15

Branching Random Walk

Branching random walk serves as the mean-field model for many interacting models involving branching, including lattice trees and percolation. In Sect. 15.1, we consider a natural mean-field model of lattice trees, which turns out to be intimately related to branching random walk with Poisson offspring distribution. In Sect. 15.2 we compute several generating functions important for a particular model of branching random walk. These will be used to derive the scaling limit of the branching random walk model in Chap. 16. In Sect. 15.3, we define a model of weakly self-avoiding lattice trees in terms of branching random walk.

15.1 A Mean-Field Model

The mean-field model for the self-avoiding walk is simple random walk, which forgets about the self-avoidance interaction. It is natural to attempt to define a mean-field model of lattice trees by somehow forgetting about the mutual avoidance of the branches in a lattice tree. In this section, we define such a mean-field model, as in [61]. For simplicity, we consider only the nearest-neighbour model.

It is convenient to switch to a site activity (or "fugacity"), rather than a bond activity. This means that we weight vertices rather than bonds by z, so that the one-point function of (7.20) becomes now $g(z) = G_z^{(1)} = \sum_{T:T\ni 0} z^{|T|+1}$. If we remove the product containing the interaction in the two-point function (8.2), we obtain $\sum_{\omega\in\mathcal{W}(0,x)}(g(z))^{|\omega|+1}$. It is also useful to keep track of the length of the path ω by associating an activity ζ to each bond in this path. This prompts us to define the two-point function of the mean-field model by

$$F_{z,\zeta}(x) = \sum_{\omega\in\mathcal{W}(0,x)} (f(z))^{|\omega|+1} \left(\frac{\zeta}{2d}\right)^{|\omega|}, \tag{15.1}$$

where the function $f(z)$, to be specified below, is the one-point function for the mean-field model, and where the factor $\frac{1}{2d}$ has been included as a convenient normalization. The sum in (15.1) is taken over simple random walks. Thus, $F_{z,\zeta}(x)$ can be written in terms of the simple random walk two-point function (1.17) as $F_{z,\zeta}(x) = f(z)C_{\zeta f(z)/2d}(0,x)$, and hence, by (1.18), its Fourier transform is

$$\hat{F}_{z,\zeta}(k) = \frac{f(z)}{1-\zeta f(z)\hat{D}(k)}, \tag{15.2}$$

with $\hat{D}(k)$ given by (1.12). The random walk two-point function has critical value $\frac{1}{2d}$, corresponding to $\zeta f(z) = 1$. We will realize the latter with $\zeta = f(z) = 1$.

For the function $f(z)$, we require by analogy with (7.22) (taking into account the switch to site activity), that $f(z)$ satisfy the differential equation

$$\hat{F}_{z,1}(0) = z\frac{\mathrm{d}f(z)}{\mathrm{d}z}. \tag{15.3}$$

Combining (15.3) and (15.2) gives

$$\frac{f(z)}{1-f(z)} = z\frac{\mathrm{d}f(z)}{\mathrm{d}z}. \tag{15.4}$$

Integrating the separable equation (15.4) over an interval $[z, z_0]$ gives

$$\frac{f(z)\mathrm{e}^{-f(z)}}{f(z_0)\mathrm{e}^{-f(z_0)}} = \frac{z}{z_0}. \tag{15.5}$$

The initial condition $z_0 = 1$ and $f(z_0) = 1$ is a choice of normalization and gives

$$f(z)\mathrm{e}^{-f(z)} = z\mathrm{e}^{-1}. \tag{15.6}$$

By (15.6), f can be written as $f(z) = -W(-z\mathrm{e}^{-1})$, where W is the principal branch of the Lambert W function defined by $W(w)\mathrm{e}^{W(w)} = w$ [56]. The latter is analytic on the w-plane with branch cut $(-\infty, -\mathrm{e}^{-1}]$, corresponding to a branch cut $[1,\infty)$ for $f(z)$ (and gives $f(0) = 0$, as it should). This uniquely specifies $f(z)$, and hence $\hat{F}_{z,\zeta}(k)$. The *same* functions $f(z)$ and $\hat{F}_{z,\zeta}(k)$ will arise below in Theorem 15.2 for a model of branching random walk.

15.2 Branching Random Walk

In this section, we define a model of branching random walk in terms of embeddings of trees into $\mathbb{Z}^d$. The presentation is based on [30]. Some related ideas can be found in [36].

The trees are the family trees of the critical Galton–Watson branching process with Poisson offspring distribution. In more detail, we begin with a single individual having ξ offspring, where ξ is a Poisson random variable of mean 1, i.e., $\mathbb{P}(\xi = m) = (em!)^{-1}$. Each of the offspring then independently has offspring of its own, with the same critical Poisson distribution. To indicate when two trees are the same, we describe them in terms of *words*. The root is the word 0. The children of the root are the words $01, 02, \dots 0\xi_0$. The children of 01 are the words $011, \dots, 01\xi_{01}$, and so on. A tree is then uniquely represented by a set of words, and two trees are the same if and only if they are represented by the same set of words. A tree T consisting of exactly n individuals, with the i^{th} individual having ξ_i offspring, has probability

$$\mathbb{P}(T) = \prod_{i \in T} \frac{1}{\mathrm{e}\, \xi_i!} = \mathrm{e}^{-n} \prod_{i \in T} \frac{1}{\xi_i!}. \tag{15.7}$$

The product in (15.7) is over the vertices of T.

We define an embedding φ of T into $\mathbb{Z}^d$ to be a mapping from the vertices of T into $\mathbb{Z}^d$, such that the root is mapped to the origin and adjacent vertices in the tree are mapped to nearest neighbours in $\mathbb{Z}^d$. There is no assumption that φ is injective, and different vertices of T can be mapped to the same vertex in $\mathbb{Z}^d$. Given a tree T having $|T|$ vertices, there are $(2d)^{|T|-1}$ possible embeddings φ of T. A branching random walk configuration is then a pair (T, φ), with associated probability

$$\mathbb{P}(T, \varphi) = \frac{1}{(2d)^{|T|-1}} \mathbb{P}(T). \tag{15.8}$$

Our aim in this section is to compute the r-point functions of this branching random walk model. These are generating functions for trees of fixed total number of vertices n, which visit a specified set of $r-1$ vertices in a specified manner (the r^{th} point is the origin, where the embedding is rooted).

We begin with the simplest case $r = 1$. For $z \in \mathbb{C}$ with $|z| \leq 1$, the one-point function is defined by

$$b_z^{(1)} = \sum_{(T,\varphi)} \mathbb{P}(T, \varphi) z^{|T|} = \sum_T \mathbb{P}(T) z^{|T|}. \tag{15.9}$$

The series on the right hand side of (15.9) is the generating function for the probability mass function for the total size of a critical Poisson tree. It converges for $|z| \leq 1$, with $b_1^{(1)} = 1$. For general z, $b_z^{(1)}$ is given in the following theorem.

We write $p_m = \mathbb{P}(\xi = m) = (\mathrm{e}m!)^{-1}$, and let

$$P(w) = \sum_{m=0}^{\infty} p_m w^m = \mathrm{e}^{w-1} \tag{15.10}$$

denote the generating function for the critical Poisson distribution.

Theorem 15.1. *For $d \geq 1$, the one-point function is given by*

$$b_z^{(1)} = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \mathrm{e}^{-n} z^n, \tag{15.11}$$

and obeys

$$b_z^{(1)} \mathrm{e}^{-b_z^{(1)}} = z\mathrm{e}^{-1}. \tag{15.12}$$

Proof. Conditioning on the number of offspring of the root gives

$$b_z^{(1)} = \sum_{m=0}^{\infty} p_m z \left(b_z^{(1)}\right)^m = zP(b_z^{(1)}) = z\mathrm{e}^{b_z^{(1)}-1}, \tag{15.13}$$

which implies (15.12). The Taylor expansion (15.11) then follows from Lagrange's inversion formula (see, e.g., [196, p.43]). ∎

Since $b_z^{(1)}$ is real for real z and $b_0^{(1)} = 0$, it follows from (15.12) that $b_z^{(1)}$ is identical to the function $f(z)$ of the mean-field model of Sect. 15.1. Theorem 15.1 rederives the standard result that for the critical Poisson branching process,

$$\mathbb{P}(|T| = n) = \frac{n^{n-1}}{n!} \mathrm{e}^{-n}. \tag{15.14}$$

By Stirling's formula,

$$\mathbb{P}(|T| = n) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{n^{3/2}}. \tag{15.15}$$

Comparing with (7.2) and (7.3), this is a statement that the critical exponent θ takes the value $\theta = \frac{5}{2}$ for branching random walk.

The two-point function is a generating function for critical Poisson branching random walk which starts at the origin, which has a family tree whose total size is dual to an activity z, and which visits the vertex x (possibly more than once) at a time dual to an activity ζ. The two-point function is defined for $z, \zeta \in \mathbb{C}$ with $|z| < 1$, $|\zeta| \leq 1$, and for $x \in \mathbb{Z}^d$, by

$$b_{z,\zeta}^{(2)}(x) = \sum_{(T,\varphi)} \mathbb{P}(T, \varphi) z^{|T|} \sum_{i \in T} I[\varphi(i) = x] \zeta^{|i|}, \tag{15.16}$$

where $|i|$ denotes the graph distance from i to the root of T. The series (15.16) clearly converges for $|z| < 1$, $|\zeta| \leq 1$, as does its sum over $x \in \mathbb{Z}^d$. The following theorem gives the Fourier transform of the two-point function, and shows that it is identical to the mean-field two-point function of Sect. 15.1.

Theorem 15.2. *For $d \geq 1$, $k \in [-\pi, \pi]^d$, $|z| < 1$, $|\zeta| \leq 1$,*

$$\hat{b}_{z,\zeta}^{(2)}(k) = \frac{b_z^{(1)}}{1 - \zeta b_z^{(1)} \hat{D}(k)}. \tag{15.17}$$

The denominator of the right hand side vanishes for $z = \zeta = 1$, $k = 0$, and in that case $\hat{b}_{1,1}^{(2)}(0) = \infty$.

Proof. The contribution to the right hand side of (15.16) arising when i is the root is simply $b_z^{(1)}\delta_{0,x}$. When i is not the root, we condition on the number of offspring of the root and on the location of the first step on the branch containing i, to obtain

$$\begin{aligned} b_{z,\zeta}^{(2)}(x) &= b_z^{(1)}\delta_{0,x} + \sum_{m=1}^{\infty} p_m m \left(b_z^{(1)}\right)^{m-1} z\big(\zeta D * b_{z,\zeta}^{(2)}\big)(x) \\ &= b_z^{(1)}\delta_{0,x} + zP'(b_z^{(1)})\big(\zeta D * b_{z,\zeta}^{(2)}\big)(x). \end{aligned} \tag{15.18}$$

In the second term of the middle member of (15.18), the factor z is associated with the root, and the factor m corresponds to choosing which of the root's offspring is an ancestor of the vertex j. The Poisson generating function obeys $zP'(b_z^{(1)}) = zP(b_z^{(1)})$, and by (15.13), this equals $b_z^{(1)}$. Taking the Fourier transform of (15.18) converts the convolution into a product, and we can then solve for $\hat{b}_{z,\zeta}^{(2)}(k)$ to obtain

$$\hat{b}_{z,\zeta}^{(2)}(k) = \frac{b_z^{(1)}}{1 - \zeta b_z^{(1)}\hat{D}(k)}. \tag{15.19}$$

Note that the denominator is zero when $z = \zeta = 1$ and $k = 0$.

Finally, we observe that the Fourier transform of (15.16) is given by

$$\hat{b}_{z,\zeta}^{(2)}(k) = \sum_{(T,\varphi)} \sum_{j\in T} \mathbb{P}(T,\varphi) z^{|T|} \mathrm{e}^{\mathrm{i}k\cdot\varphi(j)} \zeta^{|j|}. \tag{15.20}$$

We conclude from this and (15.15) that

$$\hat{b}_{1,1}^{(2)}(0) = \sum_{(T,\varphi)} \sum_{j\in T} \mathbb{P}(T,\varphi) = \sum_{n=1}^{\infty} n\mathbb{P}(|T| = n) = \infty. \tag{15.21}$$

■

The two-point function given in Theorem 15.2 can be interpreted as the two-point function of simple random walk with an activity ζ associated to each step of the walk and an activity $b_z^{(1)}$ associated to each vertex. We may therefore regard a critical Poisson branching random walk configuration containing 0 and x as corresponding to a simple random walk path from 0 to x with a one-point function attached at each vertex along the way. This was the philosophy of the mean-field model of Sect. 15.1.

Next, we define the r-point functions for $r \geq 3$. Our definition keeps track of a substantial amount of information, and requires as preparation the following definitions of *shape*, *subshape*, *skeleton* and *compatibility.*

Shape: Shapes are certain rooted binary trees. For $r \geq 2$, we give a recursive definition of the set Σ_r of r-*shapes*, as follows. Each r-shape has $2r-3$ edges,

$r-2$ vertices of degree 3 (the *branch points*) and r vertices of degree 1 (the *leaves*) labelled $0, 1, \dots, r-1$. There is a unique 2-shape given by the tree consisting of vertex 0 joined by a single edge to vertex 1. We think of this shape as indicating that vertex 0 is an ancestor of vertex 1. There is a unique 3-shape, consisting of three vertices $0, 1, 2$ each joined by an edge to a fourth vertex. We think of this shape as indicating that 0 is an ancestor of both 1 and 2. In general, for $r \geq 3$, to each $(r-1)$-shape σ, we obtain $2r-5$ r-shapes by choosing one of the $2r-5$ edges of σ, adding a vertex on that edge together with a new edge that joins the added vertex to a new leaf $r-1$. The resulting r-shapes represent the different ways in which an additional r^{th} particle can be added to the family tree of $r-1$ particles represented by σ. Thus there is a unique shape for $r=2$ and $r=3$, and $(2r-5)!!$ distinct shapes for $r \geq 4$, where we use the notation $(-1)!! = 1$, and, for $r \geq 3$,

$$(2r-5)!! = \prod_{j=3}^{r} (2j-5). \tag{15.22}$$

When r is clear from the context, we will refer to an r-shape simply as a shape. For notational convenience, we associate to each shape an arbitrary labelling of its $2r-3$ edges, with labels $1, \dots, 2r-3$. This arbitrary choice of edge labels is fixed once and for all; see Fig. 15.1.

Subshape: A subshape of a shape $\sigma \in \Sigma_r$ is a tree obtained by contracting a subset of the edges of σ to a point. This can lead to multiply-labelled vertices, and contracted edges lose their labels. The subshapes for $r=3$ are shown in Fig. 15.2. In general, there are 2^{2r-3} subshapes of a shape $\sigma \in \Sigma_r$. We denote subshapes by λ and write $\lambda \leq \sigma$ when λ is a subshape of σ. We denote the set of edge labels of λ by $e(\lambda)$.

Skeleton: We write $\bar{\imath} = (i_1, \dots, i_{r-1})$ for a sequence of $r-1$ vertices i_j (not necessarily distinct) in a tree T, and define the skeleton B of $(T, \bar{\imath})$ to be the subtree of T spanning $0, i_1, \dots, i_{r-1}$. We will distinguish $r-1$ and $2r-3$ component vectors by using $\bar{\cdot}$ and $\vec{\cdot}$, respectively.

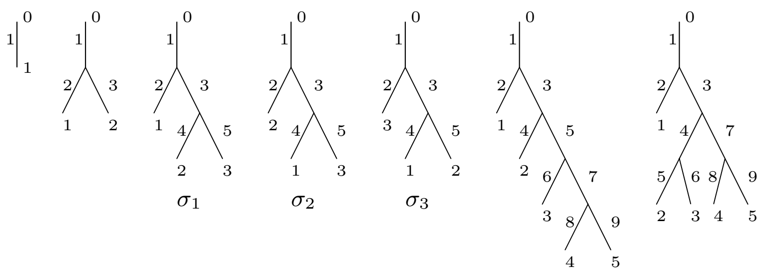

Fig. 15.1. The shapes for $r = 2, 3, 4$, and examples of the $7!! = 7 \cdot 5 \cdot 3 = 105$ shapes for $r = 6$. The shapes' edge labels are arbitrary but fixed.

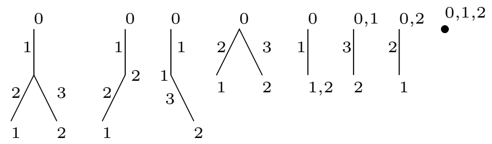

Fig. 15.2. The $2^3 = 8$ subshapes for $r = 3$.

Let β_B denote the tree obtained from B by ignoring vertices of degree 2 in B other than $0, i_1, \ldots, i_{r-1}$ (which may have degree 2 in B), and by assigning label j to vertex i_j for each j. This may lead to multiple labels at a vertex, as is the case for subshapes.

Given $\sigma \in \Sigma_r$ and a subshape $\lambda \le \sigma$, we say that β_B is isomorphic to λ if there is an edge preserving bijection from the set of all vertices of β_B to the set of all vertices of λ, which preserves the vertex labels of β_B and λ (including any multiple labels at vertices). Given such an isomorphism, the edge labels of λ induce labels on the edges of β_B and thus on the paths in T comprising the skeleton B.

Compatibility: Let $\sigma \in \Sigma_r$, $\vec{m} = (m_1, \ldots, m_{2r-3})$ for non-negative integers m_j, and $\vec{y} = (y_1, \ldots, y_{2r-3})$ for $y_j \in \mathbb{Z}^d$. Given (T, φ), fix $r-1$ vertices $\bar{\imath}$ in T. We say that $(T, \varphi, \bar{\imath})$ is *compatible* with $(\sigma; \vec{y}, \vec{m})$ if the following hold:

1. β_B is isomorphic to a subshape λ of σ (in which case the paths of the skeleton B have an induced labelling).
2. Let $l_j > 0$ denote the length of the skeleton path labelled j, and let $l_j = 0$ for any edge in σ that is not in λ. Then $l_j = m_j$ for each $j = 1, \ldots, 2r-3$.
3. The image under φ of the skeleton path (oriented away from the root) labelled j undergoes the displacement y_j for each j labelling an edge in λ, and $y_j = 0$ for any edge j in σ that is not in λ.

For example, given $(T, \bar{\imath})$ of Fig. 15.3, and any embedding φ of T, $(T, \varphi, \bar{\imath})$ is compatible with $(\sigma_3; (0, \varphi(i_3), \varphi(i_2), \varphi(i_1) - \varphi(i_2), 0), (0, 2, 1, 2, 0))$.

The r-point functions: Let $r \ge 2$, $\sigma \in \Sigma_r$, $\vec{y} = (y_1, \ldots, y_{2r-3})$ with each $y_i \in \mathbb{Z}^d$, and let $\vec{m} = (m_1, \ldots, m_{2r-3})$ with each m_i a non-negative integer. We define

$$\begin{aligned} &b_n^{(r)}(\sigma; \vec{y}, \vec{m}) && (15.23)\\ &= \sum_{(T,\varphi):|T|=n} \mathbb{P}(T, \varphi) \sum_{i_1, \ldots, i_{r-1} \in T} I[(T, \varphi, \bar{\imath}) \text{ is compatible with } (\sigma; \vec{y}, \vec{m})]. \end{aligned}$$

Then we define the r-point function by

$$b_{z,\vec{\zeta}}^{(r)}(\sigma; \vec{y}) = \sum_{n=0}^{\infty} \sum_{m_1, \ldots, m_{2r-3}=0}^{\infty} b_n^{(r)}(\sigma; \vec{y}, \vec{m}) z^n \prod_{j=1}^{2r-3} \zeta_j^{m_j}. \qquad (15.24)$$

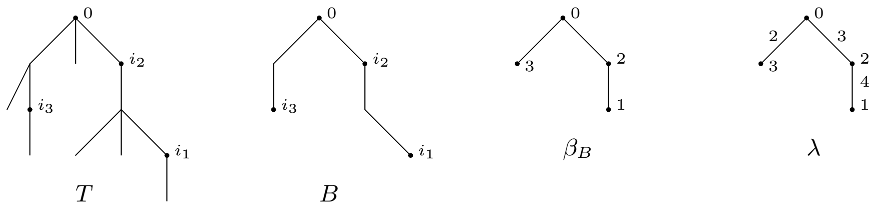

Fig. 15.3. A tree T containing i_1, i_2, i_3, its skeleton B, the reduced skeleton β_B, and the subshape λ of σ_3 (see Fig. 15.1) to which β_B is isomorphic.

Exercise 15.3. Show that (15.24) agrees with the definition (15.16) for $r = 2$.

The next theorem gives the Fourier transform of the r-point functions, where

$$\hat{f}(\vec{k}) = \sum_{y_1,\ldots,y_{2r-3}\in\mathbb{Z}^d} f(\vec{y})\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{y}} \tag{15.25}$$

with $\vec{k}\cdot\vec{y} = \sum_{j=1}^{2r-3} k_j \cdot y_j$.

Theorem 15.4. *For $d \geq 1$, $r \geq 2$, $\sigma \in \Sigma_r$, $k_j \in [-\pi,\pi]^d$, $|z| < 1$, $|\zeta_j| \leq 1$,*

$$\hat{b}^{(r)}_{z,\vec{\zeta}}(\sigma;\vec{k}) = \left(b^{(1)}_z\right)^{-2(r-2)} \prod_{j=1}^{2r-3} \hat{b}^{(2)}_{z,\zeta_j}(k_j). \tag{15.26}$$

Before proving the theorem, we remark that the factor $(b^{(1)}_z)^{-2(r-2)}$ has a combinatorial interpretation. Namely, it "corrects" for an overcounting of the branch at each of the $r-2$ shape vertices of degree 3, as this branch is counted in $\prod_{j=1}^{2r-3} \hat{b}^{(2)}_{z,\zeta_j}(k_j)$ once by each of the three two-point functions incident at that vertex. This factor is equal to 1 at the critical point $z = 1$, and does not play a role in the leading critical behaviour.

Proof of Theorem 15.4. The statement of the theorem is a tautology for $r = 2$, so we consider $r \geq 3$. Let

$$\hat{q}^{(2)}_{z,\zeta}(k) = \zeta \hat{D}(k) \hat{b}^{(2)}_{z,\zeta}(k). \tag{15.27}$$

By (15.18), $\hat{b}^{(2)}_{z,\zeta}(k) = b^{(1)}_z[1 + \hat{q}^{(2)}_{z,\zeta}(k)]$, so it suffices to show that

$$\hat{b}^{(r)}_{z,\vec{\zeta}}(\sigma;\vec{k}) = b^{(1)}_z \prod_{j=1}^{2r-3} \left(1 + \hat{q}^{(2)}_{z,\zeta_j}(k_j)\right). \tag{15.28}$$

Expanding the product, the desired identity (15.28) is equivalent to

$$\hat{b}^{(r)}_{z,\vec{\zeta}}(\sigma;\vec{k}) = b^{(1)}_z \sum_{\lambda \leq \sigma} \prod_{j \in e(\lambda)} \hat{q}^{(2)}_{z,\zeta_j}(k_j). \tag{15.29}$$

Given a subshape $\lambda \leq \sigma$, we let $b(\lambda)$ denote the result of restricting the summation in (15.24) to $m_j = 0$ (and thus $y_j = 0$) for $j \notin e(\lambda)$ and $m_j > 0$ for $j \in e(\lambda)$. Its Fourier transform will be denoted $\hat{b}(\lambda)$. We leave implicit the dependence on the variables of $\vec{k}$ and $\vec{\zeta}$, as these are determined by the edge labels of λ. Then

$$\hat{b}^{(r)}_{z,\vec{\zeta}}(\sigma;\vec{k}) = \sum_{\lambda \leq \sigma} \hat{b}(\lambda). \tag{15.30}$$

Thus it suffices to show that

$$\hat{b}(\lambda) = b^{(1)}_z \prod_{j \in e(\lambda)} \hat{q}^{(2)}_{z,\zeta_j}(k_j). \tag{15.31}$$

This is clear if λ consists of a single vertex, so it suffices to consider the case where λ contains at least one edge.

We use π to denote a subshape for which the root has degree 1, and write $\hat{q}(\pi) = (zp_1)^{-1}\hat{b}(\pi)$. The factor $(zp_1)^{-1}$ serves to cancel the factor $zp_1 = z\mathrm{e}^{-1}$ associated to the root in $\hat{b}(\pi)$. Given a subshape λ having at least one edge, let $\pi_1, \ldots, \pi_l$ be the branches emerging from its root. As in (15.18), using now that the l^{th} derivative of P obeys $zP^{(l)}(b^{(1)}_z) = b^{(1)}_z$,

$$\begin{aligned} \hat{b}(\lambda) &= \sum_{j=l}^{\infty} zp_j j(j-1)\cdots(j-l+1)\left(b^{(1)}_z\right)^{j-l} \prod_{a=1}^{l} \hat{q}(\pi_a) \\ &= b^{(1)}_z \prod_{a=1}^{l} \hat{q}(\pi_a). \end{aligned} \tag{15.32}$$

Let $\bar{\pi}$ denote the subshape obtained from π by contracting the edge incident to the root. We claim that

$$\hat{q}(\pi) = \hat{q}^{(2)}_{z,\zeta}(k) \frac{1}{b^{(1)}_z} \hat{b}(\bar{\pi}), \tag{15.33}$$

where ζ and k bear the subscript of the label of the edge incident on the root of π. From this, the desired result (15.31) then follows by substituting (15.33) into (15.32) recursively.

To prove (15.33), we condition on whether the length of the tree's skeleton path, corresponding to the edge of π incident on the root, is equal to or greater than 1. This leads, by conditioning as in (15.18), to

$$\hat{q}(\pi) = \zeta \hat{D}(k)\hat{b}(\bar{\pi}) + \zeta \hat{D}(k) b^{(1)}_z \hat{q}(\pi). \tag{15.34}$$

Solving and using (15.17) and (15.27), we obtain

$$\hat{q}(\pi) = \frac{\zeta \hat{D}(k)}{1 - \zeta \hat{D}(k) b^{(1)}_z} \hat{b}(\bar{\pi}) = \hat{q}^{(2)}_{z,\zeta}(k) \frac{1}{b^{(1)}_z} \hat{b}(\bar{\pi}), \tag{15.35}$$

which is (15.33). ■

15.3 Weakly Self-Avoiding Lattice Trees

The weakly self-avoiding walk has played an important role in the development of the theory of self-avoiding walks. There has been no parallel situation for lattice trees, perhaps because it is less obvious how to define weakly self-avoiding lattice trees. In this section we give a natural definition of weakly self-avoiding lattice trees and prove that it corresponds to usual lattice trees in the limit of infinite self-avoidance strength. Throughout the section, we follow the presentation of [30]. Presumably weakly self-avoiding lattice trees are in the same universality class as usual lattice trees, no matter how weak the self-avoidance.

Let $\mathbb{P}(T,\varphi)$ be given by (15.7)–(15.8). Given $\beta \geq 0$, let

$$Z_n^\beta = \sum_{(T,\varphi):|T|=n} \mathbb{P}(T,\varphi) \exp\left[-\tfrac{1}{2}\beta \textstyle\sum_{i,j\in T: i\neq j} I[\varphi(i)=\varphi(j)]\right], \tag{15.36}$$

and, for $|T|=n$, define

$$\mathbb{Q}_n^\beta(T,\varphi) = \frac{1}{Z_n^\beta}\mathbb{P}(T,\varphi) \exp\left[-\tfrac{1}{2}\beta \textstyle\sum_{i,j\in T: i\neq j} I[\varphi(i)=\varphi(j)]\right]. \tag{15.37}$$

The measure $\mathbb{Q}_n^\beta$ on the set of n-vertex branching random walk configurations rewards self-avoidance by giving a penalty $\mathrm{e}^{-\beta}$ to each self-intersection. For $\beta=0$, $\mathbb{Q}_n^0$ is just branching random walk conditional on $|T|=n$. The next theorem shows that the weakly self-avoiding lattice trees interpolate between branching random walk and lattice trees, in the sense that $\mathbb{Q}_n^\infty$ corresponds in an appropriate sense to the uniform measure on the set of n-vertex lattice trees containing the origin.

In the statement of the theorem $t_n^{(1)}$ denotes the number of n-vertex lattice trees containing the origin, as in Sect. 7.1. Given an injective φ and a lattice tree L, we abuse notation by writing $\varphi(T)=L$ if $\varphi(T)$ consists of the vertices in L and the edges in T are mapped to the bonds in L.

Theorem 15.5. *For $d \geq 1$ and $n \geq 1$,*

$$\lim_{\beta\to\infty} \mathbb{Q}_n^\beta(T,\varphi) = 0 \tag{15.38}$$

if φ is not injective. On the other hand, given an n-vertex lattice tree L containing the origin,

$$\lim_{\beta\to\infty} \sum_{(T,\varphi):\varphi(T)=L} \mathbb{Q}_n^\beta(T,\varphi) = \frac{1}{t_n^{(1)}}. \tag{15.39}$$

Proof. The first statement of the theorem, for non-injective φ, follows immediately from the definition of $\mathbb{Q}_n^\beta$.

For the second statement of the theorem, let $\mathcal{T}_n$ denote the set of n-vertex lattice trees containing the origin. This has cardinality $t_n^{(1)}$. We will prove that

$$\sum_{(T,\varphi):\varphi(T)=L} \mathbb{P}(T,\varphi) = (2d)^{-(n-1)}\mathrm{e}^{-n} \tag{15.40}$$

for every $L \in \mathcal{T}_n$. The important point for the proof is that the right hand side is the same for all $L \in \mathcal{T}_n$, and its particular value plays no role. In fact, given (15.40), we then have

$$Z_n^\infty = \sum_{L\in\mathcal{T}_n} \sum_{(T,\varphi):\varphi(T)=L} \mathbb{P}(T,\varphi) = t_n^{(1)}(2d)^{-(n-1)}\mathrm{e}^{-n}, \tag{15.41}$$

which gives the desired result that

$$\sum_{(T,\varphi):\varphi(T)=L} \mathbb{Q}_n^\infty(T,\varphi) = \frac{1}{Z_n^\infty} \sum_{(T,\varphi):\varphi(T)=L} \mathbb{P}(T,\varphi) = \frac{1}{t_n^{(1)}}. \tag{15.42}$$

To prove (15.40), we first note that by (15.7) and (15.8),

$$\sum_{(T,\varphi):\varphi(T)=L} \mathbb{P}(T,\varphi) = (2d)^{-(n-1)}\mathrm{e}^{-n} \sum_{(T,\varphi):\varphi(T)=L} \prod_{i\in T} \frac{1}{\xi_i!}, \tag{15.43}$$

where ξ_i is the number of offspring of vertex i. It suffices to show that

$$\sum_{(T,\varphi):\varphi(T)=L} \prod_{i\in T} \frac{1}{\xi_i!} = 1. \tag{15.44}$$

Let b_0 be the degree of 0 in L, and given nonzero $x \in L$, let b_x be the degree of x in L minus 1 (the *forward* degree of x). Then the set $\{b_x : x \in L\}$ (with repetitions) must be equal to the set of ξ_i (with repetitions) for any T that can be mapped to L. Defining $\nu(L)$ to be the cardinality of $\{(T,\varphi) : \varphi(T) = L\}$, (15.44) is therefore equivalent to

$$\nu(L) = \prod_{x\in L} b_x!. \tag{15.45}$$

We prove (15.45) by induction on the number N of generations of L. By this, we mean the length of the longest self-avoiding path in L, starting from the origin. The identity (15.45) clearly holds if $N = 0$. Our induction hypothesis is that (15.45) holds if there are $N-1$ or fewer generations. Suppose L has N generations, and let $L_1, \ldots, L_{b_0}$ denote the lattice trees resulting from deleting from L the origin and all bonds incident on the origin. We regard each L_a as rooted at the neighbour of the origin in the corresponding deleted bond. It suffices to show that $\nu(L) = b_0! \prod_{a=1}^{b_0} \nu(L_a)$, since each L_a has fewer than N generations.

To prove this, we note that each pair (T, φ) with $\varphi(T) = L$ induces a set of (T_a, φ_a) such that $\varphi_a(T_a) = L_a$. This correspondence is $b_0!$ to 1, since (T, φ) is determined by the set of (T_a, φ_a), up to permutation of the branches of T at its root. See Exercise 15.6. This proves $\nu(L) = b_0! \prod_{a=1}^{b_0} \nu(L_a)$. ■

Exercise 15.6. Let L be the 2-dimensional lattice tree consisting of bonds $\{0, e_2\}$, $\{0, -e_2\}$, $\{0, e_1\}$, $\{e_1, 2e_1\}$, $\{e_1, e_1 + e_2\}$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let T_1 be the single vertex 0_1 and T_2 be the single vertex 0_2, with $\varphi_1(0_1) = e_2$ and $\varphi_2(0_2) = -e_2$. Let T_3 be the tree consisting of the root 0 and its two offspring 00 and 01, with $\varphi_3(0) = e_1$, $\varphi_3(00) = 2e_1$, $\varphi_3(01) = e_1 + e_2$. Write down the six distinct (T, φ) that correspond to the collection (T_a, φ_a) $(a = 1, 2, 3)$ as in the last paragraph of the proof of Theorem 15.5.