# 2

# Connected Reductive Groups and Their Lie Algebras

The geometrical objects considered are defined over an algebraically closed field $k$ of characteristic $p$. In this chapter, we first introduce some notation which will be used throughout this book. We then discuss some properties about algebraic groups and their Lie algebras related to the characteristic $p$. These results will be used to give an explicit bound on $p$ for which the main result of [Lus87] applies. For any prime $r$, we choose once for all an algebraic closure $\overline{\mathbb{F}}_r$ of the finite field $\mathbb{F}_r = \mathbb{Z}/r\mathbb{Z}$. Then we denote by $\mathbb{F}_{r^n}$ the unique extension of degree $n > 0$ of $\mathbb{F}_r$ in $\overline{\mathbb{F}}_r$.

## 2.1 Notation and Background

We denote by $\mathbb{G}_m$ the one-dimensional algebraic group $(k - \{0\}, \times)$, and by $\mathbb{G}_a$ the one-dimensional algebraic group $(k, +)$. Let $H$ be a linear algebraic group over $k$, i.e. $H$ is isomorphic to a closed subgroup of some $GL_n(k)$. We denote by $1_H$ the neutral element of $H$ and by $H^o$ the connected component of $H$ containing $1_H$. We denote by $\mathrm{Lie}(H) = \mathcal{H}$ the Lie algebra of $H$ (i.e. the tangent space of $H^o$ at $1_H$) and we denote by $[,]$ the Lie product on $\mathcal{H}$. The Lie algebras of $GL_n(k)$, $SL_n(k)$ and $PGL_n(k)$ are respectively denoted by $gl_n(k)$, $sl_n(k)$ and $pgl_n(k)$. Let $Z_H = \{x \in H | \forall y \in H, xy = yx\}$ be the center of $H$, and let $z(\mathcal{H}) = \{X \in \mathcal{H} | \forall Y \in \mathcal{H}, [X, Y] = 0\}$ be the center of $\mathcal{H}$. If $x \in H$, we denote by $x_s$ the semi-simple part of $x$ and by $x_u$ its unipotent part. If $X \in \mathcal{H}$, then $X_s$ denotes the semi-simple part of $X$ and $X_n$ denotes its nilpotent part.

For an arbitrary morphism $f : X \to Y$ of algebraic varieties, we denote by $d_x f$ the differential of $f$ at $x$. If $X$ is an algebraic group, we put $df = d_{1_X} f$.

E. Letellier: LNM 1859, pp. 5–31, 2005.

### 2.1.1 $H$-Varieties and Adjoint Action of $H$ on $\mathcal{H}$

An algebraic variety on which $H$ acts morphically is called an *$H$-variety.* If $V$ is an $H$-variety and $S$ a subset of $V$, we put $C_H(S) := \{h \in H | \forall x \in S, h.x = x\}$ and we denote by $C_H^o(S)$ instead of $C_H(S)^o$ its connected component. We also put $A_H(S) := C_H(S)/C_H^o(S)$. The normalizer $\{h \in H | h.S \subset S\}$ of $S$ in $H$ is denoted by $N_H(S)$. Let $X$ be an homogeneous $H$-variety (i.e. $H$ acts transitively on $X$). Then the choice of an element $x \in X$ defines an $H$-equivariant morphism $\pi_x : H \to X$, $h \mapsto h.x$ which factors through a bijective morphism $\overline{\pi}_x : H/C_H(x) \to X$. We have the following well-known proposition.

**Proposition 2.1.2.** *The following assertions are equivalent:*
*(i) The morphism $\pi_x$ is separable.*
*(ii) The natural inclusion Lie$(C_H(x)) \subset$ Ker$(d\pi_x)$ is an equality.*
*(iii) The morphism $\overline{\pi}_x$ is an isomorphism.*

2.1.3. For any $h \in H$, let $\mathrm{Int}_h : H \to H$ be the automorphism of $H$ given by $g \mapsto hgh^{-1}$. Then the map $\mathrm{Ad} : H \to \mathrm{GL}(\mathcal{H})$, $h \mapsto d(\mathrm{Int}_h)$ is a morphism of algebraic groups and is called the *adjoint action* of $H$ on $\mathcal{H}$. We also have $[\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y] = \mathrm{Ad}(h)([X,Y])$ for any $h \in H$, $X, Y \in \mathcal{H}$. For a closed subgroup $K$ of $H$, we use the terminology "$K$-orbit of $\mathcal{H}$" for the adjoint action of $K$ on $\mathcal{H}$. If $X \in \mathcal{H}$, we denote by $\mathcal{O}_X^K$ the $K$-orbit of $X$ and if $x \in H$, we denote by $C_x^K$ the $K$-conjugacy class of $x$ in $H$. If $X, Y$ are two elements of $\mathcal{H}$, we say that they are *$K$-conjugate* if $X \in \mathcal{O}_Y^K$. The differential of $\mathrm{Ad} : H \to \mathrm{GL}(\mathcal{H})$ at 1 is denoted by ad. It satisfies $\mathrm{ad}(X)(Y) = [X, Y]$ for any $X, Y \in \mathcal{H}$. Since the restriction of Ad to $Z_H$ is trivial, we thus get that $\mathrm{Lie}(Z_H) \subset z(\mathcal{H})$. We will see later that this inclusion is not always an equality.

Let $K$ be a closed subgroup of $H$ with Lie algebra $\mathcal{K}$. For $X \in \mathcal{H}$ and $x \in H$, we define

$$C_{\mathcal{K}}(X) := \{Y \in \mathcal{K} | [Y, X] = 0\},$$

$$C_{\mathcal{K}}(x) := \{Y \in \mathcal{K} | \mathrm{Ad}(x)Y = Y\}.$$

Consider the orbit maps $\pi : K \to \mathcal{O}_X^K$, $h \mapsto \mathrm{Ad}(h)X$ and $\rho : K \to C_x^K$, $h \mapsto hxh^{-1}$. Then by [Bor, III 9.1], we have $\mathrm{Ker}(d\pi) = C_{\mathcal{K}}(X)$ and $\mathrm{Ker}(d\rho) = C_{\mathcal{K}}(x)$. Hence, by 2.1.2 the orbit map $\pi$ (resp. $\rho$ ) is separable if and only if $\mathrm{Lie}(C_K(X)) = C_{\mathcal{K}}(X)$ (resp. $\mathrm{Lie}(C_K(x)) = C_{\mathcal{K}}(x)$).

### 2.1.4 Reductive Groups

The letter $G$ will always denote a **connected reductive algebraic group** over $k$ and we will denote by $\mathcal{G}$ its Lie algebra. By a *semi-simple* algebraic group, we shall mean a **connected** reductive algebraic group whose radical is trivial, i.e. a connected reductive group whose center is finite.

*Notation* 2.1.5. We denote by $G'$ the derived subgroup of $G$, i.e. the closed subgroup of $G$ which is generated by the elements of the form $xyx^{-1}y^{-1}$ with $x, y \in G$, and by $\mathcal{G}'$ the Lie algebra of $G'$. We also denote by $\overline{G}$ the quotient $G/Z_G^o$ and by $\overline{\mathcal{G}}$ the Lie algebra of $\overline{G}$.

Recall that $G'$ and $\overline{G}$ are both semi-simple algebraic groups. Recall also that $\overline{\mathcal{G}} = \mathcal{G}/\mathrm{Lie}(Z_G^o)$. We will see that $\mathcal{G}'$ is not always the Lie subalgebra of $\mathcal{G}$ generated the elements of the form $[X, Y]$ with $X, Y \in \mathcal{G}$ (see 2.4.4).

**Definition 2.1.6.** *Let $H$ be an algebraic group and let $H_1, ..., H_n$ be closed subgroups of $H$ such that any two of them commute and each of them has a finite intersection with the product of the others. If $H = H_1...H_n$, then we say that $H$ is the* almost-direct product *of the $H_i$.*

**Theorem 2.1.7.** *[DM91, 0.38] If $G$ is a semi-simple algebraic group, then $G$ has finitely many minimal non-trivial normal connected closed subgroups and $G$ is the almost-direct product of them.*

**Definition 2.1.8.** *The minimal non-trivial normal connected closed subgroups of a semi-simple algebraic group $G$ will be called the* simple components *of $G$. We shall say that $G$ is simple if it has a unique simple component.*

The letter $B$ will usually denote a Borel subgroup of $G$, the letter $T$ a maximal torus of $B$ and $U$ the unipotent radical of $B$. Their respective Lie algebras will be denoted by $\mathcal{B}, \mathcal{T}$ and $\mathcal{U}$. The dimension of $T$ is called the *rank* of $G$ and is denoted by $rk(G)$. The rank of $\overline{G}$ is called the *semi-simple rank* of $G$ and is denoted by $rk_{ss}(G)$. If $P$ is an arbitrary parabolic subgroup of $G$, then we denote by $U_P$ the unipotent radical of $P$ and by $\mathcal{U}_P$ the Lie algebra of $U_P$. If $P = LU_P$ is a Levi decomposition of $P$ with corresponding Lie algebra decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$, then we denote by $\pi_P : P \to L$ and by $\pi_{\mathcal{P}} : \mathcal{P} \to \mathcal{L}$ the canonical projections. Throughout the book we will make the following abuse of language: by a "**Levi subgroup of** $G$", we shall mean a Levi subgroup of a parabolic subgroup of $G$.

We denote by $X(T)$ the group of algebraic group homomorphisms $T \to \mathbb{G}_m$. For any $\gamma \in X(T)$, put $\mathcal{G}_\gamma = \{v \in \mathcal{G} | \forall t \in T, \mathrm{Ad}(t)v = \gamma(t)v\}$ and $\Phi = \{\gamma \in X(T) - \{0\} | \mathcal{G}_\gamma \neq \{0\}\}$. We have

$$\mathcal{G} = \bigoplus_{\gamma \in \Phi \cup \{0\}} \mathcal{G}_\gamma = \mathcal{T} \oplus \bigoplus_{\alpha \in \Phi} \mathcal{G}_\alpha.$$

For any $\alpha \in \Phi$, we denote by $U_\alpha$ the unique closed connected one-dimensional unipotent subgroup of $G$ normalized by $T$ such that $\mathrm{Lie}(U_\alpha) = \mathcal{G}_\alpha$.

It is known that $\Phi$ forms a (reduced) root system in the subspace $V$ of $X(T) \otimes \mathbb{R}$ it generates. The set $\Phi$ is then called the *root system* of $G$ with respect to $T$ and the elements of $\Phi$ are called the roots of $G$ with respect to $T$. If there is any ambiguity, we will write $\Phi(T)$ instead of $\Phi$. We denote by $\Phi^\vee$ the set of coroots and by $X^\vee(T)$ the group of homomorphisms of algebraic groups $\mathbb{G}_m \to T$; the set $\Phi^\vee$ forms a root system in the subspace $V^\vee$ of $X^\vee(T) \otimes \mathbb{R}$ it generates. We denote by $Q(\Phi)$ the $\mathbb{Z}$-sublattice of $X(T)$ generated by $\Phi$ and by $Q(\Phi^\vee)$ the $\mathbb{Z}$-sublattice of $X^\vee(T)$ generated by $\Phi^\vee$. Recall that we have an exact pairing $\langle , \rangle : X(T) \times X^\vee(T) \to \mathbb{Z}$ such that for any $\alpha \in X(T)$, $\beta \in X^\vee(T)$ and $t \in \mathbb{G}_m$, we have $(\alpha \circ \beta^\vee)(t) = t^{\langle \alpha, \beta^\vee \rangle}$. By abuse of notation, we still denote by $\langle , \rangle$ the induced pairing between $V$ and $V^\vee$. The $\mathbb{Z}$-lattice of weights $P(\Phi)$ is defined to be $\{x \in V | \langle x, \Phi^\vee \rangle \subset \mathbb{Z}\}$. The lattice $Q(\Phi)$ is then a $\mathbb{Z}$-sublattice of $P(\Phi)$ of finite index.

If $G$ is semi-simple, we have the following inclusions of $\mathbb{Z}$-lattices $Q(\Phi) \subset X(T) \subset P(\Phi)$ and $Q(\Phi^\vee) \subset X^\vee(T) \subset P(\Phi^\vee)$; conversely if one these inclusions hold, then $G$ is semi-simple. Moreover we have $|P(\Phi)/X(T)| = |X^\vee(T)/Q(\Phi^\vee)|$ and so

$$|X(T)/Q(\Phi)||X^\vee(T)/Q(\Phi^\vee)| = |P(\Phi)/Q(\Phi)|.$$

**Definition 2.1.9.** *We say that $G$ is*

*(i)* adjoint *if $X(T) = Q(\Phi)$;*

*(ii)* simply connected *if $X^\vee(T) = Q(\Phi^\vee)$.*

It follows from Chevalley's classification theorem that each $\mathbb{Z}$-lattice between $Q(\Phi)$ and $P(\Phi)$ determines a unique (up to isomorphism) semi-simple algebraic group over $k$ with root system $\Phi$. We denote by $G_{ad}$ the adjoint group corresponding to $G$ and by $G_{sc}$ the simply connected algebraic group corresponding to $G$. Their respective Lie algebras are denoted by $\mathcal{G}_{ad}$ and $\mathcal{G}_{sc}$. When $G$ is semi-simple, the inclusions $Q(\Phi) \subset X(T) \subset P(\Phi)$ give rise to canonical isogenies (i.e surjective morphisms whose kernel is finite and so lies in the center) $\pi_{sc} : G_{sc} \to G$ and $\pi_{ad} : G \to G_{ad}$; the kernel of the later map is equal to $Z_G$ (see [Ste68, page 45]). Moreover, the canonical isogenies $\pi_{sc}$ and $\pi_{ad}$ are central, that is $\mathrm{Ker}(d\pi_{sc}) \subset z(\mathcal{G}_{sc})$ and $\mathrm{Ker}(d\pi_{ad}) \subset z(\mathcal{G})$. In fact, for the later map we have $\mathrm{Ker}(d\pi_{ad}) = z(\mathcal{G})$.

The choice of the Borel subgroup $B$ containing $T$ defines an order on $\Phi \cup \{0\}$ such that any root is positive or negative by setting $\Phi^+ := \{\gamma \in \Phi |\, \mathcal{G}_\gamma \subset \mathcal{B}\}$. The set $\Pi$ of positive roots that are indecomposable into a sum of other positive roots is called the *basis* of $\Phi$ with respect to $B$. The elements of $\Pi$ are linearly independent and any root of $\Phi$ is a $\mathbb{Z}$-linear combination of elements of $\Pi$ with coefficients all positive or all negative. If $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha \in \Phi$, then we define the *height* of $\beta$ (with respect to $\Pi$) to be the integer $\sum_{\alpha \in \Pi} n_\alpha$. The *highest root* of $\Phi$ with respect to $\Pi$ is defined to be the root of highest height. For any Levi subgroup $L$ of $G$, we denote by $W_G(L)$ the group $N_G(L)/L$. The Weyl group of $G$ relative to $T$ is $W_G(T)$. We denote by $h_o$ the Coxeter number of $W_G(T)$. It depends only on $G$, and so if there is any ambiguity, we will denote it $h_o^G$ instead of $h_o$.

### 2.1.10 About Intersections of Lie Algebras of Closed Subgroups of *G*

Let $M$ and $N$ be two closed subgroups of $G$, then we have

2.1.11. $$\mathrm{Lie}(M \cap N) \subset \mathrm{Lie}(M) \cap \mathrm{Lie}(N).$$

In general this inclusion is not an equality; it becomes an equality exactly when the quotient morphism $\pi : G \to G/N$ induces a separable morphism $M \to \pi(M)$ (see [Bor, Proposition 6.12]).

**2.1.12.** *When $M \cap N$ contains a maximal torus of $G$, the inclusion 2.1.11 is an equality.*

The above assertion follows from [Bor, Proposition 13.20]; note that [Bor, Corollary 13.21], which asserts that 2.1.11 is an equality whenever $M$ and $N$ are normalized by a maximal torus of $G$, is not correct since in positive characteristic, the intersection of two subtori of a maximal torus of $G$ may have finite intersection while their Lie algebras have an intersection of strictly positive dimension. For instance, let $G = SL_3(k)$ and let $T$ be the maximal torus of $G$ consisting of diagonal matrices, then the set $Z_G$ is finite and is the intersection of the two subtori $T_\alpha = \mathrm{Ker}(\alpha)$ and $T_\beta = \mathrm{Ker}(\beta)$ of $T$ where $\alpha : T \to k^\times$, $(t_1, t_2, t_1^{-1} t_2^{-1}) \mapsto t_1 t_2^{-1}$ and $\beta : T \to k^\times$, $(t_1, t_2, t_1^{-1} t_2^{-1}) \mapsto t_2^2 t_1$. The intersection of the Lie algebras of $T_\alpha$ and $T_\beta$ is of dimension 0 unless $p = 3$ in which case the intersection is of dimension 1.

2.1.13. We will need to deal with the question of whether the inclusion 2.1.11 is an equality or not only in the cases where the closed subgroups $M$ and $N$ involved in 2.1.11 are parabolic subgroups, Levi subgroups or unipotent radicals of parabolic subgroups.

Let $P$ and $Q$ be two parabolic subgroups $G$. Let $L$ and $M$ be two Levi subgroups of $P$ and $Q$ respectively such that $L \cap M$ contains a maximal torus $T$ of $G$ (given $P$ and $Q$, such Levi subgroups $L$ and $M$ always exists). We denote by $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{L}$ and $\mathcal{M}$ the corresponding Lie algebras of $P, Q, L$ and $M$.

**Proposition 2.1.14.** *With the above notation, we have:*
*(1)* $Lie(P \cap Q) = \mathcal{P} \cap \mathcal{Q}$,
*(2)* $Lie(L \cap M) = \mathcal{L} \cap \mathcal{M}$,
*(3)* $Lie(L \cap U_Q) = \mathcal{L} \cap \mathcal{U}_Q$,
*(4)* $Lie(U_P \cap U_Q) = \mathcal{U}_P \cap \mathcal{U}_Q$.

**Proof:** The assertions (1) and (2) are clear from 2.1.12. Let us see (3). From 2.1.11, it is enough to prove that $\dim(L \cap U_Q) = \dim(\mathcal{L} \cap \mathcal{U}_Q)$. Since $L \cap U_Q$ is a closed unipotent subgroup of $G$ normalized by $T$, by [DM91, 0.34], it is of dimension equal to the number of the $U_\alpha$, with $\alpha \in \Phi$, it contains. On the other hand the torus $T$ normalizes $\mathcal{L} \cap \mathcal{U}_Q$, therefore by full reducibility of the adjoint representation of $T$ in $\mathcal{G}$, the space $\mathcal{L} \cap \mathcal{U}_Q$ is the direct sum of the $\mathcal{G}_\alpha$ , $\alpha \in \Phi$, it contains. Hence the equality $\dim(L \cap U_Q) = \dim(\mathcal{L} \cap \mathcal{U}_Q)$ is a consequence of the fact that $\mathcal{G}_\alpha \subset \mathcal{L} \cap \mathcal{U}_Q$ if and only if $U_\alpha \subset L \cap U_Q$. The proof of (4) is completely similar. □

The above proposition together with [DM91, Proposition 2.1] has the following straightforward consequence.

**Proposition 2.1.15.** *With the above notation, we have*

$$\mathcal{P} \cap \mathcal{Q} = (\mathcal{L} \cap \mathcal{M}) \oplus (\mathcal{L} \cap \mathcal{U}_Q) \oplus (\mathcal{M} \cap \mathcal{U}_P) \oplus (\mathcal{U}_P \cap \mathcal{U}_Q).$$

### 2.1.16 $\mathbb{F}_q$-Structures

*Notation* 2.1.17. Let $r$ be a prime and let $X$ be an algebraic variety on $\overline{\mathbb{F}}_r$ defined over $\mathbb{F}_{r^n}$. If $F : X \to X$ denotes the corresponding Frobenius endomorphism, we say that $x \in X$ is *rational* if $F(x) = x$ and we denote by $X^F$ the set of rational elements of $X$.

2.1.18. Let $k = \overline{\mathbb{F}}_p$, and let $q$ be a power of $p$ such that the group $G$ is defined over $\mathbb{F}_q$. We then denote by $F : G \to G$ the corresponding Frobenius endomorphism. The Lie algebra $\mathcal{G}$ and the adjoint action of $G$ on $\mathcal{G}$ are also defined over $\mathbb{F}_q$ and we still denote by $F : \mathcal{G} \to \mathcal{G}$ the Frobenius endomorphism on $\mathcal{G}$. Assume that the maximal torus $T$ of $G$ is $F$-stable, and denote by $\tau$ the unique automorphism on $\Phi$ such that for any root $\alpha \in \Phi$, we have $F(U_\alpha) = U_{\tau(\alpha)}$; it satisfies $(\tau\alpha)(F(t)) = (\alpha(t))^q$ for any $\alpha \in X(T)$ and $t \in T$. If $B$ is also $F$-stable, then $\tau$ permutes the elements of the basis $\Pi$ of $\Phi$. Recall that an $F$-stable torus $H \subset G$ of rank $n$ is said to be *split* if there exists an isomorphism $H \xrightarrow{\sim} (\mathbb{G}_m)^n$ defined over $\mathbb{F}_q$. The $\mathbb{F}_q$-rank of an $F$-stable maximal torus $T$ of $G$ is defined to be the rank of its maximum split subtori. An $F$-stable maximal torus $T$ of $G$ is said to be *$G$-split* if it is maximally split in $G$; recall that the $G$-split maximal torus of $G$ are exactly those contained in some $F$-stable Borel subgroup of $G$. The *$\mathbb{F}_q$-rank* of $G$ is defined to be the $\mathbb{F}_q$-rank of its $G$-split maximal tori. The *semi-simple* *$\mathbb{F}_q$-rank* of $G$ is defined to be the $\mathbb{F}_q$-rank of $\overline{G}$. We say that an $F$-stable Levi subgroup $L$ of $G$ is $G$-split if it contains a $G$-split maximal torus; this is equivalent to say that there exists an $F$-stable parabolic subgroup $P$ of $G$ having $L$ as a Levi subgroup.

*Notation* 2.1.19. Let $H$ be a group with a morphism $\theta : H \to H$. We say that $x, y \in H$ are *$\theta$-conjugate* if and only if there exists $h \in H$ such that $x = hy(\theta(h))^{-1}$. We denote by $H^1(\theta, H)$ the set of $\theta$-conjugacy classes of $H$.

2.1.20. Let $k = \overline{\mathbb{F}}_q$ with $q$ a power of $p$. Let $H$ be a connected linear algebraic group acting morphically on a variety $X$. Assume that $H$, $X$ and the action of $H$ on $X$ are all defined over $\mathbb{F}_q$. Let $F : X \to X$ and $F : H \to H$ be the corresponding Frobenius endomorphisms. Let $x \in X^F$ and let $\mathcal{O}$ be the $H$-orbit of $x$. The orbit $\mathcal{O}$ is thus $F$-stable and $\mathcal{O}^F$ is a disjoint union of $H^F$-orbits. By [SS70, I, 2.7] (see also [DM91, 3.21]) we have a well-defined parametrization of the $H^F$-orbits of $\mathcal{O}$ by $H^1(F, A_H(x))$. This parametrization is given as follows. Let $y \in \mathcal{O}^F$ and let $h \in H$ be such that $y = h.x$. Then to the $H^F$-orbit of $y$, we associate the $F$-conjugacy class of the image of $h^{-1}F(h)$ in $A_H(x)$.

## 2.2 Chevalley Formulas

For any $\alpha \in \Phi$, the symbol $e_\alpha$ denotes a non-zero element of $\mathcal{G}_\alpha$ and $h_\alpha$ denotes $[e_\alpha, e_{-\alpha}]$. When $p = 0$, we assume that the $e_\alpha$ are chosen such that the set $\{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$ is a Chevalley basis of $\mathcal{G}'$ (see [Car72, 4.2] or [Ste68]). When $p > 0$ and $\mathcal{G}' = \mathcal{G}_{sc}$, then $\mathcal{G}'$ is obtained by reduction modulo $p$ from the

$\mathbb{Z}$-span of a Chevalley basis in the corresponding Lie algebra over $\mathbb{C}$. Hence in that case, we assume that the $e_\alpha$ are chosen such that $\{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$ is obtained from a Chevalley basis in the corresponding Lie algebra over $\mathbb{C}$; the set $\{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$ is then called a *Chevalley basis* of $\mathcal{G}'$. In the general case, let $\pi$ denote the canonical central isogeny $G_{sc} \to G'$; the choice of the $e_\alpha$ is made such that $B_{\mathcal{G}} := \{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$ is the image by $d\pi$ of a Chevalley basis of $\mathcal{G}_{sc}$. When it is a basis of $\mathcal{G}'$, the set $B_{\mathcal{G}}$ is called a Chevalley basis of $\mathcal{G}'$. We will see in 2.4, that the existence of Chevalley basis on $\mathcal{G}' \neq \mathcal{G}_{sc}$ is subject to some restriction on $p$. With such a choice of the $e_\alpha$, for any $r \in \Phi$, we have $dr(h_r) = 2$ and the vector $h_r$ is a linear combination of the $h_\alpha$ with $\alpha \in \Pi$. The last fact can be deduced from the simply connected case by making the use of the canonical Lie algebra homomorphism $\mathcal{G}_{sc} \to \mathcal{G}'$.

**2.2.1.** *We then have the following well-known relations:*

*(i)* $[t, h] = 0,\ t, h \in \mathcal{T}$,

*(ii)* $[t, e_r] = dr(t)e_r,\ t \in \mathcal{T}, r \in \Phi$,

*(iii)* $[e_r, e_s] = 0,\ r \in \Phi, s \in \Phi, r + s \notin \Phi \cup \{0\}$,

*(iv)* $[e_r, e_s] \in \mathcal{G}_{r+s},\ r \in \Phi, s \in \Phi, r + s \in \Phi$.

Using the decomposition $\mathcal{G} = \mathcal{T} \oplus \bigoplus_\alpha \mathcal{G}_\alpha$ and the above formulas, we see that the subspace of $\mathcal{G}'$ generated by $\{h_\alpha, e_\gamma | \alpha, \gamma \in \Phi\}$ is $[\mathcal{G}, \mathcal{G}]$. But since the vectors $h_r$ with $r \in \Phi$ are linear combinations of the $h_\alpha$ with $\alpha \in \Pi$, the Lie algebra $[\mathcal{G}, \mathcal{G}]$ is actually generated by $B_{\mathcal{G}}$. As a consequence, since $\mathcal{G}'$ is of dimension $|\Pi| + |\Phi| = |\mathcal{B}_{\mathcal{G}}|$, we see that $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ if and only if $B_{\mathcal{G}}$ is a basis of $\mathcal{G}'$, i.e. the elements of $\{h_\alpha | \alpha \in \Pi\}$ are linearly independent.

**2.2.2.** *For $r \in \Phi$, we fix an isomorphism of algebraic groups $x_r : \mathbb{G}_a \to U_r$ such that $dx_r(1) = e_r$. The following formulas give the action of $U_r$, with $r \in \Phi$, on $\mathcal{G}$:*

*(i)* $Ad(x_r(t))e_r = e_r$,

*(ii)* $Ad(x_r(t))e_{-r} = e_{-r} + th_r - t^2 e_r$,

*(iii)* $Ad(x_r(t))h = h - dr(h)te_r,\ h \in \mathcal{T}$,

*(iv)* $Ad(x_r(t))e_s = e_s + \sum_{\{i>0 | ir+s \in \Phi\}} c_{r,s,i} t^i e_{ir+s}$ *for some* $c_{r,s,i} \in k$, *if* $r \neq -s$.

## 2.3 The Lie Algebra of $Z_G$

Recall that by 2.1.3, we have an inclusion (*) $\mathrm{Lie}(Z_G) \subset z(\mathcal{G})$. In this subsection, we give among other things a necessary and sufficient condition on $p$ for (*) to be an equality. We denote by $\overline{T}$ the maximal torus $T/Z_G^o$ of $\overline{G}$ and by $T'$ the maximal torus of $G'$ which contains $T$.

We consider on $\mathrm{Lie}(Z_G)\oplus\overline{\mathcal{G}}$ the Lie product given by $[t\oplus v, h\oplus u] := [v, u]$.

**2.3.1.** *There is an isomorphism of Lie algebras $\mathcal{G} \simeq \mathrm{Lie}(Z_G) \oplus \overline{\mathcal{G}}$.*

**Proof:** It is enough to prove the existence of a $k$-subspace $V$ of $\mathcal{G}$ such that $\mathcal{G} = \mathrm{Lie}(Z_G) \oplus V$ and $[V, V] \subset V$, so that $V \simeq \overline{\mathcal{G}}$. For $\alpha \in \Pi$, denote by $\overline{h}_\alpha \in \mathrm{Lie}(\overline{T})$ the image of $h_\alpha$ under the canonical projection $\mathcal{T} \to \mathrm{Lie}(\overline{T})$. We choose a subset $I$ of $\Pi$ such that $E = \{\overline{h}_\alpha | \alpha \in I\}$ is a basis of the subspace of $\mathrm{Lie}(\overline{T})$ generated by $\{\overline{h}_\alpha | \alpha \in \Pi\}$, and we complete $E$ into a basis $E \cup \{\overline{x}_1, ..., \overline{x}_n\}$ of $\mathrm{Lie}(\overline{T})$. We choose $x_i \in \mathcal{T}$ such that its image in $\mathrm{Lie}(\overline{T})$ is $\overline{x}_i$. Now let $V$ be the subspace of $\mathcal{G}$ generated by $X := \{x_1, ..., x_n, h_\alpha, e_\gamma | \alpha \in I, \gamma \in \Phi\}$. Since the image of $X$ in $\overline{\mathcal{G}}$ is a basis of $\overline{\mathcal{G}}$, we have $\dim V = \dim \overline{\mathcal{G}}$ and $V \cap \mathrm{Lie}(Z_G) = \{0\}$. It follows that $\mathcal{G} = \mathrm{Lie}(Z_G) \oplus V$. From 2.2.1, we get that $[V, V] \subset V$. $\square$

2.3.2. It follows from 2.2.1 that

$$z(\mathcal{G}) = \bigcap_{\alpha\in\Pi} \mathrm{Ker}(d\alpha), \tag{1}$$

and from [DM91, Proposition 0.35] that

$$Z_G = \bigcap_{\alpha\in\Pi} \mathrm{Ker}(\alpha). \tag{2}$$

2.3.3. The canonical morphism $\rho : T \to \overline{T}$ induces an injective group homomorphism $\rho^* : X(\overline{T}) \to X(T)$, $\gamma \mapsto \gamma \circ \rho$ mapping bijectively the roots of $\overline{G}$ with respect to $\overline{T}$ onto $\Phi$. Hence we may identify the roots of $\overline{G}$ with respect to $\overline{T}$ with $\Phi$. Under this identification, the lattice $Q(\Phi)$ is a $\mathbb{Z}$-sublattice of $X(\overline{T})$. We have the following proposition.

**Proposition 2.3.4.** *We have $|(X(T)/Q(\Phi))_{tor}| = |X(\overline{T})/Q(\Phi)|$. The following assertions are equivalent:*

*(i) $p$ does not divide $|(X(T)/Q(\Phi))_{tor}|$,*

*(ii) $\mathrm{Lie}(Z_G) = z(\mathcal{G})$.*

**Proof:** For the sake of clarity, in this proof we prefer to differentiate the root system $\overline{\Phi}$ of $\overline{\mathcal{G}}$ with respect to $\overline{T}$ from $\Phi$. Let $r$ be the rank of $G$ and $s$ be the semi-simple rank of $G$. Let $\{\gamma_1, ..., \gamma_r\}$ be a basis of $X(T)$ such that for some integer $s$ with $s \leq r$ and some non-zero integers $m_1, ..., m_s$, the set $\{m_1\gamma_1, ..., m_s\gamma_s\}$ is a basis of $Q(\Phi)$. We have $X(T)/Q(\Phi) = \mathbb{Z}^{r-s} \times \mathbb{Z}/m_1\mathbb{Z} \times ... \times \mathbb{Z}/m_s\mathbb{Z}$ and so $|(X(T)/Q(\Phi))_{tor}| = \prod_{i=1}^{s} m_i$. Now for $i \in \{1, .., s\}$, we have $m_i\gamma_i \in Q(\Phi)$ and so, by 2.3.2(2), we have $\gamma_i(z)^{m_i} = 1$ for any $z \in Z_G^o$. Hence, if $\mu_{m_i}$ denotes the group of $m_i^{th}$ roots of unity, we get that $\gamma_i(Z_G^o) \subset \mu_{m_i}$. Since $Z_G^o$ is connected, we deduce that $\gamma_i(Z_G^o) = \{1\}$. Thus, for $i \in \{1, .., s\}$, the morphism $\gamma_i$ factors through a morphism $\overline{\gamma}_i : \overline{T} \to \mathbb{G}_m$. We see that $\{\overline{\gamma}_i\}_{i\in\{1,...,s\}}$ and $\{m_i\overline{\gamma}_i\}_i$ are respectively bases of the groups $X(\overline{T})$ and $Q(\overline{\Phi})$ (from which we see that $|(X(T)/Q(\Phi))_{tor}| = |X(\overline{T})/Q(\overline{\Phi})|$); this can be verified by using the fact that dim $X(\overline{T}) = s$ and the fact that $\rho^*$ maps $\overline{\gamma_i}$ onto $\gamma_i$ for $i \in \{1, ..., s\}$. From the fact that $\{\overline{\gamma}_i\}_i$ is a basis of $X(\overline{T})$, it results that the morphism $\overline{T} \to \mathbb{G}_m^s$ given by $t \mapsto (\overline{\gamma}_1(t), ..., \overline{\gamma}_s(t))$ is an isomorphism of algebraic groups. As a consequence, its differential Lie$(\overline{T}) \to k^s$ given by $t \mapsto (d\overline{\gamma}_1(t), ..., d\overline{\gamma}_s(t))$ is an isomorphism, i.e. the intersection of the $s$ hyperplanes Ker$(d\overline{\gamma}_i)$ of Lie$(\overline{T})$ is $\{0\}$.

We deduce that the intersection of the $s$ hyperplanes Ker$(m_i d\overline{\gamma}_i)$ of Lie$(\overline{T})$ is zero if and only if the $m_i$ are invertible in $k$ (i.e if $p$ does not divide $|(X(T)/Q(\Phi))_{tor}|$). On the other hand, since $\{m_i\overline{\gamma}_i\}_i$ is a basis of $Q(\overline{\Phi})$, by 2.3.2 (1) we have $\bigcap_{i=1}^{i=s} \mathrm{Ker}(m_i d\overline{\gamma}_i) = z(\overline{\mathcal{G}})$. We thus proved that the $m_i$ are invertible in $k$ if and only if $z(\overline{\mathcal{G}})$ is trivial.

We are now in position to see that the proposition is a consequence of the fact that any isomorphism of Lie algebras $\mathcal{G} \simeq \mathrm{Lie}(Z_G) \oplus \overline{\mathcal{G}}$ as in 2.3.1 induces an isomorphism from $z(\mathcal{G})$ onto $\mathrm{Lie}(Z_G) \oplus z(\overline{\mathcal{G}})$. □

*Remark* 2.3.5. If the assertion (i) (and so the assertion (ii)) of 2.3.4 holds for $G$, it does for any Levi subgroup of $G$.

*Remark* 2.3.6. Let $\pi : G \to G_{ad}$ be the composition morphism of the canonical projection $G \to \overline{G}$ with the canonical central isogeny $\overline{G} \to G_{ad}$, then we have $\mathrm{Ker}(\pi) = Z_G$ and $\mathrm{Ker}(d\pi) = z(\mathcal{G})$, so by 2.1.2, the morphism $\pi$ is separable if and only if $\mathrm{Lie}(Z_G) = z(\mathcal{G})$.

Using 2.3.6, we see that 2.3.4 has the following consequence.

**Corollary 2.3.7.** *The canonical morphism $G \to G_{ad}$ is separable if and only if $p$ does not divide $|(X(T)/Q(\Phi))_{tor}|$.*

**Corollary 2.3.8.** *Assume that $G$ is semi-simple and write $G = G_1 ... G_r$ where $G_1, ..., G_r$ are the simple components of $G$. Assume moreover that $p$ does not divide $|(X(T)/Q(\Phi))_{tor}| = |X(T)/Q(\Phi)|$, then $\mathcal{G} = \bigoplus_i Lie(G_i)$.*

**Proof:** For any $i$, we denote by $\mathcal{G}_i$ the Lie algebra of $G_i$. We fix $i$ and let $I$ be a subset of $\{1, .., r\}$ which does not contain $i$. Let $x \in (\sum_{j\in I} \mathcal{G}_j) \cap \mathcal{G}_i$. Since for $i \neq j$ the group $G_i$ commutes with $G_j$, we have $[\mathcal{G}_i, \mathcal{G}_j] = \{0\}$. Hence $x \in \sum_{j\in I} \mathcal{G}_j$ centralizes $\mathcal{G}_i$ and so $x \in z(\mathcal{G}_i)$. Since each element of $\mathcal{G}_i$ centralizes $\mathcal{G}_j$ for any $i \neq j$, we deduce that $x \in \mathcal{G}$. By 2.3.4 we have $x = 0$. We deduce that the sum $E = \sum_i \mathcal{G}_i$ is direct. Hence $E$ is a subspace of $\mathcal{G}$ of dimension $\sum_i \dim \mathcal{G}_i$ and so since algebraic groups are smooth, we have $\dim E = \sum_i \dim G_i = \dim G$. We deduce that $\mathcal{G} = \bigoplus_{i=1}^{i=n} \mathcal{G}_i$. □

Using the canonical map $T' \to \overline{T}$, we identify $X(\overline{T})$ with a subgroup of $X(T')$ and the root system of $\mathcal{G}'$ with respect to $T'$ with $\Phi$. Then $|(X(T)/Q(\Phi))_{tor}| = |X(\overline{T})/Q(\Phi)|$ divides $|X(T')/Q(\Phi)|$.

**Corollary 2.3.9.** *Assume that $p$ does not divide $|X(T')/Q(\Phi)|$, we have $\mathcal{G} = z(\mathcal{G}) \oplus \mathcal{G}'$.*

**Proof:** Since $\mathrm{Lie}(Z_G) \subset z(\mathcal{G})$, we have $\mathrm{Lie}(Z_G) \cap \mathcal{G}' \subset z(\mathcal{G}')$ and so by 2.3.4 applied to $G'$, we have $\mathrm{Lie}(Z_G) \cap \mathcal{G}' = \{0\}$. Hence the sum $\mathrm{Lie}(Z_G) + \mathcal{G}'$ is direct and so it is a subspace of $\mathcal{G}$ of dimension $\dim Z_G + \dim G' = \dim G$; thus we get that $\mathrm{Lie}(Z_G) \oplus \mathcal{G}' = \mathcal{G}$. Now, since $p$ does not divide $|X(T')/Q(\Phi)|$, it does not divide $|(X(T)/Q(\Phi))_{tor}|$, hence by 2.3.4, we get that $\mathcal{G} = z(\mathcal{G}) \oplus \mathcal{G}'$. □

*Remark* 2.3.10. The assumption " $p$ does not divide $|(X(T)/Q(\Phi))_{tor}|$" is not sufficient for $\mathcal{G} = z(\mathcal{G}) \oplus \mathcal{G}'$ to hold. Indeed, consider $G = GL_n(k)$; the group $(X(T)/Q(\Phi))_{tor}$ is trivial while the group $X(T')/Q(\Phi)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Assume that $p$ divides $|X(T')/Q(\Phi)| = n$. Then diagonal matrices $(a, ..., a)$ with $a \in k$ belong to the Lie algebra of $Z_G$ but also to $sl_n = \mathcal{G}'$ since $na = 0$. Hence $\mathrm{Lie}(Z_G) \cap \mathcal{G}' \neq \{0\}$ .

## 2.4 Existence of Chevalley Bases on $\mathcal{G}'$

We will need the following lemma.

**Lemma 2.4.1.** *[Bor, 8.5]*

*(i) Let $T_i : (\mathbb{G}_m)^r \to \mathbb{G}_m$ be the $i$-th projection; the maps $T_i$ form a basis of the abelian group $X(\mathbb{G}_m^r)$ of algebraic group homomorphisms $(\mathbb{G}_m)^r \to \mathbb{G}_m$, that is for any $f \in X(\mathbb{G}_m^r)$ there exists a unique tuple $(n_1, ..., n_r) \in \mathbb{Z}^r$ such*

*that* $f = T_1^{n_1} ... T_r^{n_r}$. *Let* $f = T_1^{n_1} ... T_r^{n_r} \in X(\mathbb{G}_m^r)$, *then* $df : k^r \to k$ *is given by* $df(x_1, ..., x_r) = \sum_i n_i x_i$.

*(ii) Let* $T_i^\vee : \mathbb{G}_m \to (\mathbb{G}_m)^r$ *be given by* $T_i^\vee(t) = (1, .., 1, t, 1, .., 1)$ *(t being located at the* $i-th$ *rank); the maps* $T_i^\vee$ *form a basis of the abelian group* $X^\vee(\mathbb{G}_m^r)$ *of algebraic group homomorphisms* $\mathbb{G}_m \to (\mathbb{G}_m)^r$, *that is for any* $f \in X^\vee(\mathbb{G}_m^r)$ *there exists a unique uple* $(n_1, ..., n_r) \in \mathbb{Z}^r$ *such that* $f = (T_1^\vee)^{n_1} ... (T_r^\vee)^{n_r}$. *If* $f = (T_1^\vee)^{n_1} ... (T_r^\vee)^{n_r} \in X^\vee(\mathbb{G}_m^r)$, *then* $df : k \to k^r$ *is given by* $df(t) = (n_1 t, ..., n_r t)$.

Recall that $T'$ denotes the maximal torus of $G'$ contained in $T$ and that $X(T')$ is a $\mathbb{Z}$-sublattice of $P(\Phi)$.

**Definition 2.4.2.** *The quotient* $P(\Phi)/X(T')$ *is called the* fundamental group *of* $G$ *and is denoted by* $\pi_1(G)$.

Note that $\pi_1(G_{sc}) = 1$ and $\pi_1(G_{ad}) = P(\Phi)/Q(\Phi)$.

We assume that $G$ is **semi-simple**.

By Chevalley's classification theorem, there exists a unique (up to isomorphism) connected reductive algebraic group $G^*$ over $k$ with a maximal torus $T^*$ of $G^*$ such that its root datum $(\Phi^*, X(T^*), (\Phi^*)^\vee, X^\vee(T^*))$ is $(\Phi^\vee, X^\vee(T), \Phi, X(T))$; we refer to [DM91] or [Car85] for the definition of root datum. We denote by $\mathcal{G}^*$ the Lie algebra of $G^*$. Since $G$ is assumed to be semi-simple, the group $G^*$ is also semi-simple. We denote by $\alpha^*$ the element of $X(T^*) = \mathrm{Hom}(T^*, \mathbb{G}_m)$ corresponding to $\alpha^\vee \in \Phi^\vee$ and by $\delta(\chi)$ the element of $X^\vee(T^*) = \mathrm{Hom}(\mathbb{G}_m, T^*)$ corresponding to $\chi \in X(T)$. Then for any $\chi \in X(T)$ and $\alpha \in \Phi$, we have

2.4.3.
$$\langle \chi, \alpha^\vee \rangle = \langle \alpha^*, \delta(\chi) \rangle.$$

**Proposition 2.4.4.** *The following assertions are equivalent:*

*(i)* $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$.

*(ii)* $B_\mathcal{G} = \{h_\alpha, e_\gamma | \ \alpha \in \Pi, \gamma \in \Phi\}$ *is a basis of* $\mathcal{G}$.

*(iii)* $z(\mathcal{G}^*) = \{0\}$.

*(iv)* $p$ *does not divide* $|\pi_1(G)|$.

*(v) The canonical central isogeny* $G_{sc} \to G$ *is separable.*

**Proof:** The equivalence between (i) and (ii) follows from the fact that $[\mathcal{G},\mathcal{G}]$ is generated by $B_{\mathcal{G}}$. Let $\pi : G_{sc} \to G$ be the canonical central isogeny. The equivalence between the assertions (v) and (ii) follows from the fact that $B_{\mathcal{G}}$ is the image by $d\pi$ of a Chevalley basis of $\mathcal{G}_{sc}$ and that $\pi$ is separable if and only if $d\pi$ is an isomorphism. Since we have $|P(\Phi)/X(T)| = |X^\vee(T)/Q(\Phi^\vee)|$, the equivalence between (iii) and (iv) is a consequence of 2.3.4 applied to $G^*$. We propose to prove the equivalence between the assertions (ii) and (iii).

We first prove that for any root $\alpha$ we have $d\alpha^\vee(k) = [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$ (this makes sense since $[\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] \subset \mathcal{T}$). It is known that for any root $\alpha \in \Phi$, the group $\alpha^\vee(\mathbb{G}_m)$ is contained in the subgroup $H_\alpha$ of $G$ generated by $U_\alpha$ and $U_{-\alpha}$. But the group $H_\alpha$ is a semi-simple algebraic group of rank one with maximal torus $T \cap H_\alpha$; hence it is isomorphic to $SL_2(k)$ or $PGL_2(k)$. Now a simple computation in $SL_2(k)$ or in $PGL_2(k)$ shows that we have $d\alpha^\vee(k) = [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$. Hence $d\alpha^\vee(1) = \lambda h_\alpha$ for some $\lambda \in k$. Let us see that $\lambda = 1$. Since $\langle \alpha, \alpha^\vee \rangle = 2$, we have $d\alpha \circ d\alpha^\vee(1) = 2$, and by 2.2, we also have $d\alpha(h_\alpha) = 2$. Hence

$$d\alpha^\vee(1) = h_\alpha. \tag{*}$$

Let $r$ be the rank of $G$. Let $(x_1, ..., x_r)$ be a basis of $X(T)$ and consider the isomorphisms of algebraic groups $\psi : T \to \mathbb{G}_m^r$ given by $t \mapsto (x_1(t), ..., x_r(t))$ and $\phi : \mathbb{G}_m^r \to T^*$ given by $(t_1, ..., t_r) \mapsto \prod_i \delta(x_i)(t_i)$.

Using $\phi$ and $\psi$ to identify respectively $T^*$ and $T$ with $\mathbb{G}_m^r$, we identify (as suggested by 2.4.1) the abelian groups $X(T^*)$ and $X^\vee(T)$ with $\mathbb{Z}^r$. Under these identifications, for $\alpha \in \Phi$, both $\alpha^\vee$ and $\alpha^*$ correspond to the same element $(n_1^\alpha, ..., n_r^\alpha)$ of $\mathbb{Z}^r$. Indeed, for $i \in \{1, ..., r\}$, let $T_i^\vee : \mathbb{G}_m \to \mathbb{G}_m^r$ and $T_i : \mathbb{G}_m^r \to \mathbb{G}_m$ be the morphisms of 2.4.1; then we have $\delta(x_i) = \phi \circ T_i^\vee$ and $x_i = T_i \circ \psi$. Thus we get that $\langle \alpha^* \circ \phi, T_i^\vee \rangle = \langle \alpha^*, \delta(x_i) \rangle$ and $\langle T_i, \psi \circ \alpha^\vee \rangle = \langle x_i, \alpha^\vee \rangle$ for any $\alpha \in \Phi$. We deduce from 2.4.3 that $\langle \alpha^* \circ \phi, T_i^\vee \rangle = \langle T_i, \psi \circ \alpha^\vee \rangle = n_i^\alpha$.

Let $\{\alpha_1, ..., \alpha_r\} = \Pi$, then it is clear from (*) that $\{h_\alpha, \alpha \in \Pi\}$ is a basis of $\mathcal{T}$ if and only if the matrix $M = \begin{pmatrix} n_1^{\alpha_1} & ... & n_1^{\alpha_r} \\ : & & : \\ n_r^{\alpha_1} & ... & n_r^{\alpha_r} \end{pmatrix} \in M_r(k)$ is invertible. On the other hand, since $z(\mathcal{G}^*) = \bigcap_{\alpha \in \Pi} \mathrm{Ker}(d\alpha^*)$, we have $z(\mathcal{G}^*) = \{0\}$ if and only if the linear map $f : \mathrm{Lie}(T^*) \to k^r$ given by $f(t) = (d\alpha_1^*(t), ..., d\alpha_r^*(t))$ is injective, that is if and only if ${}^tM$ (and so $M$) is invertible. $\square$

## 2.5 Existence of Non-degenerate $G$-Invariant Bilinear Forms on $\mathcal{G}$

By a $G$-invariant bilinear form $B(,)$ on $\mathcal{G}$ we shall mean a symmetric bilinear form $B(,)$ on $\mathcal{G}$ such that for any $g \in G$, $x, y \in \mathcal{G}$, we have $B(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y) = B(x, y)$. A well known example of such a form is the Killing form defined on $\mathcal{G} \times \mathcal{G}$ by $(x, y) \mapsto \mathrm{Trace}(ad(x) \circ ad(y))$. In this section, we want to discuss for which primes $p$ there exists an $G$-invariant non-degenerate bilinear form on $\mathcal{G}$. The case of simple groups has been discussed among other things in [SS70] where it has been proved that the condition "$p$ is good for $G$" (see 2.5.2) is enough to have non-degenerate invariant bilinear forms on $\mathcal{G}$ if $G$ is not of type $A_n$. On the other hand, it is known that the condition "$p$ is very good for $G$" (see 2.5.5) is sufficient if $G$ is simple of type $A_n$. By making the use of 2.3.9 and 2.3.8, we will extend the above results to the case of connected reductive groups, that is, we will see that the condition "$p$ is very good for $G$" is sufficient to have non-degenerate $G$-invariant bilinear forms on $\mathcal{G}$. However this is not completely satisfactory since if $G = GL_n(k)$, the "very good characteristics" for $G$ are the characteristics which do not divide $n$, while the trace form $(X, Y) \mapsto \mathrm{Tr}(XY)$ is always non-degenerate on $gl_n$.

As far as I know, no necessary and sufficient condition on $p$ for the existence of non-degenerate $G$-invariant bilinear forms on $\mathcal{G}$ has been given in the literature. While the above problem is not so important for reductive groups without component of type $A_n$ (indeed the "very good characteristics for $G$" are then the "good ones for $G$", and there are only few "bad characteristics", see further), it becomes more important for the others. For this reason, we will give a necessary and sufficient condition on $p$ in the case of simple groups of type $A_n$. We will also treat the cases of simply connected groups of type $B_n$, $C_n$ or $D_n$ since no extra work is required for these cases (see 2.5.11).

2.5.1. We start with some general properties of $G$-invariant bilinear forms on $\mathcal{G}$. Assume that $B(,)$ is a $G$-invariant bilinear form on $\mathcal{G}$. Then:
(1) For any $x, y, z \in \mathcal{G}$ we have

$$B(x, [y, z]) = B([x, y], z).$$

(2) Let $\alpha \in \Phi$. For any $x$ in $\mathcal{T} \oplus (\bigoplus_{\gamma \in \Phi - \{-\alpha\}} \mathcal{G}_\gamma)$, we have $B(x, e_\alpha) = 0$.

Let us prove (2). Let $x \in \mathcal{T}$; since $B(,)$ is $G$-invariant, for any $t \in T$ we have

$$B(\mathrm{Ad}(t)x, \mathrm{Ad}(t)e_\alpha) = B(x, e_\alpha),$$

that is $\alpha(t)B(x, e_\alpha) = B(x, e_\alpha)$. But $\alpha \neq 0$, thus we get that $B(x, e_\alpha) = 0$. Now let $\beta \in \Phi - \{-\alpha\}$; we have $\alpha(t)\beta(t)B(e_\beta, e_\alpha) = B(e_\beta, e_\alpha)$ for any $t \in T$. Since $\beta \neq -\alpha$, we have $B(e_\beta, e_\alpha) = 0$. □

**Definition 2.5.2 (good characteristics).** *We say that $p$ is* good *for $G$ if $p$ does not divide the coefficient of the highest root of $\Phi$, otherwise $p$ is said to be* bad *for $G$.*

Bad characteristics are $p = 2$ if the root system is of type $B_n$, $C_n$ or $D_n$, $p = 2, 3$ in type $G_2$, $F_4$, $E_6$, $E_7$ and $p = 2, 3, 5$ in type $E_8$ (see [Bou, Ch. VI, 4]).

**Definition 2.5.3.** *[Ste75, Definition 1.3] We say that $p$ is a* torsion prime *of $\Phi$ when there exists a closed root subsystem $\Phi'$ of $\Phi$ (i.e a root subsystem $\Phi'$ of $\Phi$ such that any element of $\Phi$ which is a $\mathbb{Z}$-linear combination of elements of $\Phi'$ is already in $\Phi'$) such that $Q(\Phi^\vee)/Q(\Phi'^\vee)$ has torsion of order $p$.*

**Definition 2.5.4 (torsion primes of $G$).** *We say that $p$ is a* torsion prime *of $G$, when it is a torsion prime of $\Phi$ or when $p$ divides $|\pi_1(G)|$.*

This definition is in fact [Ste75, Lemma 2.5]. For the original definition of torsion primes of $G$, see [Ste75, Definition 2.1].

Torsion primes of $\Phi$ are $p = 2$ when $\Phi$ is of type $B_n$, $D_n$ or $G_2$, $p = 2, 3$ in type $E_6$, $E_7$, $F_4$, $p = 2, 3, 5$ in type $E_8$. The fundamental group $\pi_1(G)$ is a quotient of the biggest possible fundamental group $P(\Phi)/Q(\Phi)$ whose cardinal is $r + 1$ in types $A_r$, 2 in types $B_n$, $C_n$, $E_7$, 4 in types $D_n$, 3 in types $E_6$ and 1 in types $E_8$, $F_4$ or $G_2$ (see [Slo80, page 24]).

**Definition 2.5.5 (very good characteristics).** *We say that $p$ is* very good *for $G$ when $p$ is good for $G$ and $p$ does not divide $|P(\Phi)/Q(\Phi)| = |\pi_1(G_{ad})|$.*

*Remark* 2.5.6. (a) If $\Phi$ does not have any component of type $A_n$, then $p$ is very good for $G$ if and only if $p$ is good for $G$.

(b) If $p$ is very good for $G$ then it is not a torsion prime of $G$.

(c) If $p$ is very good for $G$ and $G$ has a component of type $A_n$, then it is not necessarily very good for Levi subgroups of $G$,

(d) If $G$ is of type $A_n$, $B_n$, $C_n$ or $D_n$, then $p$ is very good if and only if $p$ does not divide $|P(\Phi)/Q(\Phi)|$.

**Proposition 2.5.7.** *[SS70, I, 5.3] Let $G$ be either an adjoint simple group not of type $A_n$ or $G = GL_n(k)$. We assume that $p$ is good for $G$. Then there exists a faithful rational representation $(\rho, V)$ of $G$ or a group isogenous to $G$ (i.e a simple group with same Dynkin diagram as $G$) such that the symmetric bilinear form $B(,)$ on $\mathcal{G}$ defined by $B(x,y) = Trace(d\rho(x) \circ d\rho(y))$ is non-degenerate. Moreover $B(,)$ is $G$-invariant.*

**Corollary 2.5.8.** *Let $G$ be simple not of type $A_n$ and assume that $p$ is good for $G$. Then there exists a non-degenerate $G$-invariant bilinear form on $\mathcal{G}$.*

**Proof:** Let $H$ be a group isogenous to $G$ (with Lie algebra $\mathcal{H}$) and let $(H_{sc} = G_{sc}, \pi)$ be the simply connected cover of $H$. Since $p$ is very good for $G$ (and so for $H$), it is not a torsion prime of $H$ (see 2.5.6 (b)), and so by 2.4.4, the differential $d\pi : \mathcal{H}_{sc} \to \mathcal{H}$ of $\pi : H_{sc} \to H$ is an isomorphism. Moreover it satisfies $d\pi \circ \mathrm{Ad}(h) = \mathrm{Ad}(\pi(h)) \circ d\pi$ for any $h \in H_{sc}$. Hence we deduce that any $H_{sc}$-invariant non-degenerate bilinear form on $\mathcal{H}_{sc}$ induces an $H$-invariant non-degenerate bilinear form on $\mathcal{H}$ and conversely. Hence, the corollary follows from 2.5.7. □

In order to do a more accurate study of the type $A_n$ we need the following well known result.

**Proposition 2.5.9.** *Let $G$ be simple of type $A_{n-1}$ $(n > 1)$. Then recall that $\mathcal{G}_{sc} = sl_n$ and $\mathcal{G}_{ad} = pgl_n$. Then we have the following assertions:*

*(1) We always have $sl_n = [sl_n, sl_n]$. Moreover $dim\, z(sl_n) \neq 0$ if and only if $p$ is not very good, in which case $dim\, z(sl_n) = 1$.*

*(2) We always have $z(pgl_n) = \{0\}$, moreover $pgl_n = [pgl_n, pgl_n]$ if and only if $p$ is very good. When $p$ is not very good, the Lie algebra $pgl_n$ is of the form $k.\sigma \oplus [pgl_n, pgl_n]$ where $\sigma$ is a semi-simple element.*

*(3) The three following situations occur:*

*(3.1) $p$ does not divide $|P(\Phi)/X(T)|$, then $\mathcal{G} \simeq sl_n$,*

*(3.2) $p$ does not divide $|X(T)/Q(\Phi)|$, then $\mathcal{G} \simeq pgl_n$,*

*(3.3) $p$ divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$, then $\mathcal{G}$ is neither isomorphic to $pgl_n$ nor to $sl_n$, and has a one-dimensional center. In fact $\mathcal{G}$ is of the form $z(\mathcal{G}) \oplus [\mathcal{G}, \mathcal{G}] \simeq z(\mathcal{G}) \oplus (sl_n/z(sl_n))$.*

**Proof:** The assertions (1) and the first sentence of (2) follow from 2.4.4 and 2.3.4; the fact that dim $z(sl_n) \leq 1$ is easy.

Now we prove the second assertion of (2). Assume that $p$ is not very good. Recall that for any semi-simple algebraic group $G$, the Lie algebra $[\mathcal{G}, \mathcal{G}]$ is generated by $\{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$, moreover by (1), we have $sl_n = [sl_n, sl_n]$. Hence if $\rho : SL_n \to PGL_n$ denotes the canonical central isogeny then $d\rho(sl_n) = [pgl_n, pgl_n]$. On the other hand since we always have $\mathrm{Ker}(d\rho) = z(sl_n)$, we have $d\rho(sl_n) \simeq sl_n/z(sl_n)$. We deduce that $[pgl_n, pgl_n] \simeq sl_n/z(sl_n)$. Now since $p$ is not very good, by (1), we have $\dim z(sl_n) = 1$ and so $[pgl_n, pgl_n]$ is of codimension one in $pgl_n$; the fact that $\sigma$ in (2) can be chosen semi-simple follows from the fact that for any connected reductive group $G$, the Lie algebra $[\mathcal{G}, \mathcal{G}]$ contains all the nilpotent elements of $\mathcal{G}$.

Now we describe the situation (3.3). First note that the situations (3.1) and (3.2) have been already studied, see equivalence between (iv) and (v) in 2.4.4 for (3.1) and in 2.3.7 for (3.2). Let $\pi : SL_n \to G$ be the canonical central isogeny.

Assume that $p$ divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$.

(i) Since $p$ divides $|P(\Phi)/X(T)|$, the map $d\pi$ is not injective. Moreover by (1), the Lie algebra $z(sl_n)$ is one-dimensional, thus we deduce from $\mathrm{Ker}(d\pi) \subset z(sl_n)$ that $\mathrm{Ker}(d\pi) = z(sl_n)$. As a consequence we have $d\pi(sl_n) \simeq sl_n/z(sl_n)$ and so $[\mathcal{G}, \mathcal{G}]$, which is equal to $d\pi([sl_n, sl_n]) = d\pi(sl_n)$, is of codimension one in $\mathcal{G}$ and has a trivial center.

(ii) Now since $p$ divides $|X(T)/Q(\Phi)|$, the Lie algebra $\mathcal{G}$ has a non-trivial center (see 2.3.4). Hence by (i), the Lie algebra $z(\mathcal{G})$ must be one-dimensional.

$\square$

We are now in position to discuss the existence of non-degenerate invariant bilinear forms on the Lie algebras of simple algebraic groups of type $A_n$. We have the following proposition.

**Proposition 2.5.10.** *Assume that $G$ is simple of type $A_n$. Then $\mathcal{G}$ is endowed with a non-degenerate $G$-invariant bilinear form if and only if $p$ is very good for $G$ or $p$ divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$.*

**Proof:** Assume that $G$ is of type $A_{n-1}$ with $n > 1$ and that $p$ is very good for $G$. Then $p$ does not divide $n$ and so the $SL_n$-invariant bilinear form $(X, Y) \mapsto \mathrm{Tr}(XY)$ on $sl_n$ is non-degenerate. Moreover the canonical morphism $sl_n \to \mathcal{G}$ is an isomorphism, hence we can proceed as in the proof of 2.5.8 to show the existence of a non-degenerate $G$-invariant bilinear form on $\mathcal{G}$.

Assume now that $p$ divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$. Then by 2.5.9 (3.3), we have $\mathcal{G} = z(\mathcal{G}) \oplus [\mathcal{G}, \mathcal{G}]$. Since $G$ acts trivially on $z(\mathcal{G})$, any $G$-invariant non-degenerate bilinear form on $[\mathcal{G}, \mathcal{G}]$ can be extended to a non-degenerate $G$-invariant bilinear form on $\mathcal{G}$. Hence, it is enough to show the existence of a non-degenerate $G$-invariant bilinear form on $[\mathcal{G}, \mathcal{G}] \simeq sl_n/z(sl_n)$. Define $\langle , \rangle$ on $sl_n/z(sl_n)$ by $\langle x + z(sl_n), y + z(sl_n)\rangle = \mathrm{Tr}(xy)$. This is well defined since $z(sl_n) \simeq k$ and for any $X \in sl_n$, $a \in k$, $\mathrm{Tr}(aX) = a\mathrm{Tr}(X) = 0$. Let $\pi : SL_n \to G$ be the canonical central isogeny, then for any $g \in SL_n$, we have $d\pi \circ \mathrm{Ad}(g) = \mathrm{Ad}(\pi(g)) \circ d\pi$ so it is not difficult to check that $\langle , \rangle$ is $G$-invariant. It remains to check that it is non-degenerate. Let $x \in sl_n$ and assume that for any $y \in sl_n$, we have $\mathrm{Tr}(xy) = 0$. Then an easy calculation shows that $x \in z(sl_n)$, that is, its image in $sl_n/z(sl_n)$ is zero. We thus proved the non-degeneracy of $\langle , \rangle$ on $[\mathcal{G}, \mathcal{G}]$.

Now assume that there exists a non-degenerate $G$-invariant bilinear form $\langle , \rangle$ on $\mathcal{G}$ and that $p$ does not divide $|X(T)/Q(\Phi)|$ or $|P(\Phi)/X(T)|$. We want to prove that $p$ is very good for $G$. Two situations occur,

(1) $p$ does not divide $|P(\Phi)/X(T)|$, then we may assume that $G = SL_n$. Let $z \in z(\mathcal{G})$, then by 2.5.1(2), for any $\alpha \in \Phi$, we have $\langle z, e_\alpha\rangle = 0$, and since $z$ is central, by 2.5.1(1) we have $\langle z, h_\alpha\rangle = 0$. But by 2.4.4, the set $\{h_\alpha, e_\gamma | \alpha \in \Pi, \gamma \in \Phi\}$ is a basis of $\mathcal{G}$, hence the non-degeneracy of $\langle , \rangle$ implies that $z = 0$. We thus proved that $z(\mathcal{G}) = \{0\}$. By 2.3.4, we deduce that $p$ does not divide $|(X(T)/Q(\Phi))_{tor}| = |P(\Phi)/Q(\Phi)|$ and so that $p$ is very good for $G$.

(2) $p$ does not divide $|X(T)/Q(\Phi)|$, then we may assume that $G = PGL_n$. Assume that $p$ is not very good (i.e $p$ divides $|P(\Phi)/X(T)|$).

Let $T \subset G$ be the set of diagonal matrices modulo $Z_{GL_n}$ and let $B$ be the set of upper triangular matrices modulo $Z_{GL_n}$. For $i \in \{1, ..., n-1\}$, let $\alpha_i : T \to k^\times$ be defined by $\alpha_i(t_1, ..., t_n) = t_i t_{i+1}^{-1}$; note that $d\alpha_i(t_1, ..., t_{n-1}) = t_i - t_{i+1}$. The basis $\Pi$ of $\Phi$ is equal to $\{\alpha_1, ..., \alpha_{n-1}\}$. Since $p$ is not very good, by 2.5.9 (2), the Lie algebra $[\mathcal{G}, \mathcal{G}]$ is of codimension one in $\mathcal{G}$. As a consequence, the vectors $h_{\alpha_i}$ with $i \in \{1, ..., n-1\}$ are linearly dependent i.e. there exists $\lambda_1, .., \lambda_{n-1}$ not all equal to zero such that $h := \lambda_1 h_{\alpha_1} + ... + \lambda_{n-1} h_{\alpha_{n-1}} = 0$. Let $r$ be the smallest integer such that $\lambda_r \neq 0$ and let $\sigma$ be the $n \times n$ matrix $(a_{ij})_{i,j}$ (modulo $z(gl_n)$) with $a_{rr} = 1$ and $a_{ij} = 0$ for $i, j \neq r$. Since $h = 0$, we have

$$\langle \sigma, h\rangle = 0. \qquad (*)$$

On the other hand, since $\langle , \rangle$ is $G$-invariant, we have $\langle \sigma, h_\alpha \rangle = \langle \sigma, [e_\alpha, e_{-\alpha}] \rangle = \langle [\sigma, e_\alpha], e_{-\alpha} \rangle = \langle d\alpha(\sigma) e_\alpha, e_{-\alpha} \rangle = d\alpha(\sigma) \langle e_\alpha, e_{-\alpha} \rangle$ for any $\alpha \in \Phi$. Since $d\alpha_r(\sigma) = 1$ and $d\alpha_i(\sigma) = 0$ for any $i > r$, we deduce that $\langle \sigma, h \rangle = \lambda_r \langle e_{\alpha_r}, e_{-\alpha_r} \rangle$. But the bilinear form $\langle , \rangle$ is non-degenerate, hence by 2.5.1(2), we have $\langle e_{\alpha_r}, e_{-\alpha_r} \rangle \neq 0$ which contradicts (*). □

*Remark* 2.5.11. Assume that $G$ is simply connected. Then we can proceed as in (1) of the proof of 2.5.10 to show that the existence of a non-degenerate $G$-invariant bilinear form on $\mathcal{G}$ implies that $p$ does not divide $|P(\Phi)/Q(\Phi)|$. Hence, when $G$ is of type $B_n$, $C_n$ or $D_n$, by 2.5.6 (a), (d) and by 2.5.8, the Lie algebra $\mathcal{G}$ admits a non-degenerate $G$-invariant bilinear form if and only if $p$ is good for $G$.

**Proposition 2.5.12.** *Let $G$ be a connected reductive group. Assume that $p$ is very good for $G$, then there exists a non-degenerate $G$-invariant bilinear form on $\mathcal{G}$.*

**Proof:** By assumption, the prime $p$ does not divide $|(X(T)/Q(\Phi))_{tor}|$. Thus, by 2.3.4 and 2.3.1, we may identify $\mathcal{G}$ with $z(\mathcal{G}) \oplus \overline{\mathcal{G}}$. Since $G$ acts trivially on $z(\mathcal{G})$, any non-degenerate $\overline{G}$-invariant bilinear form on $\overline{\mathcal{G}}$ can be extended to a non-degenerate $G$-invariant bilinear form on $z(\mathcal{G}) \oplus \overline{\mathcal{G}} \simeq \mathcal{G}$. So it is enough to show the existence of a non-degenerate $\overline{G}$-invariant bilinear form on $\overline{\mathcal{G}}$. Let $\overline{G} = G_1 ... G_n$ be a decomposition of $\overline{G}$ as the almost-direct product of its simple components. By 2.5.8 and 2.5.10, for any simple component $G_i$ of $\overline{G}$, there exists an $G_i$-invariant non-degenerate bilinear form $B_i$ on $\mathcal{G}_i := \mathrm{Lie}(G_i)$. Since $p$ is very good for $\overline{G}$, by 2.3.8, we have a decomposition $\overline{\mathcal{G}} = \bigoplus_i \mathcal{G}_i$ and so the form $B = \bigoplus_i B_i$ provides a non-degenerate $\overline{G}$-invariant bilinear form on $\overline{\mathcal{G}}$. □

*Remark* 2.5.13. We saw in the proof of 2.5.12 that a non-degenerate $\overline{G}$-invariant bilinear form on $\overline{\mathcal{G}}$ can be extended to a non-degenerate $G$-invariant bilinear form on $\mathcal{G}$. However it is not true that all non-degenerate $G$-invariant bilinear forms on $\mathcal{G}$ are obtained in this way. Indeed, the trace form $(X, Y) \mapsto \mathrm{Tr}(XY)$ is always non-degenerate on $gl_n$ while (see 2.5.10) there is no non-degenerate $PGL_n$-invariant bilinear form on $pgl_n$ unless $p$ is very good.

We have the following lemma.

**Lemma 2.5.14.** *[Leh96, proof of 4.3] If $\mathcal{G}$ admits an $G$-invariant non-degenerate bilinear form $B( , )$, then the restriction of $B( , )$ to any Levi subalgebra of $\mathcal{G}$ is still non-degenerate.*

2.5.15. Now we assume that $p$ is very good for $G$. By 2.5.12, the Lie algebra $\mathcal{G}$ is endowed with a $G$-invariant non-degenerate bilinear form $B(,)$ and in view of 2.3.1 and 2.3.4, we may write $\mathcal{G} = z(\mathcal{G}) \oplus \overline{\mathcal{G}}$. We have the following lemma.

**Lemma 2.5.16.** *The vector space $\overline{\mathcal{G}}$ is the orthogonal complement of $z(\mathcal{G})$ in $\mathcal{G}$ with respect to $B(,)$. In particular, the restrictions of $B(,)$ to $z(\mathcal{G})$ and to $\overline{\mathcal{G}}$ remain non-degenerate.*

**Proof:** Since $p$ is very good for $G$ (and so for $\overline{G}$), by 2.4.4, we have $[\overline{\mathcal{G}}, \overline{\mathcal{G}}] = \overline{\mathcal{G}}$. Thus, by 2.5.1(1), the vector space $\overline{\mathcal{G}}$ is orthogonal to $z(\mathcal{G})$. Hence the lemma follows from the non-degeneracy of $B(,)$. □

*Remark* 2.5.17. Note that if $G = GL_n(k)$, the restriction of $B(,)$ to $z(\mathcal{G})$ is non-degenerate if and only if the condition "$p$ is very good for $G$" is satisfied. However, it is not a necessary condition in the general case. For instance, if $G$ is simple of type $A_n$ and $p$ divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$. Then $z(\mathcal{G})$ is a one-dimensional vector space and we have $\mathcal{G} = z(\mathcal{G}) \oplus [\mathcal{G}, \mathcal{G}]$, see 2.5.9 (3.3). By 2.5.10, there exists a non-degenerate $G$-invariant bilinear form $B(,)$ on $\mathcal{G}$. The $G$-invariance of $B(,)$ implies that $z(\mathcal{G})$ is orthogonal to $[\mathcal{G}, \mathcal{G}]$ with respect to $B(,)$. Thus the non-degeneracy of $B(,)$ implies that its restriction to $z(\mathcal{G})$ is still non-degenerate.

## 2.6 Centralizers

Let $H$ be a closed subgroup of $G$ with Lie algebra $\mathcal{H}$. For any $X \in \mathcal{G}$, recall (see 2.1.2(iii) and 2.1.3) that we have

2.6.1. $$\mathrm{Lie}(C_H(x)) \subset C_{\mathcal{H}}(x).$$

When $H = G$, this inclusion is known to be an equality when $x$ is semi-simple [Bor, 9.1]. Due to Richardson-Springer-Steinberg, it is also known to be an equality for any $x \in \mathcal{G}$ when $H = G = GL_n$ or when $H = G$ is simple and $p$ is very good for $G$ [SS70, I, 5.6] [Slo80, 3.13]. In the following lemma, we extend the above result of R-S-S to the case where $G$ is an arbitrary reductive group and $p$ is very good for $G$.

**Lemma 2.6.2.** *Let $x \in \mathcal{G}$, then the inclusion 2.6.1 with $H = G$ is an equality (i.e the morphism $G \to \mathcal{O}_x^G$, $g \mapsto Ad(g)x$ is separable) in the following cases:*

*(i) $x$ is semi-simple,*

*(ii) $p$ is very good for $G$ or $G = GL_n$.*

**Proof:** As noticed above, the lemma is already established in the case where $X$ is semi-simple and the cases where $G = GL_n$, or $G$ is simple and $p$ is very good for $G$.

To show that $\mathrm{Lie}(C_G(x)) \subset C_{\mathcal{G}}(x)$ is an equality, it is enough to prove that $\dim(C_G(x)) = \dim(C_{\mathcal{G}}(x))$.

(a) Assume first that $G$ is semi-simple and write $G = G_1...G_n$ where $G_1, ..., G_n$ are the simple components of $G$. Since $p$ is very good for $G$, by 2.3.8 we have the following corresponding Lie algebras decomposition $\mathcal{G} = \bigoplus_i \mathcal{G}_i$ where $\mathcal{G}_i$ is the Lie algebra of $G_i$. Let $x = \sum_i x_i \in \bigoplus_i \mathcal{G}_i$, then $C_G(x)$ is the almost-direct product of the $C_{G_i}(x_i)$. We thus have

$$\dim(C_G(x)) = \sum_i \dim(C_{G_i}(x_i)). \tag{1}$$

On the other hand, let $y = y_1 + ... + y_n$ be the decomposition of $y \in \mathcal{G}$ in $\bigoplus_i \mathcal{G}_i$. Since $[\mathcal{G}_i, \mathcal{G}_j] = 0$ for $i \neq j$, we have $[y, x] = \sum_i [y_i, x_i]$. Hence $y \in C_{\mathcal{G}}(x)$ if and only if $y_i \in C_{\mathcal{G}_i}(x_i)$ for any $i$. Thus we have $C_{\mathcal{G}}(x) = \bigoplus_i C_{\mathcal{G}_i}(x_i)$ and so

$$\dim(C_{\mathcal{G}}(x)) = \sum_i \dim(C_{\mathcal{G}_i}(x_i)). \tag{2}$$

Since for any $i$, the group $G_i$ is simple and $p$ is very good for $G_i$, we have $\mathrm{Lie}(C_{G_i}(x_i)) = C_{\mathcal{G}_i}(x_i)$ and so $\dim(C_{G_i}(x_i)) = \dim(C_{\mathcal{G}_i}(x_i))$. Then we deduce from (1) and (2) that $\dim(C_{\mathcal{G}}(x)) = \dim(C_G(x))$.

(b) Assume now that $G$ is reductive. Since $p$ is very good for $G$, by 2.3.9 we have a decomposition

$$\mathcal{G} = z(\mathcal{G}) \oplus \mathcal{G}'. \tag{1}$$

Write $x = z + y$ with $z \in z(\mathcal{G})$ and $y \in \mathcal{G}'$. Since $Z_G$ acts trivially on $\mathcal{G}$ we have $C_G(x) = Z_G^o.C_{G'}(x)$. But $G$ acts trivially on $z(\mathcal{G})$, hence $C_{G'}(x) = C_{G'}(y)$. We deduce that $C_G(x) = Z_G^o.C_{G'}(y)$. Since $Z_G^o \cap C_{G'}(y)$ is finite we have

$$\dim(C_G(x)) = \dim Z_G^o + \dim(C_{G'}(y)). \tag{2}$$

On the other hand, from (1) we see that $C_{\mathcal{G}}(x) = z(\mathcal{G}) \oplus C_{\mathcal{G}'}(y)$ and so that

$$\dim(C_{\mathcal{G}}(x)) = \dim z(\mathcal{G}) + \dim(C_{\mathcal{G}'}(y)). \tag{3}$$

The group $G'$ is semi-simple, so using (a) we have $\dim(C_{\mathcal{G}'}(y)) = \dim(C_{G'}(y))$. Hence the equality $\dim(C_G(x)) = \dim(C_{\mathcal{G}}(x))$ follows from (2), (3) and 2.3.4.

$\square$

*Remark* 2.6.3. Note that 2.6.1 with $H = G$ may not be an equality if $p$ is not very good for $G$. Indeed, consider $G = PGL_2$ with $p = 2$ and $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. A simple calculation shows that $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ commutes with $e$. Hence $\dim(C_{\mathcal{G}}(e)) = 2$ while $C_G(e)$ is of dimension one.

Now we give some various results on centralizers of elements of $\mathcal{G}$ which will be used later. We first start with the following well-known characterization of the centralizers of semi-simple elements of $\mathcal{G}$ (see [SS70, II, 4.1]).

**Proposition 2.6.4.** *For each element $w \in W_G(T)$, we choose a representative $\dot{w}$ of $w$ in $N_G(T)$. Let $x \in \mathcal{T}$.*

*(i) The group $C_G(x)$ is generated by $T$, the $U_\alpha$ such that $d\alpha(x) = 0$ and the $\dot{w}$ such that $Ad(\dot{w})x = x$.*

*(ii) The group $C_G^o(x)$ is generated by $T$, and the $U_\alpha$ such that $d\alpha(x) = 0$.*

*(iii) The algebraic group $C_G^o(x)$ is reductive.*

**Lemma 2.6.5.** *[HS85, Proposition 3] Let $P = LU_P$ be a Levi decomposition in $G$ and let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the corresponding Lie algebra decomposition. Then the centralizer $C_{U_P}(x)$ is connected for any element $x$ of $\mathcal{L}$.*

We have the following standard result.

**Lemma 2.6.6.** *Let $L$ be a Levi subgroup of a parabolic subgroup $P$ of $G$ and let $\mathcal{L}$ be the Lie algebra of $L$. For any element $z$ of $\mathcal{L}$, we have $\mathcal{O}_z^{U_P} \subset z + \mathcal{U}_P$. If $C_G^o(z_s) \subset L$, then the map $U_P \to z + \mathcal{U}_P$ given by $u \mapsto Ad(u)z$ is an isomorphism.*

**Proof:** Let $z \in \mathcal{L}$, we assume that $L \supseteq T$ and that $z_s \in \mathcal{T}$ so that we can use the notation of 2.2.2. Let $\alpha, \beta \in \Phi$ be such that $\alpha \neq -\beta$ and let $u_\alpha \in U_\alpha$, then by 2.2.2 we have

(2) $\mathrm{Ad}(u_\alpha)z_s \in z_s + k.e_\alpha$,

(3) $\mathrm{Ad}(u_\alpha)e_\beta = e_\beta + \sum_{\{i>o|\beta+i\alpha\in\Phi\}} c_i e_{\beta+i\alpha}$, for some $c_i \in k$.

Note also that

**2.6.7.** *if $\beta$ is a root of $L$ with respect to $T$ (this makes sense since we assumed that $T \subset L$) and if $\alpha \in \Phi$ is a root of $U_P$ (i.e $\mathcal{G}_\alpha \subset \mathcal{U}_P$), then for any $i > 0$ such that $i\alpha + \beta \in \Phi$, the root $i\alpha + \beta$ is a root of $U_P$.*

From (2), (3) and 2.6.7, we observe that $\mathcal{O}_z^{U_P} - Z$ is a subvariety of $\mathcal{U}_P$.

Assume now that $C_G^o(z_s) \subset L$, then since $C_{U_P}(z)$ is connected by 2.6.5 and $C_{U_P}(z) \subset C_G(z_s)$, we have $C_{U_P}(z) = \{1\}$ and so $\mathcal{O}_z^{U_P} - z$ is of dimension $\dim U_P$. On the other hand, by [Bor, Proposition 4.10], the variety $\mathcal{O}_z^{U_P}$ is closed in $\mathcal{G}$. We deduce that $\mathcal{O}_z^{U_P} = z + \mathcal{U}_P$. We thus have proved that the map $U_P \to z + \mathcal{U}_P$ given by $u \mapsto \mathrm{Ad}(u)z$ is a bijective morphism. To verify that it is an isomorphism, it is sufficient to verify that it is separable, that is, we have to show that $\mathrm{Lie}\big(C_{U_P}(z)\big) = C_{\mathcal{U}_P}(z)$. Since $C_{U_P}(z) = \{1\}$, we have to show that $C_{\mathcal{U}_P}(z) = \{0\}$. Let $v \in \mathcal{U}_P$ be such that $[v, z] = 0$. We have $[v, z_s] = -[v, z_n]$. Let $B_L$ be a Borel subgroup of $L$ such that $z \in \mathrm{Lie}(B_L)$; we may assume without loss of generality that $B_L$ contains $T$. Then $B = B_L U_P$ is a Borel subgroup of $G$ containing $T$ and we denote by $\Phi^+$ the positive roots of $\Phi$ with respect to $B$. We also denote by $\Phi_L^+$ the positive roots (with respect to $B_L$) of the root system $\Phi_L$ of $L$ (with respect to $T$). Then we may write $v = \sum_{\alpha \in \Phi^+ - \Phi_L} \lambda_\alpha e_\alpha$ and $z_n = \sum_{\alpha \in \Phi_L^+} \beta_\alpha e_\alpha$. Assume that $v \neq 0$ and let $\alpha_o \in \Phi^+ - \Phi_L$ be such that $\lambda_{\alpha_o} \neq 0$ and the height of $\alpha_o$ (with respect to $B$) is minimal among the heights of the roots $\alpha \in \Phi^+ - \Phi_L$ such that $\lambda_\alpha \neq 0$. Since $C_G^o(z_s) \subset L$, from 2.6.4(ii), we have $d\alpha_o(z_s) \neq 0$, and so, from 2.2.1(ii), the vector $[v, z_s]$ has a non-zero coefficient in $e_{\alpha_o}$ while from the Chevalley relations 2.2.1(iii)(iv), we see that the vector $[v, z_n]$ does not have non-zero coefficients in $e_\alpha$ if $\alpha$ is of same height as $\alpha_o$. Hence we have $v = 0$. $\square$

*Notation* 2.6.8. For any set $J$ contained in a basis of $\Phi$, we denote by $\Phi_J$ the subroot system of $\Phi$ generated by $J$, by $L_J$ the Levi subgroup of $G$ corresponding to $\Phi_J$ (i.e the subgroup of $G$ generated by $T$ and the $U_\alpha$ such that $\alpha \in \Phi_J$) and by $\mathcal{L}_J$ the Lie algebra of $L_J$. If $I$ is a subset of a basis of $\Phi$, we denote by $B(I)$ the subset of $\Phi - \Phi_I$ consisting of the elements $\gamma$ such that the set $I \cup \{\gamma\}$ is contained in a basis of $\Phi$.

**Proposition 2.6.9.** *Let $I$ be a subset of a basis of $\Phi$. The minimal Levi subgroups of $G$ strictly containing $L_I$ are the $L_{\Phi_{I \cup \{\alpha\}}}$ with $\alpha \in B(I)$.*

**Proof:** Let $M$ be a Levi subgroup of $G$ containing $L_I$ and let $\Phi_M$ be the root system of $M$ with respect to $T$. Let $P$ be a parabolic subgroup of $M$ such that $P = L_I U_P$ is a Levi decomposition of $P$. Let $B$ be a Borel subgroup of $P$ containing $T$, then it defines a basis $\theta$ of $\Phi_M$ and since $L_I$ is the unique Levi subgroup of $P$ containing $T$, the group $L_I$ must be of the form $L_J$ for some subset $J$ of $\theta$ (cf. [DM91, Propositions 1.6, 1.15]). Now, if $\gamma \in \Phi_M$ is a $\mathbb{Q}$-linear combination of elements of $\Phi_I$, it is a $\mathbb{Z}$-linear combination of elements of $\theta$. We deduce that $\gamma$ is a $\mathbb{Z}$-linear combination of elements of $J$. We thus have $\gamma \in \Phi_I$. We proved that $\Phi_I$ is $\mathbb{Q}$-closed root subsystem of $\Phi_M$ (i.e any element of $\Phi_M$ which is a $\mathbb{Q}$-linear combination of elements of $\Phi_I$ is already in $\Phi_I$). By [Bou, VI, 1, 7, Proposition 24], we deduce that we can extend $I$ to a basis $I'$ of $\Phi_M$. Using the same argument, we can also prove that $I'$ can be extended to a basis of $\Phi$. Hence, we proved that any Levi subgroup of $G$ containing strictly $L_I$ contains a Levi subgroup of the form $L_{I\cup\{\alpha\}}$ with $\alpha \in B(I)$. It is then clear that minimal Levi subgroups containing strictly $L_I$ are of the form $L_{I\cup\{\alpha\}}$ for some $\alpha \in B(I)$. The fact that the Levi subgroups $L_{I\cup\{\alpha\}}$ with $\alpha \in B(I)$ are minimal is clear. □

**Definition 2.6.10.** *Let $L$ be a Levi subgroup of $G$, then we say that $x \in \mathcal{G}$ is $L$-*regular *in $\mathcal{G}$ if $L = C_G^o(x)$.*

**Lemma 2.6.11.** *Let $L$ be a Levi subgroup of $G$ and let $\mathcal{L}$ be its Lie algebra, then the $L$-regular elements in $\mathcal{G}$ belong to $z(\mathcal{L})$.*

**Proof:** Let $x$ be $L$-regular in $\mathcal{G}$, then $C_G(x)$ contains a maximal torus $T$ of $L$. Write $x = t + \sum_\alpha \lambda_\alpha e_\alpha \in \mathcal{G} = \mathcal{T} \oplus \bigoplus_{\alpha\in\Phi(T)} \mathcal{G}_\alpha$. Since $T$ centralizes $x$, we must have $\lambda_\alpha = 0$ for all $\alpha \in \Phi(T)$, i.e. $x \in \mathcal{T}$. Since $C_G^o(x) = L$ we deduce from 2.6.4(ii) and 2.3.2(1) that $x \in z(\mathcal{L})$. □

**Definition 2.6.12.** *Let $L$ be a Levi subgroup of $G$ and let $\mathcal{L}$ be its Lie algebra. If $x \in z(\mathcal{L})$ is not $L$-regular in $\mathcal{G}$, then $x$ is said to be $L$-*irregular*.*

**Lemma 2.6.13.** *(i) Assume that $p$ is good for $G$ and that $p$ does not divide $|(X(T)/Q(\Phi))_{tor}|$, then if $L$ is a Levi subgroup of $G$, the Lie algebra $\mathcal{G}$ contains $L$-regular elements in $\mathcal{G}$.*

*(ii) If $p$ is good for $G$, then for any semi-simple element $x \in \mathcal{G}$, the group $C_G^o(x)$ is a Levi subgroup of $G$.*

**Proof:** We first prove (ii). We may assume that $x \in \mathcal{T}$. Since $p$ is good for $G$, it follows that the set $\Phi_x := \{\alpha \in \Phi |\, d\alpha(x) = 0\}$ is a $\mathbb{Q}$-closed root subsystem of $\Phi$. Hence by [Bou, VI, 1, 7, Proposition 24], the set $\Phi_x$ is of the form $\Phi_J$ for some subset $J$ of some basis of $\Phi$. Thus by 2.6.4(ii), we have $C_G^o(x) = L_J$.

We now prove (i).

We may assume without loss of generality that $L$ is a Levi subgroup of the form $L_I$ for some subset $I$ of some basis of $\Phi$. We want to prove that $\mathcal{L}_I$ contains $L_I$-regular elements in $\mathcal{G}$. Recall first that if $J$ is a set contained in a basis of $\Phi$, then we have

$$z(\mathcal{L}_J) = \bigcap_{\alpha \in J} \operatorname{Ker}(d\alpha). \tag{1}$$

From (ii), 2.6.11, 2.6.9 and 2.6.4 (ii), we see that $x$ is $L_I$-regular in $\mathcal{G}$ if and only if

2.6.14.
$$x \in z(\mathcal{L}_I) - \bigcup_{\gamma \in B(I)} (z(\mathcal{L}_I) \cap \operatorname{Ker}(d\gamma)).$$

Since $B(I)$ is finite, the assertion (i) will follow from the fact that the subspaces $z(\mathcal{L}_I) \cap \operatorname{Ker}(d\gamma)$, where $\gamma$ runs over $B(I)$, are of dimension strictly less than $\dim z(\mathcal{L}_I)$. Hence from (1), it is enough to prove that for any basis $\Pi'$ of $\Phi$ and any inclusions $\Pi' \supseteq J \supsetneq I$ we have $z(\mathcal{L}_J) \subsetneq z(\mathcal{L}_I)$.

Consider the following inclusions $\Pi' \supseteq J \supsetneq I$ with $\Pi'$ a basis of $\Phi$, then it follows from [DM91, Proposition 1.21] that $Z^o_{L_J} \subsetneq Z^o_{L_I}$ and so (because tori are smooth) we get that $\operatorname{Lie}(Z^o_{L_J}) \subsetneq \operatorname{Lie}(Z^o_{L_I})$. From 2.3.5, we get that $p$ satisfies 2.3.4(i) applied to $L_I$ and $L_J$; thus $\operatorname{Lie}(Z^o_{L_I}) = z(\mathcal{L}_I)$ and $\operatorname{Lie}(Z^o_{L_J}) = z(\mathcal{L}_J)$. We deduce that $z(\mathcal{L}_J) \subsetneq z(\mathcal{L}_I)$. □

*Remark* 2.6.15. Let $L$ be a Levi subgroup of $G$ and let $\mathcal{L}$ be the Lie algebra of $L$. Assume that the set of $L$-regular elements in $\mathcal{G}$ is non-empty, then from 2.6.14 we see that it is an open dense subset of $z(\mathcal{L})$.

**Lemma 2.6.16.** *Let $L$ be a Levi subgroup of $G$ (with Lie algebra $\mathcal{L}$) and let $x \in \mathcal{G}$ be $L$-regular in $\mathcal{G}$. Let $g \in G$ be such that $Ad(g)x \in z(\mathcal{L})$, then $Ad(g)x$ is also $L$-regular in $\mathcal{G}$ and we have $g \in N_G(L)$.*

**Proof:** It is enough to show that $g \in N_G(L)$. We have $C^o_G(z(\mathcal{L})) \subset C^o_G(\operatorname{Ad}(g)x)$, that is $C^o_G(z(\mathcal{L})) \subset gC^o_G(x)g^{-1} = gLg^{-1}$. Since $C^o_G(z(\mathcal{L})) \supseteq L$, we deduce that $L \subset gLg^{-1}$, i.e. $L = gLg^{-1}$. □

**Lemma 2.6.17.** *We assume that $k = \overline{\mathbb{F}}_q$, that $p$ is good for $G$ and that $p$ does not divide $|(X(T)/Q(\Phi))_{tor}|$. We also assume that $T$ and $B$ are both $F$-stable. Let $I$ be a subset of $\Pi$ such that the Levi subgroup $L_I$ of $G$ is $F$-stable, i.e. the set $I$ is $\tau$-stable where $\tau$ is as in 2.1.18. If $q > |B(I)|$, then $\mathcal{L}_I^F$ contains $L_I$-regular elements in $\mathcal{G}$.*

**Proof:** Recall that the subset of $z(\mathcal{L}_I)$ consisting of the $L_I$-irregular elements of $\mathcal{G}$ is

$$\bigcup_{\gamma \in B(I)} (z(\mathcal{L}_I) \cap \mathrm{Ker}(d\gamma)).$$

Let $V = z(\mathcal{L}_I) \cap \mathrm{Ker}(d\gamma)$ for some $\gamma \in B(I)$, then for $i$ large enough, the set $V \cap F(V) \cap ... \cap F^i(V)$ is $F$-stable and contains all the rational elements of $V$. On the other hand, from the proof of 2.6.13, we have $\dim V < \dim z(\mathcal{L}_I)$. Thus by [DM91, 3.7], the number of rational $L_I$-irregular elements of $\mathcal{G}$ is $\leq |B(I)|q^{\dim(z(\mathcal{L}_I))-1}$. Hence, if $q > |B(I)|$, the number of $L_I$-irregular elements is less than $|z(\mathcal{L}_I)^F|$ and so rational $L_I$-regular elements must exist. □

**Proposition 2.6.18.** *[Ste75, Theorem 3.14] The centralizers in $G$ of the semi-simple elements of $\mathcal{G}$ are connected if and only if $p$ is not a torsion prime for $G$.*

## 2.7 The Varieties $G_{uni}$ and $\mathcal{G}_{nil}$

Let $G_{uni}$ be the subvariety of $G$ consisting of unipotent elements and let $\mathcal{G}_{nil}$ be the subvariety of $\mathcal{G}$ formed by nilpotent elements. For any $X \subset G$ and $Y \subset \mathcal{G}$, put $X_{uni} = X \cap G_{uni}$ and $Y_{nil} = Y \cap \mathcal{G}_{nil}$. Recall that the subvarieties $G_{uni} \subset G$ and $\mathcal{G}_{nil} \subset \mathcal{G}$ are closed, irreducible of codimension $rk(G)$. It has been proved [Lus76] that the number of unipotent classes of $G$ is finite for any $p$. By 2.7.5, this implies that the number of nilpotent orbits of $\mathcal{G}$ is also finite if $p$ is good for $G$. In the case of bad characteristics, the finiteness of nilpotent orbits results from a case by case argument (see [Car72, 5.11] for the classification of nilpotent orbits in bad characteristics).

The following propositions are well-known.

**Proposition 2.7.1.** *[Leh79] Let $P = LU_P$ be a Levi decomposition of a parabolic subgroup $P$ of $G$ and let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the corresponding Lie algebra decomposition.*

*(i) Let $l \in L$, then the semi-simple part of any element of $lU_P$ is $U_P$-conjugate to the semi-simple part of $l$.*

*(ii) Let $x \in \mathcal{L}$, then the semi-simple part of any element of $x + \mathcal{U}_P$ is $U_P$-conjugate to the semi-simple part of $x$. That is, for any $v \in \mathcal{U}_P$, we have $(x+v)_s = Ad(u)(x_s)$ for some $u \in U_P$.*

The following result is a straightforward consequence of the above proposition.

**Corollary 2.7.2.** *If $P = LU_P$ is a Levi decomposition in $G$ with corresponding Lie algebra decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$, then for any unipotent element $l \in L$ and any nilpotent element $x \in \mathcal{L}$, we have $lU_P \subseteq G_{uni}$ and $x+\mathcal{U}_P \subseteq \mathcal{G}_{nil}$.*

**Proposition 2.7.3.** *[Spr69] If the canonical morphism $\pi : G \to G_{ad}$ is separable (which by 2.3.7 is equivalent to $p$ does not divide the torsion of $X(T)/Q(\Phi)$), then the bijective morphism $\pi_{uni} : G_{uni} \to (G_{ad})_{uni}$ given by restricting $\pi$ to $G_{uni}$ is an isomorphism.*

*Remark* 2.7.4. Consider $G = SL_2(k)$ and assume that the morphism $\pi : G \to G_{ad}$ is not separable (i.e. $p = 2$). Then the morphism $\pi_{uni} : G_{uni} \to (G_{ad})_{uni}$ is not an isomorphism. To see that, it is enough to see that its differential $d(\pi_{uni})$ at $1 = 1_G$ is not an isomorphism. Note that $d(\pi_{uni}) : T_1(G_{uni}) \to T_1((G_{ad})_{uni})$ is the restriction morphism of $d\pi$ to the tangent space $T_1(G_{uni})$ of $G_{uni}$ at 1. On the other hand, $\dim T_1(G_{uni}) > 2$; indeed $\dim \mathcal{G}_{nil} = 2$ and the inclusion $T_1(G_{uni}) \supset \mathcal{G}_{nil}$ is strict since $\mathcal{G}_{nil}$ is not a vector space. Hence $T_1(G_{uni}) = \mathcal{G}$ since $\dim \mathcal{G} = 3$, and so we deduce that $d(\pi_{uni}) = d\pi$. Since $\pi$ is not separable, the morphism $d\pi = d(\pi_{uni})$ is not an isomorphism.

2.7.5. By a $G$-equivariant morphism $\pi : G_{uni} \to \mathcal{G}_{nil}$, we shall mean a morphism $\pi : G_{uni} \to \mathcal{G}_{nil}$ such that $\pi(gxg^{-1}) = \mathrm{Ad}(g)\pi(x)$ for all $g \in G$ and $x \in G_{uni}$. The existence of $G$-equivariant isomorphisms $G_{uni} \to \mathcal{G}_{nil}$ is discussed in [Spr69] and in [BR85]. It is proved that if $p$ is good for $G$, resp. very good for $G$, then $G$-equivariant homeomorphisms, resp. isomorphisms, $G_{uni} \to \mathcal{G}_{nil}$ exist.

We have the following lemma.

**Lemma 2.7.6.** *[Bon04, Proposition 6.1] Let $f : G_{uni} \to \mathcal{G}_{nil}$ be a $G$-equivariant homeomorphism, then for any Levi decomposition $P = LU_P$ in $G$ with $\mathcal{L} = Lie(L)$, we have*

*(i) $f(L_{uni}) = \mathcal{L}_{nil}$,*

*(ii) for any $x \in L_{uni}$, $f(xU_P) = f(x) + \mathcal{U}_P$.*