5

Admissible Vectors for Group Extensions

In this chapter we discuss a class of examples which has received considerable attention in recent years. The aim consists in making the abstract admissibility conditions developed in Chapter 4 explicit, in particular the criteria from Remark 4.30. Recall that this requires computing the Plancherel measure as well as the direct integral decomposition of the representation at hand into irreducibles. We consider certain group extensions

$$1 \to N \to G \to H \to 1 \quad ,$$

using the techniques of Kleppner and Lipsman from Section 3.6 for the computation of ν_G. Of particular interest will be the case where $G = N \rtimes H$, and π is the *quasi-regular representation* $\pi = \mathrm{Ind}_H^G 1$. The case $N = \mathbb{R}^k$ and $H < \mathrm{GL}(k, \mathbb{R})$ has been studied by a number of authors, see the list below, and we start by discussing this setting by more or less basic arguments. In fact, Examples 2.28, 2.30 and 2.36 are all special cases of this setting. It turns out that the arguments dealing with these examples extend to the general case $G = \mathbb{R}^k \rtimes H$, yielding elementary admissibility conditions which avoid explicit reference to the Plancherel transform of G (Theorem 5.8). Having done that, we show in Theorem 5.23 how the concrete admissibility conditions relate to the scheme described in Remark 4.30. As a result we obtain a very concrete description of the various objects that enter the Plancherel formula for the group under consideration. The argument is based on a general result about the containment of a quasi-regular representation in the regular representation (Theorem 5.22), which allows to conclude the existence of admissible vectors for a wide range of settings.

The result can also be applied to cases where N is nonabelian; in particular when N is a homogeneous Lie group and H a one-parameter group of dilations acting on N (Corollary 5.28). Finally, we show how the Zak transform criteria for Weyl-Heisenberg frames with integer oversampling can be regarded as admissibility conditions with respect to a certain discrete group.

Thus the results of this chapter generalize and/or complement the findings of various authors:

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- Murenzi's [93] 2D continuous wavelet transform discussed in 2.30.
- The dyadic wavelet transform of Mallat and Zhong [92] considered in 2.36
- Bohnke [25] introduced $H = \mathbb{R}^+ \cdot \mathrm{SO}_0(1,1)$ and $H = \mathbb{R}^+ \cdot \mathrm{SO}_0(2,1)$. Again the representations were irreducible.
- General characterizations of dilation groups H giving rise to *discrete series* subrepresentations of the quasiregular representation were given in [22, 50].
- Klauder, Isham and Streater [67, 74] considered $H = \mathrm{SO}(k)$ and $H = \mathrm{SO}(k-1,1)$. As a matter of fact, the representations they consider are not explicitly defined as subrepresentations of π, but they are direct integrals which are immediately recognised as subrepresentations of the representation $\widehat{\pi}$ obtained by conjugating π with the Fourier transform on $\mathbb{R}^k$.
- Cyclic and one-parameter subgroups of $\mathrm{GL}(k,\mathbb{R})$ were considered by Gröchenig, Kaniuth and Taylor [59]. A general discussion of discrete dilation groups can be found in [51].
- The characterization of dilation groups allowing the existence of admissible vectors was addressed in full generality in [77, 113], as well as in [52], and the discussion in the first two sections follows the latter paper closely.
- Lemarié [78] established the existence of wavelet orthonormal bases on $\mathrm{L}^2(N)$, where N is a stratified Lie group. These bases arise from the action of a lattice in N and a discrete group of dilations, in an entirely analogous fashion to the multiresolution wavelets on $\mathbb{R}$. Our results provide a continuous analogue of these systems, for the larger class of homogeneous Lie groups.
- Liu and Peng [83] considered a semidirect product $\mathbb{H} \rtimes \mathbb{R}$, where $\mathbb{R}$ denotes a one-parameter group of dilations, and an associated quasi-regular representation π on $\mathrm{L}^2(\mathbb{H})$. They then showed that π splits into discrete series representations, and gave admissibility conditions for each. Our results yield admissible vectors for π itself.
- Daubechies [33] (among other authors) characterized tight Gabor frames for the case of rational oversampling, making use of the Zak transform. The discussion in Section 5.5 exhibits this criterion as yet another instance of the scheme from Remark 4.30.

The standing assumptions of this chapter are: G is a type I group, $N \lhd G$ is regularly embedded and type I. Whenever G is nonunimodular, we assume that the $\mathrm{Ker}(\Delta_G)$ has the same properties as N. The assumptions are chosen to allow to apply the results from the previous chapters. As the discussion of the concrete admissibility conditions shows, the conditions are somewhat more restrictive than seems necessary. However, for all concrete examples that have so far been considered in the literature, the standing assumptions can in fact be verified.

5.1 Quasiregular Representations and the Dual Orbit Space

For this and the next section let $H < \mathrm{GL}(k,\mathbb{R})$ be a closed subgroup, and $G = \mathbb{R}^k \rtimes H$. Elements of G are denoted by (x,h) with $x \in \mathbb{R}^k$ and $h \in H$; the group law is then given by $(x_0,h_0)(x_1,h_1) = (x_0 + h_0x_1, h_0h_1)$. Left Haar measure of G is given by $d\mu_G(x,h) = |\det(h)|^{-1}dxd\mu_H(h)$, and the modular function is computed as $\Delta_G(x,h) = \Delta_H(h)|\det(h)|^{-1}$. For simplicity we will sometimes write $\Delta_G(h)$ instead of $\Delta_G(0,h)$. The quasiregular representation π of G acts on $\mathrm{L}^2(\mathbb{R}^k)$ by

$$(\pi(x,h)f)(y) = |\det(h)|^{-1/2}f(h^{-1}(y-x)).$$

The closedness of H in $\mathrm{GL}(k,\mathbb{R})$ may seem a somewhat arbitrary condition (Lie subgroups might also work), but it is in fact not a real restriction, because of the following fact. It was observed in [49, Proposition 5] for the discrete series case, but the proof does not use irreducibility. We reproduce it for the sake of completeness.

Proposition 5.1. *Let H be a subgroup of $\mathrm{GL}(k,\mathbb{R})$, endowed with some locally compact group topology. Assume that the semidirect product $\mathbb{R}^k \rtimes H$ is a topological semidirect product, and that the quasiregular representation has a nontrivial subrepresentation with an admissible vector. Then H is a closed subgroup of $\mathrm{GL}(k,\mathbb{R})$, and the topology on H is the relative topology.*

Proof. Since all wavelet coefficients vanish at infinity by 2.19, there exists a nontrivial C_0 matrix coefficient V_fg for π. We may assume $V_fg(e) = 1$. Let W_fg denote the matrix coefficient of the quasiregular representation of $\mathbb{R}^k \rtimes \mathrm{GL}(k,\mathbb{R})$ corresponding to the same pair of functions f,g. Then V_fg is the restriction of W_fg to $\mathbb{R}^k \rtimes H$. Since W_fg is continuous, there exists a compact neighborhood U of 1 in $\mathrm{GL}(k,\mathbb{R})$ with $|W_fg| > 1/2$ on U. $U \cap H$ is $\mathcal{T}$-closed in H, since U is closed and $\mathcal{T}$ is finer than the relative topology. Since H is closed in $\mathbb{R}^k \rtimes H$, the restriction of V_fg to H vanishes at infinity. By choice of U, this implies that $U \cap H$ is contained in a $\mathcal{T}$-compact set, hence is $\mathcal{T}$-compact itself. But then the inclusion map from $U \cap H$ to H is a homeomorphism onto its image, being a continuous map from a compact space to a Hausdorff space. Hence $\mathcal{T}$ coincides on $U \cap H$ with the relative topology. Since $U \cap H$ is a neighborhood of 1 for both topologies, it follows that the neighborhood filters of both topologies at unity coincide. Since a group topology is uniquely determined by the neighborhoods at unity, the topologies themselves coincide. In particular the relative topology is locally compact, which means H is closed.

Mackey's theory directs our attention towards the dual action. As it turns out, the decomposition of the quasiregular representation is also closely related to the dual orbit space. Let us quickly recall the notions of Mackey's theory for this particular setup. The dual group $\widehat{\mathbb{R}^k}$ is the character group of $\mathbb{R}^k$, identified with $\mathbb{R}^k$ itself. Denoting the usual scalar product on $\mathbb{R}^k$ by $(\omega, x) \mapsto \omega \cdot x$, the duality between $\widehat{\mathbb{R}^k}$ and $\mathbb{R}^k$ is given by

$$\langle \omega, x \rangle = \mathrm{e}^{-2\pi \mathrm{i} \omega \cdot x} \quad .$$

In this identification, the dual operation is given by

$$h.\omega = (h^t)^{-1}\omega \quad , \tag{5.1}$$

where the right-hand side denotes the product of a matrix with a column vector. $\widehat{\mathbb{R}^k}/H$ is the dual orbit space. For $\gamma \in \widehat{\mathbb{R}^k}$, H_γ denotes the stabilizer of γ in H; it is a closed subgroup of H.

For the discussion of subrepresentations of π, it is useful to introduce the representation $\widehat{\pi}$ obtained by conjugating π with the Fourier transform on $\mathbb{R}^k$. It is readily seen to operate on $\mathrm{L}^2(\widehat{\mathbb{R}^k})$ via

$$(\widehat{\pi}(x,h)\widehat{f})(\omega) = |\det(h)|^{1/2} \mathrm{e}^{-2\pi \mathrm{i} \omega \cdot x} f(h^{-1}.\omega) \quad . \tag{5.2}$$

The action of $\widehat{\pi}$ allows to identify subrepresentations in a simple way: Every invariant closed subspace $\mathcal{H} \subset \mathrm{L}^2(\mathbb{R}^k)$ is of the form

$$\mathcal{H} = \mathcal{H}_U = \{ g \in \mathrm{L}^2(\mathbb{R}^k) : \widehat{g} \text{ vanishes outside of } U \} \quad ,$$

where $U \subset \widehat{\mathbb{R}^k}$ is a measurable, H-invariant subset (see [48] for a detailed argument). We let π_U denote the subrepresentation acting on $\mathcal{H}_U$.

Remark 5.2. **The structure of the dual orbit space** As the structure of the dual orbit space is important for the decomposition of the quasi-regular representation and for the construction of admissible vectors on the one hand, but also for the computation of Plancherel measure on the other, let us take a closer look at its measure-theoretic structure. For our discussion, the following two sets will be central

$$\Omega_c = \{ \omega \in \widehat{\mathbb{R}^k} : H_\omega \text{ is compact } \} \ , \ \Omega_{rc} = \{ \omega \in \Omega_c : \mathcal{O}(\omega) \mid \text{ is locally closed } \} \ .$$

The set $\Omega_{rc} \subset \Omega_c$ consists of the "regular" orbits in Ω_c; i.e., it is the "well-behaved" part of Ω_c. Loosely speaking, Ω_c is the set we have to deal with, and Ω_{rc} is the set we can deal with. Put more precisely: While Theorem 5.8 below shows that subrepresentations with admissible vectors necessarily correspond to invariant subsets U of Ω_c, the existence result in Theorem 5.12 only considers subsets of the smaller set Ω_{rc}. However, this distinction is not due to a shortcoming of our approach: Remark 5.13 gives an example of a subset of Ω_c that does not allow admissible vectors for the corresponding subrepresentation.

The measure-theoretic properties of the two sets are summarized as follows: Ω_c can be shown to be measurable, see Corollary 5.6. But usually Ω_c is not open, even when it is conull, as is illustrated by the example of $\mathrm{SL}(2,\mathbb{Z})$: It is easy to see that Ω_c consists of all the vectors (ω_1,ω_2) such that ω_1/ω_2 is irrational. This is a conull set with dense complement in $\widehat{\mathbb{R}^2}$.

By contrast, Ω_{rc} is always open, by Proposition 5.7. A pleasant consequence of this is that Glimm's Theorem [56] applies (since Ω_{rc} is locally compact), which entails a number of useful properties of the orbit space Ω_{rc}/H: It is a standard Borel space having a measurable cross section $\Omega_{rc}/H \to \Omega_{rc}$, and there exists a measurable transversal, i.e., a Borel subset $A \subset \Omega_{rc}$ meeting each orbit in precisely one point.

Unfortunately, the example of $\mathrm{SL}(2,\mathbb{Z})$ shows that Ω_{rc} can be empty even when Ω_c is conull: Ω_c contains no nonempty open set, since its complement is dense.

The rest of the section is devoted to proving the measurability of Ω_c and the openness of Ω_{rc}. The proof for the first result uses the subgroup space of H, as introduced by Fell [44].

Definition 5.3. *Let G be a locally compact group. The* **subgroup space of G** *is the set $K(G) := \{H < G : H \text{ is closed}\}$, endowed with the topology generated by the sets*

$$U(V_1,\ldots,V_n;C) := \{H \in K(G) : H \cap V_i \neq \emptyset, \forall 1 \leq i \leq n, H \cap C = \emptyset\},$$

where $V_1,\ldots,V_n$ denotes any finite family of open subsets of G and $C \subset G$ is compact.
With this topology $K(G)$ is a compact Hausdorff space.

The motivation for introducing the $K(G)$ to our discussion is the following:

Proposition 5.4. *Let X be a countably separated Borel space, and G a locally compact group acting measurably on X. Consider the* **stabilizer map** *$s : X \to K(G)$ defined by $s(x) = \{g \in G : g.x = x\}$. Then s is Borel.*

Proof. That s indeed maps into $K(G)$ is due to [17, Chapter I, Proposition 3.7], the measurability is [17, Chapter II, Proposition 2.3].

Proposition 5.5. *Let G be a σ-compact, locally compact group. Then $K_c(G) = \{H \in K(G) : H \text{ is compact}\}$ is a Borel subset of $K(G)$.*

Proof. We first construct a sequence of compact set $(C_n)_{n\in\mathbb{N}}$ such that $A \subset G$ is relatively compact iff $A \subset C_n$ for some $n \in \mathbb{N}$. For this purpose let $G = \bigcup_{n\in\mathbb{N}} K_n$ with compact K_n. Pick a compact neighborhood V of the unit elements, and define $C_n = V \cdot \bigcup_{i=1}^n C_i$. Then the open kernels of the C_n cover G. Hence, if $A \subset G$ is compact, the fact that the C_n increase implies that $A \subset C_n$ for some n. It follows that

$$\begin{aligned} K_c(G) &= \bigcup_{n\in\mathbb{N}} \{H \in K(G) : H \cap (G \setminus C_n) = \emptyset\} \\ &= \bigcup_{n\in\mathbb{N}} K(G) \setminus U(G \setminus C_n, \emptyset) \end{aligned}$$

is an F_σ-set.

Combining 5.4 and 5.5, we obtain the desired measurability.

Corollary 5.6. Ω_c *is a Borel subset of* $\widehat{\mathbb{R}^k}$.

The proof of the following proposition uses ideas from [77].

Proposition 5.7. Ω_{rc} *is open.*

Proof. Define the ϵ-*stabilizer*

$$H_\omega^\epsilon = \{h \in H : |h.\omega - \omega| \le \epsilon\} \quad ,$$

where $|\cdot|$ denotes the euclidean norm on $\widehat{\mathbb{R}^k}$. If H_ω^ϵ is compact for some $\epsilon > 0$, then $B_\epsilon(\omega) \cap \mathcal{O}(\omega) = H_\omega^\epsilon.\omega$ is compact. Here $B_\epsilon(x)$ denotes the closed ϵ-ball around x. Hence the orbit $\mathcal{O}(\omega)$ is locally closed.

Conversely, assume that $B_\epsilon(\omega) \cap \mathcal{O}(\omega)$ is compact for some $\epsilon > 0$ and that H_ω is compact. There exists a measurable cross-section $\tau : \mathcal{O}(\omega) \to H$ which maps compact sets in $\mathcal{O}(\omega)$ to relatively compact sets in H. Hence $H_\omega^\epsilon \subset H_\omega \tau(B_\epsilon(\omega))$ is relatively compact and closed, hence compact.

In short, we have shown

$$\omega \in \Omega_{rc} \Longleftrightarrow \exists \epsilon > 0 : H_\omega^\epsilon \text{ is compact} \quad ,$$

and we are going to use this characterization to prove the openness of Ω_{rc}.

If the origin is in Ω_{rc}, then H is compact, and $\Omega_{rc} = \widehat{\mathbb{R}^k}$. In the other case, pick ω in Ω_{rc} and $\epsilon > 0$ with H_ω^ϵ compact. Since $\mathrm{GL}(k,\mathbb{R})$ acts transitively on $\widehat{\mathbb{R}^k} \setminus \{0\}$, we may (possibly after passing to a smaller ϵ) assume that there exists a continuous cross-section $\alpha : B_\epsilon(\omega) \to \mathrm{GL}(k,\mathbb{R})$ with relatively compact image, i.e., $\alpha(\gamma).\omega = \gamma$, for all $\gamma \in B_\epsilon(\omega)$, and $\alpha(B_\epsilon(\omega)) \subset U$, where U is a compact neighborhood of the identity in $\mathrm{GL}(k,\mathbb{R})$. We are going to show that $B_\epsilon(\omega) \subset \Omega_{rc}$. For this purpose let $\gamma \in B_\epsilon(\omega)$. Clearly it is enough to prove that

$$C := \{h \in H : h.\gamma \in B_\epsilon(\omega)\} = \{h \in \mathrm{GL}(k,\mathbb{R}) : h.\gamma \in B_\epsilon(\omega)\} \cap H$$

is relatively compact. By assumption,

$$H_\omega^\epsilon = \{h \in \mathrm{GL}(k,\mathbb{R}) : h.\omega \in B_\epsilon(\omega)\} \cap H$$

is compact. Hence

$$\begin{aligned} C &= \{h \in \mathrm{GL}(k,\mathbb{R}) : \omega\alpha(\gamma).h \in B_\epsilon(\omega)\} \cap H \\ &= \alpha(\gamma)^{-1}\{h \in \mathrm{GL}(k,\mathbb{R}) : h.\omega \in B_\epsilon(\omega)\} \cap H \\ &\subset U^{-1}(\{h \in \mathrm{GL}(k,\mathbb{R}) : h.\omega \in B_\epsilon(\omega)\} \cap H) \end{aligned}$$

i.e., C is contained in the product of two compact sets, and thus relatively compact. Note that we used here that H is a closed subgroup of $\mathrm{GL}(k,\mathbb{R})$, hence compactness in H is the same as compactness in $\mathrm{GL}(k,\mathbb{R})$.

5.2 Concrete Admissibility Conditions

We will now derive the admissibility condition for the quasiregular representation and its subrepresentations. The proof of the admissibility condition is fairly straightforward. It is only when we address the existence of functions fulfilling the condition that we are forced to use more involved arguments. The theorem was derived for certain concrete groups H in [92, 67, 74], and the arguments presented in this section are generalizations of those in [92, 67, 74]. The general version given here appears also in [77, 52]. Note that the admissibility condition also figures as a part of the definition of the notion of "projection generating function" in [59, Definition 2.1]. Thus the following theorem also answers a question raised in [59, Remark 2.6(b)]: There the authors observe that taking a projection generating function as wavelet gives rise to orthogonality relations among the wavelet coefficients which closely resemble those for irreducible square-integrable representations, even though the representation at hand is not irreducible. The explanation for this phenomenon is that the orthogonality relations in the discrete series case are particular instances of the orthogonality relations arising in connection with the Plancherel formula.

A comparison of the following theorem with Theorem 4.20 gives a first hint towards the connection between abstract and concrete admissibility conditions.

Theorem 5.8. *Let $(\pi_U, \mathcal{H}_U)$ be a subrepresentation of π corresponding to some invariant measurable subset U. Then*

$$g \in \mathcal{H}_U \text{ is bounded } \Leftrightarrow \int_H |\widehat{g}(h^{-1}.\omega)|^2 d\mu_H(h) \le \text{constant} \qquad (5.3)$$

$$g \in \mathcal{H}_U \text{ is cyclic } \Leftrightarrow \int_H |\widehat{g}(h^{-1}.\omega)|^2 d\mu_H(h) \neq 0 \qquad (5.4)$$

$$g \in \mathcal{H}_U \text{ is admissible } \Leftrightarrow \int_H |\widehat{g}(h^{-1}.\omega)|^2 d\mu_H(h) = 1 \qquad (5.5)$$

Here all right-hand sides are understood to hold almost everywhere. In particular, if π_U has a bounded cyclic vector, then $U \subset \Omega_c$ (up to a null set).

Proof. We start by explicitly calculating the L^2-norm of $V_g f$, for $f, g \in \mathcal{H}_U$. The following computations are generalizations of the argument used for the 1D continuous wavelet transform in Example 2.28, see also [22, 48, 113].

$$\begin{aligned}
\|V_g f\|^2_{\mathrm{L}^2(G)} &= \int_G |\langle f, \pi(x,h)g\rangle|^2 \, d\mu_G(x,h) \\
&= \int_G \left|\langle \widehat{f}, (\pi(x,h)g)^\wedge\rangle\right|^2 d\mu_G(x,h) \\
&= \int_G \left| \int_{\widehat{\mathbb{R}^k}} \widehat{f}(\omega) |\det(h)|^{1/2} \mathrm{e}^{2\pi \mathrm{i}\gamma x} \overline{\widehat{g}}(h^{-1}.\omega) d\omega \right|^2 d\mu_G(x,h) \\
&= \int_H \int_{\mathbb{R}^k} \left| \int_{\widehat{\mathbb{R}^k}} \widehat{f}(\omega) \mathrm{e}^{2\pi \mathrm{i}\omega x} \overline{\widehat{g}}(h^{-1}.\omega) d\omega \right|^2 d\lambda(x) d\mu_H(h) \\
&= \int_H \int_{\mathbb{R}^k} |\mathcal{F}(\phi_h)(-x)|^2 \, d\lambda(x) d\mu_H(h).
\end{aligned}$$

Here $\phi_h(\omega) = \widehat{f}(\omega)\overline{\widehat{g}}(h^{-1}.\omega)$, and $\mathcal{F}$ denotes the Fourier transform on $\mathrm{L}^1(\widehat{\mathbb{R}^k})$. An application of Plancherel's formula – or more precisely, the extension of 2.22 to $\mathbb{R}^k$ – to the last expression yields

$$\begin{aligned}
\int_H \int_{\widehat{\mathbb{R}^k}} |\phi_h(\omega)|^2 \, d\omega d\mu_H(h) &= \int_H \int_{\widehat{\mathbb{R}^k}} \left|\widehat{f}(\omega)\right|^2 \left|\widehat{g}(h^{-1}.\omega)\right|^2 d\omega d\mu_H(h) \\
&= \int_{\widehat{\mathbb{R}^k}} \left|\widehat{f}(\omega)\right|^2 \left(\int_H \left|\widehat{g}(h^{-1}.\omega)\right|^2 d\mu_H(h) \right) d\omega \ .
\end{aligned}$$

Now (5.3) through (5.5) are obvious. Moreover, it is easily seen that whenever the stabilizer H_ω is noncompact, we have

$$\int_H |\widehat{g}(h^{-1}.\omega)|^2 d\mu_H(h) \in \{0, \infty\},$$

(cf. also the proof of [48, Theorem 10]), hence $V_g f \in \mathrm{L}^2(G)$ entails that the pointwise product $\widehat{f}\overline{\widehat{g}}$ vanishes a.e. outside of Ω_c. In particular, a bounded cyclic vector vanishes almost everywhere outside of Ω_c, hence we obtain in such a case that $U \subset \Omega_c$ (up to a null set).

For the construction of admissible vectors we first decompose Lebesgue measure λ on Ω_{rc} into measures on the orbits and a measure on Ω_{rc}/H. Then we address the relationship of the measures on the orbits to the Haar measure of H.

Lemma 5.9. *(a) There exists a measure $\overline{\lambda}$ on Ω_{rc}/H and on each orbit $\mathcal{O}(\gamma)$ a measure $\beta_{\mathcal{O}(\gamma)}$ such that for every measurable $A \subset \Omega_{rc}$ the mapping*

$$\mathcal{O}(\gamma) \mapsto \int_{\mathcal{O}(\gamma)} \mathbf{1}_A(\omega) d\beta_{\mathcal{O}(\gamma)}(\omega)$$

is $\overline{\lambda}$-measurable, and in addition

$$\lambda(A) = \int_{\widehat{\mathbb{R}^k}/H} \int_{\mathcal{O}(\gamma)} \mathbf{1}_A(\omega) d\beta_{\mathcal{O}(\gamma)}(\omega) d\overline{\lambda}(\mathcal{O}(\gamma)).$$

(b) Let $(\overline{\lambda}, (\beta_{\mathcal{O}(\gamma)})_{\mathcal{O}(\gamma)\in\Omega_{rc}/H})$ be as in (a). For $\gamma \in \Omega_{rc}$ define $\mu_{\mathcal{O}(\gamma)}$ as the image measure of μ_H under the projection map $p_\gamma : h \mapsto h^{-1}.\gamma$. $\mu_{\mathcal{O}(\gamma)}$ is a σ-finite measure, and its definition is independent of the choice of representative γ. Then, for almost all $\gamma \in \Omega_{rc}$, the $\mu_{\mathcal{O}(\gamma)}$ and $\beta_{\mathcal{O}(\gamma)}$ are equivalent, with globally Lebesgue-measurable Radon-Nikodym-derivatives: There exists an (essentially unique) Lebesgue-measurable function $\kappa : \Omega_{rc} \to \mathbb{R}^+$ such that for $\omega \in \mathcal{O}(\gamma)$,

$$\frac{d\beta_{\mathcal{O}(\gamma)}}{d\mu_{\mathcal{O}(\gamma)}}(\omega) = \kappa(\omega) \quad .$$

(c) The function κ fulfills the **semi-invariance relation**

$$\kappa(h^{-1}.\omega) = \kappa(\omega)\Delta_G(h)^{-1} \quad . \tag{5.6}$$

In particular, κ is H-invariant iff G is unimodular. In that case, we can in fact assume that $\kappa = 1$ almost everywhere. This choice determines the measure $\overline{\lambda}$ uniquely.

Proof. Statement (a) is a special instance of Proposition 3.28.

In order to prove part (b), well-definedness and σ-finiteness of $\mu_{\mathcal{O}(\gamma)}$ follow from compactness of H_γ. The independence of the representative γ of the orbit follows from the fact that μ_H is leftinvariant. To compute the Radon-Nikodym derivative κ, we first introduce an auxiliary function $\ell : \Omega_{rc} \to \mathbb{R}_0^+$: Fix a Borel-measurable transversal $A \subset \Omega_{rc}$ of the H-orbits. Then the mapping $\tau : A \times H \to \Omega_{rc}$, $\tau(\omega, h) = h^{-1}.\omega$ is bijective and continuous, hence, since $A \times H$ is a standard Borel space, $\tau^{-1} : \Omega_{rc} \to A \times H$ is Borel as well, by [15, Theorem 3.3.2]. If we let $\tau^{-1}(\gamma)_H$ denote the H-valued coordinate of $\tau^{-1}(\gamma)$, then $\ell(\gamma) := \Delta_G(\tau^{-1}(\gamma)_H)$ is a Borel-measurable mapping. Since Δ_G is constant on every compact subgroup (in particular on all the little fixed groups of elements in Ω_{rc}), a straightforward calculation shows that ℓ satisfies the semi-invariance relation $\ell(h^{-1}.\omega) = \ell(\omega)\Delta_G(h)^{-1}$.

Next fix an orbit $\mathcal{O}(\gamma)$ and let us compare the measures $\beta_{\mathcal{O}(\gamma)}$ and $\ell\mu_{\mathcal{O}(\gamma)}$: Since

$$d\mu_{\mathcal{O}(\gamma)}(h^{-1}.\omega) = \Delta_H(h) d\mu_{\mathcal{O}(\gamma)}(\omega) \quad \text{and} \quad d\beta_{\mathcal{O}(\gamma)}(h^{-1}.\omega) = |\det(h)| d\beta_{\mathcal{O}(\gamma)}(\omega) \ ,$$

the definition of ℓ ensures that $\ell\mu_{\mathcal{O}(\gamma)}$ and $\beta_{\mathcal{O}(\gamma)}$ behave identically under the action of H. Moreover, they are σ-finite and quasi-invariant, hence equivalent. Since they have the same behaviour under the operation of H, the Radon-Nikodym derivative turns out to be a positive constant on the orbit. Summarizing, we find for $\omega \in \mathcal{O}(\gamma)$ that

$$\frac{d\beta_{\mathcal{O}(\gamma)}}{d\mu_{\mathcal{O}(\gamma)}}(\omega) = \ell(\omega) c_{\mathcal{O}(\gamma)} \quad ,$$

with $\ell, c_{\mathcal{O}(\gamma)} > 0$, and it remains to show that $c_{\mathcal{O}(\gamma)}$ depends measurably on the orbit.

For this purpose pick a relatively compact open neighborhood $B \subset H$ of the identity. Then $AB = \tau(A \times B) \subset \Omega_{rc}$ is Borel-measurable, as a continuous image of a standard space, hence $\mathbf{1}_{AB}$, the indicator function of AB, is a Borel-measurable function. Both

$$\phi_1 : \mathcal{O}(\gamma) \mapsto \int_{\mathcal{O}(\gamma)} \mathbf{1}_{AB}(\omega) d\beta_{\mathcal{O}(\gamma)}(\omega)$$

and

$$\phi_2 : \mathcal{O}(\gamma) \mapsto \int_{\mathcal{O}(\gamma)} \mathbf{1}_{AB}(\omega)\ell(\omega) d\mu_{\mathcal{O}(\gamma)}(\omega)$$

are measurable functions: The first one is by choice of the $\beta_{\mathcal{O}(\gamma)}$, see part (a). The second one is measurable by Fubini's theorem, applied to the mapping $(\omega, h) \mapsto \mathbf{1}_{AB}(h^{-1}.\omega)\ell(h^{-1}.\omega)$ on $\widehat{\mathbb{R}^k} \times H$ (recall the definition of $\mu_{\mathcal{O}(\gamma)}$).

In addition, both functions are finite and positive on Ω_{rc}. We have

$$\phi_2(\mathcal{O}(\gamma)) = \int_{\mathcal{O}(\gamma)} \mathbf{1}_{AB}(\omega)\ell(\omega) d\mu_{\mathcal{O}(\gamma)}(\omega) = \int_{p_\gamma^{-1}(AB)} \Delta_G(h) d\mu_H(h) ~,$$

and $p_\gamma^{-1}(AB)$ is relatively compact and open, hence it has finite and positive Haar measure. Since in addition Δ_G is positive and bounded on $p_\gamma^{-1}(AB)$, we find $0 < \phi_2(\mathcal{O}(\gamma)) < \infty$. Hence

$$\phi_1(\mathcal{O}(\gamma)) = c_{\mathcal{O}(\gamma)} \phi_2(\mathcal{O}(\gamma))$$

can be solved for $c_{\mathcal{O}(\gamma)}$, which thus turns out to depend measurably upon $\mathcal{O}(\gamma)$. Hence

$$\kappa(\omega) = \frac{d\beta_{\mathcal{O}(\gamma)}}{d\mu_{\mathcal{O}(\gamma)}} = \ell(\omega) c_{\mathcal{O}(\gamma)}$$

is a Lebesgue-measurable function.

The remaining part (c) is simple to prove: The semi-invariance relation of ℓ entails the relation for κ. The normalization is easily obtained: If κ is constant on the orbits, it defines a measurable mapping $\overline{\kappa}$ on Ω_{rc}/H. If we replace each $\beta_{\mathcal{O}(\gamma)}$ by $\mu_{\mathcal{O}(\gamma)}$, we can make up for it by taking $\overline{\kappa}(\mathcal{O}(\gamma)) d\overline{\lambda}(\mathcal{O}(\gamma))$ as the new measure on the orbit space. The new choice has the desired properties. The uniqueness of $\overline{\lambda}$ follows from the usual Radon-Nikodym arguments.

Remark 5.10. Let us for the next two sections fix a choice of $\overline{\lambda}$. Note that this also uniquely determines the function κ. In the unimodular case we take κ to be 1, which in turn determines $\overline{\lambda}$ uniquely.

As we shall see in Theorem 5.23, the choice of a pair $(\overline{\lambda}, \kappa)$ corresponds exactly to a choice of Plancherel measure and the associated family of Duflo-Moore operators, at least on a subset of $\widehat{G}$.

Before we turn to the construction of admissible vectors, we introduce some notation to help clarify the construction: To a function $\widehat{g}$ on U we associate two auxiliary H-invariant functions $T_H(\widehat{g})$ and $S_H(\widehat{g})$ such that admissibility of g translates to a condition on $T_H(\widehat{g})$ and square-integrability to a condition on $S_H(\widehat{g})$.

Definition 5.11. *For a measurable function $\widehat{g}$ on Ω_{rc}, let $T_H(\widehat{g})$ denote the function*

$$\begin{aligned} T_H(\widehat{g})(\omega) &:= \left(\int_{\mathcal{O}(\omega)} |\widehat{g}(\gamma)|^2 d\mu_{\mathcal{O}(\omega)}(\gamma) \right)^{1/2} \\ &= \left(\int_{\mathcal{O}(\omega)} |\kappa(\gamma)^{-1/2}\widehat{g}(\gamma)|^2 d\beta_{\mathcal{O}(\omega)}(\gamma) \right)^{1/2} . \end{aligned}$$

$T_H(\widehat{g})$ is a measurable, H-invariant mapping $\Omega_{rc} \to \mathbb{R}_0^+ \cup \{\infty\}$. The admissibility condition can then be reformulated:

$$g \in \mathrm{L}^2(U) \text{ is admissible } \Leftrightarrow T_H(\widehat{g}) \equiv 1 \quad (\text{ a.e. on } U) \quad . \tag{5.7}$$

Similarly, weak admissibility is equivalent to the requirement that $T_H(\widehat{g}) \in \mathrm{L}^\infty(U)$ and $T_H(\widehat{g}) > 0$ almost everywhere. We can also define

$$S_H(\widehat{g})(\omega) := \left(\int_{\mathcal{O}(\omega)} |\widehat{g}(\gamma)|^2 d\beta_{\mathcal{O}(\omega)}(\gamma) \right)^{1/2} .$$

By our choice of measures, S_H and T_H coincide iff G is unimodular. Both $T_H(\widehat{g})$ and $S_H(\widehat{g})$ may (and will) be regarded as functions on the quotient space U/H. By the choice of the $\beta_{\mathcal{O}(\omega)}$,

$$\int_U |\widehat{g}(\omega)|^2 d\omega = \int_{U/H} |S_H(\widehat{g})(\mathcal{O}(\omega))|^2 d\overline{\lambda}(\mathcal{O}(\omega)) \quad , \tag{5.8}$$

so that $\widehat{g}$ is square-integrable iff $S_H(\widehat{g})$ is a square-integrable function on U/H.

Now we can address the existence of admissible vectors. The following theorem is essentially the same as [77, Theorem 1.8]. Again, a comparison of this theorem with the abstract version in Theorem 4.22 is instructive. The connection will be made explicit in Theorem 5.23 below.

Theorem 5.12. *Let $U \subset \Omega_{rc}$ be measurable and H-invariant. Then π_U has a bounded cyclic vector. It has an admissible vector iff either*

(i) G is unimodular and $\overline{\lambda}(U/H) < \infty$.
(ii)G is nonunimodular.

Note that the strategy for the construction of admissible vectors in the following proof is similar to the arguments in [77], but also to the construction in Theorem 4.23. It amounts to treating the admissibility condition – involving T_H – first, and then adjusting the construction to fulfill the square-integrability condition – involving S_H – as well.

Proof. Recall that by the last remark we have for each admissible vector g that $T_H(\widehat{g})$ is constant almost everywhere. At the same time, in the unimodular case it is square-integrable as a function on U/H, because of $S_H = T_H$. This shows the necessity of (i) in the unimodular case.

To prove the existence of admissible vectors, we first construct a function $\widehat{g}$ on U fulfilling the admissibility condition (5.7), and then modify the construction to provide for square-integrability.

For this purpose we recycle the sets $A \subset \Omega_{rc}$ and $B \subset H$ from the proof of Lemma 5.9. We already observed there that $\widehat{f} = \mathbf{1}_{AB}$ is Lebesgue-measurable, and that $T_H(\widehat{f})$ is positive and finite almost everywhere on U. Hence we may define $\widehat{g} = \widehat{f}/T_H(\widehat{f})$, which fulfills the admissibility criterion. In the unimodular case, equation (5.8) together with $S_H = T_H$ shows that $\widehat{g} \in \mathrm{L}^2(U)$.

In the nonunimodular case, we modify g as follows: For every $\gamma \in U$, the compactness of $p_\gamma^{-1}(AB)$ entails that Δ_G is bounded on that set. Thus $S_H(\widehat{g})$ is positive and finite almost everywhere. Since $\overline{\lambda}$ is σ-finite, we can write $U/H = \bigcup_{n\in\mathbb{N}} V_n$, with disjoint V_n of finite measure, such that in addition $S_H(\widehat{g})$ is bounded on each V_n (here we regard $S_H(\widehat{g})$ as a function on the quotient). In particular, $S_H(\widehat{g}) \cdot \mathbf{1}_{V_n}$ is square-integrable on U/H. Now let $U_n \subset U$ be the inverse image of V_n under the quotient map, and for $h_0 \in H$ and $n, k_n \in \mathbb{N}$, denote by

$$\widehat{g_n}(\omega) := \Delta_H(h_0)^{k_n/2}\widehat{f}_2(h^{-1}.\omega_0^{k_n}) \cdot \mathbf{1}_{U_n}(\omega) \quad .$$

Then the normalization ensures that $\widehat{g_n}$ has the following properties:

$$T_H(\widehat{g_n}) = \mathbf{1}_{U_n} \tag{5.9}$$

and

$$\begin{aligned} S_H(\widehat{g_n}) &= \Delta_H(h_0)^{k_n/2}|\det(h_0)|^{-k_n/2} S_H(\widehat{g}) \cdot \mathbf{1}_{U_n} \\ &= \Delta_G(h_0)^{k_n/2} S_H(\widehat{g}) \cdot \mathbf{1}_{U_n} \quad . \end{aligned} \tag{5.10}$$

Hence the following construction gives an admissible vector: Choose $h_0 \in H$ such that $\Delta_G(h_0) < 1/2$, pick $k_n \in \mathbb{N}$ satisfying

$$2^{-k_n}\|S_H(\widehat{g}) \cdot \mathbf{1}_{U_n}\|_2^2 < 2^{-n} \tag{5.11}$$

and let $\widehat{\widetilde{g}}(\omega) := \Delta_H(h_0)^{k_n/2}\widehat{f}_2(h^{-1}.\omega_0^{k_n})$, for $\omega \in U_n$. Then (5.9) implies that $T_H(\widehat{\widetilde{g}}) = 1$ a.e., whereas (5.10) and (5.11) ensure that $S_H(\widehat{\widetilde{g}}) \in \mathrm{L}^2(U/H, \overline{\lambda})$.

A bounded cyclic vector for π_U –which is missing in the unimodular case– can be obtained by similar (somewhat simpler) methods.

Remark 5.13. In Theorem 5.12 we cannot replace Ω_{rc} by the bigger set Ω_c. To give a nonunimodular example, let $H = \{2^k h : k \in \mathbb{Z}, h \in \mathrm{SL}(2,\mathbb{Z})\}$, which is a discrete subgroup of $\mathrm{GL}(2,\mathbb{R})$. Whenever $(\gamma_1,\gamma_2) \in \widehat{\mathbb{R}^2}$ is such that γ_1/γ_2 is irrational, the stabilizer of (γ_1,γ_2) in H is finite. Hence the set Ω_c is a conull subset in $\widehat{\mathbb{R}^2}$, whereas (as we already noted) Ω_{rc} is empty. H operates ergodically on $\widehat{\mathbb{R}^2}$ (already $\mathrm{SL}(2,\mathbb{Z})$ does, [118, 2.2.9]), and we can use this fact to show that π has no admissible vectors: For every $g \in \mathrm{L}^2(\mathbb{R}^2)$, the map

$$\mathbb{R}^2 \ni \gamma \mapsto \sum_{h \in H} |\hat{g}(\gamma h)|^2$$

is measurable and H-invariant, hence, by ergodicity, it is constant almost everywhere. By the same calculation as in the proof of the admissibility condition (5.5) we see that g is admissible iff the constant is finite.

Let $g \neq 0$. Pick $\epsilon > 0$ and a Borel set A on which $|\hat{g}|^2 > \epsilon$ holds. Then

$$\sum_{h \in H} |\hat{g}(\gamma h)|^2 \geq \epsilon \sum_{h \in H} |\mathbf{1}_A(\gamma h)| \quad .$$

Choose a sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint subsets of A satisfying $\lambda(A_n) > 0$. Then, for any fixed n, the set $B_n := \{\gamma \in U : \gamma H \cap A_n \neq \emptyset\}$ is H-invariant and contains A_n, hence, by ergodicity, it is a conull set. Hence the intersection B of all B_n is a conull set, and for every $\gamma \in B$ the set $\gamma H \cap A$ is infinite, the A_n being disjoint. But this implies

$$\sum_{h \in H} |\mathbf{1}_A(\gamma h)| = \infty$$

on B, and thus a forteriori

$$\sum_{h \in H} |\hat{g}(\gamma h)|^2 = \infty \quad ,$$

hence g is not admissible.

Let us now give a short summary of the steps which have to be carried out for the construction of wavelet transforms from semidirect products:

1. Compute the H-orbits in $\widehat{\mathbb{R}^k}$, possibly by giving a parametrization of $\widehat{\mathbb{R}^k}/H$.
2. Determine the set Ω_{rc}. If $\lambda(\Omega_{rc}) = 0$, stop.
3. Parametrize each orbit in Ω_{rc} and determine the image $\mu_{\mathcal{O}(\gamma)}$ of Haar measure under the projection map $h \mapsto h^{-1}.\gamma$.
4. Compute the measure decomposition $d\lambda(\gamma) = d\beta_{\mathcal{O}(\omega)}(\gamma) d\overline{\lambda}(\mathcal{O}(\omega))$.
5. Compute the Radon-Nikodym derivative κ.
6. The admissibility condition can then be formulated for subsets of Ω_{rc} just as in Theorem 5.8. Theorem 5.12 ensures the existence of admissible vectors.

In Remark 5.17 below we carry out the steps for the Poincaré group, which was already considered in [74]. Since the final step – the actual construction of admissible vectors – is missing, the description is somewhat incomplete. Clearly the construction given in the proof of Theorem 5.12 is not very practical, but it seems doubtful to us that a more explicit method is available which works in full generality. However, in many concrete cases where parameterizations of orbits and orbit spaces are possible, they can be given differentiably. Then computing the various measures and Radon-Nikodym derivatives reduces to computing the Jacobian determinants of those parameterizations. We expect that in such a setting the construction of admissible vectors should also be facilitated.

Further simplification can be obtained by the action of a matrix group in the normalizer of H, as we will see next. For the remainder of this section we focus on the case that G is unimodular. The main motivation for the following proposition is to show that certain subrepresentations of π do not have admissible vectors. In the light of Theorem 5.12, this amounts to proving that $\overline{\lambda}(U/H)$ is infinite, for the H-invariant set $U \subset \Omega_{rc}$ under consideration.

The argument proving the following proposition employs the action of the scalars on the orbit space Ω_{rc}/H. The group of scalars could be replaced by any group $A \subset \mathrm{GL}(k,\mathbb{R})$ which normalizes H. Symmetry arguments of this type could also simplify some of the steps 1. through 6. sketched above.

The multiplicative group $(\mathbb{R}',\cdot)$ operates on $\widehat{\mathbb{R}^k}/H$ by multiplication: If $a \in \mathbb{R}^+$ then $a \cdot (\mathcal{O}(\gamma)) = \mathcal{O}(a\gamma)$ is well-defined. Obviously Ω_{rc} is invariant, so that we obtain an operation on Ω_{rc}/H. The next proposition gives the behaviour of $\overline{\lambda}$ under this action. Obviously the fixed groups are constant along $\mathbb{R}'$-orbits, i.e., $H_{a\gamma} = H_\gamma$.

Proposition 5.14. *Assume that G is unimodular. Let the measures $\overline{\lambda}$ and $\mu_{\mathcal{O}(\gamma)}$ be as in Lemma 5.9. Assume that a constant choice of Haar measure on $H_{a\gamma} = H_\gamma$ was used to compute the $\mu_{\mathcal{O}.(a\gamma)}$ $(a \in \mathbb{R}')$. For $a \in \mathbb{R}'$ and $\gamma \in \widehat{\mathbb{R}^k}$ let $a^*(\mu_{\mathcal{O}(\gamma)})$ denote the image measure of $\mu_{\mathcal{O}(\gamma)}$ on $\mathcal{O}(a\gamma)$, i.e., for measurable $B \subset \mathcal{O}(a\gamma)$ let $a^*(\mu_{\mathcal{O}(\gamma)})(B) := \mu_{\mathcal{O}(\gamma)}(a^{-1}B)$. Moreover let the measure $\overline{\lambda}_a$ be given by $\overline{\lambda}_a(B) := \overline{\lambda}(aB)$ $(B \subset \widehat{\mathbb{R}^k}/H$ measurable). Then on Ω_{rc}/H the following relations hold:*

$$\mu_{\mathcal{O}(a\gamma)} = a^*(\mu_{\mathcal{O}(\gamma)}),$$
$$\overline{\lambda}_a = |a|^n \overline{\lambda}.$$

Proof. The first equality is immediate from the definitions of $\mu_{\mathcal{O}(\gamma)}$ and $\mu_{\mathcal{O}(a\gamma)}$. For the second equation let us introduce the following notation: If $f : \Omega_{rc} \to \mathbb{R}$ is a positive, measurable function, let $q(f)$ denote the function on Ω_{rc}/H defined by

$$q(f)(\mathcal{O}(\gamma)) := \int_{\mathcal{O}(\gamma)} f(\omega) d\mu_{\mathcal{O}(\gamma)}(\omega).$$

Moreover let $f_a(\omega) := f(a^{-1}\omega)$, for all $\omega \in \Omega_{rc}$ and $a \in \mathbb{R}'$. From the first equation we obtain

$$\begin{aligned} q(f_a)(\mathcal{O}(\gamma)) &= \int_{\mathcal{O}(\gamma)} f(a^{-1}\omega) d\mu_{\mathcal{O}(\gamma)}(\omega) \\ &= \int_{\mathcal{O}(\gamma)a^{-1}} f(\omega) d\mu_{\mathcal{O}(a^{-1}\gamma)}(\omega) \\ &= q(f)(\mathcal{O}(a^{-1}\gamma)). \end{aligned}$$

Using this equation, we compute

$$\begin{aligned} \int_{\Omega_{rc}/H} q(f)(\mathcal{O}(\gamma)) d\overline{\lambda}(\mathcal{O}(\gamma)) &= a^{-n} \int_{\Omega_{rc}} f_a(\omega) d\lambda(\omega) \\ &= a^{-n} \int_{\Omega_{rc}/H} q(f_a)(\mathcal{O}(\gamma)) d\overline{\lambda}(\mathcal{O}(\gamma)) \\ &= a^{-n} \int_{\Omega_{rc}/H} q(f)(\mathcal{O}(a^{-1}\gamma)) d\overline{\lambda}(\mathcal{O}(\gamma)) \\ &= a^{-n} \int_{\Omega_{rc}/H} q(f)(\mathcal{O}(\gamma)) d\overline{\lambda}_a(\mathcal{O}(\gamma)) \end{aligned}$$

Using arguments similar to the one in the proof of Theorem 5.12, it is readily seen that for each measurable $A \subset \Omega_{rc}/H$ there exists a positive measurable f on Ω_{rc} with $q(f) = \mathbf{1}_A$. Hence we have shown the second equation.

As a first consequence we obtain that admissible vectors exist only for proper subsets of Ω_{rc}. This was already noted (in the special case where Ω_{rc} is conull in $\widehat{\mathbb{R}^k}$) in [77, Theorem 1.8].

Corollary 5.15. *Assume that G is unimodular, and that $U := \Omega_{rc}$ is not a nullset. Then the subrepresentation π_U does not have an admissible vector.*

Proof. By assumption we have $\overline{\lambda}(\Omega_{rc}/H) > 0$, and we need to show that $\overline{\lambda}(\Omega_{rc}/H) = \infty$. But for all $a \in \mathbb{R}'$, $a\Omega_{rc} = \Omega_{rc}$, and Proposition 5.14 yields $\overline{\lambda}(\Omega_{rc}/H) = \overline{\lambda}(a \cdot \Omega_{rc}/H) = |a|^{-k}\overline{\lambda}(\Omega_{rc}/H)$.

Recall from Theorem 2.25 (c) that, given a square integrable representation σ of a locally compact group G, every vector in $\mathcal{H}_\sigma$ is admissible iff σ is irreducible and G is unimodular. Hence discrete series representations of unimodular groups are particularly useful, having no restrictions at all on admissible vectors. But the following corollary excludes irreducible representations from our setting. The statement was proved first in [50] by a technique employing the Fell topology of the group.

Corollary 5.16. *Let G be unimodular. Then the quasiregular representation π does not contain any irreducible square-integrable subrepresentations.*

Proof. Assume the contrary and let π_U be an irreducible square integrable subrepresentation. Here U denotes the corresponding H-invariant subset of $\widehat{\mathbb{R}^k}$. Then, by [6, Theorem 1.1], U is (up to a null set) an orbit of positive measure, hence open by Sard's Theorem. In particular $U \subset \Omega_{rc}$, and $\overline{\lambda}(\{U\}) > 0$.

From the fact that $\mathcal{H}_U$ has admissible vectors we conclude that $\overline{\lambda}(\{U\}) < \infty$. On the other hand, an easy connectedness argument shows that for each $\gamma \in U$, the ray $\mathbb{R}^+\gamma$ is contained in the open orbit U. Hence the same argument which proved the previous corollary shows that $\overline{\lambda}(\{U\}) = \infty$, which yields the desired contradiction.

For illustration we now carry out the various steps for the Poincaré group, already considered in [74].

Remark 5.17. Admissibility conditions for the Poincaré groups. For $k \geq 3$ let L_k denote the Lorentz bilinear form on $\mathbb{R}^k$, i.e., $L_k(x,y) = -x_1y_1 + x_2y_2 + x_3y_3 + \ldots + x_ky_k$ and let $H = SO_0(k-1,1)$ denote the connected component of the linear group leaving L_k invariant. We exclude the case $k = 2$ for simplicity. The Plancherel measure was explicitly calculated in [75]. We want to determine the admissibility condition for the quasiregular representation associated with this group. For this purpose we need to compute the measure decomposition of Lebesgue-measure on $\widehat{\mathbb{R}^k}$, and we employ the symmetry arguments from 5.14 for this purpose.

1. The $SO_0(k-1,1)$-orbits in $\widehat{\mathbb{R}^k}$ can be parameterized as follows (see, e.g., [75, I, Example 2, Section 10])

$$\{0\}, \{\gamma \in \widehat{\mathbb{R}^k} : L_k(\gamma,\gamma) = 0, \gamma_1 > 0\}, \{\gamma \in \widehat{\mathbb{R}^k} : L_k(\gamma,\gamma) = 0, \gamma_1 < 0\},$$
$$\mathcal{O}_r^+ = \{\gamma \in \widehat{\mathbb{R}^k} : L_k(\gamma,\gamma) = -r, \gamma_1 < 0\} \qquad (r > 0),$$
$$\mathcal{O}_r^- = \{\gamma \in \widehat{\mathbb{R}^k} : L_k(\gamma,\gamma) = -r, \gamma_1 > 0\} \qquad (r > 0),$$
$$\mathcal{U}_r = \{\gamma \in \widehat{\mathbb{R}^k} : L_k(\gamma,\gamma) = r, \gamma_1 > 0\} \qquad (r > 0).$$

   Clearly the first three orbits may be neglected, leaving us with three families of orbits, each parameterized by a ray. Hence, under the action of $\mathbb{R}^+$ on $\widehat{\mathbb{R}^k}$ used in 5.14, we have essentially two A-orbits in $\widehat{\mathbb{R}^k}/H$, i.e., the $\mathcal{O}_r^{\pm}$-families yield one orbit, and the remaining A-orbit is obtained from the $\mathcal{U}_r$-family. Hence, $\overline{\lambda}$ is more or less completely determined by the action of A, and the $\mu_{\mathcal{O}(\gamma)}$ have to be computed for at most two H-orbits, which are represented by $\gamma_1 := (1,0,\ldots,0)$ and $\gamma_2 := (0,\ldots,0,1)$.
2. Now let us determine Ω_{rc}. It is easily seen that the fixed group H_{γ_1} is canonically isomorphic to $SO(k-1)$, whereas H_{γ_2} is isomorphic to $SO(k-2,1)$. In particular the former is compact and the latter is not. Clearly Ω_{rc} is not a null set, hence we may continue.
3. The orbit $\mathcal{O}(\gamma)_1$ can be parametrized by

$$\Psi : \mathbb{R} \times S^{k-2} \ni (u, \mathbf{y}) \mapsto (\sinh u, y_1 \cosh u, \ldots, y_{k-1} \cosh u).$$

The measure $d\mu_{\mathcal{O}(\gamma)_1} = du d\mathbf{y}$, where du is the usual measure on $\mathbb{R}$ and $d\mathbf{y}$ is the rotation-invariant surface measure on the sphere, is easily verified to be invariant under the action of H. Hence it is the image of μ_H under the quotient map, since H is unimodular. (Note that here we have in fact fixed a normalization of μ_H.) The parametrization of $\mathcal{O}(a\gamma_1)$ is $a \cdot \Psi$, and $\mu_{\mathcal{O}(a\gamma_1)} = a^*(\mu_{\mathcal{O}(\gamma_1)}$.

4. As we have seen, Ω_{rc}/H may be identified with $\mathbb{R} \setminus \{0\}$, and the measure $\overline{\lambda}$ has to fulfill the relation $\overline{\lambda}_a = |a|^k \overline{\lambda}$, whence we immediately obtain $d\overline{\lambda}(r) = |r|^{k-1} dr$.
5. G is unimodular, and we have chosen the $\mu_{\mathcal{O}(\gamma)}$ so as to ensure $\kappa \equiv 1$.
6. We obtain the following admissibility condition: An invariant subspace $\mathcal{H}_U \subset \mathrm{L}^2(\mathbb{R}^k)$ has an admissible vector iff the corresponding H-invariant subset $U \subset \widehat{\mathbb{R}^k}$ is contained in Ω_{rc} and its projection $\overline{U} := \{r \in \mathbb{R} : (r, 0, \ldots, 0) \in U\}$ satisfies
$$\int_{\overline{U}} |r|^{k-1} dr < \infty. \tag{5.12}$$
A vector $f \in \mathcal{H}_U$ is admissible for $\mathcal{H}_U$ iff
$$\int_{\mathbb{R}} \int_{S^{k-2}} |\widehat{f}(r \sinh u, r y_1 \cosh u, \ldots, r y_{k-1} \cosh u)|^2 d\mathbf{y} du = 1 \text{ for a.e. } r \in \overline{U}.$$
This normalization refers to the fixed choice of Haar measure μ_H made in step 3.
7. It is now simple to produce admissible vectors g for arbitrary H-invariant sets $U \subset \widehat{\mathbb{R}^k}$ satisfying (5.12): Fix a function g_0 on $\mathbb{R} \times S^{k-2}$ with
$$\int_{\mathbb{R}} \int_{S^{k-2}} |g_0(u, \mathbf{y})|^2 d\mathbf{y} du = 1$$
and let
$$\widehat{g}(r \sinh u, r y_1 \cosh u, \ldots, r y_{k-1} \cosh u) = g_0(u, \mathbf{y}) \quad .$$

5.3 Concrete and Abstract Admissibility Conditions

We will now work out the connection between the admissibility criteria of the previous section and the Plancherel transform of G. We consider a more general setting, i.e., $G = N \rtimes H$, where N is a unimodular group. Note that since G/N carries an invariant measure, Δ_N is the restriction of Δ_G to N. Hence $N \subset \mathrm{Ker}(\Delta_G)$, so that Δ_G can be lifted to a function on H.

We denote by $h \mapsto \alpha_h$ the associated homomorphism $H \to \mathrm{Aut}(N)$. If we identify H with a subgroup of G, then $\alpha_h(n) = hnh^{-1}$. Elements of G are pairs $(x,h) \in N \times H$, with the group law $(x,h)(x',h') = (x\alpha_h(x'), hh')$. The **quasiregular representation** $\pi = \mathrm{Ind}_H^G 1$ acts on $\mathrm{L}^2(N)$ by

$$(\pi(x,h)f)(y) = \Delta_G(h)^{-1/2} f(\alpha_h^{-1}(x^{-1}y)) \quad .$$

We want to establish when $\pi \le \lambda_G$, by decomposing π into irreducibles and checking for absolute continuity with respect to Plancherel measure.

Next we compute the decomposition of π. Since the restriction of π to N is λ_N, it is no surprise that the Plancherel transform of N is a useful tool for the decomposition of π into irreducible representations. Thus far the discussion runs completely parallel to the one in the previous two sections.

Proposition 5.18. *Let*

$$\mathcal{P}_N : \mathrm{L}^2(N) \to \int_{\widehat{N}}^{\oplus} \mathcal{B}_2(\mathcal{H}_\sigma) d\nu_N(\sigma)$$

be the Plancherel transform of N. Define a representation $\widehat{\pi}$ of G acting on the right hand side by $\widehat{\pi}(x,h) = \mathcal{P}_N \circ \pi(x,h) \circ \mathcal{P}_N^{-1}$. Then

$$(\widehat{\pi}(x,h)F)(\sigma) = \Delta_G(h)^{1/2} \sigma(x) \circ F(h^{-1}.\sigma) \tag{5.13}$$

Proof. For $F = \widehat{f}$ with $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$, and $\varphi, \eta \in \mathcal{H}_\sigma$, we compute

$$\begin{aligned}
\langle (\widehat{\pi}(x,t)F)(\sigma)\varphi, \eta \rangle &= \int_N \Delta_G^{-1/2}(h) f(\alpha_h^{-1}(x^{-1}y)) \langle \sigma(y)\varphi, \eta \rangle dy \\
&= \int_N \Delta_G^{-1/2}(h) f(\alpha_h^{-1}(y)) \langle \sigma(x)\sigma(y)\varphi, \eta \rangle dy \\
&= \Delta_G^{1/2}(h) \int_N f(y) \langle \sigma(x)\sigma(\alpha_h(y))\varphi, \eta \rangle dy \\
&= \Delta_G^{1/2}(h) \sigma(x) \widehat{f}(h^{-1}.\sigma) \quad .
\end{aligned}$$

Proposition 5.19. *Assume that for a ν_N-conull subset the Mackey obstructions are particularly trivial in the sense of 3.38. For almost every orbit $\mathcal{O}(\sigma) \subset \widehat{N}$, let $\mu_{\mathcal{O}(\sigma)}$ denote the measure arising from the measure decomposition*

$$d\nu_N = d\mu_{\mathcal{O}(\sigma)} d\overline{\nu_N} \quad . \tag{5.14}$$

By standardness of ν_N/H, these measures exist and are unique ν_N-almost everywhere. We define the representations ϱ_σ acting on $\mathrm{L}^2(\mathcal{O}(\sigma), \mu_{\mathcal{O}(\sigma)}; \mathcal{B}_2(\mathcal{H}_\sigma))$ via

$$(\varrho_\sigma(x,h)F)(\omega) = \Delta_G(h)^{1/2} \omega(x) \circ F(h^{-1}.\omega) \quad . \tag{5.15}$$

Then the following statements hold:

(a) $\varrho_\sigma \simeq \dim(\mathcal{H}_\sigma) \cdot \mathrm{Ind}_{G_\sigma}^G \sigma \times 1$.

(b) The Plancherel transform of N effects a decomposition

$$\pi \simeq \int_{\Sigma/H}^{\oplus} \dim(\mathcal{H}_\sigma) \cdot \left(\mathrm{Ind}_{G_\sigma}^G \sigma \times 1\right) d\overline{\nu_N}(\sigma) \quad . \tag{5.16}$$

This is a decomposition into irreducibles.

Proof. We first show that ϱ_σ is induced from the left action via σ on $\mathcal{B}_2(\mathcal{H}_\sigma)$, by comparing (5.15) with (3.29). First observe that by relation (5.14) and

$$\frac{d\nu_{N(x,h)^{-1}}}{d\nu_N}(\omega) = \Delta_G(h) \quad ,$$

we obtain that

$$\frac{d\mu_{\mathcal{O}(\sigma)(x,h)^{-1}}}{d\mu_{\mathcal{O}(\sigma)}}(\omega) = \Delta_G(h) \quad .$$

In other words, the Radon-Nikodym derivative used in the definition of ϱ_σ coincides with the one employed in (3.29).

Next pick a cross-section $\alpha_0 : \mathcal{O}(\sigma) \to H$, and let $\alpha : G.\sigma \to G$ be defined by $\alpha(\omega) = (e_N, \alpha_0(\omega))$, where e_N denotes the neutral element in N. Then we find for $(x,h) \in G$ that

$$\begin{aligned} \alpha(\omega)^{-1}(x,h)\alpha((n,h)^{-1}\omega)) &= (e_N, \alpha_0(\omega)^{-1})(x,h)(e_N, \alpha_0((x,h)^{-1}\omega)) \\ &= (\alpha_0(\omega)^{-1}(x), \alpha_0(\omega)^{-1}h\alpha_0(h^{-1}.\omega)) \quad , \end{aligned}$$

with $\alpha_0(\omega)^{-1}h\alpha_0(h^{-1}.\omega) \in H_\sigma$. By assumption, σ extends trivially to a representation of the big fixed group $G_\sigma = N \rtimes H_\sigma$. Hence

$$(\sigma \times 1)\left(\alpha(\omega)^{-1}(x,h)\alpha((x,h)^{-1}\omega)\right) = (\alpha_0(\omega).\sigma)(x) = \omega(x)$$

Hence the second term, $\omega(x)$, in the definition of ϱ_σ also coincides with the second term of the right hand side of (3.29). Since the same is obviously true for the third terms, we are finished.

The left action via σ on $\mathcal{B}_2(\mathcal{H}_\sigma)$ is clearly a $\dim(\mathcal{H}_\sigma)$-fold multiple of σ, and induction commutes with taking direct sums [45, 6.9], hence (a) follows. In view of this and Proposition 3.29, part (b) follows from a comparison of the definition of the ϱ_σ with (5.13). It is a decomposition into irreducibles by Mackey's theorem.

Next we show the desired containment result. As in the above discussion the set $\Omega_c = \{\sigma \in \widehat{N} : H_\sigma \text{ is compact }\}$ plays a crucial role. We have again, with the same arguments as before:

Lemma 5.20. *Ω_c is a Borel set.*

Lemma 5.21. *The mapping $\Phi : \mathcal{O}(\gamma) \mapsto \mathrm{Ind}_{G_\gamma}^G \gamma \times 1$ is a Borel isomorphism onto a Borel subset Σ of $\widehat{G}$.*

Proof. This follows from the decomposition in 5.19 and the uniqueness theorem 3.25.

Theorem 5.22. *Let $G = N \rtimes H$, and suppose that G and N are as in Theorem 3.40. Letting*

$$\pi_1 \simeq \int_{\Omega_c}^{\oplus} \dim(\mathcal{H}_\sigma) \cdot (\mathrm{Ind}_{G_\sigma}^G \sigma \times 1) d\overline{\nu}(\mathcal{O}(\gamma)) \tag{5.17}$$

and π_2 the orthogonal complement in π, then $\pi_1 < \lambda_G$, and π_2 is disjoint from λ_G.

Proof. Denote the part corresponding to Ω_c by Σ_c. The image measure of $\overline{\nu_N}$ under this map is a standard measure $\widetilde{\nu}$ on $\widehat{G}$. The key observation is now that, with respect to the measure decomposition (3.32), Σ meets each fibre in exactly one point, namely in $\sigma \times 1$. In computing $\nu_N(B)$, for subsets B of Σ, the inner integral is simply $\nu_{H_\gamma}(\{1\})$, which is positive iff $\mathcal{H}_\gamma$ is compact. In short, $\widetilde{\nu}$ is ν_G-absolutely continuous on Σ_c, and disjoint with ν_G on $\Sigma \setminus \Sigma_c$. The containment statement $\pi_1 < \lambda_G$ is obtained by checking the conditions in 3.26: The absolute continuity requirement has already been verified. For the comparison of multiplicities note that the representation space of $\mathrm{Ind}_{G_\gamma}^G \sigma \times 1$ is a nontrivial $\mathcal{H}_\sigma$-valued L^2-space, thus its dimension is necessarily $\geq \dim(\mathcal{H}_\sigma)$. The former is the multiplicity of $\mathrm{Ind}_{G_\gamma}^G \sigma \times 1$ in λ_G, the latter (smaller) is the multiplicity of the same representation in π. Thus follows the containment statement.

The disjointness part is due to Corollary 3.18.

The disjointness statement means that π_1 is the maximal subrepresentation of π which is contained in λ_G.

Let us now take a second look at the case $N = \mathbb{R}^k$. The following theorem explains how the different admissibility conditions are related:

Theorem 5.23. *Let $G = \mathbb{R}^k \rtimes H$, and assume that G and $N = \mathbb{R}^k$ fulfill all requirements of Theorem 3.40. For $\mathcal{O}(\gamma) \subset \Omega_c$ let $\mathcal{K}_{\mathcal{O}(\gamma)}$ denote the operator on $\mathrm{L}^2(\mathcal{O}(\gamma), d\beta_{\mathcal{O}(\gamma)})$ given by pointwise multiplication with $\kappa|_{\mathcal{O}(\gamma)}$. The map Φ from Lemma 5.21 gives rise to the following correspondences between the objects in Section 5.2 and those appearing in the Plancherel decomposition:*

$$\begin{aligned}
\Omega_c/H &\longleftrightarrow \Sigma \quad , \\
\mathcal{O}(\gamma) &\longleftrightarrow \sigma \quad , \\
\mathrm{L}^2(\mathcal{O}(\gamma), d\beta_{\mathcal{O}(\gamma)}) &\longleftrightarrow \mathcal{H}_\sigma \quad , \\
\widehat{f}|_{\mathcal{O}(\gamma)} &\longleftrightarrow \eta_\sigma \quad , \\
S_H(\widehat{f})(\mathcal{O}(\gamma)) &\longleftrightarrow \|\eta_\sigma\| \quad , \\
\overline{\lambda} &\longleftrightarrow \nu_G \quad , \\
\mathcal{K}_{\mathcal{O}(\gamma)} &\longleftrightarrow C_\sigma^{-2} \quad , \\
T_H(\widehat{f})(\mathcal{O}(\gamma)) &\longleftrightarrow \|C_\sigma \eta_\sigma\| \quad .
\end{aligned}$$

In particular, the admissibility criterion from Theorem 5.8 is a special case of (4.15).

Proof. It remains to check that the C_σ^{-2} corresponds to $\mathcal{K}_{\mathcal{O}(\gamma)}$, and that the Plancherel measure ν_G belonging to this particular choice of Duflo-Moore operators corresponds to $\overline{\lambda}$. Straightforward calculation, using relation (5.6) from Lemma 5.9, shows that $\mathcal{K}_{\mathcal{O}(\gamma)}$ satisfies the semi-invariance relation

$$\left(\mathrm{Ind}_{G_\gamma}^G(\gamma \times 1)(x,h)\right) \, \mathcal{K}_{\mathcal{O}(\gamma)} \, \left(\mathrm{Ind}_{G_\gamma}^G(\gamma \times 1)(x,h)\right)^* = \Delta_G(x,h)^{-1}\mathcal{K}_{\mathcal{O}(\gamma)} \quad .$$

It follows that $K_{\mathcal{O}(\gamma)}^{-1/2}$ obeys the semi-invariance relation (3.51), hence Theorem 3.48(e) entails the desired correspondence.

It remains to prove that, given this particular choice of Duflo-Moore operators, the measure $\overline{\lambda}$ is the corresponding Plancherel measure. But in view of the identifications we already established, (4.15) yields for every H-invariant measurable $U \subset \widehat{\mathbb{R}^k}$ that

$$g \in \mathcal{H}_U \text{ is admissible } \iff \left(\frac{d\overline{\lambda}}{d\nu_G}(\mathcal{O}(\gamma))\right)^{1/2} \|K_{\mathcal{O}(\gamma)}^{-1/2}(\widehat{g}|_{\mathcal{O}(\gamma)})\|_2 = 1 \;, \text{ (a.e.)} \quad .$$

On the other hand, (5.5) provides

$$g \in \mathcal{H}_U \text{ is admissible } \iff \|K_{\mathcal{O}(\gamma)}^{-1/2}(\widehat{g}|_{\mathcal{O}(\gamma)})\|_2 = 1 \;, \text{ (a.e.)} \quad .$$

Hence the Radon-Nikodym derivative is trivial.

As mentioned before, the majority of authors dealing with wavelets from semidirect products of the form $\mathbb{R}^k \rtimes H$ concentrated on discrete series representations occurring as subrepresentations of the quasiregular representation. In this context, a well-known result relates the existence of such representations to open dual orbits with associated compact fixed groups, see e.g. [22, 48]. This condition can now be retrieved, by restricting the discrete series criterion in Corollary 3.41 to representations of the form $\mathrm{Ind}_{G_\sigma}^G \sigma \times 1$, as they appear in the decomposition of the quasiregular representation. Note that a group is compact iff the Haar measure of the group is finite, iff the trivial representation is square-integrable.

Corollary 5.24. *The quasiregular representation π contains a discrete series representation iff there exists a dual orbit $\mathcal{O}(\gamma) \subset \widehat{N}$ of positive Plancherel measure, such that in addition H_γ is compact.*

Note that if N is a vector group and H a closed matrix group, then Sard's theorem implies that all orbits of positive measure are indeed open. See [49] for a proof.

5.4 Wavelets on Homogeneous Groups**

In this section we use Theorem 4.22 to prove that there exists a continuous wavelet transform on homogeneous Lie groups. Starting from a homogeneous Lie group N with a one-parameter group H of dilations, we show that the quasiregular representation of $G = N \rtimes H$ on $\mathrm{L}^2(N)$ has admissible vectors. Since G is nonunimodular, containment in λ_G will be sufficient for that.

The resolution of the identity provided by the wavelet transform can also be read as a continuous Calderon reproducing formula. Discrete versions of the Calderon reproducing formula have been employed for the analysis of pseudodifferential operators on these groups [47], and it is conceivable that the wavelet transform we present below could be useful for these purposes also. Note however that we only provide the **existence** of admissible vectors. Unless N is a vector group, the arising representation has infinite multiplicity, and an explicit characterization of admissible vectors will be a tough problem.

Now for the definition of homogeneous Lie groups.

Definition 5.25. *A connected simply connected Lie group N with Lie algebra $\mathfrak{n}$ is called* **homogeneous**, *if there exists a one-parameter group $H = \{\delta_r : r \geq 0\}$ of Lie algebra automorphisms of the form $\delta_r = \mathrm{e}^{A \log r}$, such that in addition A is diagonalizable with strictly positive eigenvalues. The elements of H are called* **dilations**.

We define $G = N \rtimes H$, and π as the quasiregular representation $\pi = \mathrm{Ind}_H^G 1$ of G. The homogenous structure of the group allows a rather nice geometric interpretation of G and π: It can be shown that N carries a "homogeneous norm" $|\cdot| : N \to \mathbb{R}_0^+$, on which the dilations act in the expected way, i.e., $|\delta_r(x)| = r|x|$. See [47, Chapter 1] for details. Hence the group G consists of shifts on N and "zooms" with respect to the norm, and the interpretation of the continuous wavelet transform as a "mathematical microscope" carries over from $\mathbb{R}$ to N. However, the "mathematical microscope" view requires some sort of decay behaviour of the admissible vectors (ideally, compact support), and whether admissible vectors exist with these properties is unclear.

Lemma 5.26. *Let N be homogeneous with dilation group H and the associated infinitesimal generator A. For $a \in \mathbb{R}$, let $E_a = \mathrm{Ker}(A - a\mathrm{Id})$. Then we have:*

(i) If $X \in E_a$ and $Y \in E_b$, then $[X, Y] \in E_{a+b}$.
(ii) N is nilpotent. H acts as a group of automorphisms of N.
(iii) The Haar measure on N is Lebesgue measure. We have $\mu_N(\delta_r(E)) = r^Q \mu_N(E)$, where $Q := \mathrm{trace}(A) > 0$. Hence G is nonunimodular, with $N = \mathrm{Ker}(\Delta_G)$.
(iv) Let $0 < a_1 \leq a_2 \leq \ldots \leq a_n$ be the eigenvalues of A, each occurring with its geometric multiplicity. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{n}$ with $AX_i = a_i X_i$. Given $Y \in \mathfrak{n}$ arbitrary, $\mathrm{Ad}(Y)$ is properly upper triangular with respect to this basis.

(v) For almost all $\sigma \in \widehat{N}$ the little fixed group H_σ is trivial.

Proof. See [47] for $(i), (ii)$ and the formula $\mu_N(\delta_r(E)) = r^Q \mu_N(E)$. But this formula entails that $\Delta_G(x,r) = r^Q$, and since $Q > 0$, we find $N = \mathrm{Ker}(\Delta_G)$. (iv) is immediate from (i). (v) follows from (iii) and Proposition 3.50(ii).

We need one more auxiliary result to show that the quasiregular representation on $\mathrm{L}^2(N)$ is contained in λ_G, namely the fact that G is type I.

Proposition 5.27. *G is an exponential Lie group. In particular, G is type I.*

Proof. Let $\mathfrak{g}$ denote the Lie algebra of G. In order to prove that G is exponential, we need to show that there does not exist a $Z \in \mathfrak{g}$ for which $\mathrm{ad}(Z)$ has a purely imaginary eigenvalue [34]. Let $a_1, \ldots, a_n$ and the associated eigenbasis $X_1, \ldots, X_n$ of $\mathfrak{n}$ be as in Lemma 5.26. Then a basis of $\mathfrak{g}$ is given by $X_1, \ldots, X_n, Y$, with $[Y, X_i] = a_i X_i$. It is therefore straightforward to compute that, for an arbitrary $Z = tY + \sum_{i=1}^n s_i X_i = tY + X$, the matrix of $\mathrm{ad}(Z)$ with respect to our basis is

$$\begin{pmatrix} tA + M & v \\ 0 & 0 \end{pmatrix} \quad , \tag{5.18}$$

where M is the matrix of $\mathrm{ad}(X)$ (acting on $\mathfrak{n}$), in particular (properly) upper triangular by Lemma 5.26 (iv), and v is some column vector. But this matrix clearly has the eigenvalues $ta_1, \ldots, ta_n$ and 0. Hence G is exponential, and [23, Chap. VI, 2.11] implies that G is type I.

Now Remark 5.26 (v) and Theorem 5.22 allow to conclude that π is contained in the regular representation. Hence 4.22 provides the desired existence result:

Corollary 5.28. *The quasi-regular representation $\pi = \mathrm{Ind}_H^G 1$ is contained in λ_G. Hence there exists a continuous wavelet transform on N arising from the action of N by left translations and the action of the dilations.*

The bad news is that, unless N is abelian (in which case we are back to the first sections of this chapter) the problem of computing concrete admissible vectors is essentially equivalent to that of computing admissible vectors for λ_G:

Proposition 5.29. *We have $\pi \approx \lambda_G$. Moreover, $\pi \simeq \lambda_G$ iff N is nonabelian.*

Proof. For the first statement we observe that ν_G is precisely the quotient measure on $\widehat{N}/H$, which also underlies the decomposition of π, by (5.16). Hence $\nu_G \approx \pi$.

If N is abelian, then π is multiplicity-free; already the restriction $\pi|_N$ is. But G is nonabelian, and therefore λ_G is not multiplicity-free.

Conversely, assume N to be nonabelian. It is enough to prove that $\pi \simeq \infty \cdot \pi$, which in view of (5.16) amounts to proving $\dim(\mathcal{H}_\sigma) = \infty$, for ν_N-almost every σ. Since N is not abelian, there exist coadjoint orbits of positive dimensions, and the coadjoint orbits of maximal dimension are a conull subset of $\mathfrak{n}^*$. Hence ν_N-almost every σ corresponds to a coadjoint orbit of positive – and necessarily even – dimension, say of dimension $2k$. By construction of the Kirillov map, σ is therefore induced from a character of a subgroup M of codimension k, hence $\mathcal{H}_\sigma = \mathrm{L}^2(N/M)$ is infinite-dimensional. Since the action of H on $\widehat{N}$ is free ν_N-almost everywhere, each dual orbit in $\widehat{N}$ contributes precisely one representation, which also occurs in (5.16).

5.5 Zak Transform Conditions for Weyl-Heisenberg Frames

This section deals with another class of examples. The results presented here are taken from [54]. We consider a characterization of tight Weyl-Heisenberg frames via the Zak transform. It turns out that it can be seen as a special instance of the Plancherel transform criterion, where the underlying group is discrete and type I. This makes the example somewhat remarkable. As in the semidirect product case, the admissibility conditions which arise are already known and obtainable by less involved machinery also.

Admissibility Conditions and Weyl-Heisenberg Frames

Weyl-Heisenberg systems are discretizations of the windowed Fourier transform introduced in Example 2.27. To make things precise, define the translation operators T_x and modulation operators M_ω on $\mathrm{L}^2(\mathbb{R})$ by

$$(T_x f)(y) = f(y-x) \quad , \quad (M_\omega f)(y) = \mathrm{e}^{2\pi \mathrm{i} \omega y} f(y) \quad .$$

Now a **Weyl-Heisenberg system** $\mathcal{G}(\alpha, \beta, g)$ of $\mathrm{L}^2(\mathbb{R})$ is a family

$$g_{k,m} = M_{\alpha k} T_{\beta m} g \quad (m, k \in \mathbb{Z}),$$

arising from a fixed vector $g \in \mathrm{L}^2(\mathbb{R})$ and $\alpha, \beta \neq 0$. A **(normalized, tight) Weyl-Heisenberg frame** is a Weyl-Heisenberg system that is a (normalized, tight) frame of $\mathrm{L}^2(\mathbb{R})$. There exist several alternative definitions, with varying indexing and ordering of operators. However, up to phase factors which clearly do not affect any of the frame properties, the resulting systems are identical.

Here we focus on normalized tight Weyl-Heisenberg frames **with integer oversampling $\boldsymbol{L}$** ($L \in \mathbb{N}$), which corresponds to choosing $\alpha = 1$ and $\beta = 1/L$. Given L, the problem is to decide for a given g whether it induces a normalized tight Weyl-Heisenberg frame or not. As we will see in the next subsection, the Zak transform allows a precise answer to this question. Our next aim is to show that the condition is in fact an admissibility condition for ψ. Note that for $L = 1$, this is obvious: The set $\{T_n M_m : n, m \in \mathbb{Z}\}$ is an abelian subgroup of the unitary group of $\mathrm{L}^2(\mathbb{R})$, and the normalized tight frame condition precisely means admissibility in this case. For $L > 1$ however, $\{T_n M_{m/L} : n, m \in \mathbb{Z}\}$ is not a subgroup, and we have to deal with the nonabelian group G generated by this set. Hence, for the duration of this section, we fix an integer oversampling rate $L \geq 1$, and define the underlying group G as

$$G = \mathbb{Z} \times \mathbb{Z} \times (\mathbb{Z}/L\mathbb{Z}) \quad ,$$

with the group law

$$(n, k, \overline{\ell})(n', k', \overline{\ell'}) = (n + n', k + k', \overline{\ell} + \overline{\ell'} + \overline{k'n}) \tag{5.19}$$

and inverse given by $(n, k, \overline{\ell})^{-1} = (-n, -k, \overline{-\ell + kn})$. Here we used the notation $\overline{n} = n + L\mathbb{Z}$. The representation π of G acts on $\mathrm{L}^2(\mathbb{R})$ by

$$\pi(n, k, \overline{\ell}) = \mathrm{e}^{2\pi \mathrm{i}(\ell - nk)/L} M_{n/L} T_k = \mathrm{e}^{2\pi \mathrm{i}\ell/L} T_k M_{n/L} \quad .$$

It is straightforward to check how normalized tight frames with oversampling L relate to admissibility for π:

Lemma 5.30. *Let $\psi \in \mathrm{L}^2(\mathbb{R})$. Then $(M_{n/L} T_k \psi)_{n,k \in \mathbb{Z}}$ is a normalized tight frame iff $\frac{1}{\sqrt{L}}\psi$ is admissible for π.*

Proof. The relation

$$M_{n/L} T_k \psi = \mathrm{e}^{-2\pi \mathrm{i}(\ell - nk)/L} \pi(k, n, \overline{\ell}) \psi$$

implies for all $g \in \mathrm{L}^2(\mathbb{R})$ that

$$\sum_{n,k,\overline{\ell}} \left| \langle g, \pi(k, n, \overline{\ell}) f \rangle \right|^2 = L \sum_{n,k} \left| \langle g, M_{n/L} T_k f \rangle \right|^2 \quad ,$$

which shows the claim.

The following lemma establishes that G is a finite extension of an abelian normal subgroup N. It is central for our purposes: It ensures that G is type I, and it allows to compute ν_G via Theorem 3.40.

Lemma 5.31. *Let*

$$N = \{(nL, k, \overline{\ell}) : k, n, \ell \in \mathbb{Z}\} \quad .$$

Then N is an abelian normal subgroup of G with $G/N \cong \mathbb{Z}/L\mathbb{Z}$. In particular, G is type I.

Proof. The statements concerning N are obvious from (5.19); for the description of G/N use the representatives $(0, 0, 0), (1, 0, 0), \ldots, (L - 1, 0, 0)$ of the N-cosets. The type I property of G follows by Theorem 3.39 (c), observing that the orbit space of a standard Borel space by a measurable finite group action is standard.

Zak Transform Criteria for Tight Weyl-Heisenberg Frames

In this subsection we introduce the Zak transform and formulate the criterion for normalized tight Weyl-Heisenberg frames. Our main reference for the following will be [58].

Definition 5.32. *For $f \in C_c(\mathbb{R})$, define the Zak transform of f as the function $\mathcal{Z}f : \mathbb{R}^2 \to \mathbb{C}$ given by*

$$\mathcal{Z}f(x,\omega) = \sum_{m\in\mathbb{Z}} f(x-m)\mathrm{e}^{2\pi \mathrm{i} m\omega} \quad .$$

The definition of the Zak transform immediately implies a quasi-periodicity condition for $F = \mathcal{Z}f$:

$$\forall m,n \in \mathbb{Z} \quad : \quad F(x+m,\omega+n) = \mathrm{e}^{2\pi \mathrm{i} m\omega} F(x,\omega) \quad . \tag{5.20}$$

In particular, the Zak transform of a function f is uniquely determined by its restriction to the unit square $[0,1]^2$. We next extend the Zak transform to a unitary operator $\mathcal{Z} : \mathrm{L}^2(\mathbb{R}) \to \mathcal{H}$, where $\mathcal{H}$ is a suitably defined Hilbert space. For the proof of the following see [58, Theorem 8.2.3]. In the proposition $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ denotes the space of all measurable functions which are square-integrable on compact sets.

Proposition 5.33. *Let the Hilbert space $\mathcal{H}$ be defined by*

$$\mathcal{H} = \{F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2) : F \textit{ satisfies (5.20) almost everywhere on } \mathbb{R}^2\} \quad ,$$

with norm

$$\|F\|_{\mathcal{H}} = \|F\|_{\mathrm{L}^2([0,1]^2)} \quad .$$

The Zak transform extends uniquely to a unitary operator $\mathcal{Z} : \mathrm{L}^2(\mathbb{R}) \to \mathcal{H}$.

The next lemma describes how the representation π operates on the Zak transform side. It is easily verified on $\mathcal{Z}(C_c(\mathbb{R}))$, and extends to $\mathcal{H}$ by density.

Proposition 5.34. *Let $\widehat{\pi}$ be the representation acting on $\mathcal{H}$, obtained by conjugating π with $\mathcal{Z}$, i.e., $\widehat{\pi}(n,k,\overline{\ell}) = \mathcal{Z} \circ \pi(n,k,\overline{\ell}) \circ \mathcal{Z}^*$. Then*

$$\widehat{\pi}(n,k,\overline{\ell})F(x,\omega) = \mathrm{e}^{2\pi \mathrm{i}(\ell-nk)/L}\mathrm{e}^{2\pi \mathrm{i} nx/L}F(x-k,\omega-n/L) \quad . \tag{5.21}$$

Now we can cite the Zak transform criterion for normalized tight Weyl-Heisenberg frames with integer oversampling. For a sketch of the proof confer [58], more details are contained in [33]. Our discussion provides an alternative proof, see Corollary 5.41.

Theorem 5.35. *Let $f \in \mathrm{L}^2(\mathbb{R})$. Then $(M_{n/L}T_k f)_{n,k\in\mathbb{Z}}$ is a normalized tight frame of $\mathrm{L}^2(\mathbb{R})$ iff*

$$\sum_{i=0}^{L-1} |\mathcal{Z}f(x,\omega+i/L)|^2 = 1 \quad \textit{almost everywhere.} \tag{5.22}$$

There exist more general versions of this criterion, which allow more complicated sets of time-frequency translations for the construction of the Gabor frames. While we have restricted our attention to the simple time-frequency lattice $\mathbb{Z}\times(1/L)\mathbb{Z}$ mostly for reasons of notational simplicity, the more general statements can be obtained by use of suitable symplectic automorphisms of the time-frequency plane.

Computing the Plancherel Measure

G is a unimodular group extension, thus ν_G is obtainable from Theorem 3.40 by computing $\widehat{G}$ with the aid of the Mackey machine, and keeping track of the various measures on duals and quotient spaces.

Note that since G/N is finite, N is regularly embedded in G. N is the direct product of three cyclic groups, hence the character group $\widehat{N}$ is conveniently parametrized by $[0,1[\times[0,1[\times\{0,1,\ldots,L-1\}$, by letting

$$\chi_{\omega_1,\omega_2,j}(nL,k,\overline{\ell}) = \mathrm{e}^{2\pi \mathrm{i}(\omega_1 n+\omega_2 k+j\ell/L)} \quad .$$

Since

$$(n,k,\overline{\ell})(n'L,k',\overline{\ell'})(n,k,\overline{\ell})^{-1} = (n'L,k',\overline{\ell'+k'n}) \quad ,$$

we compute the dual action as

$$(\omega_1,\omega_2,j).(n,k,\overline{\ell}) = (\omega_1,\omega_2+jn/L-\lfloor\omega_2+jn/L\rfloor,j) \quad .$$

Here $\lfloor x\rfloor$ denotes the largest integer $\le x$. Hence, defining

$$\Omega_j = [0,1[\times[0,\gcd(j,L)/L[\times\{j\} \quad ,$$

a measurable transversal of the orbits under the dual action is given by $\Omega = \bigcup_{j=0}^{L-1}\Omega_j$. Here $\gcd(j,L)$ is the greatest common divisor of j and L. The fact that the subgroup of $\mathbb{Z}/L\mathbb{Z}$ generated by $\overline{j}$ coincides with the subgroup generated by $\overline{\gcd(j,L)}$ accounts for this choice of transversal. With the respect to the dual action, $(\omega_1,\omega_2,j)\in\Omega_j$ has $N_j=\{(nL/\gcd(j,L),k,\overline{\ell}) : k,n,\ell\in\mathbb{Z}\}$ as fixed group. The associated little fixed group is $N_j/N\cong\mathbb{Z}/\gcd(j,L)\mathbb{Z}$. For a convenient parametrization of $\widehat{G}$ in terms of Ω and the duals of the N_j we need to establish that every $(\omega_1,\omega_2,j)\in\widehat{N}$ has trivial Mackey obstruction.

Lemma 5.36. *Let* $(\omega_1,\omega_2,j)\in\Omega_j$ *and* $m\in\{0,1,\ldots,\gcd(j,L)-1\}$. *Then*

$$\varrho_{m,\omega_1,\omega_2,j}(nL/\gcd(j,L),k,\overline{\ell}) = \mathrm{e}^{2\pi\mathrm{i}((\omega_1+m)n/\gcd(j,L)+\omega_2 k+j\ell/L)}$$

defines a character of N_j *with* $\varrho_{m,\omega_1,\omega_2,j}|_N = \chi_{\omega_1,\omega_2,j}$. *Moreover, every irreducible representation of* N_j *whose restriction to* N *is a multiple of* $\chi_{\omega_1,\omega_2,j}$ *is equivalent to some* $\varrho_{m,\omega_1,\omega_2,j}$.

Proof. The character property is verified by straightforward computation. The last statement is [45, Proposition 6.40].

Note that the additional parameter m indexes the characters of the little fixed group N_j/N. Now Theorem 3.39 implies that $\widehat{G}$ is obtained by inducing the $\varrho_{m,\boldsymbol{\omega},j}$.

Proposition 5.37. *Define, for* $(\omega_1,\omega_2,j) \in \Omega$ *and* $m \in \{0,\ldots,\gcd(j,L)-1\}$ *the representation*

$$\sigma_{m,\omega_1,\omega_2,j} = \mathrm{Ind}_{N_j}^{G} \varrho_{m,\omega_1,\omega_2,j} \quad .$$

If we let

$$\Sigma_j = \{\sigma_{m,\omega_1,\omega_2,j} : (\omega_1,\omega_2,j) \in \Omega_j, m \in \{0,1,\ldots,\gcd(j,L)-1\} \quad ,$$

then the dual of G *is the disjoint union*

$$\widehat{G} = \bigcup_{j=0}^{L-1} \Sigma_j$$

We normalize all Haar measures on discrete groups occurring in the following by $|\{e\}| = 1$. This choice fixes the Plancherel measures uniquely, and implies in particular for all abelian groups H arising in the following that $\nu_H(\widehat{H}) = 1$. Moreover, Weil's integral formulae are automatically ensured by these choices, whence we will obtain the correct normalizations.

Recall that we have the identification

$$\widehat{G} = \bigcup_{j=0}^{L-1} \Sigma_j = \bigcup_{j=0}^{L-1} (N_j/N)^\wedge \times \Omega_j \quad .$$

On each of the Σ_j, Plancherel measure is a product measure: The $(N_j/N)^\wedge$ carry the Plancherel measure of the finite quotient group, which is simply counting measure weighted with $1/|N_j/N| = 1/\gcd(j,L)$. For the missing parts, we decompose Plancherel measure of N on $\widehat{N}$ along orbits of the dual action. This results in a measure on $\Omega \simeq \widehat{N}/G$, and the restrictions to the Ω_j provide the second factors. In order to explicitly compute these we note that the Plancherel measure on $\widehat{N} \cong [0,1[\times[0,1[\times\{0,1,\ldots,L-1\}$ is $1/L$ times the product measure of Lebesgue measure on the first two factors and counting measure on the third. Since each orbit carries counting measure, the measure on the quotient is simply Lebesgue measure on the transversal $[0,1[\times[0,\gcd(j,L)/L[$, for each j. Thus we arrive at:

Proposition 5.38. *The Plancherel measure of* G *is given by*

$$d\nu_G(\sigma_{m,\omega_1,\omega_2,j}) = \frac{1}{L\gcd(j,L)} dm\, d\omega_1\, d\omega_2\, dj \quad . \tag{5.23}$$

Here $d\omega_1$ *and* $d\omega_2$ *are Lebesgue measure on the intervals* $[0,1[$ *and* $[0,\gcd(j,L)/L[$, *and* dm, dj *are counting measure on* $\{0,\ldots,\gcd(j,L)-1\}$ *and* $\{0,\ldots,L-1\}$, *respectively.*

As we will see in the next subsection, only the set Σ_1 will be of interest for the Weyl-Heisenberg frame setting. Here the indexing somewhat simplifies: $N_1 = N$, and m can only take the value 0. So we can identify Σ_1 with $\{0\} \times [0,1[\times[0,1/L[\times\{1\} \cong [0,1[\times[0,1/L[$.

Zak Transform and Plancherel Transform

The aim in this subsection is to exhibit the representation $\widehat{\pi}$ obtained by conjugating π with the Zak transform as a direct integral of irreducibles. This is done by taking a second look at (5.21), which is a twisted action by translations along $\mathbb{Z} \times (1/L)\mathbb{Z}$. Hence a decomposition of Lebesgue measure along cosets of $\mathbb{Z} \times (1/L)\mathbb{Z}$ gives rise to a decomposition into representations acting on the cosets, and the twisted action of the latter representations reveals them as induced representations.

To make this more precise, we let for $\boldsymbol{\omega} \in [0,1[\times[0,1/L[$ denote $\mathcal{O}_{\boldsymbol{\omega}} = \boldsymbol{\omega} + \mathbb{Z} \times (1/L)\mathbb{Z}$. The following lemma exhibits the direct integral structure of $\widehat{\pi}$; it is a direct consequence of Proposition 3.29.

Lemma 5.39. *Define for* $\boldsymbol{\omega} \in [0,1[\times[0,1/L[$ *the Hilbert space*

$$\mathcal{H}_{\boldsymbol{\omega}} = \{F : \mathcal{O}_{\boldsymbol{\omega}} \to \mathbb{C} : F \textit{ fulfills } (5.20)\} \quad ,$$

with the norm defined by

$$\|F\|^2_{\mathcal{H}_{\boldsymbol{\omega}}} = \sum_{i=0}^{L-1} |F(\boldsymbol{\omega} + (0, i/L))|^2. \tag{5.24}$$

Let $\widehat{\pi}_{\boldsymbol{\omega}}$ *be the representation acting on* $\mathcal{H}_{\boldsymbol{\omega}}$ *by*

$$\widehat{\pi}_{\boldsymbol{\omega}}(k, n, \overline{\ell})F(\boldsymbol{\gamma}) = \mathrm{e}^{2\pi \mathrm{i}(\ell+nk)/L} \mathrm{e}^{2\pi \mathrm{i} nx/L} F(\boldsymbol{\gamma} - (k, n/L)) \quad .$$

Then

$$\widehat{\pi} \simeq \int^{\oplus}_{[0,1[\times[0,1/L[} \widehat{\pi}_{\boldsymbol{\omega}} \, d\boldsymbol{\omega} \quad , \tag{5.25}$$

via the map

$$F \mapsto (F|_{\mathcal{O}_{\boldsymbol{\omega}}})_{\boldsymbol{\omega} \in [0,1[\times[0,1/L[} \tag{5.26}$$

As a first glimpse of the connection between conditions (5.22) and (4.15) note that the right-hand side of (5.22) can now be reformulated as

$$\|(\mathcal{Z}f)\,|_{\mathcal{O}_{\boldsymbol{\omega}}}\|_{\mathcal{H}_{\boldsymbol{\omega}}} = 1, \quad \text{for almost every } \boldsymbol{\omega} \in [0,1[\times[0,1/L[\quad .$$

Hence the final step is to note that (5.25) is in fact a decomposition into irreducibles:

Lemma 5.40. *If* $\boldsymbol{\omega} \in [0,1[\times[0,1/L[$*, then* $\widehat{\pi}_{\boldsymbol{\omega}} \simeq \sigma_{0,\boldsymbol{\omega},1} \in \Sigma_1$.

Proof. We will use the imprimitivity theorem to show that $\widehat{\pi}_{\boldsymbol{\omega}}$ is induced from a character of N. For this purpose consider the set $S = \{\boldsymbol{\omega} + (0, i/L) : i = 0, \ldots, L-1\}$, with an action of G on S given by

$$\boldsymbol{\gamma}.(n, k, \overline{\ell}) = (\gamma_1, \gamma_2 - n/L - \lfloor \gamma_2 - n/L \rfloor) \quad .$$

The action is transitive with N as associated stabilizer. To any subset $A \subset S$ we associate a projection operator P_A on $\mathcal{H}_{\boldsymbol{\omega}}$ defined by pointwise multiplication with the characteristic function of $A + \mathbb{Z} \times \mathbb{Z}$. It is then straightforward to check that $A \mapsto P_A$ is a projection-valued measure on S satisfying

$$\widehat{\pi}_{\boldsymbol{\omega}}(n, k, \overline{\ell}) P_A \widehat{\pi}_{\boldsymbol{\omega}}(n, k, \overline{\ell})^* = P_{A.(n,k,\ell)} \quad .$$

In other words, $A \mapsto P_A$ defines a transitive system of imprimitivity. Hence the imprimitivity theorem [45, Theorem 6.31] applies to show that $\widehat{\pi}_{\boldsymbol{\omega}} \simeq \mathrm{Ind}_N^G \varrho$ for a suitable representation ϱ of N. Since the system of imprimitivity is based on a discrete set, we can follow the procedure outlined in [45] immediately after Theorem 6.31, which identifies ϱ as the representation of N acting on $P_{\{\boldsymbol{\omega}\}}(\mathcal{H}_\omega)$. For this purpose consider the function $F \in \mathcal{H}_{\boldsymbol{\omega}}$ defined by

$$F(\boldsymbol{\omega} + (0, m/L)) = \delta_{m,0} \quad \text{for } m = 0, \ldots, L-1 \quad .$$

Now the fact that

$$\widehat{\pi}_{\boldsymbol{\omega}}(nL, k, \overline{\ell}) F = \mathrm{e}^{2\pi \mathrm{i} \ell / L} \mathrm{e}^{2\pi \mathrm{i} \omega_1 n} \mathrm{e}^{2\pi \mathrm{i} \omega_2 k} F = \chi_{\omega_1, \omega_2, 1}(nL, k, \overline{\ell}) F$$

shows that

$$\widehat{\pi}_{\boldsymbol{\omega}} \simeq \mathrm{Ind}_N^G \chi_{\omega_1, \omega_2, 1} = \sigma_{0, \omega_1, \omega_2, 1}, \quad .$$

By the last lemma and Mackey's theory, no two representations appearing in (5.25) are equivalent. Since G is type I, (5.25) is central, and 3.20 implies that π is multiplicity-free.

Summarizing our findings in the language of Remark 4.30, we have verified the following:

Corollary 5.41. *(a) $\mathcal{Z}$ implements a unitary equivalence $\pi \simeq \int_{\Sigma_1}^{\oplus} \sigma d\nu_G(\sigma)$. The multiplicity function is computed as $m_\pi(\sigma) = \mathbf{1}_{\Sigma_1}(\sigma)$.*
(b) The necessary conditions for the existence of admissible vectors, in terms of absolute continuity of the underlying measure, the multiplicity function and the finite Plancherel measure condition are trivially fulfilled.
(c) The Zak transform criterion (5.22) is a special instance of (4.17).

In view of Theorem 4.32, it is in fact enough to establish the direct integral decomposition of π. However, the calculations in this section also serve as an illustration of the use of Theorem 3.40 for the explicit computation of ν_G.