4 Classification at Infinity and Global Solutions

A classification similar to that given in Chapter 2 can be performed at $+\infty$. This is the topic of Sections 4.1–4.3.

The results of Chapters 2, 3 apply to local solutions, i.e., solutions up to a random time. In Sections 4.4, 4.5, we study the existence and uniqueness of a global solution, i.e., a solution in the sense of Definition 1.28. This is done for the SDEs that have no more than one singular point.

Throughout this chapter, we assume that $\sigma(x) \neq 0$ for all $x \in \mathbb{R}$.

4.1 Classification at Infinity: The Results

Throughout this section, we assume that

$$\frac{1+|b|}{\sigma^2} \in L^1_{\text{loc}}([a,\infty)) \tag{4.1}$$

for some $a \in \mathbb{R}$.

We will use the functions

$$\rho(x) = \exp\left(-\int_a^x \frac{2b(y)}{\sigma^2(y)}dy\right), \quad x \in [a,\infty), \tag{4.2}$$

$$s(x) = -\int_x^\infty \rho(y)dy, \quad x \in [a,\infty) \tag{4.3}$$

and the notation

$$\overline{T}_\infty = \lim_{n\to\infty} T_n,$$
$$\overline{T}_{a,\infty} = \overline{T}_a \wedge \overline{T}_\infty.$$

Theorem 4.1. *Suppose that*

$$\int_a^\infty \rho(x)dx = \infty.$$

If $x_0 \in [a,\infty)$, *then there exists a unique solution* P *defined up to* T_a. *We have* $T_a < \infty$ P-*a.s.*

A.S. Cherny and H.-J. Engelbert: LNM 1858, pp. 81–91, 2005.

If the conditions of Theorem 4.1 are satisfied, we will say that $+\infty$ has *type A*.

Theorem 4.2. *Suppose that*

$$\int_a^\infty \rho(x)dx < \infty, \quad \int_a^\infty \frac{|s(x)|}{\rho(x)\sigma^2(x)}dx = \infty.$$

If $x_0 \in [a, \infty)$, then there exists a unique solution P defined up to T_a. If moreover $x_0 > a$, then $\mathsf{P}\{T_a = \infty\} > 0$ and $\lim_{t\to\infty} X_t = +\infty$ P-a.s. on $\{T_a = \infty\}$.

If the conditions of Theorem 4.2 are satisfied, we will say that $+\infty$ has *type B*.

Theorem 4.3. *Suppose that*

$$\int_a^\infty \rho(x)dx < \infty, \quad \int_a^\infty \frac{|s(x)|}{\rho(x)\sigma^2(x)}dx < \infty.$$

If $x_0 \in (a, \infty)$, then there exists a unique solution P defined up to $\overline{T}_{a,\infty}-$. We have $\mathsf{P}\{\overline{T}_\infty < \infty\} > 0$. (In other words, the solution explodes into $+\infty$ with strictly positive probability.)

If the conditions of Theorem 4.3 are satisfied, we will say that $+\infty$ has *type C*.

As a consequence of the above results, we obtain *Feller's criterion for explosions* (see [16], [29, Ch. 5, Th. 5.29], or [34, § 3.6]).

Corollary 4.4. *Suppose that $x_0 \in (a, \infty)$ and P is a solution defined up to $\overline{T}_{a,\infty}-$. Then it explodes into $+\infty$ with strictly positive probability (i.e., $\mathsf{P}\{\overline{T}_\infty < \infty\} > 0$) if and only if*

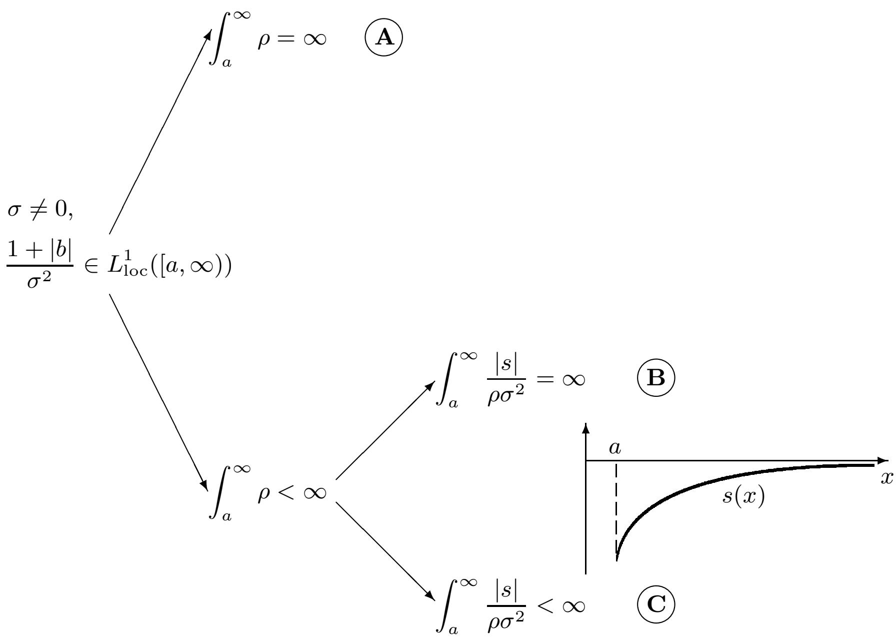
$$\int_a^\infty \rho(x)dx < \infty, \quad \int_a^\infty \frac{|s(x)|}{\rho(x)\sigma^2(x)}dx < \infty.$$

4.2 Classification at Infinity: Informal Description

If $+\infty$ has type **A**, then a solution cannot explode into $+\infty$. Moreover, a solution is *recurrent* in the following sense. If there are no singular points between the starting point x_0 and a point $a < x_0$, then the solution reaches the level a a.s. An example of a SDE, for which $+\infty$ has type A, is provided by the equation

$$dX_t = dB_t, \quad X_0 = x_0.$$

If $+\infty$ has type **B**, then there is no explosion into $+\infty$ and a solution tends to $+\infty$ with strictly positive probability. In other words, a solution is *transient*. For the SDE



Type	Behaviour
A	recurrent
B	transient
C	explosion

$$\rho(x) = \exp\left(-\int_a^x \frac{2b(y)}{\sigma^2(y)}dy\right),$$

$$s(x) = -\int_x^\infty \rho(y)dy$$

Fig. 4.1. Classification at infinity

$$dX_t = \mu dt + \sigma dB_t, \quad X_0 = x_0$$

with $\mu > 0$, $+\infty$ has type B (this follows from Theorem 5.5).

If $+\infty$ has type **C**, then a solution explodes into $+\infty$ (i.e., it reaches $+\infty$ within a finite time) with strictly positive probability. A corresponding example is provided by the equation

$$dX_t = \varepsilon|X_t|^{1+\varepsilon}dt + dB_t, \quad X_0 = x_0$$

with $\varepsilon > 0$ (this follows from Theorem 5.5).

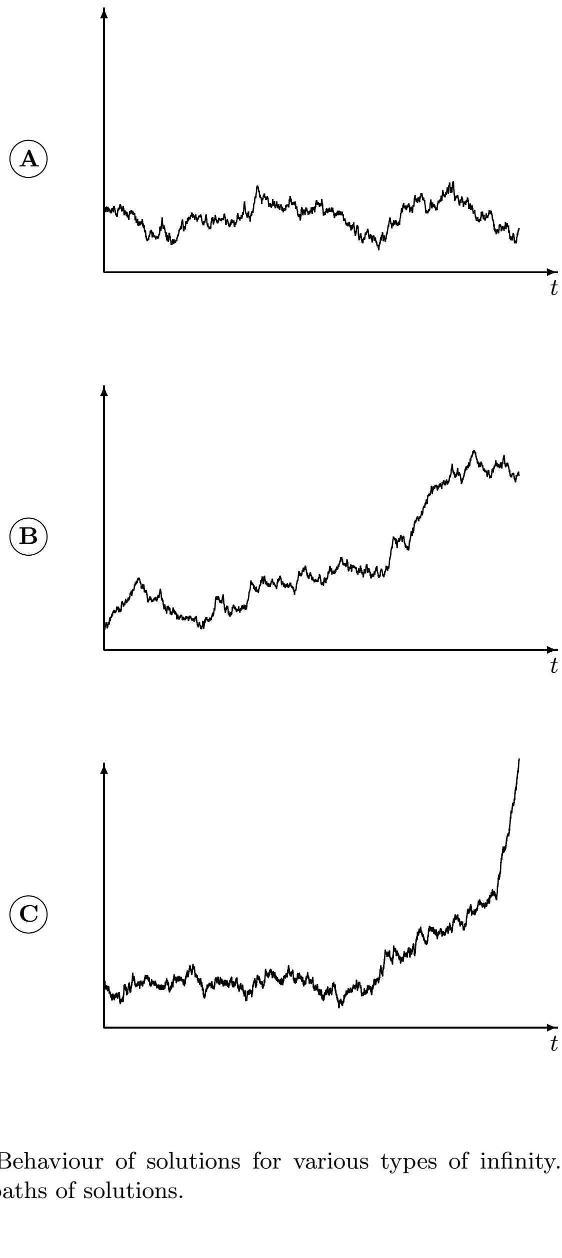


Fig. 4.2. Behaviour of solutions for various types of infinity. The graphs show simulated paths of solutions.

4.3 Classification at Infinity: The Proofs

Proof of Theorem 4.1. *Existence.* Consider the function

$$r(x) = \int_a^x \rho(y)dy, \quad x \in [a, \infty).$$

Let B be a $(\mathcal{G}_t)$-Brownian motion started at $r(x_0)$ on a filtered probability space $\big(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathsf{Q}\big)$. Let us consider

$$\begin{aligned}
\varkappa(y) &= \rho(r^{-1}(y))\sigma(r^{-1}(y)), \quad y \in [0, \infty),\\
A_t &= \begin{cases} \int_0^t \varkappa^{-2}(B_s)ds & \text{if } t < T_0(B),\\ \infty & \text{if } t \ge T_0(B),\end{cases}\\
\tau_t &= \inf\{s \ge 0 : A_s > t\},\\
Y_t &= B_{\tau_t}, \quad t \ge 0.
\end{aligned}$$

Arguing in the same way as in the proof of Theorem 2.11, we check that $A_{T_0(B)-} = T_0(Y) < \infty$ Q-a.s. Set $Z = s^{-1}(Y)$. The estimates used in (2.23) show that, for any $c > x_0$,

$$\mathsf{E}_{\mathsf{Q}} \int_0^{T_{a,c}(Z)} \big(1 + |b(Z_t)| + \sigma^2(Z_t)\big)dt < \infty. \tag{4.4}$$

Furthermore, $T_a(Z) = T_0(Y) < \infty$ Q-a.s. Letting $c \to +\infty$ in (4.4), we get

$$\int_0^{T_a(Z)} \big(1 + |b(Z_t)| + \sigma^2(Z_t)\big)dt < \infty \quad \mathsf{Q}\text{-a.s.} \tag{4.5}$$

The proof of existence is now completed in the same way as in Theorem 2.11.

Uniqueness. Uniqueness follows from Lemma B.6 applied to the stopping times $T_{a,n}$.

The property $T_a < \infty$ P-a.s. is a consequence of (4.5). □

Proof of Theorem 4.2. *Existence.* Let B be a $(\mathcal{G}_t)$-Brownian motion started at $s(x_0)$ on a filtered probability space $\big(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathsf{Q}\big)$. Let us consider

$$\begin{aligned}
\varkappa(y) &= \rho(s^{-1}(y))\sigma(s^{-1}(y)), \quad y \in [\alpha, 0),\\
A_t &= \begin{cases} \int_0^t \varkappa^{-2}(B_s)ds & \text{if } t < T_{\alpha,0}(B),\\ \infty & \text{if } t \ge T_{\alpha,0}(B),\end{cases}\\
\tau_t &= \inf\{s \ge 0 : A_s > t\},\\
Y_t &= B_{\tau_t}, \quad t \ge 0,
\end{aligned}$$

where $\alpha = s(a)$. With the same arguments as in the proof of Theorem 2.11 we check that $A_{T_{\alpha,0}(B)-} = T_{\alpha,0}(Y)$ Q-a.s. Furthermore, for any $\varepsilon > 0$,

$$\int_{-\varepsilon}^{0} \frac{|y|}{\varkappa^2(y)} dy = \int_{s^{-1}(-\varepsilon)}^{\infty} \frac{|s(x)|}{\rho(x)\sigma^2(x)} dx = \infty.$$

By Corollary A.24, $A_{T_{\alpha,0}(B)-}$ is Q-a.s. infinite on the set $\{T_0(B) < T_\alpha(B)\}$. Hence, $T_0(Y) = \infty$ Q-a.s. Thus, the process $Z = s^{-1}(Y)$ is correctly defined. The arguments used in (2.23) show that, for any $c > x_0$,

$$\mathsf{E}_{\mathsf{Q}} \int_0^{T_{a,c}(Z)} \big(1 + |b(Z_t)| + \sigma^2(Z_t)\big) dt < \infty.$$

By letting $c \to +\infty$, we get, for any $t \geq 0$,

$$\int_0^{t \wedge T_a(Z)} \big(1 + |b(Z_t)| + \sigma^2(Z_t)\big) dt < \infty \quad \mathsf{Q}\text{-a.s.}$$

The proof of existence is now completed in the same way as in Theorem 2.11.

Uniqueness. The uniqueness of a solution follows from Lemma B.6 applied to the stopping times $T_{a,n}$.

The properties $\mathsf{P}\{T_a = \infty\} > 0$ and $\lim_{t\to\infty} X_t = +\infty$ P-a.s. on $\{T_a = \infty\}$ follow from the properties that $\mathsf{Q}\{T_0(B) < T_\alpha(B)\} > 0$, and on the set $\{T_0(B) < T_\alpha(B)\}$ we have $Y_t \xrightarrow[t\to\infty]{\mathsf{Q}\text{-a.s.}} 0$. □

Proof of Theorem 4.3. The proof is similar to the proof of Theorem 2.14. □

4.4 Global Solutions: The Results

Throughout this section, we consider global solutions, i.e., solutions in the sense of Definition 1.28.

Theorem 4.5. *Suppose that SDE* (1) *has no singular points, i.e.,*

$$\frac{1 + |b|}{\sigma^2} \in L^1_{\text{loc}}(\mathbb{R}).$$

(i) *If $-\infty$ and $+\infty$ have types A or B, then there exists a unique solution* P*. For any point $a \in \mathbb{R}$, we have* $\mathsf{P}\{T_a < \infty\} > 0$.

(ii) *If $-\infty$ or $+\infty$ has type C, then there exists no solution.*

Theorem 4.6. *Suppose that zero is the unique singular point for* (1)*. Let $x_0 > 0$.*

(i) *If $+\infty$ has type C, then there exists no solution.*

(ii) *If zero has type (i,j) with $i = 0,1,4,5,6$, $j = 0,1$ (we exclude the case $i = j = 0$), then there exists no solution.*

(iii) *If zero has type (i,j) with $i = 2,3$, $j = 0,1$, $-\infty$ has type A or B, and $+\infty$ has type A or B, then there exists a unique solution P. We have $\mathsf{P}\{T_0 < \infty\} > 0$ and $X \le 0$ on $[\![T_0,\infty[\![$ P-a.s.*

(iv) *If zero has type (i,j) with $i = 2,3$, $j = 0,1$ and $-\infty$ has type C, then there exists no solution.*

(v) *If zero has type (i,j) with $i = 0,1,4,5,6$, $j = 2$ and $+\infty$ has type A or B, then there exists a unique solution P. It is positive and $\mathsf{P}\{T_0 < \infty\} > 0$.*

(vi) *If zero has type (i,j) with $i = 2,3$, $j = 2$, $-\infty$ has type A or B, and $+\infty$ has type A or B, then there exist different solutions.*

(vii) *If zero has type (i,j) with $i = 2,3$, $j = 2$, $-\infty$ has type C, and $+\infty$ has type A or B, then there exists a unique solution P. It is positive and $\mathsf{P}\{T_0 < \infty\} > 0$.*

(viii) *If zero has type (i,j) with $j = 3,4,5$ and $+\infty$ has type A or B, then there exists a unique solution. It is strictly positive.*

(ix) *If zero has type (i,j) with $j = 6$, then there exists no solution.*

Theorem 4.7. *Suppose that zero is the unique singular point for (1). Let $x_0 = 0$.*

(i) *If zero has type (i,j) with $i = 0,1,4,5,6$, $j = 0,1,4,5,6$ (we exclude the case $i = j = 0$), then there exists no solution.*

(ii) *If zero has type (i,j) with $i = 0,1,4,5,6$, $j = 2,3$ and $+\infty$ has type A or B, then there exists a unique solution. It is positive.*

(iii) *If zero has type (i,j) with $i = 0,1,4,5,6$, $j = 2,3$ and $+\infty$ has type C, then there exists no solution.*

(iv) *If zero has type (i,j) with $i = 2,3$, $j = 0,1,4,5,6$ and $-\infty$ has type A or B, then there exists a unique solution. It is negative.*

(v) *If zero has type (i,j) with $i = 2,3$, $j = 0,1,4,5,6$ and $-\infty$ has type C, then there exists no solution.*

(vi) *If zero has type (i,j) with $i = 2,3$, $j = 2,3$, $-\infty$ has type A or B, and $+\infty$ has type A or B, then there exist different solutions.*

(vii) *If zero has type (i,j) with $i = 2,3$, $j = 2,3$, $-\infty$ has type A or B, and $+\infty$ has type C, then there exists a unique solution. It is negative.*

(viii) *If zero has type (i,j) with $i = 2,3$, $j = 2,3$, $-\infty$ has type C, and $+\infty$ has type A or B, then there exists a unique solution. It is positive.*

(ix) *If zero has type (i,j) with $i = 2,3$, $j = 2,3$, $-\infty$ has type C, and $+\infty$ has type C, then there exists no solution.*

Remark. Theorems 4.6, 4.7 reveal an interesting effect. It may happen that a branch point does *not* disturb the uniqueness of a global solution. (As we have seen in Chapter 3, a branch point always disturbs the uniqueness of a local solution.) The explanation of this effect is as follows. Suppose, for example, that zero is a branch point, $-\infty$ has type C, $+\infty$ has type A or B, and $x_0 \ge 0$. If a solution becomes strictly negative with strictly positive

Table 4.1. Existence and uniqueness in the case with no singular points. As an example, line 2 corresponds to the situation, where $-\infty$ has type C and there are no restrictions on the type of $+\infty$. The table shows that in this case there exists no solution because there is an explosion into $-\infty$.

Type of $-\infty$	**Type of $+\infty$**	**Existence**	**Uniqueness**	**Comments**
A B	A B	+	+	solution can reach any point
C		−	+	explosion into $-\infty$
	C	−	+	explosion into $+\infty$

probability, then it explodes into $-\infty$ with strictly positive probability, and hence, it is not a global solution. Thus, any global solution should be positive. But there exists a unique positive solution.

4.5 Global Solutions: The Proofs

Proof of Theorem 4.5. **(i)** This statement is proved similarly to Theorems 4.1, 4.2.

(ii) Without loss of generality, we may assume that $+\infty$ has type C. Suppose that there exists a solution P. Fix $a < x_0$. Let Q be the solution defined up to $\overline{T}_{a,\infty}-$ (it is provided by Theorem 4.3). Then $\mathsf{Q}\{\overline{T}_\infty < \infty\} > 0$. Hence, there exist $t > 0$ and $c > a$ such that $\mathsf{Q}\{\overline{T}_\infty < t \wedge T_c\} = \theta > 0$. Then, for any $n > c$, we have $\mathsf{Q}\{T_n < t \wedge T_c\} \geq \theta$. The set $\{T_n < t \wedge T_c\}$ belongs to $\mathcal{F}_{T_{c,n}}$, and $\mathsf{Q}|\mathcal{F}_{T_{c,n}}$ is a solution up to $T_{c,n}$. It follows from the uniqueness of a solution that, for any $n > c$, $\mathsf{P}\{T_n < t \wedge T_c\} \geq \theta$. But this is a contradiction. □

Proof of Theorem 4.6. **(i)** The proof is similar to the proof of Theorem 4.5 (ii).

(ii) Suppose that there exists a solution P. Fix $a > x_0$. Then $\mathsf{P}|\mathcal{F}_{T_{0,a}}$ is a solution up to $T_{0,a}$. It follows from the results of Section 2.3 that $\mathsf{P}\{T_{0,a} < \infty \text{ and } X_{T_{0,a}} = 0\} > 0$. Hence, $\mathsf{P}\{T_0 < \infty\} > 0$. But this contradicts Theorem 3.2.

(iii) *Existence.* The results of Section 2.3 ensure that there exists a solution R_0 with $X_0 = 0$ defined up to T_{-1}. Employing similar arguments as in the proofs of Theorems 2.12 and 2.16 (ii), we construct a solution R_{-1} with $X_0 = -1$ defined up to ∞. Let R'_{-1} be the image of R_{-1} under the map $\omega \mapsto \omega + 1$. We consider R'_{-1} as a measure on $C_0(\mathbb{R}_+)$. Set $\widetilde{\mathsf{R}}_0 = \mathsf{R}_0 \circ \Phi_{T_{-1}}^{-1}$ (Φ is defined by (B.1)). Let Q_0 be the image of $\widetilde{\mathsf{R}}_0 \times \mathsf{R}'_{-1}$ under the map

Table 4.2. Existence and uniqueness in the case, where zero is the unique singular point. The starting point is greater than zero.

Left type of zero	**Right type of zero**	**Type of $-\infty$**	**Type of $+\infty$**	**Existence**	**Uniqueness**	**Comments**
			C	−	+	explosion into $+\infty$
0 1 4 5 6	0 1			−	+	killing at zero
2 3	0 1	A B	A B	+	+	passing through zero
2 3	0 1	C		−	+	explosion into $-\infty$
0 1 4 5 6	2		A B	+	+	reflection at zero
2 3	2	A B	A B	+	−	branching at zero
2 3	2	C	A B	+	+	reflection at zero
	3 4 5		A B	+	+	solution is strictly positive
	6			−	+	killing at zero

Table 4.3. Existence and uniqueness in the case, where zero is the unique singular point. The starting point is equal to zero.

Left type of zero	**Right type of zero**	**Type of $-\infty$**	**Type of $+\infty$**	**Existence**	**Uniqueness**	**Comments**
0 1 4 5 6	0 1 4 5 6			−	+	killing at zero
0 1 4 5 6	2 3		A B	+	+	solution is positive
0 1 4 5 6	2 3		C	−	+	explosion into $+\infty$
2 3	0 1 4 5 6	A B		+	+	solution is negative
2 3	0 1 4 5 6	C		−	+	explosion into $-\infty$
2 3	2 3	A B	A B	+	−	branching at zero
2 3	2 3	A B	C	+	+	solution is negative
2 3	2 3	C	A B	+	+	solution is positive
2 3	2 3	C	C	−	+	explosion into $-\infty$ or $+\infty$

$$C(\mathbb{R}_+) \times C_0(\mathbb{R}_+) \ni (\omega_1, \omega_2) \longmapsto G(\omega_1, \omega_2, T_{-1}(\omega_1)) \in C(\mathbb{R}_+).$$

Using Lemma B.9, one can verify that Q_0 is a solution of (1) with $X_0 = 0$.

Arguing in the same way as in the proofs of Theorems 4.1, 4.2, we deduce that there exists a solution Q with $X_0 = x_0$ defined up to T_0. For this solution, $\mathsf{Q}\{T_0 < \infty\} > 0$. Set $\widetilde{\mathsf{Q}} = \mathsf{Q} \circ \Phi_{T_0}^{-1}$ (Φ is defined by (B.1)). Let P be the image of $\widetilde{\mathsf{Q}} \times \mathsf{Q}_0$ under the map

$$C(\mathbb{R}_+) \times C_0(\mathbb{R}_+) \ni (\omega_1, \omega_2) \longmapsto G(\omega_1, \omega_2, T_0(\omega_1)) \in C(\mathbb{R}_+).$$

Using Lemma B.9, one can verify that P is a solution of (1).

Uniqueness. The uniqueness of a solution follows from Theorem 3.3 (ii) and Lemma B.6 applied to the stopping times $T_{-n,n}$.

The properties $\mathsf{P}\{T_0 < \infty\} > 0$ and $X \le 0$ on $[\![T_0, \infty[\![$ P-a.s. follow from the construction of the solution.

(iv) Suppose that there exists a solution P. For any $a > x_0$, $\mathsf{P}|\mathcal{F}_{T_{0,a}}$ is a solution up to $T_{0,a}$. Applying the results of Section 2.3, we conclude that $\mathsf{P}\{T_0 < \infty\} > 0$. Set $\mathsf{P}' = \mathsf{P}(\,\cdot \mid T_0 < \infty)$, $\mathsf{P}_0 = \mathsf{P}' \circ \Theta_{T_0}^{-1}$, where Θ is defined by (B.2). By Lemma B.7, P_0 is a solution of (1) with $X_0 = 0$. Thus, $\mathsf{P}_0|\mathcal{F}_{T_{-1,1}}$ is a solution with $X_0 = 0$ up to $T_{-1,1}$. Applying Theorem 3.3 (i), we deduce that $X \le 0$ on $[\![0, T_{-1}]\!]$ P_0-a.s. Moreover, $\mathsf{P}_0\{\forall t \ge T_0,\ X_t = 0\} = 0$ (see the proof of Theorem 3.2). Therefore, there exists $c < 0$ such that $\mathsf{P}_0\{T_c < \infty\} > c$. Consider $\mathsf{P}_0' = \mathsf{P}_0(\,\cdot \mid T_c < \infty)$, $\mathsf{P}_c' = \mathsf{P}_0' \circ \Theta_{T_c}^{-1}$. By Lemma B.7, P_c' is a solution of (1) with $X_0 = c$. But this contradicts point (i) of this theorem.

(v) *Existence.* Using similar arguments as in the proof of Theorem 2.12, we conclude that there exists a positive solution P.

Uniqueness. Suppose that there exists another solution P'. Then, for any $n > x_0$, $\mathsf{P}'|\mathcal{F}_{T_n}$ is a solution up to T_n. It follows from the results of Section 2.3 that $\mathsf{P}'|\mathcal{F}_{T_n}$ is positive. Due to Theorem 2.12, $\mathsf{P}'|\mathcal{F}_{T_n} = \mathsf{P}|\mathcal{F}_{T_n}$. Lemma B.6 yields that $\mathsf{P}' = \mathsf{P}$.

The property $\mathsf{P}\{T_0 < \infty\} > 0$ follows from Theorem 2.12.

(vi) Similar arguments as in the proof of Theorem 2.12 allow us to deduce that there exists a positive solution P.

Arguing in the same way as in the proof of point (iii) above, we construct a solution P' such that $\mathsf{P}'\{T_0 < \infty\} > 0$ and $X \le 0$ on $[\![T_0, \infty[\![$ P'-a.s. Moreover, $\mathsf{P}'\{\forall t \ge T_0,\, X_t = 0\} = 0$ (see the proof of Theorem 3.2). Hence, P' is not positive, and therefore, P and P' are two different solutions.

(vii) *Existence.* Using similar arguments as in the proof of Theorem 2.12, we deduce that there exists a positive solution P.

Uniqueness. Suppose that there exists another solution P'. Assume first that it is not positive. Then there exists $c < 0$ such that $\mathsf{P}\{T_c < \infty\} > 0$. Set $\mathsf{P}' = \mathsf{P}(\,\cdot \mid T_c < \infty)$, $\mathsf{P}_c = \mathsf{P}' \circ \Theta_{T_c}^{-1}$. By Lemma B.7, P_c is a solution of (1) with $X_0 = c$. But this contradicts point (i) of this theorem.

Assume now that P' is positive. By Theorem 2.12, for any $n > x_0$, $\mathsf{P}'|\mathcal{F}_{T_n} = \mathsf{P}|\mathcal{F}_{T_n}$. Lemma B.6 yields that $\mathsf{P}' = \mathsf{P}$.

The property $\mathsf{P}\{T_0 < \infty\} > 0$ follows from Theorem 2.12.

(viii) *Existence.* Using the same arguments as in the proofs of Theorems 2.15–2.17, we deduce that there exists a strictly positive solution P.

Uniqueness. Suppose that there exists another solution P'. It follows from the results of Section 2.3 that, for any $n > x_0$, $\mathsf{P}'|\mathcal{F}_{T_n} = \mathsf{P}|\mathcal{F}_{T_n}$. By Lemma B.6, $\mathsf{P}' = \mathsf{P}$.

(ix) This statement follows from Theorem 2.14. □

Proof of Theorem 4.7. **(i)** This statement follows from Theorem 3.2.

(ii) This statement is proved in the same way as Theorem 4.6 (v).

(iii) Suppose that there exists a solution P. It follows from the results of Section 2.3 that P is positive. Moreover, $\mathsf{P}\{\forall t \geq 0,\, X_t = 0\} = 0$ (see the proof of Theorem 3.2). Hence, there exists $a > 0$ such that $\mathsf{P}\{T_a < \infty\} > 0$. Set $\mathsf{P}' = \mathsf{P}(\,\cdot \mid T_a < \infty)$, $\mathsf{P}_a = \mathsf{P}' \circ \Theta_{T_a}^{-1}$. By Lemma B.7, P_a is a solution of (1) with $X_0 = a$. But this contradicts Theorem 4.6 (i).

(vi) Using similar arguments as in Section 2.5, one can construct both a positive solution and a negative solution.

(vii) This statement is proved in the same way as Theorem 4.6 (vii).

(ix) The proof of this statement is similar to the proof of point (iii). □