6

Sampling Theorems for the Heisenberg Group

In this chapter we characterize the sampling spaces of the Heisenberg group by use of the Plancherel transform. We are aware of two previous authors, Dooley and Pesenson, dealing with sampling theorems on nonabelian groups, for two rather different settings [37, 98, 99, 100]. What these two approaches have in common with each other (and with the one developed here) is the fact that they deal with the problem of reconstructing elements of a fixed leftinvariant subspace of $\mathrm{L}^2(G)$ from sampled values.

The difference between the approaches of Dooley and Pesenson initiate from different notions of bandlimitation. Dooley [37] considers groups of the form $G = \mathbb{R}^k \rtimes H$, with H compact. His concept of bandlimitedness is representation-theoretic, that is, defined as a condition on the support of the subspace on the Plancherel transform side. The condition was inspired by the Mackey picture. Recall that by Mackey's theory the dual has the form

$$\widehat{G} = \bigcup_{\mathcal{O}(\gamma) \in \mathbb{R}^k/H} \{\mathcal{O}(\gamma)\} \times \widehat{H_\gamma} \quad .$$

Now, given a leftinvariant subspace $\mathcal{H} \subset \mathrm{L}^2(G)$ with associated projection field $(P_{\mathcal{O}(\gamma),\sigma})_{\mathcal{O}(\gamma),\sigma}$, the space $\mathcal{H}$ is declared bandlimited if there exists a bounded set $B \subset \widehat{\mathbb{R}^k}$ such that $P_{\mathcal{O}(\gamma),\sigma} = 0$ for almost all $\gamma \notin B$.

By contrast, Pesenson [98, 99, 100] studies stratified Lie groups G, and his notion of bandlimitedness is of a geometric rather than a representation-theoretic nature. It involves a (leftinvariant) sub-Laplacian L on G, which is a particular selfadjoint differential operator on G. This time, a leftinvariant space $\mathcal{H} \subset \mathrm{L}^2(G)$ is called bandlimited if the projection onto $\mathcal{H}$ was given by a spectral projection of L corresponding to a bounded interval in $\mathbb{R}$.

In both cases, the sampling theorems state that the elements of the spaces under consideration are uniquely determined by their sampled values. Put differently, the restriction maps were shown to be injective, but no other functional-analytic property of the restriction map was studied. Accordingly,

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the reconstruction procedures, if they were at all given, were not shown to be stable with respect to the L^2-norm.

The approach presented here is rather different in this respect: We start from our notion of sampling space, which is tied to the existence of a sampling expansion with convergence in the norm, and our goal is to characterize the subspaces that admit this type of expansions; i.e., to solve **T5**. We only consider one particular group, the Heisenberg group, but we will be able to give a complete characterization of sampling spaces in terms of conditions on the associated projection fields. These conditions, in turn, can be read as bandlimitation requirements.

It is instructive to compare Dooley's notion of bandlimited subspaces with our notion of sampling spaces. It is easy to check by Theorem 3.39(c) that the semidirect product group $G = \mathbb{R}^k \rtimes H$ is type I if H is compact. Picking any bounded measurable set $B \subset \widehat{\mathbb{R}^k}$, we can consider the twosided invariant subspace $\mathcal{H} \subset \mathrm{L}^2(G)$, supported by

$$\widetilde{B} = \{\mathrm{Ind}_{G_\gamma}^G(\gamma \times \sigma) : \gamma \in B, \sigma \in \widehat{H_\gamma}\} \subset \widehat{G} \quad ,$$

which is bandlimited in the sense of [37]. We will show that if $\mathcal{H}$ is a sampling space in the sense of 2.51, then H is finite.

By Theorem 2.56, $\mathcal{H}$ is generated by an L^2-convolution idempotent. Theorem 4.22 then implies that

$$\int_{\widetilde{B}} \dim(\mathcal{H}_\pi) d\nu_G(\pi) < \infty \quad .$$

Before we continue the computation, observe that the representation space of $\mathrm{Ind}_{G_\gamma}^G(\gamma \times \sigma)$ has Hilbert space dimension $\min(\infty, [G : G_\gamma]) \cdot \dim(\mathcal{H}_\sigma)$. Theorem 3.40 by Kleppner and Lipsman allows to compute

$$\begin{aligned} \infty &> \int_{\widetilde{B}} \dim(\mathcal{H}_\pi) d\nu_G(\pi) \\ &= \int_B \int_{\widehat{H_\gamma}} \min(\infty, [G : G_\gamma]) \cdot \dim(\mathcal{H}_\sigma) d\nu_{H_\gamma}(\sigma) d\overline{\lambda}(\gamma) && (6.1) \\ &\geq \int_B \int_{\widehat{H_\gamma}} \dim(\mathcal{H}_\sigma) d\nu_{H_\gamma}(\sigma) d\overline{\lambda}(\gamma) && (6.2) \end{aligned}$$

The finiteness of (6.1) entails that almost all G_γ have finite index, which means that almost all orbits are finite. In addition, the finiteness of (6.2) requires that for almost all γ the inner integral is finite. Then Corollary 4.25 yields for these γ that H_γ is discrete (and compact of course), hence finite. But now the orbits and the fixed groups are finite for almost all γ, i.e., H is finite. Hence we see that Dooley's concept and ours are identical only for finite extensions of $\mathbb{R}^k$.

6.1 The Heisenberg Group and Its Lattices

Let us quickly recall the main properties of the Heisenberg group, outlined in Section 4.5. As a set $\mathbb{H} = \mathbb{R}^3$, with group law

$$(p,q,t) * (p',q',t') = (p+p', q+q', t+t'+(pq'-qp')/2) \quad .$$

It is a unimodular group, with the usual Lebesgue measure on as Haar measure.

The center of $\mathbb{H}$ is given by $Z(\mathbb{H}) = \{(0,0,t) : t \in \mathbb{R}\}$. We denote the group of topological automorphisms of $\mathbb{H}$ by $\mathrm{Aut}(\mathbb{H})$. Recall the definition of the Schrödinger representations, acting via

$$[\varrho_h(p,q,t)f](x) = \mathrm{e}^{2\pi \mathrm{i} ht} \mathrm{e}^{2\pi \mathrm{i} qx} \mathrm{e}^{\pi \mathrm{i} hpq} f(x+hp) \quad .$$

The Schrödinger representations do not exhaust the dual of $\mathbb{H}$, which in addition contains the characters of the abelian factor group $\mathbb{H}/Z(\mathbb{H})$. However, for the decomposition of the regular representation of $\mathbb{H}$, we may concentrate on the Schrödinger representations. More precisely, the set $(\varrho_h)_{h \in \mathbb{R}'}$ is a $\nu_{\mathbb{H}}$-conull subset of $\widehat{\mathbb{H}}$, and Plancherel measure is given by $|h|dh$, where dh denotes Lebesgue measure [45, Section 7.6].

To close our survey of the Heisenberg group, we cite a result classifying the lattices of $\mathbb{H}$. We associate to such a lattice Γ two numbers $d(\Gamma) \in \mathbb{N}'$, $r(\Gamma) \in \mathbb{R}^+$ which contain sufficient information for our purposes. The two parameters provide a measure of the density of Γ in $\mathbb{H}$. We first single out a particular family of lattices, which turns out to be exhaustive (up to automorphisms of $\mathbb{H}$).

Definition 6.1. *For any positive integer d let Γ_d be the subgroup generated by $(1,0,0), (0,d,0), (0,0,1)$. Γ_d is a lattice, with*

$$\Gamma_d = \left\{ (m, dk, \ell + \frac{1}{2} dmk) : m,k,\ell \in \mathbb{Z} \right\} \quad .$$

It is convenient to introduce the **reduced lattice** *Γ_d^r which is the subset*

$$\Gamma_d^r = \{(m, dk, dmk/2) : m,k \in \mathbb{Z}\} \quad .$$

Note that Γ_d^r is not a lattice, not even a subgroup.

Let us next give a classification of lattices. It has been attributed (in more generality) to Maltsev. Since we were not able to locate the original source, we sketch a short proof for the sake of completeness.

Theorem 6.2. *Let Γ be a lattice of $\mathbb{H}$. Then there exists a strictly positive integer d and $\alpha \in \mathrm{Aut}(\mathbb{H})$ with $\alpha(\Gamma_d) = \Gamma$. The integer d is uniquely determined by these properties.*

Proof. By [30, Theorem 5.1.6], there exist a basis $\widetilde{P}, \widetilde{Q}, \widetilde{Z}$ of $\mathfrak{h}$ with $\widetilde{Z} \in Z(\mathbb{H})$, and $\Gamma = \mathbb{Z}\widetilde{Z} * \mathbb{Z}\widetilde{P} * \mathbb{Z}\widetilde{Q}$. Now $[\widetilde{P}, \widetilde{Q}] = \widetilde{P}\widetilde{Q}\widetilde{P}^{-1}\widetilde{Q}^{-1} \in \Gamma \cap Z(\mathbb{H}) = \mathbb{Z}\widetilde{Z}$ implies $[\widetilde{P}, \widetilde{Q}] = d\widetilde{Z}$ for some $d \in \mathbb{Z}$, w.l.o.g. $d \geq 0$ (otherwise exchange $\widetilde{Q}, \widetilde{P}$). In fact, $d > 0$ since $\mathfrak{h}$ is not abelian. It is immediately checked that the linear isomorphism defined by

$$(1,0,0) \mapsto \widetilde{P} \quad , \quad (0,d,0) \mapsto \widetilde{Q} \quad , \quad (0,0,1) \mapsto \widetilde{Z}$$

is in $\mathrm{Aut}(\mathbb{H})$. That d is unique is due to the fact that each automorphism α mapping Γ_d to $\Gamma_{d'}$ maps Z onto $\pm Z$: Indeed, from Proposition 6.17 (a) below follows that α leaves the Haar measure of $\mathbb{H}$ invariant, and this implies that $\mathrm{covol}(\Gamma_d) = \mathrm{covol}(\Gamma_{d'})$. On the other hand, $\mathrm{covol}(\Gamma_d) = d$, hence $d = d'$.

We denote by $d(\Gamma)$ the unique integer d from the theorem. For the definition of $r(\Gamma)$ we take the unique positive real satisfying $\Gamma \cap Z(\mathbb{H}) = r(\Gamma)\mathbb{Z}Z$.

6.2 Main Results

Now we can state the main results of this chapter. In this section, $\mathcal{H}$ always denotes a closed, leftinvariant subspace of $\mathrm{L}^2(\mathbb{H})$, and $\Gamma < \mathbb{H}$ a lattice. Recall from Theorem 2.56 that we may assume $\mathcal{H} = \mathrm{L}^2(G)*S$, where S is a selfadjoint convolution idempotent, and that $\mathcal{H}$ is a sampling space iff $\lambda_{\mathbb{H}}(\Gamma)S$ is a tight frame of $\mathcal{H}$.

Definition 6.3. *We associate to a leftinvariant subspace $\mathcal{H} \subset \mathrm{L}^2(\mathbb{H})$ with associated projection field $(\widehat{P}_h)_{h\in\mathbb{R}'}$ the multiplicity function $m_{\mathcal{H}} : \mathbb{R}' \to \mathbb{N}_0 \cup \{\infty\}$ of the associated subrepresentation. This function is given by $m_{\mathcal{H}}(h) = \mathrm{rank}(\widehat{P}_h)$. The support of $m_{\mathcal{H}}$ is denoted by $\Sigma(\mathcal{H})$. $\mathcal{H}$ is called* **bandlimited** *is $\Sigma(\mathcal{H})$ is bounded in $\mathbb{R}$.*

Similarly, if a representation π is equivalent to a subrepresentation of $\lambda_{\mathbb{H}}$, say to the restriction of $\lambda_{\mathbb{H}}$ to $\mathcal{H}$, we let $m_\pi = m_{\mathcal{H}}$ and $\Sigma(\pi) = \Sigma(\mathcal{H})$. This is obviously well-defined.

The main theorem characterizes the subspaces admitting tight frames.

Theorem 6.4. *(i) There exists a tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$ with suitable $\Phi \in \mathcal{H}$ iff the multiplicity function m associated to $\mathcal{H}$ satisfies almost everywhere*

$$m(h) \cdot |h| + m\left(h - \frac{1}{r(\Gamma)}\right) \cdot \left|h - \frac{1}{r(\Gamma)}\right| \leq \frac{1}{d(\Gamma)r(\Gamma)} \quad . \tag{6.3}$$

In particular, $\Sigma(\mathcal{H}) \subset \left[-\frac{1}{d(\Gamma)r(\Gamma)}, \frac{1}{d(\Gamma)r(\Gamma)}\right]$ (up to a set of measure zero).
(ii) There does not exist an orthonormal basis of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$ for $\mathcal{H}$.

Remark 6.5. Note that if $d(\Gamma) > 1$ and $m(h) > 0$ and $m\left(h - \frac{1}{r(\Gamma)}\right) > 0$, inequality (6.3) entails the inequality

$$|h| + \left|h - \frac{1}{r(\Gamma)}\right| \leq \frac{1}{2r(\Gamma)}$$

which is impossible to satisfy. Hence in the case $d(\Gamma) > 1$ relation (6.3) simplifies to

$$m(h) \cdot |h| \leq \frac{1}{d(\Gamma)r(\Gamma)} \quad . \tag{6.4}$$

Corollary 6.6. *Assume that $m_{\mathcal{H}}$ is essentially bounded. There exists a tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$, (with a suitable lattice Γ and suitable $\Phi \in \mathcal{H}$) iff $\mathcal{H}$ is bandlimited.*

The following is a rephrasing for discretization of continuous wavelet transforms.

Corollary 6.7. *Let $(\pi, \mathcal{H}_\pi)$ be a representation of $\mathbb{H}$ with admissible vector. There exists a tight frame of the form $\pi(\Gamma)\eta$, (with suitable lattice Γ and suitable $\eta \in \mathcal{H}_\pi$) if $\Sigma(\pi)$ and m_π are bounded.*

That bounded multiplicity cannot be dispensed with in the previous corollary is shown by the next result:

Corollary 6.8. *There exists a bandlimited leftinvariant subspace $\mathcal{H} = \mathrm{L}^2(G) * S$, with a selfadjoint convolution idempotent $S \in \mathrm{L}^2(\mathbb{H})$, admitting no tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$.*

With regard to the existence of sampling subspaces, we have:

Corollary 6.9. *Not every space admitting a tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)S$ is a sampling subspace for Γ. However, for such a space $\mathcal{H}$ there exists $\Phi \in \mathcal{H}$ such that $f \mapsto f * \Phi^*$ is an isometry on $\mathcal{H}$, mapping $\mathcal{H}$ onto a sampling space. There does not exist a sampling space $\mathcal{H}$ with the interpolation property.*

The proofs for these results will be given in Section 6.5 below. The following remarks discuss similarities and differences to the case of the reals.

Remark 6.10.
1. The main similarity lies in the notion of bandwidth, and the fact that it can be interpreted as inversely proportional to the density of the lattice. Note that over $\mathbb{H}$ the bandwidth restriction is much more rigid: The set $\Sigma(\mathcal{H})$ is contained in a fixed interval, whereas the analog of that set in the real case can be shifted arbitrarily and still give a sampling subspace; compare Theorem 2.71.
2. Corollaries 6.8 and 6.9 provide an interesting contrast between the sampling theories of $\mathbb{H}$ and $\mathbb{R}$. None of the counterexamples given in the corollaries has an analog in the real setting. In particular, in the Heisenberg group case the question whether a given space is a sampling space

is much more subtle than deciding whether it has a frame. For the first problem, a close inspection of the projection operator field $(\widehat{P}_h)_{h\in\mathbb{R}'}$ is necessary (using the criteria in Proposition 6.11 below), for the second, only the ranks of these operators are needed. By contrast, 2.71 (a) $\Leftrightarrow$ (a') shows that for the reals the two properties are equivalent. This is not so surprising after all: Projection fields on $\int_{\widehat{\mathbb{R}}}^{\oplus} \mathbb{C} d\omega$ are obviously uniquely determined by their supports.

3. While Theorem 6.4 shows that the Plancherel transform can be used to characterize sampling spaces and frames, it is not clear how it can be generalized to a larger class of locally compact groups. Indeed, as far as we are aware, among the entities entering the central relation (6.3), only the multiplicity function m has an abstract interpretation. Also, the techniques proving Theorem 6.4 are rather specific to the Heisenberg group, combining known results concerning Weyl-Heisenberg with the Plancherel transform of $\mathbb{H}$ (see the arguments in the next section), which is a further illustration how the
4. We use lattices as sampling sets simply because they are easily accessible. In particular, we do not exploit the representation theory of the lattices at all. Recall that the tight frame condition is nothing but an admissibility condition for the restriction of $\lambda_{\mathbb{H}}$. However, the lattices in $\mathbb{H}$ are not type I, hence a better understanding of the non-type I setting will be needed.

6.3 Reduction to Weyl-Heisenberg Systems*

In this section we start the discussion of normalized tight frames for left-invariant subspaces. On the Plancherel transform side, the space $\mathcal{H}$ under consideration decomposes into a direct integral. In this section, we reduce the complexity of the problem in two ways: We get rid of the direct integral on the one hand, and the central variable of the lattice on the other, and are faced with the problem of constructing certain normalized tight frames in the fibres, arising from the action of the reduced lattice. The latter problem is equivalent to the construction of Weyl-Heisenberg (super-)frames.

Proposition 6.11. *Let $\Gamma = \Gamma_d$. Let $\Phi \in \mathcal{H}$ be such that $\lambda_G(\Gamma)\Phi$ is a normalized tight frame of $\mathcal{H}$. Then, for almost every $h \in \Sigma(\mathcal{H})$, the reduced lattice satisfies the following condition:*

$$\left(|h|^{1/2} \varrho_h(\gamma) \widehat{\Phi}(h) \right)_{\gamma \in \Gamma_d^r} \text{ is a normalized tight frame of } \mathcal{B}_2(\mathrm{L}^2(\mathbb{R})) \circ \widehat{P}_h \ . \quad (6.5)$$

Conversely, if both (6.5) (for almost every h) and the support condition

$$\forall m \in \mathbb{Z} \setminus \{0\} \quad : \quad \Sigma(\mathcal{H}) \cap m + \Sigma(\mathcal{H}) \text{ has measure zero} \qquad (6.6)$$

hold, then $\lambda_G(\Gamma)\Phi$ is a normalized tight frame of $\mathcal{H}$.

Proof. Assume first that $|\Sigma(f) \cap m + \Sigma(f)| = 0$. We calculate

$$\begin{aligned}\sum_{\gamma \in \Gamma} |\langle f, \lambda_{\mathbb{H}}(\gamma)\Phi\rangle|^2 &= \sum_{\gamma \in \Gamma} \left| \int_{\Sigma(f)} \langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h)\rangle |h| dh \right|^2 \\ &= \sum_{\gamma \in \Gamma_d^r} \sum_{\ell \in \mathbb{Z}} \left| \int_{\Sigma(f)} \mathrm{e}^{-2\pi \mathrm{i} h \ell} \langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h)\rangle |h| dh \right|^2 \\ &= \sum_{\gamma \in \Gamma_d^r} \int_{\Sigma(f)} \left| \langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h)\rangle \right|^2 |h|^2 dh \\ &= \int_{\Sigma(f)} \sum_{\gamma \in \Gamma_d^r} \left| \langle \widehat{f}(h), \varrho_h(\gamma)|h|^{1/2}\widehat{\Phi}(h)\rangle \right|^2 |h| dh \quad .\end{aligned}$$

Here we used the assumption on $\Sigma(f)$ to apply the Plancherel Theorem for Fourier series on $\Sigma(f)$ and thereby discard the summation over ℓ. On the other hand, the tight frame condition together with the Plancherel formula for $\mathbb{H}$ implies that

$$\sum_{\gamma \in \Gamma} |\langle f, \lambda(\gamma)s\rangle|^2 = \int_{\Sigma(f)} \|\widehat{f}(h)\|^2 |h| dh \quad ,$$

and thus

$$\int_{\Sigma(f)} \sum_{\gamma \in \Gamma_d^r} \left| \langle \widehat{f}(h), \varrho_h(\gamma)|h|^{1/2}\widehat{\Phi}(h)\rangle \right|^2 |h| dh = \int_{\Sigma(f)} \|\widehat{f}(h)\|^2 |h| dh \quad . \tag{6.7}$$

Replacing f by g with $\widehat{g}(h) = \mathbf{1}_B(h)\widehat{f}(h)$, we see that we may replace $\Sigma(f)$ in (6.7) by any Borel subset B. Hence the integrands must be equal almost everywhere:

$$\sum_{\gamma \in \Gamma_d^r} \left| \langle \widehat{f}(h), \varrho_h(\gamma)|h|^{1/2}\widehat{\Phi}(h)\rangle \right|^2 = \|\widehat{f}(h)\|^2 \quad . \tag{6.8}$$

By covering $\Sigma(\mathcal{H})$ with sets of the form $\Sigma(f)$ fulfilling the initial support condition we see that (6.8) holds for every $f \in \mathcal{H}$ and almost every $h \in \mathbb{R}'$. However, it remains to show that the relation holds for all h in a common conull subset, independent of f. For this purpose we pick a countable dense $\mathbb{Q}$-subspace $\mathcal{A} \subset \mathrm{L}^2(\mathbb{H})$. Then there exists a conull subset $C \subset \mathbb{R}'$ such that, for all $h \in C$, $\{\widehat{f}(h) : f \in \mathcal{A}\}$ is dense in $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R})) \circ \widehat{P}_h$, and in addition (6.8) holds for all $f \in \mathcal{A}$ and $h \in C$. Now, for every $h \in C$, the coefficient map

$$\widehat{g}(h) \mapsto \Big(\langle \widehat{g}(h), \varrho_h(\gamma)|h|^{1/2}\widehat{\Phi}(h)\rangle \Big)_{\gamma \in \Gamma_d^r}$$

is a closed linear operator, by Proposition 2.53 (d), coinciding with an isometry on a dense subset, hence it is an isometry.

Finally, we note that the argument can be reversed to prove the sufficiency of condition (6.5) under the additional assumption (6.6).

6.4 Weyl-Heisenberg Frames*

For any $g \in \mathrm{L}^2(\mathbb{R})$, the operation of the reduced lattice Γ_d^r on g via ϱ_h gives the system

$$(\varrho_h(m, dk, dmk/2)g)(x) = \mathrm{e}^{\pi \mathrm{i} hmdk}\mathrm{e}^{2\pi \mathrm{i} kx} g(x+hm) \quad ,$$

hence $\varrho_h(\Gamma_d^r)g$ and the Weyl-Heisenberg system $\mathcal{G}(d,h,g)$, as defined in Section 5.5, only differ up to phase factors. Clearly these phase factors do not influence any normalized tight frame or ONB properties of the system, hence we may and will switch freely between the Weyl-Heisenberg system and the orbit of the reduced lattice.

We cite the following existence result:

Theorem 6.12. *A normalized tight Weyl-Heisenberg frame $\mathcal{G}(d,h,g)$ of $\mathrm{L}^2(\mathbb{R})$ exists iff $|h|d \le 1$. For any such frame we have $\|g\|_2^2 = |h|d$.*

Proof. The "only-if"-part is [58, Corollary 7.5.1]. The "if"-part follows from [58, Theorem 6.4.1], applied to a suitably chosen characteristic function. The norm equality is due to [58, Corollary 7.3.2].

In dealing with subspaces of Hilbert-Schmidt spaces, we have to consider a more general setting: We will be interested in normalized tight frames of $\left(\mathrm{L}^2(\mathbb{R})\right)^r$ consisting of vectors of the type

$$g_{k,m} = (\mathrm{e}^{2\pi \mathrm{i} dkx} g^j(x+h_j m))_{j=1,\dots,r} = (g^j_{k,m})_{j=1,\dots,r} \tag{6.9}$$

where $g = (g^j)_{j=1,\dots,r} \in \left(\mathrm{L}^2(\mathbb{R})\right)^r$ is suitably chosen, and $\mathbf{h} = (h_j)_j \in \mathbb{R}^r$ is a vector of nonzero real numbers. This problem has already been considered by other authors, see [21] and the references therein. The following two lemmata extend the results on $\mathrm{L}^2(\mathbb{R})$ to the more general situation. The first one is quite obvious and does not reflect the special structure of Weyl-Heisenberg frames. We already used a similar argument for the proof of 2.23. A similar result for arbitrary frames is given in [21].

Lemma 6.13. *Let $\mathbf{h} = (h_1,\dots,h_r) \in \mathbb{R}^r$ and $g = (g^j)_{j=1,\dots,r} \in \left(\mathrm{L}^2(\mathbb{R})\right)^r$. Then $(g_{k,m})_{k,m\in\mathbb{Z}}$, defined as in equation (6.9), is a normalized tight frame of $\left(\mathrm{L}^2(\mathbb{R})\right)^r$ iff*

(i) for $j = 1,\dots,r$, $\mathcal{G}(h_j,d,g^j)$ is a normalized tight frame of $\mathrm{L}^2(\mathbb{R})$; and
(ii) for $i \neq j$, and for all $f_1, f_2 \in \mathrm{L}^2(\mathbb{R})$,

$$\left\langle \left(\langle f_1, g^j_{m,n}\rangle\right)_{m,n}, \left(\langle f_2, g^i_{m,n}\rangle\right)_{m,n} \right\rangle_{\ell^2(\mathbb{Z}\times\mathbb{Z})} = 0 \quad . \tag{6.10}$$

i.e., the coefficient operators belonging to $\mathcal{G}(h_j,d,g^j)$ and $\mathcal{G}(h_i,d,g^i)$ have orthogonal ranges in $\ell^2(\mathbb{Z}\times\mathbb{Z})$.

Proof. Consider the subspace $\mathcal{H}_j \subset \left(\mathrm{L}^2(\mathbb{R})\right)^r$ whose elements are nonzero at most on the jth component. The necessity of property (i) follows immediately from Proposition 2.53(a), applied to the $\mathcal{H}_j$. Property (ii) is necessary because the (pairwise orthogonal) $\mathcal{H}_j$ need to have orthogonal images in $\ell^2(\mathbb{Z} \times \mathbb{Z})$. The converse is clear.

Necessary and sufficient conditions for the existence of such frames are given in the next proposition.

Proposition 6.14. *Let $(h_j)_{j=1,\dots,r}$, $d \in \mathbb{N}'$ be given.*

(a) There exists a normalized tight frame of $\left(\mathrm{L}^2(\mathbb{R})\right)^r$ of the form (6.9) iff $d\sum_{j=1}^r |h_j| \leq 1$.
(b) Assume that $h_j = h$, for all $j = 1, \dots, r$, and $g = (g^j)_{j=1,\dots,r}$ is such that (6.9) is a normalized tight frame. Then $g^i \perp g^j$, for $i \neq j$.

Proof. For the necessity in part (a), observe that Lemma 6.13 together with Theorem 6.12 yields that $\|g^j\|^2 = |h_j|d$, and thus $\|g\|^2 = d\sum_{j=1}^r |h_j|$. Now Proposition 2.53 (c) entails the desired inequality.

The proof for sufficiency is a slight modification of a construction given by Balan [21, Example 13]. Define $c_i = \sum_{j=1}^i |h_j|$, and let $g^i = \sqrt{d}\mathbf{1}_{[c_{i-1},c_i]}$. Given $f = (f^i) \in \left(\mathrm{L}^2(\mathbb{R})\right)^r$, we compute

$$
\begin{aligned}
\langle f, g_{k,m} \rangle &= \sum_{i=1}^r \langle f^i, g^i_{k,m} \rangle \\
&= \sum_{i=1}^r \sqrt{d} \int_{c_{i-1}}^{c_i} \mathrm{e}^{-2\pi \mathrm{i} m d x} f^i(x + h_i k) dx \\
&= \sqrt{d} \int_0^{1/d} \mathrm{e}^{-2\pi \mathrm{i} m d x} H_k(x) dx \quad ,
\end{aligned}
$$

where

$$
H_k(x) = \begin{cases} f^i(x - h_i k) & x \in [c_{i-1}, c_i] \\ 0 & \text{elsewhere} \end{cases} \quad .
$$

Fixing k, we compute

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}} |\langle f, g_{k,m} \rangle|^2 &= \sum_{m\in\mathbb{Z}} d \left| \int_0^{1/d} \mathrm{e}^{-2\pi \mathrm{i} m d x} H_k(x) dx \right|^2 \\
&= \int_0^{1/d} |H_k(x)|^2 dx \\
&= \sum_{i=1,\dots,r} \int_{c_{i-1}}^{c_i} |f^i(x - h_i k)|^2 dx \quad .
\end{aligned}
$$

Since the $h_i\mathbb{Z}$-translates of $[c_{i-1}, c_i]$ tile $\mathbb{R}$, summing over k yields the desired normequality. This closes the proof of (a).

For the proof of (b), pick $f_1, f_2 \in L^\infty(\mathbb{R})$ with supports in $[0, |h|]$. Then we calculate

$$\sum_{m,k\in\mathbb{Z}} \langle f_1, g^j_{m,k}\rangle \overline{\langle f_2, g^i_{m,k\in\mathbb{Z}}\rangle} =$$

$$= \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} \langle f_1 g^j_{m,0}, \mathrm{e}^{2\pi \mathrm{i} dk\cdot}\rangle \mathrm{e}^{2\pi \mathrm{i} dkx} \right) \overline{f_2(x)} g^i(x+hm)dx$$

$$= \sum_{m\in\mathbb{Z}} d^{-1} \int_0^{|h|} f_1(x)\overline{f_2(x) g^j(x+hm)} g^i(x+hm)dx$$

$$= d^{-1} \int_0^{|h|} \left(\sum_{m\in\mathbb{Z}} \overline{g^j(x+hm)} g^i(x+hm) \right) f_1(x)\overline{f_2(x)}dx$$

Here the Fourier series

$$\sum_{k\in\mathbb{Z}} \langle f_1 g^j_{m,0}, \mathrm{e}^{2\pi \mathrm{i} dk\cdot}\rangle \mathrm{e}^{2\pi dkx} = d^{-1} f_1(x) g^j(x+hm)$$

is valid on $[0, |h|]$, at least in the L^2-sense, because of $|h| \le d^{-1}$, the latter being a consequence of Theorem 6.12. Now, for arbitrary f_1, f_2, the scalar product we started with has to be zero, whence we obtain for almost every $x \in [0, |h|]$,

$$\sum_{m\in\mathbb{Z}} g^i(x+hm)\overline{g^j(x+hm)} = 0 \quad .$$

Integrating over $[0, |h|]$ and applying Fubini's theorem yields $\langle g^i, g^j\rangle = 0$.

Remark 6.15. Note that the vectors $(g^i)_{i=1,\dots,r}$ constructed in the proof of part (a) depend measurably on $\mathbf{h}$, i.e., if we let $(g^i_{\mathbf{h}})$ be the vector of functions constructed from $\mathbf{h}$, then $(x, \mathbf{h}) \mapsto (g^i_{\mathbf{h}}(x))_{i=1,\dots,r}$ is a measurable mapping.

6.5 Proofs of the Main Results*

The general proof strategy consists in explicit calculation for the Γ_d and then transferring the results to arbitrary lattices by the action of $\mathrm{Aut}(\mathbb{H})$. For this purpose we need a more detailed description of $\mathrm{Aut}(\mathbb{H})$ and its action on the Plancherel transform side. Most of the results are standard, and we only sketch the proofs.

Proposition 6.16. *(a) For $r > 0$ let $\alpha_r(p,q,t) := (\sqrt{r}p, \sqrt{r}q, rt)$. Then $\alpha_r \in \mathrm{Aut}(\mathbb{H})$. In addition, $\alpha_{inv} : (p,q,t) \mapsto (q,p,-t)$ defines an involutory automorphism of $\mathbb{H}$.*

(b) Each $\alpha \in \mathrm{Aut}(\mathbb{H})$ can be written uniquely as $\alpha = \alpha_r \alpha^i_{inv} \alpha'$, where $r \in \mathbb{R}'$, $i \in \{0,1\}$ and α' leaves the center of $\mathbb{H}$ pointwise fixed.

(c) *Suppose that* $\alpha(\Gamma_d) = \Gamma$ *for some* d, α, *and let* $\alpha = \alpha_s \alpha_{inv}^i \alpha'$ *be the decomposition from part (b). Then* $r(\Gamma) = s$.

Proof. For parts (a), (b) see [46, Theorem 1.22]. Part (c) follows directly from the definition of $r(\Gamma)$ and the fact that α' and α_{inv} map every discrete subgroup of $Z(\mathbb{H})$ onto itself.

Next let us consider the action on the Fourier transform side.

Proposition 6.17. *(a) Define* $\Delta : \mathrm{Aut}(\mathbb{H}) \to \mathbb{R}^+$ *by*

$$\Delta(\alpha) = \frac{\mu_{\mathbb{H}}(\alpha(B))}{\mu_{\mathbb{H}}(B)} \quad ,$$

where B *is a measurable set of positive Haar measure.* Δ *does not depend on the choice of* B, *and it is a continuous group homomorphism. For* $\alpha = \alpha_r \alpha_{inv}^i \alpha'$ *as in 6.16(b),* $\Delta(\alpha) = r^2$.

(b) *For* $\alpha \in \mathrm{Aut}(\mathbb{H})$, *let* $\mathcal{D}_\alpha : \mathrm{L}^2(\mathbb{H}) \to \mathrm{L}^2(\mathbb{H})$ *be defined as* $(\mathcal{D}_\alpha f)(x) := \Delta(\alpha)^{1/2} f(\alpha(x))$. *This defines a unitary operator.*

(c) *Let* $\mathcal{H} \subset \mathrm{L}^2(G)$ *be a closed, leftinvariant subspace with multiplicity function* m. *Then* $\widetilde{\mathcal{H}} = \mathcal{D}_\alpha(\mathcal{H})$ *is closed and leftinvariant as well. Let* $\widetilde{m}$ *denote the multiplicity function related to* $\widetilde{\mathcal{H}}$. *If* $\alpha = \alpha_r \alpha_{inv}^i \alpha'$ *then* $\widetilde{m}$ *satisfies*

$$\widetilde{m}(h) = m((-1)^i r^{-1} h) \text{ (almost everywhere)} \quad . \tag{6.11}$$

(d) *Let* Γ *be a lattice,* $\alpha \in \mathrm{Aut}(\mathbb{H})$ *such that* $\alpha(\Gamma_d) = \Gamma$. *Let* $\mathcal{H} \subset \mathrm{L}^2(\mathbb{H})$ *be a closed, leftinvariant subspace. Then* $\lambda_{\mathbb{H}}(\Gamma)\Phi$ *is a normalized tight frame (an ONB) for* $\mathcal{H}$ *iff* $\lambda_{\mathbb{H}}(\Gamma_d)(\mathcal{D}_\alpha \Phi)$ *is a normalized tight frame (an ONB) for* $\mathcal{D}_\alpha(\mathcal{H})$.

Proof. Parts (a) and (b) are standard results concerning the action of automorphisms on locally compact groups, see [64]. The explicit formula for $\Delta(\alpha)$ follows from the fact that every automorphism leaving the center invariant factors into an inner and a symplectic automorphism [46, Theorem 1.22]; both do not affect the Haar measure.

For part (c), we first note that by the Stone-von Neumann theorem [46, Theorem 1.50], any automorphism α' keeping the center pointwise fixed acts trivially on the dual of $\mathbb{H}$. Hence,

$$\left(\mathcal{D}_{\alpha'} f\right)^\wedge (h) = U_{\alpha',h} \circ \widehat{f}(h) \circ U_{\alpha',h}^* \quad ,$$

where $U_{\alpha',h}$ is a unitary operator on $\mathrm{L}^2(\mathbb{R})$. Hence the action of α' does not affect the multiplicity function, and from now on, we only consider $\alpha = \alpha_r \alpha_{inv}^i$. In this case, letting

$$(D_r f)(x) = r^{1/2} f(rx) \quad ,$$

we obtain by straightforward computation that

$$\left(\mathcal{D}_\alpha f\right)^\wedge(h) = r^{-1} \cdot D_r \circ \widehat{f}((-1)^i r^{-1} h) \circ D_r^* \quad . \tag{6.12}$$

This immediately implies (6.11).

To prove (d), observe that the unitarity of $\mathcal{D}_\alpha$ implies that $\mathcal{D}_\alpha\left(\lambda_{\mathbb{H}}(\Gamma)\right)$ is a normalized tight frame of $\mathcal{D}_\alpha(\mathcal{H})$, and check the equality

$$\mathcal{D}_\alpha(\lambda_{\mathbb{H}}(x)S) = \lambda_{\mathbb{H}}(\alpha^{-1}(x))(\mathcal{D}_\alpha S) \quad .$$

Proof of Theorem 6.4. We first prove the theorem for the case $\Gamma = \Gamma_d$. Writing

$$\widehat{\Phi}(h) = \sum_{i \in I_h} \varphi_i^h \otimes \eta_i^h \quad ,$$

we find by Proposition 6.11, that for almost every h, $(\varrho_h(\gamma) \circ |h|^{1/2}\widehat{\Phi}(h))_{\gamma \in \Gamma}$ has to be a normalized tight frame of $\mathrm{L}^2(\mathbb{R}) \circ \widehat{P}_h$, or equivalently, that the vector $(\varphi_i^h)_{i=1,\ldots,m(h)}$ generates a frame of $\left(\mathrm{L}^2(\mathbb{R})\right)^{m(h)}$, for $\mathbf{h} = (h, \ldots, h)$. Then 6.13 (a) implies that $\mathcal{G}(h, d, |h|^{1/2}\varphi_i^h)$ is a normalized tight frame of $\mathrm{L}^2(\mathbb{R})$. In particular, Theorem 6.12 entails

$$\|\varphi_i^h\|^2 = d \quad , \tag{6.13}$$

as well as $\Sigma(\mathcal{H}) \subset \left[-\frac{1}{d}, \frac{1}{d}\right]$. If $d > 1$, the support condition (6.6) in Proposition 6.11 is fulfilled. Hence Proposition 6.14 (a), applied to $\mathbf{h} = (h, \ldots, h)$, shows that (6.4) is necessary and sufficient for the existence of a normalized tight frame for $\mathcal{H}$. (Note that by Remark 6.15, 6.14 (a) provides a *measurable* vector field.)

The case $d = 1$ requires a somewhat more involved argument. Assume that $\lambda_{\mathbb{H}}(\Gamma)\Phi$ is a normalized tight frame, and let $f \in \mathcal{H}$. Condition (6.5) from Proposition 6.11 yields

$$\begin{aligned} \|f\|^2 = \int_0^1 \sum_{\gamma \in \Gamma_d^r} & \left|\langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h)\rangle\right|^2 |h|^2 \\ & + \left|\langle \widehat{f}(h-1), \varrho_{h-1}(\gamma)\widehat{\Phi}(h-1)\rangle\right|^2 |h-1|^2 dh \quad . \end{aligned} \tag{6.14}$$

On the other hand,

$$\begin{aligned} \|f\|^2 &= \sum_{\gamma \in \Gamma_d^r} \sum_{\ell \in \mathbb{Z}} \left| \int_0^1 \mathrm{e}^{-2\pi \mathrm{i} h \ell} \left(\langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h)\rangle |h| + \right.\right. \\ & \qquad \left.\left. + \langle \widehat{f}(h-1), \varrho_{h-1}(\gamma)\widehat{\Phi}(h-1)\rangle |h-1| \right) dh \right|^2 \\ &= \int_0^1 \left(\sum_{\gamma \in \Gamma_d^r} |\langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h)\rangle |h| \right. \\ & \qquad \left. + \langle \widehat{f}(h-1), \varrho_{h-1}(\gamma)\widehat{\Phi}(h-1)\rangle |h-1| \;|^2 \right) dh \quad . \end{aligned} \tag{6.15}$$

As in the proof of Proposition 6.11, the fact that the two equations hold for all $f \in \mathcal{H}$ allows to equate the integrands of (6.14) and (6.15). But this implies the orthogonality of the coefficient families:

$$\left\langle \left(\langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h) \rangle \right)_{\gamma \in \Gamma_d^r}, \left(\langle \widehat{f}(h-1), \varrho_{h-1}(\gamma)\widehat{\Phi}(h-1) \rangle \right)_{\gamma \in \Gamma_d^r} \right\rangle_{\ell^2(\Gamma_d^r)} = 0 \quad .$$

Plugging this fact, together with condition (6.5) from Proposition 6.11, into Proposition 6.13, we finally obtain that the system

$$\Big(\varrho_{h-1}(\gamma)|h-1|^{1/2}\varphi_1^{h-1} \;, \ldots, \; \varrho_{h-1}(\gamma)|h-1|^{1/2}\varphi_{m(h-1)}^{h-1}, \varrho_h(\gamma)|h|^{1/2}\varphi_1^h, \\ , \ldots, \; \varrho_h(\gamma)|h|^{1/2}\varphi_{m(h)}^h \Big)_{\gamma \in \Gamma_d^r}$$

has to be a normalized tight frame of $\left(\mathrm{L}^2(\mathbb{R})\right)^{m(h)+m(h-1)}$. An application of Proposition 6.14 (a) with $\mathbf{h} = (h-1, \ldots, h-1, h, \ldots, h)$ yields that such a frame exists iff $m(h)|h| + m(h-1)|h-1| \le 1$. This shows the necessity of (6.3). The sufficiency is obtained by running the proof backward; the measurability of the constructed operator field is again ensured by Remark 6.15.

For the proof of (ii) we need to show, by 2.53(c), that $\|\Phi\| < 1$, for every Φ for which $\lambda_{\mathbb{H}}(\Gamma)\Phi$ is a normalized tight frame. Recalling that

$$|h| \|\widehat{\Phi}(h)\|_{\mathcal{B}_2}^2 = |h| m(h) d \quad ,$$

and using the fact that the inequality $m(h)|h|d + m(h-1)|h-1|d \le 1$ is strict almost everywhere (say, for h irrational) we can estimate

$$\begin{aligned} \|\Phi\|^2 &= \int_{-1}^{1} \|\widehat{\Phi}(h)\|_{\mathcal{B}_2}^2 |h| dh \\ &= \int_0^1 m(h)|h|d + m(h-1)|h-1|d \; dh \\ &< 1 \quad . \end{aligned}$$

This closes the proof for $\Gamma = \Gamma_d$. For $\Gamma = \alpha(\Gamma_d)$, write $\alpha = \alpha_{r(\Gamma)} \alpha_{inv}^i \alpha'$ as in Proposition 6.16 (c). By 6.17 (d), we may consider Γ_d and $\widetilde{\mathcal{H}} = \mathcal{D}_\alpha(\mathcal{H})$ instead of Γ and $\mathcal{H}$. Part (ii) immediately follows from this observation. For part (i), we find by Proposition 6.17(c) that the associated multiplicity function $\widetilde{m}$ fulfills $\widetilde{m}(h) = m((-1)^i r(\Gamma)^{-1} h)$. Hence, (6.3) for $\Gamma_d, \widetilde{\mathcal{H}}$ becomes

$$m((-1)^i r(\Gamma)^{-1} h)|h| + m((-1)^i r(\Gamma)^{-1}(h-1))|h-1| \le \frac{1}{d} \quad \text{(almost everywhere)}$$

which after dividing both sides by $r(\Gamma)$ and passing to the variable $\widetilde{h} = (-1)^i r(\Gamma)^{-1} h$ is the desired inequality (6.3). $\square$

Proof of Corollary 6.6. The assumptions imply that $m(h)|h| \leq c$, for all $h \in \mathbb{R}'$, and c a constant. Hence picking $s \geq \frac{2c}{d}$ and defining $\Gamma = \alpha_s(\Gamma_d)$ ensures that (6.3) is fulfilled. □

Proof of Corollary 6.7. Pick an admissible vector η and transfer the corresponding results from Theorem 6.4 and Corollary 6.6 to $\mathcal{H}_\pi$ via V_η^{-1}. □

Proof of Corollary 6.8. Pick any measurable function $m : [-1, 1] \to \mathbb{N}'$ such that $h \mapsto m(h)|h|$ is integrable but unbounded. Pick a closed, leftinvariant space $\mathcal{H}$ with multiplicity function m. The space can be constructed by realizing that each representation in the Plancherel decomposition of $\mathbb{H}$ enters with infinite multiplicity, hence the projection field

$$P_h = \sum_{n=1}^{m(h)} e_n(h) \otimes e_n(h)$$

constructed from a measurable field $(e_n)_{n\in\mathbb{N}}$ of ONB's is well-defined and measurable. $\mathcal{H}$ is of the desired form, but violates (6.3), for all lattices Γ. □

Proof of Corollary 6.9. To give an example proving the first statement, let $\Gamma = \Gamma_d$; using the appropriate $\alpha \in \mathrm{Aut}(\mathbb{H})$ the argument can be adapted to suit any other lattice. For $h \in \left[0, \frac{1}{d}\right]$, define

$$\eta^h = \frac{1}{\sqrt{h/2}} \mathbf{1}_{[0,h/2]} \quad .$$

and $S \in \mathrm{L}^2(\mathbb{H})$ with $\widehat{S}(h) = \eta_h \otimes \eta_h$. Then S is a selfadjoint convolution idempotent, and $\mathcal{H} = \mathrm{L}^2(\mathbb{H}) * S$ has a tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$. However, for $\mathcal{H}$ to be a sampling space, $\lambda_{\mathbb{H}}(\Gamma)S$ must be a tight frame, and condition (6.5) implies that $\mathcal{G}(h, d, \eta_h)$ is a tight frame of $\mathrm{L}^2(\mathbb{R})$, for almost every h. But $\mathbf{1}_{[h/2,h]}$ has disjoint support with all elements of that system, hence $\mathcal{G}(h, d, \eta_h)$ is not even total.

The second statement is obvious from Proposition 2.54. The last statement follows from Theorem 6.4 (ii). □

6.6 A Concrete Example

In this section we explicitly compute a sinc-type function for $\Gamma = \Gamma_1$. The construction proceeds backwards, starting on the Plancherel transform side by giving a field of rank-one projection operators fulfilling the additional requirements for the sampling space property. Fourier inversion yields the sinc-type function S. As a consequence, the sampling space is given as $\mathrm{L}^2(\mathbb{H}) * S$. In order to minimize tedium, we have shortened some of the more straightforward calculations. The three steps carry out the abstract program developed above.

1. Construction on the Plancherel transform side.
For $h \in [-0.5, 0.5]$ let $\eta_h = |h|^{-1/2}\mathbf{1}_{[-|h|/2,|h|/2]}$, and

$$\widehat{S}(h) = \eta_h \otimes \eta_h \quad ,$$

and let $\widehat{S}$ be zero outside of $[-0.5, 0.5]$. $\widehat{S}$ is a measurable field of rank-one projection operators, with integrable trace, hence has an inverse image $S \in \mathrm{L}^2(\mathbb{H})$ which is a selfadjoint convolution idempotent. Moreover, it is straightforward to check that $\varrho_h(\Gamma_1^r)|h|^{1/2}\eta_h = \varrho_h(\Gamma_1^r)\mathbf{1}_{[-0.5,0.5]}$ is a normalized tight frame of $\mathrm{L}^2(\mathbb{R})$, (compare the proof of Proposition 6.14 (a)). Hence, by Proposition 6.11, $\lambda_{\mathbb{H}}(\Gamma)S$ is a normalized tight frame of $\mathcal{H} = \mathrm{L}^2(\mathbb{H}) * S$, and $\mathcal{H}$ is a sampling space.

2. Plancherel inversion.

An application of the Plancherel inversion formula (4.5) yields

$$\begin{aligned}
&S(p,q,t) \\
&= \int_{-0.5}^{0.5} \langle \eta_h, \varrho_h(p,q,t)\eta_h \rangle |h| dh \\
&= \int_{-0.5}^{0.5} \mathrm{e}^{-2\pi \mathrm{i} h(t+pq/2)} \int_{-\frac{|h|}{2}}^{\frac{|h|}{2}} \mathrm{e}^{-2\pi \mathrm{i} q x} \mathbf{1}_{[-|h|/2,|h|/2]}(x+hp) dx\ dh \quad . \qquad (6.16)
\end{aligned}$$

3. Computing integrals.

Let $\widetilde{S}(p,q,h)$ denote the inner integral. In the following, we assume that $q \neq 0$ and $p \geq 0$. The missing values will be obtained by taking limits (for $q = 0$) and reflection (for $p < 0$). Observe further that $S(p,q,t) = 0$ for $|p| > 1$, hence we will use $|p| \leq 1$ wherever we may need it. Integration yields

$$\widetilde{S}(p,q,h) = \begin{cases} \dfrac{\mathrm{e}^{2\pi \mathrm{i} q|h|/2} - \mathrm{e}^{-2\pi \mathrm{i} q(|h|/2 - hp)}}{2\pi \mathrm{i} q} & h \geq 0 \\ \dfrac{\mathrm{e}^{2\pi \mathrm{i} q(|h|/2 + hp)} - \mathrm{e}^{-2\pi \mathrm{i} q|h|/2}}{2\pi \mathrm{i} q} & h < 0 \end{cases} ,$$

After plugging this into (6.16) and integrating, straightforward simplifications lead to

$$S(p,q,t) = \frac{1}{2\pi q} \left(\frac{\cos(\pi(t+(p-1)q/2)) - 1}{\pi(t+(p-1)q/2)} - \frac{\cos(\pi(t-(p-1)q/2)) - 1}{\pi(t-(p-1)q/2)} \right) .$$

In order to further simplify this expression, we use the relation

$$\frac{\cos(\pi\alpha) - 1}{\pi\alpha} = -\frac{\pi\alpha}{2} \mathrm{sinc}^2\left(\frac{\alpha}{2}\right)$$

by which means we finally arrive at

$$\begin{aligned}
S(p,q,t) = \frac{1}{4} \Bigg[&\left(\frac{t}{q} + \frac{1-p}{2} \right) \mathrm{sinc}^2 \left(\frac{t}{2} + \frac{1-p}{4} q \right) \\
&- \left(\frac{t}{q} - \frac{1-p}{2} \right) \mathrm{sinc}^2 \left(\frac{t}{2} - \frac{1-p}{4} q \right) \Bigg] \quad . \qquad (6.17)
\end{aligned}$$

For $p < 0$ we use that $S(p,q,t) = S^*(p,q,t) = S(-p,-q,-t)$. It turns out that replacing p by $|p|$ in (6.17) is the only necessary adjustment for the formula to hold in the general case. Finally, sending q to 0 allows to compute the values $S(p,0,t)$, since S is continuous. The following theorem summarizes our calculations:

Theorem 6.18. *Define* $S \in \mathrm{L}^2(\mathbb{H})$ *by*

$$S(p,q,t) = \begin{cases} 0 & \text{for } |p| > 1 \\ \frac{1}{4}\left[\left(\frac{t}{q} + \frac{1-|p|}{2}\right)\mathrm{sinc}^2\left(\frac{t}{2} + \frac{1-|p|}{4}q\right)\right. & \\ \left. - \left(\frac{t}{q} - \frac{1-|p|}{2}\right)\mathrm{sinc}^2\left(\frac{t}{2} - \frac{1-|p|}{4}q\right)\right] & \text{for } |p| \le 1, q \neq 0 \\ \frac{1-|p|}{4}(2\mathrm{sinc}(t) - \mathrm{sinc}^2(t/2)) & \text{for } |p| \le 1, q = 0 \end{cases}$$

Let $\mathcal{H} \subset \mathrm{L}^2(\mathbb{H})$ *be the leftinvariant closed subspace generated by* S, $\mathcal{H} = \mathrm{L}^2(\mathbb{H}) * S$. *Then* $\mathcal{H}$ *is a sampling space for the lattice* Γ_1, *with* $c_{\mathcal{H}} = 1$, *and* S *the associated sinc-type function.* $\lambda_{\mathbb{H}}(\Gamma_1)S$ *is a normalized tight frame, but not an orthonormal basis of* $\mathcal{H}$, *because of* $\|S\|_2 = \frac{1}{2}$.