^{\alpha_1} \cdots \mathfrak{z}_k^{\alpha_k}$ with $\alpha_i \neq 0$ for some j. We prove (b) by weak induction over n. If $n = 0$, then either $l = 0$ and $f_{\mathfrak{I}} = 0$, or $l = 1$, $\mathfrak{d}_1 = \{1\}$ and $f_{\mathfrak{I}} = f$.

Assume now that $n > 0$ and order the variables $\mathfrak{z}_1, \ldots, \mathfrak{z}_k$ in such a way that $\mathfrak{z}_k$ effectively occurs in one of the $\mathfrak{d}_i$. For each $\alpha \in \mathbb{N}$, let

$$\begin{aligned} \mathfrak{I}_\alpha &= \{\mathfrak{m} \in \mathfrak{z}_1^{\mathbb{N}} \cdots \mathfrak{z}_{k-1}^{\mathbb{N}} : \mathfrak{m} \, \mathfrak{z}_k^\alpha \in \mathfrak{I}\}; \\ \mathfrak{D}_\alpha &= \{\mathfrak{m} \in \mathfrak{z}_1^{\mathbb{N}} \cdots \mathfrak{z}_{k-1}^{\mathbb{N}} : \mathfrak{m} \, \mathfrak{z}_k^\alpha \in \mathfrak{D}\}. \end{aligned}$$

We observe that

$$\mathfrak{I}_\alpha = \operatorname{in}(\mathfrak{D}_0 \amalg \cdots \amalg \mathfrak{D}_\alpha) \cap \mathfrak{z}_1^{\mathbb{N}} \cdots \mathfrak{z}_{k-1}^{\mathbb{N}}.$$

In particular, if ν is maximal with $\mathfrak{D}_\nu \neq \varnothing$, then $\mathfrak{I}_\alpha = \mathfrak{I}_\nu$ for all $\alpha \geqslant \nu$ and

$$\mathfrak{I} = \mathfrak{I}_0 \amalg \cdots \amalg \mathfrak{I}_{\nu-1} \, \mathfrak{z}_k^{\nu-1} \amalg \mathfrak{I}_\nu \, \mathfrak{z}_k^{\nu+\mathbb{N}},$$

so that

$$f_{\mathfrak{I}} = f_{0,\mathfrak{I}_0}\mathfrak{z}_k^0 + \cdots + f_{\nu-1,\mathfrak{I}_{\nu-1}}\mathfrak{z}_k^{\nu-1} + \left(\frac{f - f_0\mathfrak{z}_k^0 - \cdots - f_{k-1}\mathfrak{z}_k^{\nu-1}}{\mathfrak{z}_k^\nu}\right)_{\mathfrak{I}_\nu\mathfrak{z}_k^{\mathbb{N}}} \mathfrak{z}_k^\nu.$$

Moreover, for each α, at most $n-1$ variables effectively occur in the set $\mathfrak{D}_0 \amalg \cdots \amalg \mathfrak{D}_\alpha$ of dominant monomials of $\mathfrak{I}_\alpha$. Therefore $f_{\mathfrak{I}} \in A_k$, by the induction hypothesis. □

Proposition 3.19. *Given a Cartesian representation*

$$f \in A_k\,\mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_k^{\mathbb{Z}}$$

of a series $\hat{f} \in C[[\mathfrak{M}]]$, *its truncation*

$$\tilde{f} = f_{\{\mathfrak{m}\in\mathfrak{z}_1^{\mathbb{N}}\cdots\mathfrak{z}_k^{\mathbb{N}} : \exists \hat{\mathfrak{n}}\in\operatorname{supp}\hat{f},\ \hat{\mathfrak{m}}\preccurlyeq\hat{\mathfrak{n}}\}} \in A_k\,\mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_k^{\mathbb{Z}}$$

is a faithful Cartesian representation of the same series $\hat{f}$. □

3.5.4 Applications of faithful Cartesian representations

Proposition 3.20. *Let* $\hat{f} \in C[[\mathfrak{M}]]_A$ *be series, which is either*

a) *infinitesimal,*
b) *bounded, or*
c) *regular.*

Then $\hat{f}$ *admits a Cartesian representation in* $A_k\mathfrak{z}_1^{\mathbb{Z}}\cdots\mathfrak{z}_k^{\mathbb{Z}}$ *for some* $k\in\mathbb{N}$, *which is also infinitesimal, bounded resp. regular.*

Proof. Assume that $\hat{f}$ is infinitesimal and let $f \in A_k\,\mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_k^{\mathbb{Z}}$ be a faithful Cartesian representation of $\hat{f}$, with dominant monomials $\mathfrak{d}_1, \ldots, \mathfrak{d}_l \prec 1$. For each $i \in \{1, \ldots, l\}$, let

$$f_i = f_{\operatorname{in}(\mathfrak{d}_1,\ldots,\mathfrak{d}_i)} - f_{\operatorname{in}(\mathfrak{d}_1,\ldots,\mathfrak{d}_{i-1})} \in A_k\,\mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_k^{\mathbb{Z}},$$

with $\mathfrak{d}_{f_i} = \mathfrak{d}_i$. Then $f = f_1 + \cdots + f_l$ and

$$\tilde{f} = \sum_{i=1}^{l} f_i\,\frac{\mathfrak{d}_1}{\mathfrak{d}_i}\,\mathfrak{z}_{k+i}$$

is an infinitesimal Cartesian representation of $\hat{f}$ in A_{k+l}, when setting $\hat{\mathfrak{z}}_{k+i} = \hat{\mathfrak{d}}_i/\hat{\mathfrak{d}}_1$ for each $i \in \{1, \ldots, l\}$. This proves (a).

If $\hat{f}$ is bounded, then let $g \in A_k$ be an infinitesimal Cartesian representation of $\hat{g} = \hat{f} - \hat{f}_{\{1\}}$. Now $f = g + \hat{f}_{\{1\}}\,\mathfrak{z}_1^0 \cdots \mathfrak{z}_k^0 \in A_k$ is a bounded Cartesian representation of $\hat{f}$. This proves (b).

Assume finally that $\hat{f} \neq 0$ is regular, with dominant monomial $\hat{\mathfrak{d}}$. Let $g \in A_k$ be a bounded Cartesian representation of $\hat{g} = \hat{f}/\hat{\mathfrak{d}}$. Since $\hat{g}_0 \neq 0$, the series g is necessarily regular. Now take a Cartesian monomial $\mathfrak{d}$ which represents $\hat{\mathfrak{d}}$ (e.g. among the dominant monomials of a faithful Cartesian representation of $\hat{\mathfrak{d}}$). Then $f = g\,\mathfrak{d}$ is a regular Cartesian representation of $\hat{f}$. □

3.5.5 The Newton polygon method revisited

Theorem 3.21. *Let $(A_k)_{k\in\mathbb{N}}$ be a local community over a ring C and let $\mathfrak{M}$ be a monomial monoid. Consider the polynomial equation*

$$\hat{P}_n \hat{f}^n + \cdots + \hat{P}_0 = 0 \tag{3.20}$$

with coefficients $\hat{P}_0, \ldots, \hat{P}_n \in C[\![\mathfrak{M}_{\preccurlyeq}]\!]_A$, such that $(\hat{P}_0)_1 = 0$ and $(\hat{P}_1)_1 \in C^*$. Then (3.20) *admits a unique solution in $C[\![\mathfrak{M}_{\prec}]\!]_A$.*

Proof. By proposition 3.20, there exist bounded Cartesian representations $P_0, \ldots, P_n \in A_k$ for certain $\hat{\mathfrak{z}}_1, \ldots, \hat{\mathfrak{z}}_k \in \mathfrak{M}$. Now consider the series

$$P = P_0 + P_1\,\mathfrak{z}_{k+1} + \cdots + P_n\,\mathfrak{z}_{k+1}^n \in A_{k+1}.$$

We have $[\mathfrak{z}_1^0 \cdots \mathfrak{z}_{k+1}^0] P = 0$ and $[\mathfrak{z}_1^0 \cdots \mathfrak{z}_k^0 \mathfrak{z}_{k+1}^1] P \in C^*$, so there exists a $f \in A_k$ with

$$P \circ (\mathfrak{z}_1, \ldots, \mathfrak{z}_k, f) = P_0 + P_1 f + \cdots + P_n f^n = 0,$$

by **LC3**. We conclude that $\hat{f} \in C[\![\mathfrak{M}]\!]_A$ satisfies $\hat{P}_n \hat{f}^n + \cdots + \hat{P}_0 = 0$. The uniqueness of $\hat{f}$ follows from theorem 3.3. □

Theorem 3.22. *Let $(A_k)_{k\in\mathbb{N}}$ be a local community over a (real) algebraically closed field C and $\mathfrak{M}$ a totally ordered monomial group with $\mathbb{Q}$-powers. Then $C[\![\mathfrak{M}]\!]_A$ is a (real) algebraically closed field.*

Proof. The proof is analogous to the proof of theorem 3.8. In the present case, theorem 3.21 ensures that $\varphi \in C[\![\mathfrak{M}]\!]_A$ in step 2 of `polynomial_solve`. □

Exercise 3.9. Let C be a ring, $\mathfrak{M}$ a monomial monoid and $(A_k)_{k\in\mathbb{N}}$ a local community. We define $C[\![\mathfrak{M}]\!]_A$ to be the set of series f in $C[\![\mathfrak{M}]\!]$, which admit a semi-Cartesian representation

$$f = \hat{\varphi}(\check{f}_1)\,\mathfrak{m}_1 + \cdots + \hat{\varphi}(\check{f}_p)\,\mathfrak{m}_p$$

with $\check{f}_1, \ldots, \check{f}_p \in A_k$ for some $k \in \mathbb{N}$, $\varphi\colon \check{\mathfrak{z}}_1^{\mathbb{N}} \cdots \check{\mathfrak{z}}_k^{\mathbb{N}} \to \mathfrak{M}$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_p \in \mathfrak{M}$. Which results from this section generalize to this more general setting?

Exercise 3.10. Let C be a field. A series f in $C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]$ is said to be *differentially algebraic*, if the field generated by its partial derivatives $\partial^{i_1+\cdots+i_k} f / (\partial\mathfrak{z}_1)^{i_1} \cdots (\partial\mathfrak{z}_k)^{i_k}$ has finite transcendence degree over C. Prove that the collection of such series forms a local community over C.

Exercise 3.11. Assume that C is an effective field, i.e. all field operations can be performed by algorithm. In what follows, we will measure the complexity of algorithms in terms of the number of such field operations.

a) A series $f \in C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]$ is said to be *effective*, if there is an algorithm which takes α_1, ..., $\alpha_k \in \mathbb{N}$ on input, and which outputs $f_{\alpha_1, \ldots, \alpha_k}$. Show that the collection of effective series form a local community.
b) An effective series $f \in C[[\mathfrak{z}_1, \ldots, \mathfrak{z}_k]]$ is said to be of *polynomial time complexity*, if there is an algorithm, which takes $n \in \mathbb{N}$ on input and which computes $f_{\alpha_1, \ldots, \alpha_n}$ for all $\alpha_1, \ldots, \alpha_n$ with $\alpha_1 + \cdots + \alpha_n \leqslant k$ in time $\binom{n+k}{n}^{O(1)}$. Show that the collection of such series forms a local community. What about even better time complexities?

Exercise 3.12. Let $(A_k)_{k \in \mathbb{N}}$ be a local community and let

$$f \in A_k \, \mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_k^{\mathbb{Z}}$$

be a Cartesian representation of an infinitesimal, bounded or regular grid-based series $\hat{f}$ in $C[[\mathfrak{M}]]$. Show that, modulo widenings, there exists an infinitesimal, bounded resp. regular Cartesian representation of $\hat{f}$, with respect to a Cartesian basis with at most k elements.

Exercise 3.13. Let $(A_k)_{k \in \mathbb{N}}$ be a local community over a field C.

a) If $f \in C[[\mathfrak{M}]]_{A, \prec}$ and $g \in A_1$, then show that $g \circ f \in C[[\mathfrak{M}]]_A$.
b) If $\mathfrak{M}$ is totally ordered, then prove that $C[[\mathfrak{M}]]_A$ is a field.

Exercise 3.14. Let $(A_k)_{k \in \mathbb{N}}$ be a local community over a field C and let $\mathfrak{M}$ be a totally ordered monomial group. Prove that $f_{\succ}, f_{\asymp}, f_{\prec} \in C[[\mathfrak{M}]]_A$ for any $f \in C[[\mathfrak{M}]]_A$, and

$$C[[\mathfrak{M}]]_A = C[[\mathfrak{M}]]_{A, \succ} \oplus C \oplus C[[\mathfrak{M}]]_{A, \prec}.$$

Exercise 3.15. Let $(A_k)_{k \in \mathbb{N}}$ be a Cartesian community. Given monomial groups $\mathfrak{M}$ and $\mathfrak{N}$, let $\mathscr{A}(C[[\mathfrak{M}]], C[[\mathfrak{N}]])$ be the set of strong C-algebra morphisms from $C[[\mathfrak{M}]]$ into $C[[\mathfrak{N}]]$ and $\mathscr{A}(C[[\mathfrak{M}]], C[[\mathfrak{N}]])_A$ the set of $\varphi \in \mathscr{A}(C[[\mathfrak{M}]], C[[\mathfrak{N}]])$, such that $\varphi(\mathfrak{m}) \in C[[\mathfrak{N}]]_A$ for all $\mathfrak{m} \in \mathfrak{M}$.

a) Given $\varphi \in \mathscr{A}(C[[\mathfrak{M}]], C[[\mathfrak{N}]])_A$ and $\psi \in \mathscr{A}(C[[\mathfrak{N}]], C[[\mathfrak{V}]])_A$, where $\mathfrak{V}$ is a third monomial group, prove that $\psi \circ \varphi \in \mathscr{A}(C[[\mathfrak{M}]], C[[\mathfrak{V}]])_A$.
b) Given $\varphi \in \mathscr{A}(C[[\mathfrak{M}]], C[[\mathfrak{N}]])_A$ and $\psi \in \mathscr{A}(C[[\mathfrak{N}]], C[[\mathfrak{M}]])$ such that $\psi \circ \varphi = \mathrm{Id}_{C[[\mathfrak{M}]]}$, prove that $\psi \in \mathscr{A}(C[[\mathfrak{N}]], C[[\mathfrak{M}]])_A$.

Exercise 3.16. Let C be a subfield of $\mathbb{C}$ and let $\mathfrak{M}$ and $\mathfrak{N}$ be monomial groups with $\mathfrak{M} \subseteq \mathfrak{N}$. Prove that $C\{\{\mathfrak{M}\}\} = C[[\mathfrak{M}]] \cap C\{\{\mathfrak{N}\}\}$. Does this property generalize to other local communities?

Exercise 3.17. Let $(A_k)_{k \in \mathbb{N}}$ be the local community from example 3.17 and let $\mathfrak{M}$ be a totally ordered monomial group. Prove that $C[[\mathfrak{M}]]_A$ is isomorphic to the algebraic closure of $C[\mathfrak{M}]$.

Exercise 3.18. Does theorem 3.22 still hold if we remove condition **LC2** in the definition of local communities?

Exercise 3.19. Consider the resolution of $P(f)=0$ $(f \prec \mathfrak{v})$, with $P \in C[[\mathfrak{M}]]_A$ and $\mathfrak{v} \in \mathfrak{M}$.

a) Given a starting term $c\,\mathfrak{m}$ of multiplicity d, let $\mathfrak{n}$ be minimal for $\preccurlyeq$ such that $P_i\,\mathfrak{m}^i \preccurlyeq \mathfrak{n}$ for all i. Show that there exist Cartesian coordinates $\mathfrak{z}_1, \ldots, \mathfrak{z}_k$ with $\mathfrak{m}, \mathfrak{n} \in \mathfrak{z}_1^{\mathbb{Z}} \cdots \mathfrak{z}_k^{\mathbb{Z}}$, in which $P_i\,\mathfrak{m}^i/\mathfrak{n}$ admits a bounded Cartesian representations u_i for all $0 \leqslant i \leqslant n = \deg P$.

b) Consider a bounded Cartesian representation $\varphi \in A_k$ with $\varphi \sim c$ and let $\tilde{u}_i = \sum_{k=i}^{n} \binom{k}{i} u_k\,\varphi^{k-i}$. Given $\mathfrak{w} \in \mathfrak{z}_1^{\mathbb{Q}^{\geqslant}} \cdots \mathfrak{z}_k^{\mathbb{Q}^{\geqslant}}$, let

$$Q_{\mathfrak{w}} = \sum_{i=0}^{n} \tilde{u}_{i,\mathfrak{w}^{d-i}}\,F^i.$$

Show that $Q = \sum_{\mathfrak{w}} Q_{\mathfrak{w}}\,\mathfrak{w}$ is a series in $C[F][[\mathfrak{z}_1^{1/d!}, \ldots, \mathfrak{z}_k^{1/d!}]]_A$.

c) For each $\mu \in \{0, \ldots, d\}$, let $\mathfrak{I}_\mu$ be initial segment generated by the $\mathfrak{w}$ such that $\operatorname{val} Q_{\mathfrak{w}} < \mu$, and $\mathfrak{F}_\mu$ its complement. We say that $\varphi_{\mathfrak{F}_\mu}$ is the part of multiplicity $\geqslant \mu$ of φ as a zero of $u_0 + \cdots + u_n F^n$. Show that $\varphi_{\mathfrak{F}_\mu} \in A_k$ can be determined effectively for all μ.

d) In `polynomial_solve`, show that refinements of the type

$$f = \hat{\varphi}\,\mathfrak{m} + \tilde{f} \;\; (\tilde{f} \prec \mathfrak{m}),$$

where $\varphi \in C_k$ is the unique solution to $\partial^{d-1}(u_0 + \cdots + u_n F^n)/\partial F^{d-1}$, may be replaced by refinements

$$f = \widehat{\varphi_{\mathfrak{F}_{d-1}}}\,\mathfrak{m} + \tilde{f} \;\; (\tilde{f} \prec \mathfrak{m}).$$