9

Subbranches of Types A_l, B_l, C_l

In this chapter, we introduce important notions "subbranches of types A_l, B_l, and C_l". In the first section, we summarize their properties often without proof, and the subsequent sections are devoted to the proofs of these properties. The proofs are routine and technical in nature. For the first reading, we recommend the reader to read only the first section (and assuming it) to skip to the next chapter.

9.1 Subbranches of types A_l, B_l, C_l

Let lY be a subbranch of a branch X where l is a positive integer and Y is a subbranch of X. Here Y itself is possibly multiple. We express $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ and $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ ($e \leq \lambda$), and then set

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}, \quad (i = 1, 2 \ldots, \lambda - 1), \qquad r_\lambda := \frac{m_{\lambda-1}}{m_\lambda}.$$

Recall that r_i ($i = 1, 2, \ldots, \lambda$) are positive integers satisfying $r_i \geq 2$. Next we recall the deformation atlas $DA_{e-1}(lY, k)$ associated with lY. First we define a sequence of integers p_i ($i = 0, 1, \ldots, \lambda + 1$) inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1, 2, \ldots, \lambda. \end{cases}$$

Then $p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0$ (6.2.4). Let $f(z)$ be a non-vanishing holomorphic function defined around $z = 0$, and we set $f_i = f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f}_i = f(z^{p_{i+1}}\zeta^{p_i})$ (see (6.2.7)). Then $DA_{e-1}(lY, d)$ is given by the following data (see Lemma 7.1.1): for $i = 1, 2, \ldots, e - 1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1} - ln_{i-1}}\eta^{m_i - ln_i}(w^{n_{i-1}}\eta^{n_i} + t^k f_i)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1} - ln_{i+1}}\zeta^{m_i - ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^k \widehat{f}_i)^l - s = 0 \\ g_i : & \text{the transition function } z = 1/w, \ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

Then we ask:

Problem When does $DA_{e-1}(lY, k)$ admit a complete propagation?

As we will show later, there are exactly three types of Y for which $DA_{e-1}(lY, k)$ admits a complete propagation (Theorem 13.1.1). Now we introduce these three types. Below, the notation $lY \le X$ means $ln_i \le m_i$ for $i = 0, 1, \dots, e$.

Definition 9.1.1 Let l be a positive integer and let X be a branch.

Type A_l A subbranch Y of X is of *type A_l* if one of the following conditions holds: (In fact, these conditions are equivalent. See Lemma 9.2.3.)

- **(A.1)** $lY \le X$ and $\frac{n_{e-1}}{n_e} \ge r_e$.
- **(A.2)** $lY \le X$ and lY is dominant tame.
- **(A.3)** $lY \le X$ and Y is dominant tame.

Type B_l A subbranch Y of X is of *type B_l* if $lY \le X$, $m_e = l$, and $n_e = 1$

Type C_l A subbranch Y of X is of *type C_l* if one of the following conditions holds: (In fact, these conditions are equivalent. See Lemma 9.4.2.)

- **(C.1)** $lY \le X$, n_e divides n_{e-1}, and $\frac{n_{e-1}}{n_e} < r_e$, and u divides l where $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$. (As in (C.3) "Note", $u > 0$.)
- **(C.2)** $lY \le X$, $n_e = r_e n_e - n_{e-1}$, and u divides l where u is in (C.1).
- **(C.3)** $lY \le X$, $n_e = r_e n_e - n_{e-1}$, and $m_e - m_{e+1}$ divides l.

  (Note: $\lambda \ge e+1$ holds for type C_l. See Corollary 9.4.4. Also note that by Lemma 9.1.5 below, $m_e - m_{e+1}$ is equal to u in (C.1); so $u > 0$.)

We provide respective examples of types A_l, B_l, C_l:

Example A_l $l = 2$, $\mathbf{m} = (12, 9, 6, 3)$ and $\mathbf{n} = (3, 2, 1)$.
Example B_l $l = 2$, $\mathbf{m} = (12, 7, 2, 1)$ and $\mathbf{n} = (3, 2, 1)$.
Example C_l $l = 5$, $\mathbf{m} = (30, 25, 20, 15, 10, 5)$ and $\mathbf{n} = (3, 3, 3, 3)$.
(In Example C_l, $m_e - ln_e = 0$ and so $u = 5$.)

Note: Take $l = 7$, $\mathbf{m} = (57, 16, 7, 5, 3, 1)$, and $\mathbf{n} = (7, 2, 1)$. Then Y satisfies the conditions of type C_l except that u divides l. Indeed $u = 2$, and so u does *not* divide $l = 7$. Consequently Y is not of type C_l.

Recall that a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ ($e \le \lambda$) of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is proportional if $\frac{m_0}{n_0} = \frac{m_1}{n_1} = \cdots = \frac{m_e}{n_e}$.

Lemma 9.1.2 *Any subbranch Y of type C_l is "not" proportional.*

Proof. In fact, when $e = \lambda$, from a condition in (C.1), we have $\frac{n_{\lambda-1}}{n_\lambda} < r_\lambda = \frac{m_{\lambda-1}}{m_\lambda}$ and so $\frac{m_{\lambda-1}}{n_{\lambda-1}} > \frac{m_\lambda}{n_\lambda}$; this confirms the non-proportionality of Y. When $e < \lambda$, we show the non-proportionality of Y by contradiction; if Y is proportional, then $(m_{e-1}, m_e) = (cn_{e-1}, cn_e)$ for some rational number c. By (C.3), $n_e = r_e n_e - n_{e-1}$, and hence $cn_e = r_e cn_e - cn_{e-1}$, that is,

$m_e = r_e m_e - m_{e-1}$. Thus we have

$$\frac{m_e + m_{e-1}}{m_e} = r_e.$$

However, from the definition of a branch,

$$\frac{m_{e-1} + m_{e+1}}{m_e} = r_e,$$

and the comparison of the above two equations gives $m_{e+1} = m_e$. This is a contradiction. Therefore any subbranch of type C_l is not proportional. □

On the other hand, types A_l and B_l may be proportional. For instance if $X = lY$;

$$\mathbf{m} = (ln_0, ln_1, \ldots, ln_\lambda), \quad \mathbf{n} = (n_0, n_1, \ldots, n_\lambda),$$

then Y is of proportional type A_l; for a special case $n_\lambda = 1$ and $m_\lambda = l$, this is of proportional type B_l at the same time. A subbranch both of type A_l and B_l is simply referred to as of *type* AB_l.

Lemma 9.1.3 *Suppose that Y is a dominant subbranch of a branch X. Then Y is of type AB_l if and only if Y is of proportional type B_l.*

Proof. $\Longrightarrow$: Trivial.

$\Longleftarrow$: By proportionality, $\frac{m_0}{n_0} = \frac{m_1}{n_1} = \cdots = \frac{m_e}{n_e}$. Since $m_e = l$ and $n_e = 1$ (type B_l), these common fractions are equal to l. Namely

$$\big(m_0, m_1, \ldots, m_e\big) = l\big(n_0, n_1, \ldots, n_e\,(=1)\big). \tag{9.1.1}$$

Next we insist that $e = \lambda$; assuming $e < \lambda$, we derive a contradiction. Note that (9.1.1) with the equations $m_{i+1} = r_i m_i - m_{i-1}$ $(i = 1, 2, \ldots, \lambda - 1)$ implies that l divides all m_i $(i = 0, 1, \ldots, \lambda)$. We "define" $n_{e+1}, n_{e+2}, \ldots, n_\lambda$ by $n_i := \frac{m_i}{l}$ $(i = e+1, e+2, \ldots, \lambda)$. Then $(m_0, m_1, \ldots, m_\lambda) = l(n_0, n_1, \ldots, n_\lambda)$. In particular the sequence $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ is contained in a dominant sequence $\mathbf{n}' = (n_0, n_1, \ldots, n_\lambda)$, and so Y is not dominant (a contradiction!). Thus $e = \lambda$ and

$$\big(m_0, m_1, \ldots, m_\lambda\big) = l\big(n_0, n_1, \ldots, n_\lambda\,(=1)\big).$$

This shows that Y is of type AB_l. □

From this lemma, type AB_l coincides with proportional type B_l; so the arithmetic property of the latter is the same as that of type A_l — dominant tame. Thus as long as we are concerned with the arithmetic property of type B_l, it is enough to investigate that of non-proportional one. We remark that when we later construct deformations from subbranches of types A_l, B_l, and C_l, a subbranch of proportional type B_l (i.e. type AB_l) produces two different deformations according to the application of the respective constructions for types A_l and B_l.

We point out that a subbranch Y both of type B_l and C_l also exists; $l = 2$, $\mathbf{m} = (4, 3, 2, 1)$ and $\mathbf{n} = (1, 1, 1)$ is such an example. As we will see later, a subbranch Y both of type B_l and C_l produces the same deformation regardless of the application of the respective constructions for type B_l and C_l, and thus there is no reason to distinguish them; we adopt the following convention.

Convention 9.1.4 To avoid overlapping of type C_l with type B_l, we exclude the case $m_e = l$ and $n_e = 1$ from type C_l.

Now we give several comments on (C.1), (C.2), and (C.3) in the definition of type C_l.

Lemma 9.1.5 *The integer* $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$ *in* (C.1) *is equal to* $m_e - m_{e+1}$ *in* (C.3). (Note: since $m_e > m_{e+1}$, we have $u = m_e - m_{e+1} > 0$.)

Proof. In fact, we may write

$$\begin{aligned} u &= (m_{e-1} - r_e m_e) + m_e + l(r_e n_e - n_{e-1} - n_e) \\ &= (m_{e-1} - r_e m_e) + m_e \\ &= m_e - m_{e+1}. \end{aligned}$$

where the second and third equalities respectively follows from $n_e = r_e n_e - n_{e-1}$ (a condition in (C.2)) and $\dfrac{m_{e-1} + m_{e+1}}{m_e} = r_e$. □

By the above lemma, $u = m_e - m_{e+1} > 0$. We remark that "$lY \le X$, n_e divides n_{e-1}, and $\frac{n_{e-1}}{n_e} < r_e$" (cf. (C.1)) implies $u > 0$ (Proposition 9.4.8). However, if we drop "$\frac{n_{e-1}}{n_e} < r_e$", then $u > 0$ fails; for example,

$l = 1$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (5, 3, 1)$. Then n_e divides n_{e-1}, but $\frac{n_{e-1}}{n_e} = 3 > r_e = 2$. In this case $u = -1$. (Actually $\mathbf{n}$ is of type A_1.)

$l = 1$, $\mathbf{m} = (4, 3, 2, 1)$ and $\mathbf{n} = (3, 2, 1)$. Then n_e divides n_{e-1}, but $\frac{n_{e-1}}{n_e} = 2 = r_e = 2$. In this case $u = 0$. (Actually $\mathbf{n}$ is of type A_1.)

Secondly we point out that the condition (C.1) (or all other conditions) of type C_l implies that

(C′) $lY \le X$, n_e divides n_{e-1}, and $\frac{n_{e-1}}{n_e} < r_e$, and $m_e - m_{e+1}$ divides l.

But the converse is *not* true; namely (C.1) is not equivalent to (C′). In fact, under the condition (C'), $m_e - m_{e+1}$ does not necessarily equal u in (C.1). (cf. Lemma 9.1.5.) For instance, $l = 1$, $\mathbf{m} = (13, 4, 3, 2, 1)$ and $\mathbf{n} = (2, 1)$, which satisfies all conditions of (C'). However $m_e - m_{e+1} = 1$, while $u = 2$. In particular, $m_e - m_{e+1}$ divides l, while u does not, and thus this example is not of type C_l.

Remark 9.1.6 *For type C_l, from the condition that n_e divides n_{e-1} and Y is a subbranch, it is easy to deduce that n_e divides n_i $(i = 0, 1, \dots, e-1)$. Namely, when $n_e \geq 2$, a subbranch Y of type C_l itself is multiple. See the proof of Lemma* 8.1.4.

It is worth pointing out the following property (type $B_l^\sharp$ means non-proportional type B_l):

Type A_l	lY is dominant tame
Type $B_l^\sharp$	lY is dominant wild (Proposition 9.3.2)
Type C_l	lY is wild (Proposition 9.4.11)

As we explained above, proportional type B_l (i.e. type AB_l) is dominant tame. We also note that type $B_l^\sharp$ (non-proportional type B_l) and type C_l are wild, but in contrast with type $B_l^\sharp$, type C_l is in general *not* dominant, e.g.

$$l = 1, \quad \mathbf{m} = (6, 5, 4, 3, 2, 1) \quad \text{and} \quad \mathbf{n} = (1, 1, 1).$$

This is not dominant; $\mathbf{n}$ is contained in a dominant sequence $\mathbf{n}' = (1, 1, 1, 1, 1, 1)$. (Interesting enough, $\mathbf{n}'$ is not of type C_l but of type B_l where $l = 1$.) A more complicated example is the following:

$$l = 10, \quad \mathbf{m} = (40, 26, 12, 10, 8, 6, 4, 2) \quad \text{and} \quad \mathbf{n} = (3, 2, 1).$$
(In this case $u = 2$.)

This example is also of type C_l but not dominant; $\mathbf{n}$ is contained in $\mathbf{n}' := (3, 2, 1, 1)$ (type B_l where $l = 10$). Another curious example is: $l = 2$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (2, 2)$. Then $\mathbf{n}$ is of type C_l contained in $\mathbf{n}' = (2, 2, 2)$, which is again of type C_l. See also Remark 20.2.4, p357 for this example.

Remark 9.1.7 If $n_{e-1} < n_e$, then Y is none of types A_l, B_l and C_l. (1) Y is not type A_l: In fact, $n_{e-1} < n_e$ implies $\frac{n_{e-1}}{n_e} < 1$, and so $\frac{n_{e-1}}{n_e} < r_e$ because $r_e \geq 2$. Thus Y does not fulfill (A.1). (2) Noting that $1 \leq n_{e-1} < n_e$, we have $1 < n_e$, and so Y is not of type B_l. (3) As $n_{e-1} < n_e$, the integer n_e does not divide n_{e-1}, and hence Y is not of type C_l.

Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. If Y is of type C_l, then $\lambda \geq e+1$ by Corollary 9.4.4 below. On the other hand, this is not necessarily true for types A_l and B_l. It may occur that $\lambda = e$; for example,

Example A_l $\quad l = 1$, $\mathbf{m} = (9, 6, 3)$ and $\mathbf{n} = (3, 2, 1)$.
Example B_l $\quad l = 3$, $\mathbf{m} = (9, 6, 3)$ and $\mathbf{n} = (1, 1, 1)$.

Now setting

$$a := m_{e-1} - ln_{e-1}, \quad b := m_e - ln_e, \quad c := n_{e-1}, \quad d := n_e,$$

we restate the definitions of types A_l, B_l, and C_l as follows:

Type A_l A subbranch Y is of *type* A_l if $lY \leq X$ and $\frac{c}{d} \geq r_e$.
Type B_l A subbranch Y is of *type* B_l if $lY \leq X$, $b = 0$ and $d = 1$.
Type C_l A subbranch Y is of *type* C_l if one of the following conditions holds:

(C.1) $lY \leq X$, d divides c, and $\frac{c}{d} < r_e$, and u divides l where $u := a - (r_e - 1)b$.

(C.2) $lY \leq X$, $d = r_e d - c$, and u divides l where $u := a - (r_e - 1)b$.

We next summarize signs for some quantities concerning with types A_l, $B_l^\sharp$ and C_l, where type $B_l^\sharp$ means non-proportional type B_l.

Type A_l	$a \geq 0$	$b \geq 0$	$c > 0$	$d > 0$
Type $B_l^\sharp$	$a > 0$	$b = 0$	$c > 0$	$d = 1$
Type C_l	$a > 0$	$b \geq 0$	$c > 0$	$d > 0$

Here note that for any type, $c = n_{e-1} > 0$ and $d = n_e > 0$, and for type B_l, $d = 1$. In general $a \geq 0$ and $b \geq 0$ hold; the strict inequality $a > 0$ is valid for types $B_l^\sharp$ and C_l, which will be proved in Proposition 9.3.2 and Proposition 9.4.11 respectively. On the other hand, $b > 0$ is not true for type B_l because $b = m_e - ln_e = l - l = 0$. We also remark that for type A_l, $a = 0$ if and only if $b = 0$. Moreover $a = 0$ (equivalently $b = 0$) occurs precisely when $X = lY$ and in this case, Y is proportional (Corollary 9.2.8).

Next we provide the table for the signs of quantities $a - r_e b$, $r_e d - c$ and $u := a - (r_e - 1)b$; this table are useful for our later construction of deformations associated with subbranches of types A_l, B_l, and C_l. In the table, type $B_l^\sharp$ means non-proportional type B_l, and for type A_l, if $e = \lambda$, then we formally set $m_{e+1} := 0$. For a subbranch Y of type C_l such that lY is not dominant, we formally set $n_{e+1} := r_e n_e - n_{e-1}$; then $m_{e+1} \geq ln_{e+1}$ by non-dominance, and hence $ln_{e+1} - m_{e+1} \leq 0$.

Table 9.1.8

Type A_l	$a - r_e b \leq -m_{e+1} < 0$	$r_e d - c \leq 0$	$u \leq b - m_{e+1}$
Type $B_l^\sharp$	$a - r_e b > 0$	$r_e d - c > 0$	$u > 0$
Type C_l	$a - r_e b > 0$ if lY is dominant $a - r_e b = ln_{e+1} - m_{e+1} \leq 0$ if lY is not dominant	$r_e d - c = d > 0$	$u = m_e - m_{e+1} > 0$

(The inequalities in the above table will be shown in Proposition 9.2.5, Proposition 9.3.2, and Proposition 9.4.11 for types A_l, $B_l^\sharp$, and C_l respectively.)

The following table for type C_l, to be proved in Lemma 9.4.10, will be used later in the construction of deformations.

Table 9.1.9

Type C_l	$u > b$	if lY is dominant
	$u \le b$	if lY is not dominant

For a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + m_e\Theta_e$ of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, recall that Θ_i is a (-2)-*curve* if the self-intersection number $\Theta_i \cdot \Theta_i = -2$; a *chain of* (-2)-*curve* is a set of (-2)-curves of the form $\Theta_a + \Theta_{a+1} + \cdots + \Theta_b$ where $a \le b$. If Y is of type C_l, then in most cases the complement of Y in X contains a chain of (-2)-curves, where by the "complement of Y in X", we mean $\Theta_{e+1} + \Theta_{e+2} + \cdots + \Theta_\lambda$ (note $\lambda \ge e+1$ for type C_l by Corollary 9.4.4). To explain this result, we set $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$, and then u divides l by the definition of type C_l, and so we write $l = Nu$ where N is a positive integer. Next we set $b := m_e - ln_e$ and $d := n_e$, and if $u \le b$, considering the division of b by u, we let v be the integer such that $b - vu \ge 0$ and $b - (v+1)u < 0$. According to whether $u > b$ or $u \le b$, we have the following information about chains of (-2)-curves in the complement of Y in X. (Note: $r_i = 2$ is equivalent to Θ_i being a (-2)-curve.)

Table 9.1.10 (Type C_l) Refer Proposition 9.4.12 for the proof.

$b = 0$	$r_{e+1} = r_{e+2} = \cdots = r_\lambda = 2, \quad \lambda = e + Nd - 1$
$b \ge 1, \ u > b$	$r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2, \quad \lambda = e + Nd$
$b \ge 1, \ u \le b$, u does not divide v	$r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2, \quad \lambda = e + Nd + v$
$b \ge 1, \ u \le b$, u divides v	$r_{e+1} = r_{e+2} = \cdots = r_\lambda = 2, \quad \lambda = e + Nd + v - 1$

Example 9.1.11 (Exceptional example) In the above table, for the case $b \ge 1$ and $u > b$, if $\lambda = e + 1$, then the complement of Y in X may *not* contain a chain of (-2)-curves at all; $l = 2$, $\mathbf{m} = (5, 3, 1)$ and $\mathbf{n} = (1, 1)$ is such an example of type C_l, in which case $u = 2$ and so $N = 1$ in $l = Nu$, and consequently $\lambda = e + 1 = e + Nd = 2$. Then $r_\lambda\,(= 3) \ne 2$ and hence the complement Θ_λ of Y in X is not a (-2)-curve.

The following criterion for Y to be of type C_1 is useful.

Lemma 9.1.12 *When $l = 1$, a dominant subbranch Y is of type C_l if and only if the following conditions are fulfilled:* (1) n_e *divides* n_{e-1}*, and* $\frac{n_{e-1}}{n_e} < r_e$*,* (2) $m_{e-1} - n_{e-1} = 1$*, and* (3) $m_e = n_e$*.*

Proof. $\Longrightarrow$: Suppose that Y is of type C_1. Then by definition, (1) is satisfied. To show (2) and (3), we set $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$, and then u divides l by the definition of type C_l. In the present case, $l = 1$ and so $u = \pm 1$. As $u = m_e - m_{e+1} > 0$ for type C_l (Lemma 9.1.5), we have $u = 1$.

Now for simplicity, we set $a := m_{e-1} - n_{e-1}$ and $b := m_e - n_e$. Of course $b \geq 0$. Since Y $(= lY)$ is dominant and of type C_l, we have $a - r_e b \geq 1$ by Table 9.1.8. Thus

$$b \geq 0, \qquad a - r_e b \geq 1. \tag{9.1.2}$$

Our goal is to show that (2) $a = 1$ and (3) $b = 0$. Note that since $l = 1$,

$$u = (m_{e-1} - n_{e-1}) - (r_e - 1)(m_e - n_e) = a - (r_e - 1)b = (a - r_e b) + b.$$

On the other hand, as we saw above, $u = 1$ and thus $(a - r_e b) + b = 1$. Taking (9.1.2) into consideration, this equation holds exactly when $a - r_e b = 1$ and $b = 0$, that is, $a = 1$ and $b = 0$. Hence (2) and (3) hold.

$\Longleftarrow$: Taking into account (1), we only have to show that $u = a - (r_e - 1)b$ divides $l = 1$. But $a = 1$ (2) and $b = 0$ (3), and so $u = 1$, which obviously divides l. Therefore the condition (C.1) of type C_l is satisfied; so Y is of type C_l. □

Remark 9.1.13 Recall that a subbranch Y is of ripple type if $n_{e-1} = n_e = m_e$ (see (8.1.3), p146); then Y is dominant by Lemma 8.1.5. Clearly the three conditions of the above lemma are fulfilled and thus Y is of type C_1. However, the converse is not true; even if the conditions (1), (2) and (3) of Lemma 9.1.12 are fulfilled, it does *not* imply that Y is of ripple type. The following examples are of type C_1 but not of ripple type because $n_{e-1} \neq n_e$.

(1) $l = 1$, $\mathbf{m} = (15, 11, 7, 3, 2, 1)$ and $\mathbf{n} = (12, 9, 6, 3)$.

(2) $l = 1$, $\mathbf{m} = (22, 17, 12, 7, 2, 1)$ and $\mathbf{n} = (18, 14, 10, 6, 2)$.

A recipe to produce subbranches of type C_l

We close this section by giving a recipe to produce examples of subbranches of type C_l. Given $\mathbf{m} = (m_0, m_1, \ldots, m_\lambda)$, take two positive integers $n_e := 1$ and $n_{e-1} := r_e - 1$ where $r_e = \dfrac{m_{e-1} + m_{e+1}}{m_e}$. Then clearly $n_e = r_e n_e - n_{e-1}$, and hence the first condition in (C.3) of Definition 9.1.1 is fulfilled. Thus Y is of type C_l precisely when $m_e - m_{e+1}$ divides l. For example, (1) $m_e - m_{e+1} = 1$ or (2) $m_e - m_{e+1} = 2$ and l is even. If this is the case, we define $n_{e-2}, n_{e-3}, \ldots, n_\lambda$ inductively by $n_{i-1} := r_i n_i - n_{i+1}$ for $i = e-1, e-2, \ldots, 1$. This yields a sequence $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ of type C_l.

9.2 Demonstration of properties of type A_l

In this section, we demonstrate the properties of type A_l. We begin by recalling the definition of dominance. Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + m_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. We set $n_{e+1} := r_e n_e - n_{e-1}$ formally, where $r_e := \dfrac{m_{e-1} + m_{e+1}}{m_e}$. Then Y is said to be dominant if either

(i) $n_{e+1} \leq 0$ or (ii) $n_{e+1} > m_{e+1}$ holds. According to (i) or (ii), Y is called tame or wild respectively. The condition (i) is rewritten as $\frac{n_{e-1}}{n_e} \geq r_e$, and so we have

Lemma 9.2.1 *A subbranch Y is dominant tame if and only if $\frac{n_{e-1}}{n_e} \geq r_e$ holds.*

We then note

Lemma 9.2.2 *Let lY be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. Then the following statements hold:*

(1) *Y is dominant $\Longrightarrow$ lY is dominant.*
(2) *Y is dominant tame $\Longleftrightarrow$ lY is dominant tame.*

Proof. We write $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$. First we show (1) by contradiction. Suppose that lY is not dominant. Then there exists an integer k_{e+1} $(0 < k_{e+1} \leq m_{e+1})$ satisfying

$$\frac{ln_{e-1} + k_{e+1}}{ln_e} = r_e. \tag{9.2.1}$$

Thus $ln_{e-1} + k_{e+1} = ln_e r_e$. It follows that l divides k_{e+1}. We write $k_{e+1} = ln_{e+1}$ where n_{e+1} is a positive integer, and then (9.2.1) is

$$\frac{ln_{e-1} + ln_{e+1}}{ln_e} = r_e.$$

Thus $\frac{n_{e-1} + n_{e+1}}{n_e} = r_e$. This implies that Y is not dominant, because the sequence $\mathbf{n} = (n_0, n_1, \dots, n_e)$ is contained in a longer sequence $(n_0, n_1, \dots, n_{e+1})$. This contradicts that Y is dominant. Hence lY is dominant, and so (1) is confirmed.

Next we show (2). Remember that a subbranch is dominant tame if and only if $\frac{n_{e-1}}{n_e} \geq r_e$ (Lemma 9.2.1). Obviously $\frac{n_{e-1}}{n_e} \geq r_e$ is equivalent to $\frac{ln_{e-1}}{ln_e} \geq r_e$, and so we confirm the equivalence in (2). □

We remark that in (1) of the above lemma, the converse is *not* true in general. For instance, $l = 4$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (1, 1, 1)$. Then $l\mathbf{n} = (4, 4, 4)$ is dominant wild, whereas $\mathbf{n}$ is not dominant; indeed, $\mathbf{n}$ is contained in a dominant sequence $(1, 1, 1, 1, 1, 1)$.

Next we show the equivalence of conditions of type A_l.

Lemma 9.2.3 *The following conditions are equivalent:*

(A1) *$lY \leq X$ and $\frac{n_{e-1}}{n_e} \geq r_e$.*
(A2) *$lY \leq X$ and lY is dominant tame.*
(A3) *$lY \leq X$ and Y is dominant tame.*

Proof. The equivalence of (A1) and (A.2) follows from Lemma 9.2.2, while that of (A.2) and (A.3) follows from Lemma 9.2.2 (2). □

Next we derive a formula needed for later use.

Lemma 9.2.4 *Let lY be a subbranch* (note: Y is not assumed to be of type A_l). *Set $a := m_{e-1} - ln_{e-1}$ and $b := m_e - ln_e$. Then*

$$a - r_e b = -m_{e+1} - l(n_{e-1} - r_e n_e).$$

In fact,

$$\begin{aligned} a - r_e b &= (m_{e-1} - ln_{e-1}) - r_e(m_e - ln_e) \\ &= m_{e-1} - r_e m_e - ln_{e-1} + r_e ln_e \\ &= -m_{e+1} - ln_{e-1} + r_e ln_e, \end{aligned}$$

where the last equality follows from $\dfrac{m_{e-1} + m_{e+1}}{m_e} = r_e$.

Proposition 9.2.5 *Let Y be a subbranch of type A_l, and set*

$$a := m_{e-1} - ln_{e-1}, \quad b := m_e - ln_e, \quad c := n_{e-1} \quad \text{and} \quad d := n_e.$$

Then the following inequalities hold:

(1) $a - r_e b \le -m_{e+1}$ (2) $r_e d - c \le 0$ (3) $u \le b - m_{e+1}$, *where* $u := a - (r_e - 1)b$.

Proof. (1): If Y is of type A_l, then $\dfrac{n_{e-1}}{n_e} \ge r_e$, i.e.

$$n_{e-1} - r_e n_e \ge 0. \tag{9.2.2}$$

By Lemma 9.2.4, $a - r_e b = -m_{e+1} - l(n_{e-1} - r_e n_e)$, and so from (9.2.2), we derive $a - r_e b \le -m_{e+1}$. This proves (1).
(2): As $d = n_e$ and $c = n_{e-1}$, (2) is nothing but (9.2.2).
(3): By (1), $a - r_e b \le -m_{e+1}$, and hence together with $b \ge 0$, we have

$$u = b + (a - r_e b) \le b - m_{e+1}.$$

This proves (3). □

We gather several basic lemmas for subbranches (not necessarily of type A_l):

Lemma 9.2.6 *Let lY be a subbranch with the multiplicities $l\mathbf{n} = (ln_0, ln_1, \dots, ln_e)$. Let Z be the dominant subbranch containing lY, and write its multiplicities as*

$$\mathbf{k} = (ln_1, ln_2, \dots, ln_e, k_{e+1}, k_{e+2}, \dots, k_f).$$

Then l divides k_i for $i = e+1, e+2, \dots, f$. (In particular, "defining" n_i $(i = e+1, e+2, \dots, f)$ by $n_i := \dfrac{k_i}{l}$, then $Z = lY'$ where $Y' = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_f\Theta_f$.)

Proof. Since Z is a subbranch, we have

$$\frac{ln_{e-1} + k_{e+1}}{ln_e} = r_e, \tag{9.2.3}$$

$$\frac{ln_e + k_{e+2}}{k_{e+1}} = r_{e+1}, \tag{9.2.4}$$

$$\frac{k_{i-1} + k_{i+1}}{lk_i} = r_i, \qquad i = e+2, e+3, \dots, f-1. \tag{9.2.5}$$

By (9.2.3), we have $n_{e-1} + \frac{k_{e+1}}{l} = r_e n_e$, and hence l divides k_{e+1}. Set $n_{e+1} := \frac{k_{e+1}}{l}$, i.e. $k_{e+1} = ln_{e+1}$ which we substitute into (9.2.4): $n_e + \frac{k_{e+2}}{l} = r_e n_{e+1}$. Hence l divides k_{e+2}. Repeating this argument, we see that l divides k_i for $i = e+1, e+2, \dots, f$. □

Lemma 9.2.7 *Suppose that $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is a branch. Let lY be a subbranch with the multiplicities $l\mathbf{n} = (ln_0, ln_1, \dots, ln_e)$, and let Z be the dominant subbranch containing lY* (note: $Z = lY'$ for some Y' by Lemma 9.2.6). *Set $a := m_{e-1} - ln_{e-1}$ and $b := m_e - ln_e$. Then the following statements hold:*

(I) *If $a = 0$, then $Z = X$* (and so Z is “trivially” dominant tame.)
(II) *If Y is dominant tame, then the following equivalences hold:*

$$a = 0 \iff b = 0 \iff X = lY.$$

(Note: If lY is dominant but not tame, then (II) is *not* valid. For example, $l = 4$, $\mathbf{m} = (6,5,4,3,2,1)$ and $\mathbf{n} = (1,1,1)$. Then $a = 0$ but $b \neq 0$.)

Proof. (I): By Lemma 9.2.6, we may express $Z = lY'$ where $Y' = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_f\Theta_f$ $(e \le f)$. It is enough to show that $m_i = ln_i$ for $i = 0, 1, \dots, f$; in fact, once this is shown, we have $Z = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_f\Theta_f$, and Z of this form is dominant precisely when $f = \lambda$, i.e. $Z = X$. Now we show that $m_i = ln_i$ firstly for $i = 0, 1, \dots, e$. From the definition of a subbranch,

$$\frac{ln_{e-2} + ln_e}{ln_{e-1}} = \frac{m_{e-2} + m_e}{m_{e-1}} \ (= r_{e-1}).$$

In particular if $a = 0$, i.e. $m_{e-1} = ln_{e-1}$, then

$$ln_{e-2} + ln_e = m_{e-2} + m_e. \tag{9.2.6}$$

Taking into account $ln_{e-2} \le m_{e-2}$ and $ln_e \le m_e$, (9.2.6) implies that $ln_{e-2} = m_{e-2}$ and $ln_e = m_e$. Next, again from the definition of a subbranch,

$$\frac{ln_{e-3} + ln_{e-1}}{ln_{e-2}} = \frac{m_{e-3} + m_{e-1}}{m_{e-2}} \ (= r_{e-2}).$$

From $ln_{e-2} = m_{e-2}$ (we showed this just above), we have

$$ln_{e-3} + ln_{e-1} = m_{e-3} + m_{e-1}. \tag{9.2.7}$$

Taking into account $ln_{e-3} \leq m_{e-3}$ and $ln_{e-1} \leq m_{e-1}$, (9.2.7) implies that $ln_{e-3} = m_{e-3}$ and $ln_{e-1} = m_{e-1}$. Repeating this argument, we deduce $m_i = ln_i$ for $i = 0, 1, \ldots, e$. Similarly, we can show that $m_i = ln_i$ for $i = e+1, e+2, \ldots, f$. Therefore $m_i = ln_i$ holds for $i = 0, 1, \ldots, f$. This proves (I).
(II): The equivalence "$a = 0 \iff b = 0$" is already shown in Lemma 6.3.1, p107. To show the equivalence "$a = 0 \iff X = lY$", we note that if Y is dominant tame, then lY is also dominant tame by Lemma 9.2.3; thus the dominant subbranch Z containing lY is lY itself; $Z = lY$. We now show "$a = 0 \iff X = lY$".

$\Longrightarrow$: If $a = 0$, then $X = Z$ by the assertion (I). Since $Z = lY$, we have $X = lY$.

$\Longleftarrow$: Trivial. This completes the proof of the assertion (II). □

As a corollary, we have the following result.

Corollary 9.2.8 *Let Y be a subbranch of type A_l, and set $a := m_{e-1} - ln_{e-1}$ and $b := m_e - ln_e$. Then the following equivalences hold: $a = 0 \iff b = 0 \iff X = lY$.*

Proof. By Lemma 9.2.3, if Y is of type A_l, then Y is dominant tame, and so the assertion follows from the above lemma. □

9.3 Demonstration of properties of type B_l

We begin with the following lemma for subbranches not necessarily of type B_l.

Lemma 9.3.1 *Let l be a positive integer and let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ such that lY is a dominant wild subbranch of X. Set $a := m_{e-1} - ln_{e-1}$, $b := m_e - ln_e$, $c := n_{e-1}$, $d := n_e$ and $u := a - (r_e - 1)b$. Then the following inequalities hold:*

(1) $a, c, d > 0$, (2) $b \geq 0$, (3) $a - r_e b > 0$, (4) $r_e d - c > 0$, (5) $u > 0$.

Proof. We first verify (1) and (2). Since lY is a subbranch of X, we have $m_{e-1} \geq ln_{e-1}$ and $m_e \geq ln_e$, and so $a, b \geq 0$. Since Y is a subbranch of X, we have $n_{e-1}, n_e > 0$, and so $c, d > 0$. Hence to prove (1) and (2), it remains to show $a > 0$, which is carried out by contradiction. Suppose that $a = 0$, namely $m_{e-1} = ln_{e-1}$. Then

$$\begin{aligned} r_e &> \frac{ln_{e-1} + m_{e+1}}{ln_e} && \text{because } lY \text{ is wild} \\ &= \frac{m_{e-1} + m_{e+1}}{ln_e} && \text{by } m_{e-1} = ln_{e-1} \end{aligned}$$

$$\begin{aligned} &\geq \frac{m_{e-1}+m_{e+1}}{m_e} \qquad \text{by } m_e \geq l n_e \\ &= r_e, \end{aligned}$$

and thus $r_e > r_e$, giving a contradiction. This proves $a > 0$. To show (3), we first note that

$$\begin{aligned} a - r_e b &:= (m_{e-1} - ln_{e-1}) - r_e(m_e - ln_e) \\ &= (m_{e-1} - r_e m_e) - ln_e + r_e ln_e, \end{aligned}$$

where $m_{e-1} - r_e m_e = -m_{e+1}$ by $\dfrac{m_{e-1}+m_{e+1}}{m_e} = r_e$, and therefore

$$a - r_e b = -m_{e+1} - ln_e + r_e ln_e.$$

Since lY is wild, we have $r_e > \dfrac{ln_{e-1}+m_{e+1}}{ln_e}$, and so

$$\begin{aligned} a - r_e b &= -m_{e+1} - ln_{e-1} + r_e ln_e \\ &> -m_{e+1} - ln_{e-1} + \frac{ln_{e-1}+m_{e+1}}{ln_e} ln_e \\ &= 0. \end{aligned}$$

Thus $a - r_e b > 0$, and (3) is proved. Similarly, (4) is shown as follows:

$$r_e d - c = r_e n_e - n_{e-1} > \frac{ln_{e-1}+m_{e+1}}{ln_e} n_e - n_{e-1} = \frac{m_{e+1}}{l} > 0.$$

Finally, it is immediate to show (5). Indeed, $a - r_e b > 0$ by (3) and $b \geq 0$ by (2), and so we have $u = a - (r_e - 1)b = (a - r_e b) + b > 0$. Thus (5) is proved.□

Recall that a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is proportional if $\frac{m_0}{n_0} = \frac{m_1}{n_1} = \cdots = \frac{m_e}{n_e}$. By Lemma 9.1.3, a dominant subbranch Y is both of type A_l and of type B_l (i.e. type AB_l) if and only if Y is of proportional type B_l; explicitly this is the case

$$\mathbf{m} = (ln_0, ln_1, \ldots, ln_\lambda), \quad \mathbf{n} = (n_0, n_1, \ldots, n_\lambda), \quad \text{and} \quad n_\lambda = 1.$$

The arithmetic property of proportional type B_l is the same as that of type A_l, namely dominant tame. Next we investigate the arithmetic property of non-proportional type B_l; remember that a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is of type B_l provided that $m_e = l$ and $n_e = 1$.

Proposition 9.3.2 *Let $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ be a branch, and suppose that $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + m_e\Theta_e$ is a subbranch of non-proportional type B_l of X. Set $a := m_{e-1} - ln_{e-1}$, $b := m_e - ln_e$, $c := n_{e-1}$, $d := n_e\ (= 1)$ and $u := a - (r_e - 1)b$. Then*

(1) *lY is dominant wild, and*
(2) *$a > 0$, $\quad a - r_e b > 0$, $\quad r_e d - c > 0$, $\quad u > 0$.*

Proof. The proof of (1) consists of two steps:
Step 1 We demonstrate that lY is dominant by contradiction. Suppose that lY is not dominant. Then there exists an integer k_{e+1} $(0 < k_{e+1} \leq m_{e+1})$ satisfying

$$\frac{ln_{e-1} + k_{e+1}}{ln_e} = r_e, \tag{9.3.1}$$

and so

$$\frac{ln_{e-1} + k_{e+1}}{ln_e} = \frac{m_{e-1} + m_{e+1}}{m_e} \ (= r_e).$$

Since $m_e = ln_e \,(= l)$ by the definition of type B_l, we have

$$ln_{e-1} + k_{e+1} = m_{e-1} + m_{e+1}.$$

As $ln_{e-1} \leq m_{e-1}$ and $k_{e+1} \leq m_{e+1}$, this holds exactly when

$$ln_{e-1} = m_{e-1}, \qquad k_{e+1} = m_{e+1}. \tag{9.3.2}$$

Note that from (9.3.1), we have $n_{e-1} + \frac{k_{e+1}}{l} = r_e n_e$. So l divides k_{e+1}, and in particular, $l \leq k_{e+1}$. Namely $m_e \leq m_{e+1}$ by $m_e = l$ (the definition of type B_l) and $m_{e+1} = k_{e+1}$ (9.3.2). This yields a contradiction because the sequence $m_0, m_1, \ldots, m_\lambda$ is strictly decreasing. Therefore lY is dominant.
Step 2 We next show that lY is wild, that is, $\frac{ln_{e-1} + m_{e+1}}{ln_e} < r_e$ as follows:

$$\begin{aligned} \frac{ln_{e-1} + m_{e+1}}{ln_e} &< \frac{m_{e-1} + m_{e+1}}{ln_e} && \text{by } ln_{e-1} < m_{e-1} \\ &= \frac{m_{e-1} + m_{e+1}}{m_e} && \text{by } m_e = ln_e \,(= l) \\ &= r_e. && (9.3.3) \end{aligned}$$

Thus lY is dominant wild, and so (1) is confirmed. The assertion (2) follows immediately from Lemma 9.3.1 because lY is dominant wild. (Note: In (9.3.3), "$ln_{e-1} < m_{e-1}$" is *not* valid for proportional type B_l, as $ln_{e-1} = m_{e-1}$.) □

9.4 Demonstration of properties of type C_l

Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, where we set

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}, \quad (i = 1, 2 \ldots, \lambda - 1), \qquad r_\lambda := \frac{m_{\lambda-1}}{m_\lambda}.$$

For a while, we do *not* assume that Y is of type C_l; Y is an arbitrary subbranch.

Lemma 9.4.1 *Set* $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$. *Then*

$$u = m_e - m_{e+1} + l(r_e n_e - n_{e-1} - n_e).$$

(In particular, if $n_e = r_e n_e - n_{e-1}$, then $u = m_e - m_{e+1}$.)

Proof. In fact,

$$\begin{aligned} u &= (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e) \\ &= m_e + m_{e-1} - r_e m_e + l(r_e n_e - n_{e-1} - n_e) \\ &= m_e - m_{e+1} + l(r_e n_e - n_{e-1} - n_e), \end{aligned}$$

where in the last equality we used $r_e m_e = m_{e-1} + m_{e+1}$. □

Next we show the equivalence of three conditions of type C_l.

Lemma 9.4.2 *The following conditions are equivalent:*

(C.1) $lY \leq X$, n_e *divides* n_{e-1}, *and* $\frac{n_{e-1}}{n_e} < r_e$, *and* u *divides* l *where* $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$.
(C.2) $lY \leq X$, $n_e = r_e n_e - n_{e-1}$, *and* u *divides* l *where* u *is in* (C.1).
(C.3) $lY \leq X$, $n_e = r_e n_e - n_{e-1}$, *and* $m_e - m_{e+1}$ *divides* l, *where by convention,* $m_{e+1} = 0$ *if* $\lambda = e$.

(Note: By Lemma 9.4.1, $m_e - m_{e+1}$ equals u in (C.1) and (C.2). In (C.3), actually $m_{e+1} = 0$ does *not* occur as we will see in Corollary 9.4.4 below.)

Proof. We first show that (C.2) is equivalent to (C.1).
(C.2) $\Longrightarrow$ (C.1): This is easy. If $n_e = r_e n_e - n_{e-1}$, then n_e divides n_{e-1}, and $\frac{n_{e-1}}{n_e} = r_e - 1 < r_e$, hence (C.1) holds.
(C.2) $\Longleftarrow$ (C.1): Under the assumption that n_e divides n_{e-1} and $\frac{n_{e-1}}{n_e} < r_e$, it suffices to prove that $n_e = r_e n_e - n_{e-1}$, that is, $r_e - \frac{n_{e-1}}{n_e} = 1$ holds; setting $q := r_e - \frac{n_{e-1}}{n_e}$, we show $q = 1$. We first note that (i) q is an integer because n_e divides n_{e-1}, and (ii) q is positive because $\frac{n_{e-1}}{n_e} < r_e$. Therefore q is a positive integer. We then prove $q = 1$ by contradiction. Suppose that

$$q \geq 2. \tag{9.4.1}$$

We note

$$\begin{aligned} u &= (m_e - m_{e+1}) + l(r_e n_e - n_{e-1} - n_e) && \text{by Lemma 9.4.1} \\ &= (m_e - m_{e+1}) + ln_e\left(r_e - \frac{n_{e-1}}{n_e} - 1\right) \\ &= (m_e - m_{e+1}) + ln_e(q - 1). \end{aligned}$$

Here $m_e - m_{e+1} > 0$ because the sequence $m_0, m_1, \ldots, m_\lambda$ strictly decreases. On the other hand, $n_e \geq 1$ and $q - 1 \geq 1$ (9.4.1). Hence

$$u = (m_e - m_{e+1}) + ln_e(q-1) > l.$$

But u divides l by assumption, and so $l \geq u$. This is a contradiction. Therefore $q = 1$, and the claim is confirmed.

Finally, we show that (C.2) is equivalent to (C.3). This is evident. Indeed, $u = (m_e - m_{e+1}) + l(r_e n_e - n_{e-1} - n_e)$ by Lemma 9.4.1, and hence if $n_e = r_e n_e - n_{e-1}$, then $u = m_e - m_{e+1}$. □

Recall that a subbranch Y is of type C_l provided that Y satisfies one of the equivalent conditions of Lemma 9.4.2.

Corollary 9.4.3 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. Set $b := m_e - ln_e$ and $d := n_e$, and then*

(1) $m_e = ld + b$, *and*
(2) *if furthermore Y is of type C_l, then $m_{e+1} = ld + b - u$.*

Proof. From $d = n_e$ and $b = m_e - ln_e$, we have $m_e = ld + b$, and so (1) is confirmed. Next we show (2). If Y is of type C_l, we have $u = m_e - m_{e+1}$ (Lemma 9.1.5). Substituting (1) $m_e = ld + b$ into $u = m_e - m_{e+1}$, we obtain $u = ld + b - m_{e+1}$. This confirms (2). □

We also note the following.

Corollary 9.4.4 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. If Y is of type C_l, then $e + 1 \leq \lambda$.*

Proof. We show this by contradiction. Suppose that $m_{e+1}(:= r_e m_e - m_{e-1}) = 0$. We first claim that m_e divides both l and $m_e - ln_e$. In fact, since $m_e - m_{e+1}$ divides l by the definition (C.3) of type C_l and $m_{e+1} = 0$ by assumption, we see that m_e divides l; clearly this also assures that m_e divides $m_e - ln_e$. Next setting $b := m_e - ln_e$, we write $l = l'm_e$ and $b = b'm_e$ where l' (resp. b') is a positive (resp. nonnegative) integer. Then

$$b' = \frac{b}{m_e} = \frac{m_e - ln_e}{m_e} = \frac{m_e - l'm_e n_e}{m_e} = 1 - l'n_e.$$

Namely

$$b' = 1 - l'n_e. \tag{9.4.2}$$

From $l' \geq 1$ and $n_e \geq 1$, we have $b' \leq 0$. Since b' is nonnegative, we obtain $b' = 0$ (and so $b = 0$), and then by (9.4.2), $l' = n_e = 1$. Here note that $b = 0$ implies that $m_e = ln_e$, and thus together with $n_e = 1$, we have $m_e = l$. This means that Y is not only of type C_l but also of type B_l. But this contradicts Convention 9.1.4 (we excluded this case from type C_l). □

We collect several lemmas needed for later discussion.

Lemma 9.4.5 *Let lY be a subbranch with the multiplicities $l\mathbf{n} = (ln_0, ln_1, \ldots, ln_e)$. Let Z be the dominant subbranch containing lY, and write its multiplicities as*

$$\mathbf{k} = (ln_1, ln_2, \ldots, ln_e, k_{e+1}, k_{e+2}, \ldots, k_f).$$

Then

(I) *l divides k_i $(i = e{+}1, e{+}2, \ldots, f)$.* (So "defining" n_i $(i = e{+}1, e{+}2, \ldots, f)$ by $n_i := \frac{k_i}{l}$, then $Z = lY'$ and $\mathbf{k} = l\mathbf{n}'$ where $Y' := n_0\Delta_0 + n_1\Theta_1 + \cdots + n_f\Theta_f$ and $\mathbf{n}' := (n_0, n_1, \ldots, n_f)$.)

(II) *if n_e divides n_{e-1}, then n_e also divides n_i $(i = e+1, e+2, \ldots, f)$, and moreover $n_e \le n_{e+1} \le n_{e+2} \le \cdots \le n_f$.*

Proof. (I) is nothing but Lemma 9.2.6. We show (II); we first prove that n_e divides n_{e+1}. Since $Z = ln_0\Delta_0 + ln_1\Theta_1 + \cdots + ln_f\Theta_f$ is a subbranch, we have $ln_{i+1} = r_i ln_i - ln_{i-1}$, so $n_{i+1} = r_i n_i - n_{i-1}$ for $i = 1, 2, \ldots, f$. In particular $n_{e+1} = r_e n_e - n_{e-1}$. Hence n_e divides n_{e+1} (note: by assumption, n_e divides n_{e-1}), and consequently $n_e \le n_{e+1}$. Next since n_e divides n_{e+1}, from $n_{e+2} = r_{e+1}n_{e+1} - n_e$, it follows that n_e also divides n_{e+2}. Furthermore

$$\begin{aligned} n_{e+2} &= r_{e+1}n_{e+1} - n_e \\ &\ge 2n_{e+1} - n_e && \text{by } r_{e+1} \ge 2 \\ &= n_{e+1} + (n_{e+1} - n_e) \\ &\ge n_{e+1} && \text{by } n_{e+1} \ge n_e. \end{aligned} \tag{9.4.3}$$

Namely, $n_{e+1} \le n_{e+2}$. Then using the fact (as shown above) that n_e divides both n_{e+1} and n_{e+2}, it follows from $n_{e+3} = r_{e+2}n_{e+2} - n_{e+1}$ that n_e divides n_{e+3}. Also we can show $n_{e+2} \le n_{e+3}$ as in (9.4.3). Repeat this argument, and then (II) is shown. □

Lemma 9.4.6 *Let Y be a subbranch of type C_l. Then Y and lY are* (*not necessarily dominant*) *wild.*

Proof. We first verify the wildness of lY. We separate into two cases according to whether lY is dominant or not.

Case 1 lY is dominant: Since Y is of type C_l, $\frac{n_{e-1}}{n_e} < r_e$ and so $\frac{ln_{e-1}}{ln_e} < r_e$, which means that lY is wild.

Case 2 lY is not dominant: Let Z be the dominant subbranch containing lY. Then by Lemma 9.4.5 (I), the multiplicities of Z are of the form:

$$\mathbf{k} = (ln_1, ln_2, \ldots, ln_f).$$

Since Y is of type C_l, n_e divides n_{e-1} and so by Lemma 9.4.5 (II), we have

$$n_e \le n_{e+1} \le n_{e+2} \le \cdots \le n_f.$$

In particular $\frac{n_{f-1}}{n_f} \leq 1$. Since $r_f \geq 2$, we have $\frac{n_{f-1}}{n_f} < r_f$, and so $\frac{ln_{f-1}}{ln_f} < r_f$ This implies that Z is wild, and consequently (by definition) lY is wild. Similarly we can show the wildness of Y. □

For subsequent discussion, we need some result on Y not necessarily of type C_l.

Lemma 9.4.7 *Let lY be a subbranch which is not dominant. Set*

$$a := m_{e-1} - ln_{e-1}, \quad b := m_e - ln_e \quad \textit{and} \quad u := a - (r_e - 1)b.$$

If n_e divides n_{e-1}, then

(I) $a - r_e b = ln_{e+1} - m_{e+1}$ *where* $n_{e+1} := r_e n_e - n_{e-1}$. (In particular, from $ln_{e+1} \leq m_{e+1}$, we have $a - r_e b \leq 0$).
(II) $u > 0$ *and* $a > 0$.

Proof. The statement (I) is derived as follows:

$$\begin{aligned} a - r_e b &= m_{e-1} - ln_{e-1} - r_e(m_e - ln_e) \\ &= (m_{e-1} - r_e m_e) - ln_{e-1} + r_e ln_e \\ &= (-m_{e+1}) - ln_{e-1} + \frac{ln_{e-1} + ln_{e+1}}{ln_e} ln_e \\ &= -m_{e+1} + ln_{e+1}, \end{aligned}$$

where in the third equality we used $m_{e-1} - r_e m_e = -m_{e+1}$ and $r_e = \frac{ln_{e-1} + ln_{e+1}}{ln_e}$ (note that by assumption, lY is not dominant and so $0 < ln_{e+1} \leq m_{e+1}$).

Next we show (II). We first prove $u > 0$. By assumption, n_e divides n_{e-1} and thus by Lemma 9.4.5, we have

$$n_{e+1} \geq n_e. \tag{9.4.4}$$

Then

$$\begin{aligned} u &= b + (a - r_e b) \\ &= (m_e - ln_e) + (ln_{e+1} - m_{e+1}) && \text{by (I)} \\ &= (m_e - m_{e+1}) + l(n_{e+1} - n_e) \\ &> 0 && \text{by } m_e > m_{e+1} \text{ and (9.4.4).} \end{aligned}$$

This proves $u > 0$. Finally, we show $a > 0$. We divide into two cases according to whether $b = 0$ or $b > 0$.

Case $b = 0$: In this case we have $u = a - (r_e - 1)b = a$. Thus $a > 0$ because $u > 0$ as shown above.

Case $b > 0$: Noting that $a \geq 0$, we assume that $a = 0$ and deduce a contradiction. If $a = 0$, then we have $u = a - (r_e - 1)b = -(r_e - 1)b$. Since $r_e \geq 2$, together with $b > 0$, we obtain $u < 0$. This contradicts $u > 0$, and we conclude that $a > 0$. □

Proposition 9.4.8 *Let lY be a subbranch such that* (i) n_e *divides* n_{e-1} *and* (ii) $\frac{n_{e-1}}{n_e} < r_e$. *Then* $u > 0$ *where* $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$.

Proof. According to whether lY is dominant or not, we separate into two cases. If lY is dominant, then from the assumption (ii), we have $\frac{ln_{e-1}}{ln_e} < r_e$ which means that lY is (dominant) wild and hence by Lemma 9.3.1, we have $u > 0$. (Notice that in this case we do not need the assumption (i).) Next if lY is not dominant, together with (i), we conclude that $u > 0$ by Lemma 9.4.7 (II). (Notice that in this case we do not need (ii).) □

Remark 9.4.9 As is clear from the proof, $u > 0$ holds under a weaker assumption: (1) lY is dominant wild or (2) n_e divides n_{e-1}.

The above proposition will be often used later (e.g. for the proofs of Lemma 13.3.5, p242 and Lemma 13.4.5, p247). For $u := a - (r_e - 1)b$ where $a := m_{e-1} - ln_{e-1}$ and $b := m_e - ln_e$, the following inequalities are also valid.

Lemma 9.4.10 *Let Y be a subbranch of type C_l. Then*

$$\begin{cases} u > b & \text{if } lY \text{ is dominant} \\ u \leq b & \text{if } lY \text{ is not dominant.} \end{cases}$$

Proof. If lY is dominant, then (noting that type C_l is wild by Lemma 9.4.6), we have $a - r_e b > 0$ by Lemma 9.3.1, and so $u = b + (a - r_e b) > b$.

If lY is not dominant, then $a - r_e b \leq 0$ by Lemma 9.4.7 (I), and thus $u = b + (a - r_e b) \leq b$. □

We provide examples for the respective cases of the above lemma.

Example ($u > b$)
$l = 1$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (4, 4, 4)$.
Then Y is of type C_l and lY is dominant; in this case $u = 1 > b = 0$.

Example ($u \leq b$)
$l = 1$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (1, 1, 1)$.
Then Y is of type C_l and lY is not dominant; in this case $u = 1 < b = 3$.

Now we summarize the properties of type C_l obtained thus far.

Proposition 9.4.11 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. Set*

$a := m_{e-1} - ln_{e-1}$, $b := m_e - ln_e$, $c := n_{e-1}$, $d := n_e$ *and*
$u := a - (r_e - 1)b$.

Suppose that Y is of type C_l. Then the following statements hold:

(1) *Y and lY are (not necessarily dominant) wild* (Lemma 9.4.6).
(2) $a > 0$ (Lemma 9.4.7 (II)).

(3) $a - r_e b > 0$ *if* lY *is dominant* (Lemma[1] 9.3.1), *and*
$a - r_e b = ln_{e+1} - m_{e+1} \le 0$ *if* lY *is not dominant* (Lemma 9.4.7 (I)).
(4) $r_e d - c = d > 0$ (the definition (C.2) or (C.3) of type C_l).
(5) $u = m_e - m_{e+1} > 0$ (Lemma 9.1.5).
(6) $u > b$ *if* lY *is dominant, and* $u \le b$ *if* lY *is not dominant* (Lemma 9.4.10).
(7) $e + 1 \le \lambda$ (Corollary 9.4.4).

Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + m_e\Theta_e$ be a subbranch of of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$; recall that Θ_i is a (-2)-*curve* if the self-intersection number $\Theta_i \cdot \Theta_i = -2$, and a *chain of* (-2)-*curve* is a set of (-2)-curves of the form $\Theta_a + \Theta_{a+1} + \cdots + \Theta_b$ where $a \le b$. (*It is valuable to keep in mind that the existence of a chain of* (-2)-*curves often implies the existence of various deformations.*)

We shall show that if Y is of type C_l, then in most cases, the complement of a subbranch Y of X contains a chain of (-2)-curves where the "complement" is $\Theta_{e+1} + \Theta_{e+2} + \cdots + \Theta_\lambda$ (note that $e + 1 \le \lambda$ for type C_l as shown in Corollary 9.4.4). cf. Example 9.1.11 for an exceptional case where the complement contains no (-2)-curves.

Now we give the information on chains of (-2)-curves in the complement of Y in X. Below we note that $r_i = 2$ is equivalent to Θ_i being a (-2)-curve.

Proposition 9.4.12 *Let* $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ *be a subbranch of type* C_l. *Set*

$$a := m_{e-1} - ln_{e-1}, \quad b := m_e - ln_e, \quad c := n_{e-1}, \quad d := n_e \quad \text{and}$$
$$u := a - (r_e - 1)b,$$

and (*noting that* u *divides* l *by the definition of type* C_l), *write* $l = Nu$ *where* N *is a positive integer, and if* $u \le b$, *then* (*considering the division of* b *by* u), *let* v *be the positive integer such that* $b - vu \ge 0$ *and* $b - (v+1)u < 0$. *Then the following holds:*

(I) *if* $b = 0$, *then* $\lambda = e + Nd - 1$ *and*
$r_{e+1} = r_{e+2} = \cdots = r_\lambda = 2, \quad m_{\lambda-1} = 2u, \quad$ *and* $\quad m_\lambda = 2,$
(II) *if* $b \ge 1$ *and* $u > b$, *then* $\lambda = e + Nd$ *and*
$r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2, \quad m_{\lambda-1} = b + u, \quad$ *and* $\quad m_\lambda = b,$
(Note: $r_\lambda := \dfrac{m_{\lambda-1}}{m_\lambda} = \dfrac{b+u}{b}$ is an integer and so in this case, b divides u.)
(III) *if* $b \ge 1$, $u \le b$, *and* u *does not divide* b, *then* $\lambda = e + Nd + v$ *and*
$r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2, \quad m_{\lambda-1} = b - (v-1)u, \quad$ *and* $\quad m_\lambda = b - vu.$
(IV) *if* $b \ge 1$, $u \le b$, *and* u *divides* b (so $b = vu$), *then* $\lambda = e + Nd + v - 1$ *and* $r_{e+1} = r_{e+2} = \cdots = r_\lambda = 2, \quad m_{\lambda-1} = 2u, \quad$ *and* $\quad m_\lambda = u.$

[1] If Y is of type C_l, then lY is wild by (1), and so we can apply Lemma 9.3.1 for dominant wild Y.

Remark 9.4.13 Note that $u > b$ if lY is dominant, and $u \leq b$ otherwise. See Proposition 9.4.11 (6).

Proof. (I): By Corollary 9.4.3 applied for $b = 0$, we have

$$m_e = ld, \qquad m_{e+1} = ld - u, \qquad \text{where } d := n_e. \tag{9.4.5}$$

Since a sequence $(m_e, m_{e+1}, \ldots, m_\lambda)$ is inductively determined from m_e and m_{e+1} by the division algorithm, this sequence is uniquely characterized by the following properties:

(a) $m_e > m_{e+1} > \cdots > m_\lambda > 0$,
(b) $\dfrac{m_{e+i-1} + m_{e+i+1}}{m_{e+i}}$, $(i = 0, 1, \ldots, \lambda - e - 1)$ is an integer, and m_λ divides $m_{\lambda-1}$.

Therefore (9.4.5) implies that $m_{e+i} = ld - iu$ $(i = 0, 1, \ldots, \lambda - e - 1)$, an arithmetic progression. Note that

$$\begin{aligned} m_{e+(Nd)} &= ld - (Nd)u = ld - ld \qquad \text{by } l = Nu \\ &= 0, \end{aligned}$$

whereas $m_{e+(Nd-1)} = u$. Thus we conclude that $\lambda = e + (Nd - 1)$ and

$$(m_e, m_{e+1}, \ldots, m_\lambda) = (ld,\ ld - u,\ ld - 2u,\ \ldots\ ,\ 2u,\ u),$$

from which we deduce $r_{e+1} = r_{e+2} = \cdots = r_\lambda = 2$. This proves the assertion (I).

(II) $b \geq 1$ and $u > b$: The proof is essentially the same as that for (I). By Corollary 9.4.3, we have

$$m_e = ld + b, \qquad m_{e+1} = ld + b - u. \tag{9.4.6}$$

As in (I), this implies that $m_{e+i} = ld + b - iu$, an arithmetic progression. When $i = Nd - 1$, we have

$$\begin{aligned} m_{e+(Nd-1)} &= ld + b - (Nd - 1)u \\ &= ld + b - ld + u \qquad \text{by } l = Nu \\ &= b + u, \end{aligned}$$

and likewise $m_{e+(Nd)} = b$. On the other hand, we have $m_{e+(Nd+1)} = b - u < 0$ (note $u > b$ by assumption), and thus $\lambda = e + Nd$. Therefore

$$m_{e+i} = \begin{cases} ld + b - iu, & i = 0, 1, \ldots, Nd - 1 \\ b, & i = Nd, \end{cases} \tag{9.4.7}$$

and $r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2$.

(III) $b \geq 1$, $u \leq b$, and u does not divide b: The proof is similar to that of (II); by Corollary 9.4.3, we have

$$m_e = ld + b, \qquad m_{e+1} = ld + b - u, \tag{9.4.8}$$

which implies that $m_{e+i} = ld + b - iu$, an arithmetic progression. Let v be the positive integer such that $b - vu \geq 0$ and $b - (v+1)u < 0$. Then for $i = Nd + v - 1$, we have

$$\begin{aligned} m_{e+(Nd+v-1)} &= ld + b - (Nd + v - 1)u \\ &= ld + b - ld - vu + u \qquad\qquad \text{by } l = Nu \\ &= b - vu + u, \end{aligned}$$

and likewise $m_{e+(Nd+v)} = b - vu$. Similarly we obtain $m_{e+(Nd+v+1)} = b - (v+1)u$. As we took the positive integer v in such a way that $b - vu \geq 0$ and $b - (v+1)u < 0$, we have $m_{e+(Nd+v)} = b - vu > 0$ and $m_{e+(Nd+v+1)} < 0$; hence $\lambda = e + (Nd + v)$. We thus conclude that

$$m_{e+i} = \begin{cases} ld + b - iu, & i = 0, 1, \ldots, Nd + v - 1 \\ b - vu, & i = Nd + v, \end{cases}$$

from which we derive $r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2$. This proves the assertion.

(IV) $b \geq 1$, $u \leq b$, and u divides b (i.e. $b = vu$): Using the computation of (III), we have $m_{e+(Nd+v)} = b - vu = 0$ in the present case, and thus $\lambda = e + (Nd + v - 1)$, and $m_{\lambda-1} = 2u$ and $m_\lambda = u$. The remaining statement follows from the same argument as in (III). □

Supplement

In the proof of Proposition 9.4.12, we only used the fact "u divides ld". The reader may wonder that in the definition of type C_l, we can replace "u divides l" by a weaker condition "u divides ld". Unfortunately this is not true, because in that case the deformation atlas associated with lY does *not* necessarily admit a complete propagation. This is confirmed by the following example, which illustrates the essential role of the condition "u divides l" in type C_l.

Example 9.4.14 Let $X = 32\Delta_0 + 24\Theta_1 + 16\Theta_2 + 8\Theta_3$. We take $Y = 2\Delta_0 + 2\Theta_1$ and $l = 12$. Then lY satisfies the condition of type C_l except "u divides l"; indeed $u = 8$ and $d\,(:= n_e) = 2$, hence u does not divide l but divides $ld = 24$.

We show that the deformation atlas associated with lY does *not* admit a complete propagation. First note that

$$\mathcal{H}_1 : \quad w^8(w^2\eta^2 + t^2)^{12} - s = 0.$$

(The exponent 2 of t^2 is necessary for making a first propagation possible. See the second equality of (9.4.9) below.) We take $g_1 :\ z = 1/w,\ \zeta = w^2\eta -$

$\sqrt{-1}tw$, where we note that there is no other choice of g_1 which transforms $\mathcal{H}_1$ to a hypersurface. Since

$$w^8(w^2\eta^2+t^2)^{12} = w^8\left[\frac{1}{w^2}(w^2\eta)^2+t^2\right]^{12},$$

the map g_1 transforms the polynomial $w^8(w^2\eta^2+t^2)^{12}$ to

$$\begin{aligned}\frac{1}{z^8}\left[z^2\left(\zeta+t\sqrt{-1}\frac{1}{z}\right)^2+t^2\right]^{12} &= \frac{1}{z^8}\left[\left(z^2\zeta^2+2\sqrt{-1}tz\zeta-t^2\right)+t^2\right]^{12}\\ &= \frac{1}{z^8}\left[z^2\zeta^2+2\sqrt{-1}tz\zeta\right]^{12}\\ &= z^4[\,z\zeta^2+2\sqrt{-1}t\zeta\,]^{12}.\end{aligned} \tag{9.4.9}$$

Thus the following data gives a first propagation.

$$\begin{cases}\mathcal{H}_1: & w^8(w^2\eta^2+t^2)^{12}-s=0\\ \mathcal{H}_1': & z^4(z\zeta^2+2\sqrt{-1}t\zeta)^{12}-s=0\\ g_1: & z=1/w,\ \zeta=w^2\eta-\sqrt{-1}tw.\end{cases}$$

Similarly, we can construct a second propagation as follows: Noting that

$$\mathcal{H}_2: \quad \eta^4\,(w^2\eta+2\sqrt{-1}tw)^{12}-s=0,$$

we take $g_2:\ z=1/w,\ \zeta=w^2\eta+2\sqrt{-1}tw$. (Note: there is no other choice of g_2 which transforms $\mathcal{H}_2$ to a hypersurface. See the second equality of (9.4.10) below.) Since

$$\eta^4(w^2\eta+2\sqrt{-1}tw)^{12} = \frac{1}{w^8}(w^2\eta)^4\left[(w^2\eta)+2\sqrt{-1}tw\right]^{12},$$

the map g_2 transforms a polynomial $\eta^4\,(\,w^2\eta+2\sqrt{-1}tw\,)^{12}$ to

$$\begin{aligned}z^8\left(\zeta-2\sqrt{-1}t\frac{1}{z}\right)^4\left[\left(\zeta-2\sqrt{-1}t\frac{1}{z}\right)+2\sqrt{-1}t\frac{1}{z}\right]^{12} &= z^8\left(\zeta-2\sqrt{-1}t\frac{1}{z}\right)^4\zeta^{12}\\ &= z^4\zeta^{12}(z\zeta-2\sqrt{-1}t)^4.\end{aligned} \tag{9.4.10}$$

Hence the following data gives a second propagation:

$$\begin{cases}\mathcal{H}_2: & \eta^4(\,w^2\eta+2\sqrt{-1}tw\,)^{12}-s=0\\ \mathcal{H}_2': & z^4\zeta^{12}(z\zeta-2\sqrt{-1}t)^4-s=0\\ g_2: & z=1/w,\ \zeta=w^2\eta+2\sqrt{-1}w.\end{cases}$$

It remains to construct a third propagation. However this is impossible, which is seen as follows. Note that $\mathcal{H}_3:\ w^{12}\eta^4(w\eta-2\sqrt{-1}t)^4-s=0$, and a standard form of a deformation g_3 of $z = 1/w,\ \zeta = w^2\eta$ is given by $z = 1/w,\ \zeta = w^2\eta + f(t)w$ where $f(t)$ is a holomorphic function in t. For brevity, we only consider the case $f(t) = \alpha t^k$ where $\alpha \in \mathbb{C}$ and k is a positive integer (the discussion below is valid for general $f(t)$). We claim that for any α and k, the map g_3 cannot transform

$$\mathcal{H}_3 : \quad w^{12}\eta^4(w\eta - 2\sqrt{-1}t)^4 - s = 0$$

to a hypersurface. In fact, since

$$w^{12}\eta^4(w\eta - 2\sqrt{-1}t)^4 = w^4(w^2\eta)^4\left(\frac{1}{w}(w^2\eta) - 2\sqrt{-1}t\right)^4,$$

the map g_3 transforms $\mathcal{H}_3$ to

$$\frac{1}{z^4}\left(\zeta - \alpha t^k\frac{1}{z}\right)^4\left(z\zeta - \alpha t^k - 2\sqrt{-1}t\right)^4 - s = 0.$$

Clearly for any choice of $\alpha \in \mathbb{C}$ and a positive integer k, the left hand side, after expansion, contains a fractional term. So a further propagation is impossible, and consequently the deformation atlas associated with lY does not admit a complete propagation. (For a non-standard form of g_3 containing higher or lower order terms, the argument is essentially the same though the computation becomes complicated. cf. Example 5.5.12, p96.)