10

Two-sided estimates

In this chapter we harvest the crop of the earlier chapters. In order to show the conjunction of the upper and lower estimates, we use key observations on the Einstein relation, on the elliptic Harnack inequality and on upper and lower heat kernel estimates.

Corollary 10.1. *Assume that (Γ,μ) satisfies $(p_0),(VD)$ and (H), then*

$$(wTC)\Leftrightarrow(aD\rho v)\Leftrightarrow(TC)\Leftrightarrow(ER)\Leftrightarrow RLE\,(E)\overset{(i)}{\Leftrightarrow}g\,(F)\,,\tag{10.1}$$

where $F\in W_0$ is a consequence in the direction $\overset{(i)}{\Longrightarrow}$ and an assumption for $\overset{(i)}{\Longleftarrow}$.

Proof All the implications are established in Remark 7.2, except the last one which is given there for E, i.e., for $g\,(E)$ and $RLE\,(E)$. The implication $(ER)\Longrightarrow RLE\,(E)$ is trivial. Its reverse is similarly simple; $c\rho v\leq E$ follows as usual, and $E<C\rho v$ is $RLE\,(E)$ itself. It is clear that the $\overset{(i)}{\Longrightarrow}$ direction holds and only its reverse needs some additional arguments. Let us assume that $r_i=2^i,r_{n-1}<2R\leq r_n,B_i=B\,(x,r_i)$, $A_i=B_i\backslash B_{i-1}$ and $V_i=V\,(x,r_i)$. We have derived in Proposition 7.5 from $(p_0),(VD)$ and (H) that

$$E\,(x,2R)\leq C\sum_{i=0}^{n-1}V_{i+1}\rho\,(x,r_i,r_{i+1})\,.$$

Now we use a consequence (3.39) of (H):

$$\rho\,(x,r_i,r_{i+1})\leq C\max_{y\in A_{i+1}}g^{B_{i+1}}\,(x,y)$$

to obtain

$$\begin{aligned} E(x,2R) &\leq C\sum_{i=0}^{n-1} V_{i+1} \max_{y\in A_{i+1}} g^{B_{i+1}}(x,y) \\ &\leq C\sum_{i=0}^{n-1} F(x,r_{i+1}) \leq CF(x,r_n)\sum_{i=0}^{n-1} 2^{-i\beta'} \\ &\leq CF(x,2R), \end{aligned}$$

where (7.12) was used to get the second inequality.

On the other hand, from (7.13) we obtain

$$\begin{aligned} &c\frac{F(x,2R)}{V(x,2R)}\left(V(x,R)-V(x,R/2)\right) \\ &\leq \min_{y\in B(x,R)\setminus B(x,R/2)} g^B(x,y) \sum_{z\in B(x,R)\setminus B(x,R/2)} \mu(z) \\ &\leq \sum_{z\in B(x,R)\setminus B(x,R/2)} g^B(x,z)\,\mu(z) \leq E(x,2R), \end{aligned}$$

which means that

$$cF(x,2R) \leq E(x,2R).$$

Consequently, $F \simeq E$, $E \in W_0$ and (TC) is satisfied which implies (ER) and all the other equivalent conditions. ■

Remark 10.1. Let us remark here that as a side result it follows that $RLE(E)$ or for $F \in W_0$, $g(F)$ implies $\rho v \simeq F$ and $E \simeq F$ as well.

We have seen that

$$(wTC) \Leftrightarrow (aD\rho v) \Leftrightarrow (TC) \Leftrightarrow (ER) \Leftrightarrow RLE(E). \tag{10.2}$$

Let $(*)$ denote any of the equivalent conditions. Using this convention, we can state the main result on weakly homogeneous graphs.

Theorem 10.1. *If a weighted graph (Γ,μ) satisfies (p_0) and (VD), then the following statements are equivalent:*

1. *there is an $F \in W_0$ such that $g(F)$ is satisfied;*
2. *(H) and (wTC) hold;*
3. *(H) and (TC) hold;*
4. *(H) and $(aD\rho v)$ hold;*
5. *(H) and (ER) hold;*
6. *(H) and $RLE(E)$ hold;*
7. *there is an $F \in W_0$ such that $UE(F)$ and $PLE(F)$ are satisfied;*
8. *there is an $F \in W_0$ such that $PMV(F)$ and $PSMV(F)$ are satisfied.*

The proof of Theorem 10.1 contains two autonomous results. The first one is $UE(F) \Longleftrightarrow PMV(F)$, the second one is $PLE(F) \Longleftrightarrow PSMV(F)$.

Let us emphasize the importance of the condition $(aD\rho v)$. It entirely relies on volume and resistance properties, no assumption of stochastic nature is involved, so the result is in the spirit of Einstein's observation on heat propagation. This condition in conjunction with (VD) and (H) provides the characterization of heat kernel estimates in terms of volume and resistance properties. Of course, the elliptic Harnack inequality is not easy to verify. We learn from $g(F)$ that the main properties ensured by the elliptic Harnack inequality that the equipotential surfaces of the local Green kernel $g^{B(x,R)}$ are basically spherical and that the potential growth is regular.

10.1 Time comparison (the return route)

In this section we summarize the results which lead to the equivalence of 1...6, and $6 \Longrightarrow 7$ in Theorem 10.1, and we prove the return route from $8 \Longrightarrow 2$. The equivalence of 1 and 6 is established by Theorem 7.1, 7.4 and 10.1, see also Remark 10.1. The implication $6 \Longrightarrow 7$ is given by Theorem 8.2, and $7 \Longleftrightarrow 8$ is a combination of Theorem 8.2 and 9.1.

Now we prove the return route $8 \Longrightarrow 2$ of Theorem 10.1.

Our task is to verify the implications in the diagram below under the assumption $F \in W_0, (p_0)$ and (VD).

$$\left.\begin{array}{c} PMV_1(F) \\ PSMV(F) \end{array}\right\} \Longrightarrow \left.\begin{array}{c} PMV_{\delta^*}(F) \\ PSMV(F) \end{array}\right\} \Longrightarrow (H) \tag{10.3}$$

$$\left.\begin{array}{c} PMV_1(F) \\ PSMV(F) \end{array}\right\} \Longrightarrow \left.\begin{array}{c} DUE(F) \\ PLE(F) \\ (H) \end{array}\right\} \Longrightarrow \begin{array}{c} \rho v \simeq F \\ (H) \end{array} \Longrightarrow \begin{array}{c} (TC) \\ (H) \end{array} \tag{10.4}$$

The heat kernel estimates are established as we indicated above. Now we deal with the proof of the elliptic Harnack inequality (H) and the time comparison principle (TC).

Theorem 10.2. *If Γ satisfies $(p_0), (VD)$ and there is an $F \in W_0$ for which $PMV(F)$ and $PSMV(F)$ are satisfied, then the elliptic Harnack inequality holds.*

Lemma 10.1. *If (Γ, μ) satisfies $(p_0), (VD)$ and $PMV_1(F)$ for an $F \in W_0$, then for a given $\varepsilon, \delta > 0, 0 < \delta^* \leq \frac{1}{C_F}\varepsilon^{\frac{1}{\beta'}}\frac{\delta}{2}$ there are $c_1 < ... < c_4$ such that $PMV_{\delta^*}(F)$ holds for ε and c_is.*

Proof We would like to derive $PMV_{\delta^*}(F)$ for c_i from $PMV_1(F)$ which holds for some other constants a_i. We will apply $PMV_1(F)$ on the ball $B = B(x, \delta R)$ and re-scale the time accordingly. We have $PMV_{\delta^*}(F)$ on $B(x, R)$ by

$$\max_{\substack{c_3F(x,R)\leq i\leq c_4F(x,R)\\ y\in B}} u_i \leq \max_{\substack{a_3F(x,\delta R)\leq i\leq a_4F(x,\delta R)\\ y\in B}} u_i$$
$$\leq \sum_{j=a_1F(x,\delta R)}^{a_2F(x,\delta R)} \sum_{y\in B} u_j(z)\,\mu(z) \leq \sum_{j=c_1F(x,R)}^{c_2F(x,R)} \sum_{y\in B} u_j(z)\,\mu(z)$$

if the inequalities $c_1 < ... < c_4, a_1 < ... < a_4$

$$\begin{aligned} a_4F(x,\delta R) &\geq c_4F(x,R)\\ a_3F(x,\delta R) &\leq c_3F(x,R)\\ a_2F(x,\delta R) &\leq c_2F(x,R)\\ a_1F(x,\delta R) &\geq c_1F(x,R)\end{aligned}$$

are satisfied. In addition we require $c_4 \leq \varepsilon$ and $\frac{1}{C_F}(c_3 - c_2)^{\frac{1}{\beta'}}\frac{\delta}{2} \geq \delta^*$. We can see that the following choice satisfies these restrictions. Let $p = C_F(\delta^*)^{\beta}$ and $q = c_F(\delta^*)^{\beta'}$. Let us choose

$$\begin{aligned} &c_4 = \varepsilon, a_4 = \frac{2q}{p}c_4,\\ &c_3 < c_4, a_3 = qc_3,\\ &c_2 < c_3, a_2 = \frac{1}{2}\min\{pc_2, a_3\},\\ &c_1 = \frac{1}{2}\min\left\{\frac{a_2}{q}, c_2\right\}, a_1 = qc_1.\end{aligned}$$

Let us observe that c_1 can be arbitrarily small since $c_4 \leq \varepsilon$, and if the sub-solution is not given from an m up to $a_4F(x,\delta R)$, it can be extended simply by $u_{i+m} = P_i^{B(x,R)}u_m$. ■

Proof [of Theorem 10.2] Let us fix a set of constants $c_1 < c_2 < c_3 < c_4 = \varepsilon$ as in Lemma 10.1 and apply $PSMV(F)$ for them. Let us apply Lemma 10.1 for δ^* to receive $PMV_{\delta^*}(F)$ on $B = B(x,R)$. As a consequence for $D = B(x,\delta^*R)$, $u_k(y) = h(y)$ we obtain

$$\max_D h \leq \frac{C}{\mu(D)}\sum_{y\in D} h(y)\,\mu(y). \tag{10.5}$$

Similarly $PSMV(F)$ yields

$$\min_D h \geq \frac{c}{\mu(D)}\sum_{y\in D} h(y)\,\mu(y). \tag{10.6}$$

The combination of (10.5) and (10.6) gives the elliptic Harnack inequality for the shrinking parameter δ^*. Finally (H) can be shown by using the standard chaining argument along a finite chain of balls. The finiteness of the number of balls follows from volume doubling via the bounded covering principle. ■

Theorem 10.3. *If (p_0) and (VD) hold, and there is an $F \in W_0$ for which $PMV(F)$ and $PSMV(F)$ are satisfied, then $E \simeq F$ and (TC) is true.*

Proposition 10.1. *Assume (p_0) and (VD). If $PLE(F)$ for $F \in W_0$ holds, then*

$$E(x,R) \geq cF(x,R).$$

Proof It follows from $PLE(F)$ that there are $c, C, 1 > \delta > \delta' > 0, 1 > \varepsilon > \varepsilon' > 0$ such that for all $x \in \Gamma, R > 1$, $A = B(x,2R)$ and $n : \varepsilon' F(x,R) < n < \varepsilon F(x,R)$, $r = \delta' R$, $y \in B = B(x,r)$

$$\widetilde{P}_n^A(x,y) = P_n^A(x,y) + P_{n+1}^A(x,y) \geq \frac{c\mu(y)}{V(x,R)}.$$

It follows for $F = \varepsilon F(x,R)$ and $F' = \varepsilon' F(x,R)$ that

$$\begin{aligned} E(x,2R) &= \sum_{k=0}^{\infty} \sum_{y \in B(x,2R)} P_k^A(x,y) \geq \sum_{k=0}^{\infty} \sum_{y \in B} \frac{1}{2} \widetilde{P}_k^A(x,y) \\ &\geq \sum_{k=F'}^{F} \sum_{y \in B} \frac{1}{2} \widetilde{P}_k^A(x,y) \geq c \frac{V(x,r)}{V(x,R)} F(x,R) \geq cF(x,R). \end{aligned}$$

■

Proposition 10.2. *If $DUE(F)$ holds for an $F \in W_0$, then*

$$\rho(x,2R)\, v(x,2R) \leq CF(x,2R).$$

The first step towards the upper estimate of ρv is to show an upper estimate for λ^{-1}.

Proposition 10.3. *If $(p_0), DUE(F)$, (VD) hold and $F \in W_0$, then*

$$\lambda(x,R) \geq cF^{-1}(x,R). \tag{10.7}$$

Proof Assume that $C_1 > 1, n = F(x, C_1 R)$, $y, z \in B = B(x,R)$. Let us use Lemma 8.8 and $DUE(F)$ to obtain

$$P_{2n}(y,z) \leq C \frac{\mu(z)}{\left(V(y, f(y,2n)) V(z, f(z,2n))\right)^{1/2}}.$$

From (VD) and $F \in W_0$ it follows for $w = y$ or z $d(x,w) \leq R < C_1 R = f(x,n)$ that

$$\frac{V(x, C_1 R)}{V(w, C_1 R)} \leq C,$$

which by using (p_0) yields that for all n,

$$P_n(y,z) \leq C \frac{\mu(z)}{V(x, f(x,n))}.$$

If ϕ is the left eigenvector (measure) belonging to the smallest eigenvalue λ of $I - P^B$ and normalized to $(\phi 1) = 1$, then

$$\begin{aligned}(1-\lambda)^{2n} &= \phi P_{2n}^B 1 = \sum_{y,z\in B(x,R)} \phi(z) P_{2n}^B(z,y) \le \sum_{y\in B(x,R)} \frac{C\mu(y)}{\min_{z\in B(x,R)} V(z,f(z,2n))} \\ &\le C \max_{z\in B(x,R)} \left(\frac{R}{f\,(z,2n)}\right)^{\alpha} = C \max_{z\in B(x,R)} \left(\frac{1}{C_1}\frac{f\,(x,2n)}{f\,(z,2n)}\right)^{\alpha} \\ &\le C\left(\frac{1}{C_1}C_f\right)^{\alpha} \le \frac{1}{2}\end{aligned}$$

if $C_1 = 2C^{1/\alpha}C_f$. Using the inequality and $1-\xi \ge \frac{1}{2}\log\frac{1}{\xi}$ for $\xi \in [\frac{1}{2}, 1]$, where $\xi = 1 - \lambda(x,R)$, we have

$$\lambda(x,R) \ge \frac{\log 2}{2n} \ge cF(x, C_1 R)^{-1} > cF(x,R)^{-1}.$$

■

Proof [of Proposition 10.2] Let us recall from (3.9) that

$$\lambda\,(x,2R)\,\rho\,(x,R,2R)\,V\,(x,R) \le 1$$

in general and the application of (VD) and (10.7) immediately yields the statement. ■

Proposition 10.4. *Assume* (p_0). *If* $PLE\,(F)$ *for an* $F \in W_0$ *holds, then there is a* $c > 0$ *such that for all* $R > 0, x \in \Gamma$

$$\rho\,(x,R,2R)\,v\,(x,R,2R) \ge cF\,(x,2R)$$

Proof The inequality (3.16) establishes that

$$\rho\,(x,R,2R)\,v\,(x,R,2R) \ge \min_{z\in\partial B\left(x,\frac{3}{2}R\right)} E\,(z,R/2)\,.$$

From Proposition 10.1 we know that

$$\min_{z\in\partial B\left(x,\frac{3}{2}R\right)} E\,(z,R/2) \ge c \min_{z\in\partial B\left(x,\frac{3}{2}R\right)} F\,(z,R/2)\,,$$

and from $F \in W_0$, it follows that

$$\rho\,(x,R,2R)\,v\,(x,R,2R) \ge \min_{z\in\partial B\left(x,\frac{3}{2}R\right)} F\,(z,R/2) \ge cF\,(x,2R)\,.$$

■

Proof [of Theorem 10.3] From Proposition 10.2 we have that $\rho v < CF$ which together with Proposition 10.4, yields that

$$\rho(x,R,2R)\, v(x,R,2R) \simeq F(x,2R).$$

Since $F \in W_0$, we have that $\rho v \in W_0$. From Proposition 7.1 and from $(aD\rho v)$ the Einstein relation follows:

$$E(x,2R) \simeq \rho(x,R,2R)\, v(x,R,2R) \simeq F(x,2R) \tag{10.8}$$

since (H) is ensured by $PMV(F) + PSMV(F)$. Since $F \in W_0$ and $E \simeq F$, it follows that $E \in W_0$ which includes (TC) and, of course, (wTC) too, and the proof of $8 \Longrightarrow 2$ in Theorem 10.1 is completed. ■

10.2 Off-diagonal lower estimate

Now we are ready to show the off-diagonal lower estimate $LE(F)$:

$$\widetilde{p}_n(x,y) \geq \frac{c}{V(x, f(x,n))} \exp\left(-C\left[\frac{F(x,d)}{n}\right]^{\frac{1}{\beta'-1}}\right),$$

where $d = d(x,y)$, $F \in W_1$. The proof of the off-diagonal lower estimate uses the modified Aronson's chaining argument. We have shown that

$$(VD) + (TC) + (H) \Longrightarrow DUE(E), \tag{10.9}$$

$$(\overline{E}) \Longrightarrow DLE(E),$$

furthermore for $F \in W_0$,

$$(VD) + DUE(F) + DLE(F) + (H) \Longrightarrow NDLE(F). \tag{10.10}$$

The lower estimate will follow if we show for $F \in W_1$ that

$$(VD) + NDLE(F) \Longrightarrow LE(F). \tag{10.11}$$

It results from $(10.9 - 10.11)$ that our final conclusion is

$$(VD) + (TC) + (H) \Longrightarrow LE(E)$$

if $\beta' > 1$ for E.

Theorem 10.4. *Assume that (Γ,μ) satisfies (p_0). Then for an $F \in W_1$*

$$(VD) + NDLE(F) \Longrightarrow LE(F),$$

and if $E \in W_1$

$$(VD) + (TC) + (H) \Longrightarrow LE(E). \tag{10.12}$$

Let us recall that

$$P_n P_m = P_{n+m}. \tag{10.13}$$

We need a replacement of this property for the operator $\tilde{P}_n$ which is stated below in Lemma 10.5.

Lemma 10.2. *Assume that (p_0) holds on (Γ, μ), then for all integers $n \geq l \geq 1$ such that*

$$n \equiv l \pmod 2, \tag{10.14}$$

we have for all $x, y \in \Gamma$

$$P_l(x, y) \leq C^{n-l} P_n(x, y)\,, \tag{10.15}$$

with a constant $C = C(p_0)$.

Proof Due to the semigroup property (5.22), we have

$$P_{k+2}(x, y) = \sum_{z \in \Gamma} P_k(x, z) P_2(z, y) \geq P_k(x, y) P_2(y, y).$$

Using (p_0), we obtain

$$P_2(y, y) = \sum_{z \sim y} P(y, z) P(z, y) \geq p_0 \sum_{z \sim y} P(y, z) = p_0,$$

whence $P_{k+2}(x, y) \geq p_0 P_k(x, y)$. Iterating this inequality, we obtain (10.15) with $C = p_0^{-1/2}$. ■

Lemma 10.3. *Assume that (Γ, μ) satisfies (p_0). Then for all integers $n \geq l \geq 1$ and all $x, y \in \Gamma$,*

$$\tilde{P}_l(x, y) \leq C^{n-l} \tilde{P}_n(x, y), \tag{10.16}$$

where $C = C(p_0)$.

Proof This is an immediate consequence of Lemma 10.2 because both $P_l(x, y)$ and $P_{l+1}(x, y)$ can be estimated from above via either $P_n(x, y)$ or $P_{n+1}(x, y)$, depending on the parity of n and l. ■

Remark 10.2. Note that no parity condition is required here in contrast to the condition (10.14) of Lemma 10.2.

Lemma 10.4. *Assume that (Γ, μ) satisfies (p_0). Then for all $n, m \in \mathbb{N}$ and $x, y \in \Gamma$, we have the following inequality*

$$\tilde{P}_n \tilde{P}_m(x, y) \leq C \tilde{P}_{n+m+1}(x, y), \tag{10.17}$$

where $C = C(p_0)$.

Proof Observe that, by (10.13),

$$\tilde{P}_n \tilde{P}_m = (P_n + P_{n+1})(P_m + P_{m+1}) = P_{n+m} + 2P_{n+m+1} + P_{n+m+2}.$$

By Lemma 10.2, $P_{n+m}(x, y) \leq C P_{n+m+2}$, whence

$$\tilde{P}_n \tilde{P}_m\,(x, y) \leq C(P_{n+m+1} + P_{n+m+2}) = C \tilde{P}_{n+m+1}.$$

■

Lemma 10.5. *Assume that* (Γ, μ) *satisfies* (p_0). *Then for all* $x, y \in \Gamma$ *and* $k, m, n \in \mathbb{N}$ *such that* $n \geq km + k - 1$, *we have the following inequality*

$$\left(\tilde{P}_m\right)^k (x, y) \leq C^{n-km} \tilde{P}_n(x, y). \tag{10.18}$$

Proof By induction, (10.17) implies

$$\left(\tilde{P}_m\right)^k (x, y) \leq C^{k-1} \tilde{P}_{km+k-1}(x, y).$$

From inequality (10.16) with $l = km + k - 1$, we obtain

$$\tilde{P}_{km+k-1}(x, y) \leq C^{n-km-(k-1)} \tilde{P}_n(x, y),$$

whence (10.18) follows. ■

Proof [of Theorem 10.4] The proof starts with separation of three cases according to different regions for $d = d\,(x, y)$.

1. $d(x, y) \leq \delta f\,(x, n)$,
2. $\delta f\,(x, n) < d(x, y) \leq \delta' n$,
3. $\delta' n < d(x, y) \leq n$.

In Case 1 $l = n$ by definition and cp_0^{Cn} is a trivial bound. In Case 3

$$\frac{n}{l} \geq QF\left(x, \frac{d}{l}\right) \geq c\left(\frac{n}{l}\right)^2$$

which results in $l > cn$ and again the lower estimate is smaller than $exp\,(-Cn)$ which can be received from (p_0).

Case 2

The proof uses varying radii for a chain of balls.

Assume that $\delta f\,(x, n) < d\,(x, y) < \delta' n$. Consider a shortest path π between x and y, write $d = d\,(x, y)$,

$$m = \left\lfloor \frac{n}{l\,(n, R, A)} \right\rfloor - 1, \tag{10.19}$$

$R = f\,(x, n)$, $S = f\,(y, n)$, $A = B\,(x, d + R) \cup B\,(y, d + S)$. Let $o_1 = x$ and

$$r_1 = \lceil \delta c_0 f\,(o_1, m) \rceil\,,$$

and choose $o_2 \in \pi : d\,(o_1, o_2) = r_1 - 1$ and recursively

$$r_{i+1} = \lceil \delta c_0 f\,(o_{i+1}, m) \rceil \tag{10.20}$$

and $o_{i+1} \in \pi : d\,(o_i, o_{i+1}) = r_{i+1} - 1$ and $d\,(y, o_{i+1}) < d\,(y, o_i)$. Write $B_i = B\,(o_i, r_i)$. The iteration ends with the first j for which $y \in B_j$. From $F \in W_0$ and $z_{i+1} \in B_i$ it follows that

$$c_1 \le \frac{f(z_i,m)}{f(z_{i+1},m)} \le C_2, \tag{10.21}$$

and the from triangle inequality it is evident that

$$d(z_i,z_{i+1}) \le 2r_i + r_{i+1} \le \left(2+\frac{1}{c_1}\right)\delta c_0 f(z_i,m). \tag{10.22}$$

Here we specify $c_0 = (2+1/c_1)^{-1}$. Let us recall the definition of $l = l(n,d,A)$:

$$\frac{n}{l} \ge \max_{z\in A} CE\left(z,\frac{d}{l}\right), \tag{10.23}$$

and taking the inverse, we obtain

$$\min_{z\in A} f\left(z,\frac{1}{C}\frac{n}{l}\right) \ge \frac{d}{l}. \tag{10.24}$$

Let us choose $C > C_F\left(\frac{1}{\delta}\right)^{\beta}$ in (10.23) (using $F\in W_1$) such that

$$f\left(o_i,\frac{1}{C}\frac{n}{l}\right) \le \delta c_0 f\left(o_i,\frac{n}{l}\right) = r_i.$$

By the definition of j,

$$d > \sum_{i=1}^{j-1} r_i \ge (j-1)\frac{d}{l},$$

consequently, $j-1 \le l$

$$\left(\widetilde{P}_m\right)^j(x,y) \ge \sum_{z_1\in B_0} \cdots \sum_{z_{j-1}\in B_{j-2}} \widetilde{P}_m(x,z_1)\,\widetilde{P}_m(z_1,z_2)\ldots\widetilde{P}_m(z_{j-1},y).$$

Now we use $NDLE$ to obtain

$$\begin{aligned}
\left(\widetilde{P}_m\right)^j(x,y) &\ge \sum_{z_1\in B_0} \cdots \sum_{z_{j-1}\in B_{j-2}} \frac{c\mu(z_1)}{V(x,f(x,m))} \cdots \frac{c\mu(y)}{V(z_{j-1},f(z_{j-1},m))} \\
&\ge c^{j-1}\frac{V(o_1,r_1)}{V(x,f(x,m))}\cdots\frac{V(o_{j-1},r_{j-1})}{V(z_{j-2},f(z_{j-2},m))}\frac{\mu(y)}{V(z_{j-1},f(z_{j-1},m))} \\
&\ge c^{j-1}\frac{\mu(y)}{V(x,f(x,m))}\frac{V(o_1,r_1)}{V(z_2,f(z_2,m))}\cdot\cdot\frac{V(o_{j-2},r_{j-2})}{V(z_{j-1},f(z_{j-1},m))}.
\end{aligned}$$

If we use (10.20), (10.21) and (VD) it follows that

$$\begin{aligned}
\left(\widetilde{P}_m\right)^j(x,y) &\geq \frac{c^{j-1}\mu(y)}{V(x,f(x,m))}\frac{V(o_1,r_1)}{V\left(z_2,\frac{1}{\delta c_0 c_1}r_1\right)}..\frac{V(o_{j-2},r_{j-2})}{V\left(z_{j-1},\frac{1}{\delta c_0 c_1}r_{j-2}\right)} \\
&\geq \frac{c^{j-1}\mu(y)}{V(x,f(x,m))}(c')^{j-2} \\
&\geq \frac{c\mu(y)}{V(x,f(x,n))}\exp\left[-C(j-1)\right] \\
&\geq \frac{c\mu(y)}{V(x,f(x,n))}\exp\left[-Cl\right] \qquad (10.25)
\end{aligned}$$

Finally from Lemma 10.5 we know that there is a $c>0$ such that

$$\widetilde{P}_n \geq c^{n-lm}\left(\widetilde{P}_m\right)^l$$

if $n \geq lm+l-1$. Let us note that from (10.19) it follows that $n-lm+l\leq 3l$ which results in

$$\begin{aligned}
\widetilde{P}_n \geq c^{n-lm}\left(\widetilde{P}_m\right)^l &\geq c'\frac{c^{3l}}{V(x,f(x,n))}\exp(-Cl) \\
&\geq \frac{c}{V(x,f(x,n))}\exp\left[-C\left(\frac{F(x,d(x,y))}{n}\right)^{\frac{1}{\beta'-1}}\right].
\end{aligned}$$

This completes the proof of the lower estimate. ■