Introduction

The basis of the theory of diffusion processes was formed by Kolmogorov [30] (the Chapman–Kolmogorov equation, forward and backward partial differential equations). This theory was further developed in a series of papers by Feller (see, for example, [16], [17]).

Both Kolmogorov and Feller considered diffusion processes from the point of view of their finite-dimensional distributions. Itô [24], [25] proposed an approach to the "pathwise" construction of diffusion processes. He introduced the notion of a stochastic differential equation (abbreviated below as *SDE*). At about the same time and independently of Itô, SDEs were considered by Gikhman [18], [19]. Stroock and Varadhan [44], [45] introduced the notion of a martingale problem that is closely connected with the notion of a SDE.

Many investigations were devoted to the problems of existence, uniqueness, and properties of solutions of SDEs. Sufficient conditions for existence and uniqueness were obtained by Girsanov [21], Itô [25], Krylov [31], [32], Skorokhod [42], Stroock and Varadhan [44], Zvonkin [49], and others. The evolution of the theory has shown that it is reasonable to introduce different types of solutions (weak and strong solutions) and different types of uniqueness (uniqueness in law and pathwise uniqueness); see Liptser and Shiryaev [33], Yamada and Watanabe [48], Zvonkin and Krylov [50]. More information on SDEs and their applications can be found in the books [20], [23], [28, Ch. 18], [29, Ch. 5], [33, Ch. IV], [36], [38, Ch. IX], [39, Ch. V], [45].

For one-dimensional homogeneous SDEs, i.e., the SDEs of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0, \tag{1}$$

one of the weakest sufficient conditions for weak existence and uniqueness in law was obtained by Engelbert and Schmidt [12]–[15]. (In the case, where $b = 0$, there exist even necessary and sufficient conditions; see the paper [12] by Engelbert and Schmidt and the paper [1] by Assing and Senf.) Engelbert and Schmidt proved that if $\sigma(x) \neq 0$ for any $x \in \mathbb{R}$ and

$$\frac{1 + |b|}{\sigma^2} \in L^1_{\text{loc}}(\mathbb{R}), \tag{2}$$

then there exists a unique solution of (1). (More precisely, there exists a unique solution defined up to the time of explosion.)

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Condition (2) is rather weak. Nevertheless, SDEs that do not satisfy this condition often arise in theory and in practice. Such are, for instance, the SDE for a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad X_0 = x_0$$

(the Black-Scholes model!) and the SDE for a δ-dimensional Bessel process ($\delta > 1$):

$$dX_t = \frac{\delta - 1}{2X_t} dt + dB_t, \quad X_0 = x_0.$$

In practice, SDEs that do not satisfy (2) arise, for example, in the following situation. Suppose that we model some process as a solution of (1). Assume that this process is positive by its nature (for instance, this is the price of a stock or the size of a population). Then a SDE used to model such a process should *not* satisfy condition (2). The reason is as follows. If condition (2) is satisfied, then, for any $a \in \mathbb{R}$, the solution reaches the level a with strictly positive probability. (This follows from the results of Engelbert and Schmidt.)

The SDEs that do not satisfy condition (2) are called in this monograph *singular SDEs*. The study of these equations is the subject of the monograph. We investigate three main problems:

(i) *Does there exist a solution of* (1)*?*

(ii) *Is it unique?*

(iii) *What is the qualitative behaviour of a solution?*

In order to investigate singular SDEs, we introduce the following definition. A point $d \in \mathbb{R}$ is called a *singular point* for SDE (1) if

$$\frac{1 + |b|}{\sigma^2} \notin L^1_{\text{loc}}(d).$$

We always assume that $\sigma(x) \neq 0$ for any $x \in \mathbb{R}$. This is motivated by the desire to exclude solutions which have sojourn time in any single point. (Indeed, it is easy to verify that if $\sigma \neq 0$ at a point $z \in \mathbb{R}$, then any solution of (1) spends no time at z. This, in turn, implies that any solution of (1) also solves the SDE with the same drift and the diffusion coefficient $\sigma - \sigma(z) I_{\{z\}}$. "Conversely", if $\sigma = 0$ at a point $z \in \mathbb{R}$ and a solution of (1) spends no time at z, then, for any $\eta \in \mathbb{R}$, it also solves the SDE with the same drift and the diffusion coefficient $\sigma + \eta I_{\{z\}}$.)

The first question that arises in connection with this definition is: Why are these points indeed "singular"? The answer is given in Section 2.1, where we explain the qualitative difference between the singular points and the regular points in terms of the behaviour of solutions.

Using the above terminology, we can say that a SDE is singular if and only if the set of its singular points is nonempty. It is worth noting that in practice one often comes across SDEs that have only one singular point (usually, it is zero). Thus, the most important subclass of singular points is formed by the *isolated singular points*. (We call $d \in \mathbb{R}$ an isolated singular point if d is singular and there exists a deleted neighbourhood of d that consists of regular points.)

In this monograph, we perform a complete qualitative classification of the isolated singular points. The classification shows whether a solution can leave an isolated singular point, whether it can reach this point, whether it can be extended after having reached this point, and so on. According to this classification, an isolated singular point can have one of 48 possible types. The type of a point is easily computed through the coefficients b and σ. The constructed classification may be viewed as a counterpart (for SDEs) of Feller's classification of boundary behaviour of continuous strong Markov processes.

The monograph is arranged as follows.

Chapter 1 is an overview of basic definitions and facts related to SDEs, more precisely, to the problems of the existence and the uniqueness of solutions. In particular, we describe the relationship between different types of existence and uniqueness (see Figure 1.1 on p. 8) and cite some classical conditions that guarantee existence and uniqueness. This chapter also includes several important examples of SDEs. Moreover, we characterize all the possible combinations of existence and uniqueness (see Table 1.1 on p. 18).

In Chapter 2, we introduce the notion of a singular point and give the arguments why these points are indeed "singular". Then we study the existence, the uniqueness, and the qualitative behaviour of a solution in the right-hand neighbourhood of an isolated singular point. This leads to the one-sided classification of isolated singular points. According to this classification, an isolated singular point can have one of 7 possible *right types* (see Figure 2.2 on p. 39).

In Chapter 3, we investigate the existence, the uniqueness, and the qualitative behaviour of a solution in the two-sided neighbourhood of an isolated singular point. We consider the effects brought by the combination of right and left types. Since there exist 7 possible right types and 7 possible left types, there are 49 feasible combinations. One of these combinations corresponds to a regular point, and therefore, an isolated singular point can have one of 48 possible types. It turns out that the isolated singular points of only 4 types can disturb the uniqueness of a solution. We call them the *branch points* and characterize all the strong Markov solutions in the neighbourhood of such a point.

In Chapter 4, we investigate the behaviour of a solution "in the neighbourhood of $+\infty$". This leads to the classification at infinity. According to this classification, $+\infty$ can have one of 3 possible types (see Figure 4.1 on p. 83). The classification shows, in particular, whether a solution can explode into $+\infty$. Thus, the well known Feller's test for explosions is a consequence of this classification.

All the results of Chapters 2 and 3 apply to local solutions, i.e., *solutions up to a random time* (this concept is introduced in Chapter 1). In the second part of Chapter 4, we use the obtained results to study the existence, the uniqueness, and the qualitative behaviour of global solutions, i.e., solutions in the classical sense. This is done for the SDEs that have no more than one singular point (see Tables 4.1–4.3 on pp. 88, 89).

In Chapter 5, we consider the power equations, i.e., the equations of the form

$$dX_t = \mu|X_t|^\alpha dt + \nu|X_t|^\beta dB_t$$

and propose a simple procedure to determine the type of zero and the type of infinity for these SDEs (see Figure 5.1 on p. 94 and Figure 5.2 on p. 98). Moreover, we study which types of zero and which types of infinity are possible for the SDEs with a constant-sign drift (see Table 5.1 on p. 101 and Table 5.2 on p. 103).

The known results from the stochastic calculus used in the proofs are contained in Appendix A, while the auxiliary lemmas are given in Appendix B.

The monograph includes 7 figures with simulated paths of solutions of singular SDEs.