2

Wavelet Transforms and Group Representations

In this chapter we present the representation-theoretic approach to continuous wavelet transforms. Only basic representation theory and functional analysis (including the spectral theorem) are required. The main purpose is to clarify the role of the regular representation, and to develop some related notions, such as selfadjoint convolution idempotents, which are then used for the formulation of the problems which the book addresses in the sequel. Most of the results in this chapter may be considered well-known, or are more or less straightforward extensions of known results, with the exception of the last two sections: The notion of sampling space and the related results presented in Section 2.6 are apparently new. Section 2.7 contains the discussion of an example which is crucial for the following: It motivates the use of Fourier analysis and thus serves as a blueprint for the arguments in the following chapters.

2.1 Haar Measure and the Regular Representation

Given a second countable locally compact group G, we denote by μ_G a **left Haar measure** on G, i.e. a Radon measure on the Borel σ-algebra of G which is invariant under left translations: $\mu_G(xE) = \mu_G(E)$. Since G is σ-compact, any Radon measure ν on G is inner and outer regular, i.e., for all Borel sets $A \subset G$ and $\epsilon > 0$ there exist sets $C \subset A \subset V$ with C compact, V open such that $\nu(V \setminus C) < \epsilon$.

One of the pillars of representation theory of locally compact groups is the fact that Haar measure always exists and is unique up to normalization. We use a simple dx to denote integration against μ_G, and $|A| = \mu_G(A)$ for Borel subsets $A \subset G$. An associated rightinvariant measure, the so called **right Haar measure** is obtained by letting $\mu_{G,r}(A) = |A^{-1}|$. The modular function $\Delta_G : G \to \mathbb{R}^+$ measures the rightinvariance of the left Haar measure. It is given by $\Delta_G(x) = \frac{|Ex|}{|E|}$, for an arbitrary Borel set E of finite positive measure. Using the fact that μ_G is unique up to normalization, one can show

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that Δ_G is a well-defined continuous homomorphism, and independent of the choice of E. The homomorphism property entails that Δ_G is either trivial or unbounded: $\Delta_G(G)$ is a subgroup of the multiplicative $\mathbb{R}^+$, and all nontrivial subgroup of the latter are unbounded. Δ_G can also be viewed as a Radon-Nikodym derivative, namely

$$\Delta_G = \frac{d\mu_G}{d\mu_{G,r}} \quad ,$$

see [45, Proposition 2.31]. Hence the following formula, which will be used repeatedly [45, (2.32)]:

$$\int_G f(x)dx = \int_G f(x^{-1})\Delta_G(x^{-1})dx \quad . \tag{2.1}$$

G is called **unimodular** if $\Delta_G \equiv 1$, which is the case iff μ_G is rightinvariant also.

We will frequently use invariant and quasi-invariant measures on quotient spaces. If $H < G$ is a closed subgroup, we let $G/H = \{xH : x \in G\}$, which is a Hausdorff locally compact topological space. G acts on this space by $y.(xH) = yxH$, and the question of invariance of measures on G/H arises naturally. Given any measure ν on G/H let ν_g be the measure given by $\nu_g(A) = \nu(gA)$. Then ν is called **invariant** if $\nu_g = \nu$ for all $g \in G$, and **quasi-invariant** if ν_g and ν are equivalent. The following lemma collects the basic results concerning quasi-invariant measures on quotients.

Lemma 2.1. *Let G be a locally compact group, and $H < G$.*

(a) There exists a quasi-invariant Radon measure on G/H. All quasi-invariant Radon measures on G/H are equivalent.
(b) There exists an invariant Radon measure on G/H iff Δ_H is the restriction of Δ_G to H.
(c) If there exists an invariant Radon measure $\mu_{G/H}$ on G/H, it is unique up to normalization. After picking Haar measures on G and H, the normalization of $\mu_{G/H}$ can be chosen such as to ensure **Weil's integral formula**

$$\int_G f(x)dx = \int_{G/H}\int_H f(xh)dhd\mu_{G/H}(xH) \quad . \tag{2.2}$$

The invariance property of Haar measure implies that the left translation action of the group on itself gives rise to a unitary representation on $\mathrm{L}^2(G)$. The result is the **regular representation** defined next.

Definition 2.2. *Let G be a locally compact group. The* **left (resp. right) regular representation** *λ_G (ϱ_G) acts on $\mathrm{L}^2(G)$ by*

$$(\lambda_G(x)f)(y) = f(x^{-1}y) \quad \textit{resp. } (\varrho_G(x)f)(y) = \Delta_G(x)^{1/2}f(yx)$$

The **two-sided representation** *of the product group $G \times G$ is defined as*

$$(\lambda_G \times \varrho_G)(x,y) = \lambda_G(x)\varrho_G(y) \quad .$$

λ_G-invariant subspaces are called **leftinvariant**.

The convolution of two functions f, g on G is defined as the integral

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) dy \quad . \tag{2.3}$$

This is well-defined, with absolute convergence for almost every $x \in G$, whenever $f, g \in \mathrm{L}^1(G)$. But L^2-functions can be convolved also, if we employ a certain involution.

Definition 2.3. *Given any function f on G, define $f^*(x) = \overline{f(x^{-1})}$.*

Remark 2.4. If f is p-integrable with respect to left Haar measure, then f^* is p-integrable with respect to right Haar measure, and vice versa. In general, f^* will not be in $\mathrm{L}^p(G)$ if f is. Notable exceptions are given by the (trivial) case that G is unimodular, or more generally, that f is supported in a set on which Δ_G^{-1} is bounded.

The mapping $f \mapsto f^*$ is obviously a conjugate-linear involution. With respect to convolution, the involution turns out to be an antihomomorphism:

$$\begin{aligned}
(g * f)^*(x) &= \overline{\int_G g(y) f(y^{-1}x^{-1}) dy} \\
&= \int_G g^*(y^{-1}) f^*(xy) dy \\
&= \int_G f^*(y) g^*((x^{-1}y)^{-1}) dy \\
&= \int_G f^*(x) g^*(y^{-1}x) dy \\
&= f^* * g^*(x) \quad .
\end{aligned}$$

Note that our definition differs from the notation in [45, 35]. Our choice is motivated by proposition 2.19 below which clarifies the connection between the involution and taking adjoints of coefficient operators.

The following simple observation relates convolution to coefficient functions:

Proposition 2.5. *For $f, g \in \mathrm{L}^2(G)$,*

$$(g * f^*)(x) = \int_G g(y) \overline{f(x^{-1}y)} dy = \langle g, \lambda_G(x) f \rangle \quad , \tag{2.4}$$

in particular the convolution integral $g * f^*$ converges absolutely for every x, yielding a continuous function which vanishes at infinity.

Proof. Equation (2.4) is self-explanatory, and it yields pointwise absolute convergence of the convolution product. Continuity follows from the continuity of the regular representation. Recall that a function f on G vanishes at infinity if for every $\epsilon > 0$ there exists a compact set $C \subset G$ such that $|f| < \epsilon$ outside of C. If f and g are compactly supported, it is clear that $g * f^*$ also has compact support, hence vanishes at infinity. For arbitrary L^2-functions f and g pick sequences $f_n \to f$ and $g_n \to g$ with $f_n, g_n \in C_c(G)$. Then the Cauchy-Schwarz inequality implies $g_m * f_n^* \to g * f^*$ uniformly, as $m, n \to \infty$. But then the limit vanishes at infinity also.

The von Neumann algebras generated by the regular representation are the *left and right group von Neumann algebras.*

Definition 2.6. *Let G be a locally compact group. The von Neumann algebras generated by the left and right regular representations are*

$$VN_l(G) = \lambda_G(G)'' \text{ and } VN_r(G) = \varrho_G(G)'' \quad .$$

$VN_l(G)$ and $VN_r(G)$ obviously commute; in fact $VN_l(G)' = VN_r(G)$. If the group is abelian, $VN_l(G) = VN_r(G) =: VN(G)$.

The equality $VN_l(G)' = VN_r(G)$ is a surprisingly deep result, known as the commutation theorem. For a proof, see [109].

2.2 Coherent States and Resolutions of the Identity

In this section we present a general notion of coherent state systems. Basically, the setup discussed in this section yields a formalization for the expansion of Hilbert space elements with respect to certain systems of vectors. The blueprint for this type of expansions is provided by ONB's: If $\eta = (\eta_i)_{i \in I}$ is an ONB of a Hilbert space $\mathcal{H}$, it is well-known that the **coefficient operator**

$$V_\eta : \mathcal{H} \ni \varphi \mapsto (\langle \varphi, \eta_i \rangle)_{i \in I} \in \ell^2(I) \tag{2.5}$$

is unitary, and that every $\varphi \in \mathcal{H}$ may be written as

$$\varphi = \sum_{i \in I} \langle \varphi, \eta_i \rangle \eta_i \quad . \tag{2.6}$$

The generalization discussed here consists in replacing I by a **measure space** $(X, \mathcal{B}, \mu)$, and summation by integration. In the following sections we will mostly specialize to the case $X = G$, a locally compact group, endowed with left Haar measure. However, in connection with sampling we will also need to discuss tight frames (obtained by taking a discrete space with counting measure), which is why have chosen to base the discussion on a slightly more abstract level.

Definition 2.7. *Let $\mathcal{H}$ be a Hilbert space. Let $\eta = (\eta_x)_{x\in X}$ denote a family of vectors, indexed by the elements of a measure space $(X, \mathcal{B}, \mu)$.*

(a) If for all $\varphi \in \mathcal{H}$, the **coefficient function**

$$V_\eta \varphi : X \ni x \mapsto \langle \varphi, \eta_x \rangle$$

is μ-measurable, we call η a **coherent state system**.

(b) Let $(\eta_x)_{x\in X}$ be a coherent state system, and define

$$\mathrm{dom}(V_\eta) := \{\varphi \in \mathcal{H} : V_\eta\varphi \in \mathrm{L}^2(X,\mu)\} \quad ,$$

which may be trivial. Denote by $V_\eta : \mathcal{H} \to \mathrm{L}^2(X,\mu)$ the (possibly unbounded) **coefficient operator** *or* **analysis operator** *with domain $\mathcal{D}_\eta$.*

(c) The coherent state system $(\eta_x)_{x\in X}$ is called **admissible** *if the associated coefficient operator $V_\eta : \varphi \mapsto V_\eta\varphi$ is an isometry, with* $\mathrm{dom}(V_\eta) = \mathcal{H}$.

It would be more precise to speak of μ-admissibility, since obviously the property depends on the measure. However, we treat the measure space $(X, \mathcal{B}, \mu)$ as given; it will either be a locally compact group with left Haar measure, or a discrete set with counting measure.

We next collect a few basic functional-analytic properties of coherent state systems. The following observation will frequently allow density arguments in connection with coefficient operators:

Proposition 2.8. *For any coherent state system $(\eta_x)_{x\in X}$, the associated coefficient operator is a closed operator.*

Proof. Let $\varphi_n \to \varphi$, where $\varphi_n \in \mathrm{dom}(V_\eta)$. Assume in addition that $V_\eta\varphi_n \to F$ in $\mathrm{L}^2(X,\mu)$. After passing to a suitable subsequence we may assume in addition pointwise almost everywhere convergence. Now the Cauchy-Schwarz-inequality entails

$$|V_\eta\varphi_n(x) - \langle\varphi,\eta_x\rangle| = |\langle \varphi_n - \varphi, \eta_x\rangle| \le \|\varphi_n - \varphi\| \, \|\eta_x\| \to 0 \quad ,$$

hence $F = V_\eta\varphi$ a.e., in particular the right hand side is in $\mathrm{L}^2(X,\mu)$.

Next we want to describe adjoint operators. For this purpose weak integrals will be needed.

Definition 2.9. *Let $(\eta_x)_{x\in X}$ be a coherent state system. If the right-hand side of*

$$\varphi \mapsto \int_X \langle \varphi, \eta_x\rangle d\mu(x)$$

converges absolutely for all φ, and defines a continuous linear functional on $\mathcal{H}$, we let the element of $\mathcal{H}$ corresponding to the functional by the Fischer-Riesz theorem be denoted by the **weak integral**

$$\int_X \eta_x d\mu(x) \quad .$$

Hence we obtain the following defining relation for $\int_X \eta_x d\mu(x)$*:*

$$\left\langle \varphi, \int_X \eta_x d\mu(x) \right\rangle = \int_X \langle \varphi, \eta_x \rangle d\mu(x) \tag{2.7}$$

For a family of operators $(T_x)_{x\in X}$ *we define the weak operator integral* $\int_X T_x dx$ *pointwise as*

$$\left(\int_X T_x dx \right)(\varphi) = \int_X T_x(\varphi)\ dx \quad ,$$

whenever the right-hand sides converges weakly for every φ*.*

Proposition 2.10. *Let* $(\eta_x)_{x\in X}$ *be a coherent state system. The associated coefficient operator* V_η *is bounded on* $\mathcal{H}$ *iff* $\mathrm{dom}(V_\eta) = \mathcal{H}$*. In that case, its adjoint operator is the* **synthesis operator***, given pointwise by the weak integral*

$$V_\eta^*(F) = \int_X F(x)\eta_x d\mu(x) \quad . \tag{2.8}$$

Proof. The first statement follows from the closed graph theorem and the previous proposition. For (2.8) we compute

$$\begin{aligned} \langle V_\eta \varphi, F \rangle &= \int_X \langle \varphi, \eta_x \rangle \overline{F(x)} d\mu(x) = \int_X \langle \varphi, F(x)\eta_x \rangle d\mu(x) \\ &= \left\langle \varphi, \int_X F(x)\eta_x d\mu(X) \right\rangle \quad . \end{aligned}$$

We will next apply the proposition to admissible coherent state systems. Note that for such systems η the isometry property entails that $V_\eta^* V_\eta$ is the identity operator on $\mathcal{H}$, and $V_\eta V_\eta^*$ is the projection onto the range of V_η. The first formula, the inversion formula, can then be read as a (usually continuous and redundant) expansion of a given vector in terms of the coherent state system. An alternative way of describing this property, commonly used in mathematical physics, expresses the identity operator as the (usually continuous) superposition of rank-one operators. In order to present this formulation, we use the **bracket notation** for rank-one operators:

$$|\eta\rangle\langle\psi| : \varphi \mapsto \langle\psi|\varphi\rangle\eta \quad . \tag{2.9}$$

Note the attempt to reconcile mathematics and physics notation by letting $\langle\eta|\varphi\rangle = \langle\varphi,\eta\rangle$. In particular, the bracket (2.9) is linear in η and antilinear in ψ. Outside the following proposition, we will however favor the tensor product notation $\eta \otimes \psi$ over the bracket notation.

Proposition 2.11. *If $(\eta_x)_{x \in X}$ is an admissible coherent state system, then for every $\varphi \in \mathcal{H}$, the following (weak-sense) reconstruction formula (or* **coherent state expansion***) holds:*

$$\varphi = \int_X \langle \eta_x | \varphi \rangle \eta_x d\mu(x) \quad . \tag{2.10}$$

Equivalently, we obtain the **resolution of the identity** *as a weak operator integral*

$$\int_X |\eta_x\rangle\langle\eta_x| d\mu(x) = \mathrm{Id}_{\mathcal{H}} \quad . \tag{2.11}$$

Proof. Recall that by the defining relation (2.7) the right hand side of (2.10) denotes the Hilbert space element $\psi \in \mathcal{H}$ satisfying for all $z \in \mathcal{H}$ the equation

$$\langle \psi, z \rangle = \int_X \langle \varphi, \eta_x \rangle \langle \eta_x, z \rangle d\mu(x) \quad .$$

But the right-hand side of this equation is just $\langle V_\eta \varphi, V_\eta z \rangle_{\mathrm{L}^2(X)} = \langle \varphi, z \rangle$, by the isometry property of V_η. Hence $\psi = \varphi$. Equation (2.11) is just a rephrasing of (2.10).

As a special case of (2.10) we retrieve (2.6) (with a somewhat weaker sense of convergence), observing that by (2.5) ONB's are admissible coherent state systems. Next we identify the ranges of coefficient mappings.

Proposition 2.12. *Let $(\eta_x)_{x \in X}$ be an admissible coherent state system. Then the image space $\widetilde{\mathcal{K}} = V_\eta(\mathcal{H}) \subset \mathrm{L}^2(X, \mu)$ is a reproducing kernel Hilbert space, i.e., the projection P onto $\mathcal{K}$ is given by*

$$PF(x) = \int_X F(y) \langle \eta_y, \eta_x \rangle d\mu(y) \quad .$$

Proof. Note that the integral converges absolutely since $V_\eta(\eta_y) \in \mathrm{L}^2(X)$. If we assume that V_η is an isometry, then $P = V_\eta V_\eta^*$. Plugging in (2.8) gives the desired equation:

$$\begin{aligned} V_\eta V_\eta^* F(x) &= \langle V_\eta^* F, \eta_x \rangle \\ &= \int_X F(y) \langle \eta_y, \eta_x \rangle d\mu(y) \quad . \end{aligned}$$

2.3 Continuous Wavelet Transforms and the Regular Representation

We now introduce the particular class of coherent state expansions associated to group representations which this book studies in detail. We first exhibit the close relation to the regular representation of the group. After that we investigate the functional-analytic basics of the coefficient operators in this setting, i.e., domains and adjoints.

Definition 2.13. *Let $(\pi, \mathcal{H}_\pi)$ denote a strongly continuous unitary representation of the locally compact group G. In the following, we endow G with left Haar measure. Associate to $\eta \in \mathcal{H}_\pi$ the orbit $(\eta_x)_{x\in G} = (\pi(x)\eta)_{x\in G}$. This is clearly a coherent state system in the sense of Definition 2.7(a), in particular the coefficient operators V_η can be defined according to 2.7(b).*

(a) *η is called* **admissible** *iff $(\pi(x)\eta)_{x\in G}$ is admissible.*
(b) *If η is admissible, then $V_\eta : \mathcal{H}_\pi \hookrightarrow \mathrm{L}^2(G)$ is called* **(generalized) continuous wavelet transform**.
(c) *η is called* **a bounded vector** *if $V_\eta : \mathcal{H}_\pi \to \mathrm{L}^2(G)$ is bounded on $\mathcal{H}_\pi$.*

We note in passing that η is cyclic iff V_η, this time viewed as an operator $\mathcal{H}_\pi \to C_b(G)$, is injective: Indeed, $V_\eta\varphi = 0$ iff $\varphi \perp \pi(G)\eta$, and that is equivalent to the fact that φ is orthogonal to the subspace spanned by $\pi(G)\eta$.

A straightforward but important consequence of the definitions is that

$$V_\eta(\pi(x)\varphi)(y) = \langle \pi(x)\varphi, \pi(y)\eta\rangle = \langle \varphi, \pi(x^{-1}y)\eta\rangle = (V_\eta\varphi)(x^{-1}y) \quad , \qquad (2.12)$$

i.e., coefficient operators intertwine π with the action by left translations on the argument. The same calculation shows that $\mathrm{dom}(V_\eta)$ is invariant under π.

Our next aim is to shift the focus from general representations of G to subrepresentations of λ_G. For this purpose the following simple proposition concerning the action of the commuting algebra on admissible (resp. bounded, cyclic) vectors is useful.

Proposition 2.14. *Let $(\pi, \mathcal{H}_\pi)$ be a representation of G and $\eta \in \mathcal{H}_\pi$. If $T \in \pi(G)'$, then*

$$V_{T\eta} = V_\eta \circ T^* \quad . \qquad (2.13)$$

In particular, suppose that $\mathcal{K}$ is an invariant closed subspace of $\mathcal{H}_\pi$, with projection $P_\mathcal{K}$. If $\eta \in \mathcal{H}_\pi$ is admissible (resp. bounded or cyclic) for $(\pi, \mathcal{H}_\pi)$, then $P_\mathcal{K}\eta$ has the same property for $(\pi|_\mathcal{K}, \mathcal{K})$.

Proof. $V_{T\eta}\varphi(x) = \langle \varphi, \pi(x)T\eta\rangle = \langle T^*\varphi, \pi(x)\eta\rangle$ shows (2.13), in particular the natural domain of $V_\eta \circ T^*$ coincides with $\mathrm{dom}(V_{T\eta})$. As a consequence $V_{P_\mathcal{K}\eta}$ is the restriction of V_η to $\mathcal{K}$. The remaining statements are immediate from this: The restriction of an isometry (resp. bounded or injective operator) has the same property.

The following rather obvious fact, which follows from similar arguments, will be used repeatedly.

Corollary 2.15. *Let T be a unitary operator intertwining the representations π and σ. Then $\eta \in \mathcal{H}_\pi$ is admissible (cyclic, bounded) iff $T\eta$ has the same property.*

We will next exhibit the central role of the regular representation for wavelet transforms. In view of the intertwining property (2.12), the remaining problems have more to do with functional analysis. The chief tool for this is the generalization of Schur's lemma given in 1.2.

Proposition 2.16. *(a) If π has a cyclic vector η for which V_η is densely defined, there exists an isometric intertwining operator $T : \mathcal{H}_\pi \hookrightarrow \mathrm{L}^2(G)$. Hence $\pi < \lambda_G$.*
(b) If $\varphi \in \mathcal{H}_\pi$ is such that $V_\varphi : \mathcal{H}_\pi \to \mathrm{L}^2(G)$ is a topological embedding, there exists an admissible vector $\eta \in \mathcal{H}_\pi$.
(c) Suppose that η is admissible and define $\mathcal{H} = V_\eta(\mathcal{H}_\pi)$. Then $\mathcal{H} \subset \mathrm{L}^2(G)$ is a closed, leftinvariant subspace, and the projection onto $\mathcal{H}$ is given by right convolution with $V_\eta \eta$.

Proof. For part (a) note that by assumption V_η is densely defined, and it intertwines π and λ_G on its domain, by (2.12). Hence Lemma 1.2 applies. Since η is cyclic, $\ker V_\eta = 0$, yielding $\pi < \lambda_G$.

For (b) define $U = V_\eta^* V_\eta$ and $\eta = U^{-1/2}\varphi$. Note that by assumption U is a selfadjoint bounded operator with bounded inverse, hence $U^{-1/2}$ is bounded also. Moreover, $U \in \pi(G)'$, hence 1.4 implies $U^{-1/2} \in \pi(G)'$.

Then by (2.13), $V_\eta^* V_\eta = U^{-1/2} U U^{-1/2} = \mathrm{Id}_{\mathcal{H}_\pi}$. The statements in (c) are obvious; for the calculation of the projection confer Proposition 2.12.

The proposition shows that up to unitary equivalence all representations of interest are subrepresentations of the left regular representation. In this setting, wavelet transforms are right convolution operators. We next want to discuss adjoint operators in this setting. Before we do this, we need to insert a small lemma.

Lemma 2.17. *Let a be a measurable bounded function, $b \in \mathrm{L}^2(G)$ such that for all $g \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$,*

$$\int_G a(x)\overline{g(x)}dx = \langle b, g \rangle \quad .$$

Then $g = f$ almost everywhere.

Proof. Assuming that a and b differ on a Borel set M of positive, finite measure, we find a measurable function g supported on M, with modulus 1 and such that $g(x)\overline{(b(x) - a(x))} > 0$ on M. But then $g \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ yields the desired contradiction.

Remark 2.18. The nontrivial aspect of this lemma is that its proof is not just a density argument. Initially it is not even clear whether a is square-integrable. For this type of argument, replacing $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ by some dense subspace generally does not work, as the following example shows: Consider the constant function $a(x) = 1$ on G and the subspace $\mathcal{H} = \{g \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G) : \int_G g(x)dx = 0\} \subset \mathrm{L}^2(G)$. $\mathcal{H}$ is dense if G is noncompact, and for all $g \in \mathcal{H}$,

$$\int_{\mathbb{R}} g(x)\overline{a(x)}dx = 0 \quad ,$$

with absolute convergence, but of course $a \neq 0 \in \mathrm{L}^2(\mathbb{R})$.

One of the reasons we single this argument out is that we will meet it again in connection with the Plancherel Inversion Theorem 4.15.

Proposition 2.19. *Suppose that* $f \in \mathrm{L}^2(G)$.

(a) $V_f : \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ *is a closed operator with domain*

$$\mathrm{dom}(V_f) = \{g \in \mathrm{L}^2(G) : g * f^* \in \mathrm{L}^2(G)\} \quad ,$$

and acts by $V_f g = g * f^*$. *The subspace* $\mathrm{dom}(V_f)$ *is invariant under left translations.*

(b) If $f\Delta^{-1/2} \in \mathrm{L}^1(G)$, *then* f *is a bounded vector, with* $\|V_f\| \le \|f\Delta^{-1/2}\|_1$. *This holds in particular when* f *has compact support.*

(c) If $f^* \in \mathrm{L}^2(G)$ *then* $\mathrm{L}^1(G) \cap \mathrm{L}^2(G) \subset \mathrm{dom}(V_f)$.

(d) Suppose that $f^* \in \mathrm{L}^2(G)$. *Then* $V_f^* \subset V_{f^*}$. *If one of the operators is bounded, so is the other, and they coincide.*

Proof. The first part of (a) was shown in Proposition 2.8. $V_f g = g * f^*$ was observed in equation (2.4). (b) and (c) are nonabelian versions of Young's inequality. We prove (b) along the lines of [45, Proposition 2.39], the proof of part (c) is similar (and can be found in [45]). We write

$$\begin{aligned} g * f^*(x) &= \int_G g(y)\overline{f(x^{-1}y)}dy \\ &= \int_G g(xy)\overline{f(y)}dy \\ &= \int_G (R_y g)(x)\overline{f(y)}dy \quad , \end{aligned}$$

where $(R_y g)(x) = g(xy)$. An application of the generalized Minkowski inequality then yields

$$\begin{aligned} \|g * f^*\|_2 &\le \int_G \|R_y g\|_2 |f(y)| dy = \int_G \|g\|_2 \Delta_G(y)^{-1/2} |f(y)| dy \\ &= \|g\|_2 \|f\Delta_G^{-1/2}\|_1 \quad . \end{aligned}$$

For the computation of the adjoint operator in (d), let $g \in \mathrm{dom}(V_f^*)$. For all $h \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G) \subset \mathrm{dom}(V_f)$, we note that

$$\int_G \int_G |h(x)\overline{f(y^{-1}x)g(y)}| dy dx \le \int_G |h(x)| \|f^*\|_2 \|g\|_2 dx < \infty \quad ,$$

hence we may apply Fubini's theorem to compute

$$\begin{aligned}\langle h, V_f^* g\rangle &= \langle V_f h, g\rangle \\ &= \int_G \int_G h(x)\overline{f(y^{-1}x)}dx\overline{g(y)}dy \\ &= \int_G h(x)\overline{\int_G g(y)f(y^{-1}x)dy}dx \\ &= \int_G h(x)\overline{\int_G g(y)\overline{f^*(x^{-1}y)}dy}dx \\ &= \int_G h(x)\overline{V_{f^*}g(x)}dx\end{aligned}$$

Note that $V_{f^*}g$ here denotes the coefficient function as an element of $C_b(G)$; we have yet to establish that $g \in \mathrm{dom}(V_{f^*}$. Here Lemma 2.17 applies to prove $V_{f^*}g = V_f^* g \in \mathrm{L}^2(G)$ and thus $V_f^* \subset V_{f^*}$. Assuming that V_f is bounded, it follows that $V_{f^*} \supset V_f^*$ is everywhere defined and closed, hence bounded. Conversely, V_f^* being contained in a bounded operator clearly implies that V_f^* is bounded.

Remark 2.20. Part (c) of the proposition implies that V_f is densely defined for arbitrary $f \in \mathrm{L}^2(G)$, when G is unimodular. This need not be true in the nonunimodular case, see example 2.29 below.

We note the following existence theorem for bounded cyclic vectors.

Theorem 2.21. *There exists a bounded cyclic vector for λ_G. Hence, an arbitrary representation π has a bounded cyclic vector iff $\pi < \lambda_G$.*

Proof. Losert and Rindler [84] proved for arbitrary locally compact groups the following statement: There exists $f \in C_c(G)$ which is a cyclic vector for λ_G iff G is first countable. Thus second countable groups have a cyclic vector $f \in C_c(G)$. But then 2.19 (b) entails that V_f is bounded on $\mathrm{L}^2(G)$, i.e. f is a bounded cyclic vector for $\mathrm{L}^2(G)$. Propositions 2.14 and 2.16 (a) yield the second statement.

Remark 2.22. When dealing with subrepresentations $\pi_1 < \pi_2$ and a vector $\eta \in \mathcal{H}_{\pi_1} \subset \mathcal{H}_{\pi_2}$, the notation V_η is somewhat ambiguous. Nonetheless, we refrain from introducing extra notation, since no serious confusion can occur: Denoting $V_\eta^{\pi_i}$ for the operator on $\mathcal{H}_{\pi_i}$ $(i = 1, 2)$, we find that $V_\eta^{\pi_2} = V_\eta^{\pi_1}$ on $\mathcal{H}_{\pi_1}$, and $V_\eta^{\pi_2} = 0$ on $\mathcal{H}_{\pi_1}^{\perp}$. Hence $V_\eta^{\pi_2}$ is just the trivial extension of $V_\eta^{\pi_1}$.

We close the section with a first short discussion of direct sum representations.

Proposition 2.23. *Let $\pi = \bigoplus_{i\in I} \pi_i$, and $\eta \in \mathcal{H}$. Let P_i denote the projection onto $\mathcal{H}_{\pi_i}$, and $\eta_i = P_i\eta$. Then the following are equivalent:*

(a) η is admissible.

(b) $\eta_i := P_i\eta$ *is admissible for* π_i*, for all* $i \in I$*, and* $\mathrm{Im}(V_{\eta_i}) \perp \mathrm{Im}(V_{\eta_j})$*, for all* $i \neq j$*.*

Proof. For $(a) \Rightarrow (b)$, the admissibility of η_i is due to Proposition 2.14. Moreover, if V_η is isometric, then it respects scalar products; in particular, the pairwise orthogonal subspaces $(P_i(\mathcal{H}))_{i\in I}$ have orthogonal images. But since $V_\eta \circ P_i = V_{\eta_i}$, this is precisely the second condition. The converse direction is similar.

One way of ensuring the pairwise orthogonality of image spaces in part (b) of the proposition is to choose the representations π_i as pairwise disjoint:

Lemma 2.24. *Let* π_1 *and* π_2 *be disjoint representations, and* $\eta_i \in \mathcal{H}_{\pi_i}$ *be bounded vectors (*$i = 1, 2$*). Then* $V_{\eta_1}(\mathcal{H}_{\pi_1}) \perp V_{\eta_2}(\mathcal{H}_{\pi_2})$ *in* $\mathrm{L}^2(G)$*.*

Proof. $V_{\eta_2}^* V_{\eta_1} : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ is an intertwining operator, hence zero. Therefore, for all $\varphi_1 \in \mathcal{H}_{\pi_1}$ and $\varphi_2 \in \mathcal{H}_{\pi_2}$,

$$0 = \langle V_{\eta_2}^* V_{\eta_1}\varphi_1, \varphi_2\rangle = \langle V_{\eta_1}\varphi_1, V_{\eta_2}\varphi_2\rangle \quad ,$$

which is the desired orthogonality relation.

2.4 Discrete Series Representations

The major part of this book is concerned with the following two questions:

- Which representations π have admissible vectors?
- How can the admissible vectors be characterized?

For *irreducible* representations (such as the above mentioned examples), these questions have been answered by Grossmann, Morlet and Paul [60]; the key results can already be found in [38]. Irreducible subrepresentations of λ_G are called **discrete series representations**. The complete characterization of admissible vectors is contained in the following theorem. We will not present a full proof here, since the theorem is a special case of the more general results proved later on. However, some of the aspects of more general phenomena encountered later on can be studied here in a somewhat simpler setting, and we will focus on these.

Theorem 2.25. *Let* π *be an irreducible representation of* G*.*

(a) π *has admissible vectors iff* $\pi < \lambda_G$*.*
(b) A nonzero $\eta \in \mathcal{H}_\pi$ *is admissible (up to normalization) if* $V_\eta\eta \in \mathrm{L}^2(G)$*, or equivalently, if* $V_\eta\varphi \in \mathrm{L}^2(G)$*, for some nonzero* $\varphi \in \mathcal{H}_\pi$*.*

(c) *There exists a unique, densely defined positive operator C_π with densely defined inverse, such that*

$$\eta \in \mathcal{H}_\pi \text{ is admissible } \Longleftrightarrow \eta \in \mathrm{dom}(C_\pi), \text{ with } \|C_\pi \eta\| = 1 \quad . \tag{2.14}$$

This condition follows from the **orthogonality relation**

$$\langle C_\pi \eta', C_\pi \eta \rangle \langle \varphi, \varphi' \rangle = \langle V_\eta \varphi, V_{\eta'} \varphi' \rangle \quad , \tag{2.15}$$

which holds for all $\varphi, \varphi' \in \mathcal{H}_\pi$ and $\eta, \eta' \in \mathrm{dom}(C_\pi)$. Conversely, $V_\psi \varphi \notin \mathrm{L}^2(G)$ whenever $\psi \notin \mathrm{dom}(C_\pi)$ and $0 \neq \varphi \in \mathcal{H}_\pi$.

(c) *$C_\pi = c_\pi \times \mathrm{Id}_{\mathcal{H}_\pi}$ for a suitable $c_\pi > 0$ iff G is unimodular, or equivalently, if every nonzero vector is admissible up to normalization.*

(d) *Up to normalization, C_π is uniquely characterized by the* **semi-invariance relation**

$$\pi(x) C_\pi \pi(x)^* = \Delta_G(x)^{1/2} C_\pi \quad . \tag{2.16}$$

The normalization of C_π is fixed by (2.15).

Proof. The "only-if" part of (a) is noted in Proposition 2.16 (a). For the converse direction assume $\pi < \lambda_G$, w.l.o.g. π acts by left translation on a closed subspace of $\mathrm{L}^2(G)$. Then projecting any $\eta \in C_c(G)$ into $\mathcal{H}_\pi$ yields a bounded vector, by 2.19(b) and 2.14. Since $C_c(G)$ is dense in $\mathrm{L}^2(G)$, we thus obtain a nonzero bounded vector η. Since π is irreducible, it follows that V_η is isometric up to a constant (by Lemma 1.2), hence we have found the admissible vector.

For the proof of part (b) note that the following chain of implications is trivial:

$$\begin{aligned} \eta \text{ is admissible up to normalization } &\Rightarrow V_\eta \eta \in \mathrm{L}^2(G) \\ &\Rightarrow (\exists \varphi \in \mathcal{H}_\pi \setminus \{0\} \ : \ V_\eta \varphi \in \mathrm{L}^2(G)) \quad . \end{aligned}$$

For the converse direction, assume $V_\eta \varphi \in \mathrm{L}^2(G)$ for a nonzero φ. Then $\mathrm{dom}(V_\eta)$ is nonzero and invariant, hence it is dense by irreducibility of π. But then Lemma 1.2 applies to yield that V_η is isometric up to a constant. Since $V_\eta \eta \neq 0$, the constant is nonzero, and thus η is admissible up to normalization.

The construction of the operators C_π requires additional tools from functional analysis. The basic idea is the following: Fix a normalized vector $\varphi \in \mathcal{H}_\pi$ and consider the positive definite sesquilinear form

$$B_\varphi : (\eta, \eta') \mapsto \langle V_{\eta'} \varphi, V_\eta \varphi \rangle \quad ,$$

which is the right hand side of (2.15) for the special case that $\varphi = \varphi'$. The domain of this form is $\mathcal{D} \times \mathcal{D}$, where $\mathcal{D}$ is the space of vectors η which are admissible up to normalization. Note that $\mathcal{D}$ is dense, being nonzero and invariant.

Recalling from linear algebra the representation theorem establishing a close connection between quadratic forms and symmetric matrices, we are looking for a positive selfadjoint operator A such that

$$B_\varphi(\eta,\eta') = \langle A\eta,\eta'\rangle \quad ,$$

and then letting $C_\pi = A^{1/2}$ should do the trick. Here we are in the situation that the domain is only a dense subset. We intend to use the representation theorem [101, Theorem VIII.6], and for this we need to show that B_φ is **closed**. This amounts to checking the following condition, for every sequence $(\eta_n)_{n\in\mathbb{N}}$ and $\eta \in \mathcal{H}_\pi$ such that $\eta_n \to \eta$: If

$$B_\varphi(\eta_n-\eta_m,\eta_n-\eta_m) \to 0 \quad , \quad \text{as } n,m\to\infty \tag{2.17}$$

then $\eta \in \mathcal{D}$ and $B_\varphi(\eta_n-\eta,\eta_n-\eta)\to 0$. It turns out that this is precisely the argument from the proof of Proposition 2.8: Observing that

$$B_\varphi(\eta-\eta',\eta-\eta') = \|V_{\eta-\eta'}\varphi\|_2^2 = \|V_\eta\varphi - V_{\eta'}\varphi\|_2^2 \quad ,$$

we see that condition (2.17) is equivalent to saying that $(V_{\eta_n}\varphi)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^2(G)$. Hence after passing to a suitable subsequence we find that $V_{\eta_n}\varphi \to F \in \mathrm{L}^2(G)$, both in L^2 and pointwise almost everywhere. On the other hand, $\eta_n \to \eta$ entails $V_{\eta_n}\varphi \to V_\eta\varphi$ uniformly, by Cauchy-Schwarz. Hence $F = V_\eta$, and $\eta\in\mathcal{D}$ by part (a). Therefore we obtain the operator A, and letting $C_\pi = A^{1/2}$ yields

$$\langle V_{\eta'}\varphi, V_\eta\varphi\rangle = \langle C_\pi\eta, C_\pi\eta'\rangle \tag{2.18}$$

The first step for deriving the general orthogonality relations consists in observing that B_φ (and consequently C_π) is independent of the choice of normed vector φ: Fixing an arbitrary admissible η, the fact that V_η is the multiple of an isometry yields for all normed φ

$$B_\varphi(\eta,\eta) = \|V_\eta\varphi\|_2^2 = c_\eta\|\varphi\|^2$$

where c_η is a constant independent of φ. By polarization this implies that B_φ is independent of φ. Hence we obtain for arbitrary $\varphi\in\mathcal{H}$ and admissible vectors η,η'

$$\langle V_{\eta'}\varphi, V_\eta\varphi\rangle = \|\varphi\|^2\langle C_\pi\eta, C_\pi\eta\rangle \quad .$$

Polarization with respect to φ yields (2.15).
Part (c) follows from (d), for (d) we refer to [38].

We note that (2.16) entails that C_π is unbounded in the nonunimodular case, since the operator norm on $\mathcal{B}(\mathcal{H}_\pi)$ is invariant under conjugation with unitaries. The operators C_π are called *Duflo-Moore* operators. More details on these operators can be found in Section 3.8. The proof given here basically follows the argument in [60]. The main reason we have reproduced it in part is to demonstrate the close connection between the admissibility condition and the construction of the operators: The admissibility criterion (2.14) implies the orthogonality criterion (2.15) by polarization, and the latter was used to define C_π. Let us also point out the crucial role of irreducibility, which particularly implies that the space of admissible vectors (up to normalization) is dense in $\mathcal{H}_\pi$.

Remark 2.26. Note that the Duflo-Moore operators C_π studied here relate to the **formal dimension operators** K_π in [38] as $K_\pi^{-1/2} = C_\pi$. The terminology "formal dimension operator" is best understood by considering compact groups: Let π be an irreducible representation of a compact group G. Since coordinate functions are bounded, it is obvious that π is square-integrable. G is unimodular, thus C_π is scalar. Now the Schur orthogonality relations for compact groups [45, 5.8] yield for a normalized vector φ that

$$\|V_\varphi \varphi\|_2^2 = d_\pi^{-1} \|\varphi\|^2$$

where $d_\pi = \dim(\mathcal{H}_\pi)$. Thus $C_\pi = d_\pi^{-1/2} \cdot \mathcal{H}_\pi$, and the formal dimension operator $K_\pi = C_\pi^{-2}$ is multiplication with the Hilbert space dimension of $\mathcal{H}_\pi$.

The theorem of Grossmann, Morlet and Paul provides a rich reservoir of cases. In fact the large majority of papers dealing with the construction of wavelet transforms refers to this result. We give a small sample which contains the most popular examples.

Example 2.27. **Windowed Fourier transform.** Consider the **reduced Heisenberg group**, given as the set $\mathbb{H}_r = \mathbb{R}^2 \times \mathbb{T}$, with the group law

$$(p,q,z)(p',q',z') = (p+p', q+q', zz' \mathrm{e}^{\pi \mathrm{i}(pq'-qp')}) \ .$$

Haar measure here is given by $dpdqdz$, where dz is the rotation-invariant measure on the torus, normalized to one. G is unimodular. It acts on $\mathrm{L}^2(\mathbb{R})$ via the **Schrödinger representation** given by

$$(\pi(p,q,z)f)(x) = z\mathrm{e}^{2\pi \mathrm{i} q(x+p/2)} f(x+p) \quad . \tag{2.19}$$

Straightforward calculation allows to establish that

$$\begin{aligned}
\|V_\eta g\|_2^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{T}} \left| \int_{\mathbb{R}} g(x) \overline{z} \mathrm{e}^{-2\pi \mathrm{i} q(x+p/2)} \overline{\eta(x+p)} dx \right|^2 dz dq dp \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} g(x) \mathrm{e}^{-2\pi \mathrm{i} q(x+p/2)} \overline{\eta(x+p)} dx \right|^2 dq dp \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} g(x) \mathrm{e}^{-2\pi \mathrm{i} qx} \overline{\eta(x+p)} dx \right|^2 dq dp \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{H_p}(q)|^2 dp dq \quad ,
\end{aligned}$$

where $H_p(x) = g(x)\overline{\eta(x+p)}$, which for fixed $p \in \mathbb{R}$ is an integrable function. An application of Fubini's and Plancherel's theorem for the reals yields

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{H_p}(q)|^2 dpdq = \int_{\mathbb{R}} \int_{\mathbb{R}} |g(x)\overline{\eta(x+p)}|^2 dxdp$$
$$= \|\eta\|_2^2 \|g\|_2^2 \quad .$$

This relation implies first of all that π is irreducible: V_η is injective for every nonzero η, i.e., η is cyclic. Moreover, every $\eta \in \mathrm{L}^2(\mathbb{R})$ is admissible up to normalization; more precisely, iff $\|\eta\| = 1$. This is what we are to expect by Theorem 2.25: G is unimodular, hence the formal dimension operator is a scalar multiple of the identity. In addition, we have established by elementary calculation that the scalar equals one.

Since the torus acts by multiplication, we have $|V_f(p,q,z)| = |V_f(p,q,1)|$, for all $z \in \mathbb{T}$. Hence the map $\mathcal{W}_f : g \mapsto (V_f g)|_{\mathbb{R}^2 \times \{1\}}$ is isometric as well. $\mathcal{W}_f$ is the **windowed Fourier transform** associated to the window f.

Hence we have derived for all $f \in \mathrm{L}^2(\mathbb{R})$ with $\|f\| = 1$ the transform

$$\mathcal{W}_f g(p,q) = \int_{\mathbb{R}} g(x) \mathrm{e}^{2\pi \mathrm{i} q(x+p/2)} \overline{f(x+p)} \, dx \quad ,$$

with inversion formula

$$g(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{W}_f g(p,q) \; \mathrm{e}^{-2\pi \mathrm{i} q(x+p/2)} \overline{f(x+p)} \, dpdq \quad .$$

Note that this inversion is to be understood in the weak sense and usually does not hold pointwise.

Example 2.28. **1-D CWT.** This is the original "continuous wavelet transform" introduced in [60]. It is based on the $ax + b$ group, the semidirect product $\mathbb{R} \rtimes \mathbb{R}'$. As a set G is given as $G = \mathbb{R} \times \mathbb{R}'$, with group law

$$(b,a)(b',a') = (b + ab', aa') \quad .$$

The left Haar measure is $db|a|^{-2}da$, which is distinct from the right Haar measure $db|a|^{-1}da$. Wavelets arise from the **quasi-regular representation** π acting on $\mathrm{L}^2(\mathbb{R})$ via

$$(\pi(b,a)f)(x) = |a|^{-1/2} f\left(\frac{x-b}{a}\right) \quad .$$

Again, computing L^2-norms of wavelet coefficients turns out to be an exercise in real Fourier analysis. First observe that on the Fourier transform side π acts as

$$(\pi(b,a)f)^\wedge(\omega) = |a|^{1/2} \mathrm{e}^{-2\pi \mathrm{i}\omega b} \widehat{f}(a\omega)$$

Hence, using the Plancherel theorem for the reals we can compute

$$\begin{aligned}
\|V_\eta g\|_2^2 &= \int_G |\langle g, \pi(b,a)\eta\rangle|^2 \, d\mu_G(b,a) \\
&= \int_G |\langle \hat{g}, (\pi(b,a)\eta)^\wedge\rangle|^2 \, d\mu_G(b,a) \\
&= \int_{\mathbb{R}'} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \hat{f}(\gamma)|a|^{1/2} \mathrm{e}^{2\pi \mathrm{i}\gamma b} \overline{\hat{\eta}}(a\gamma) d\gamma \right|^2 |a|^{-2} dbda \\
&= \int_{\mathbb{R}'} \int_{\mathbb{R}} \left| \int_{\widehat{\mathbb{R}^k}} \hat{f}(\gamma) \mathrm{e}^{2\pi \mathrm{i}\gamma b} \overline{\hat{\eta}}(a\gamma) d\gamma \right|^2 |a|^{-1} dbda \\
&= \int_{\mathbb{R}'} \int_{\mathbb{R}} \left| \widehat{\phi_a}(-b) \right|^2 |a|^{-1} dbda,
\end{aligned}$$

where $\phi_a(\gamma) = \hat{g}(\gamma)\overline{\hat{\eta}}(a\gamma)$. The Plancherel theorem allows thus to continue

$$\begin{aligned}
\int_{\mathbb{R}'} \int_{\mathbb{R}} \left| \widehat{\phi_a}(-b) \right|^2 |a|^{-1} dbda &= \int_{\mathbb{R}'} \int_{\mathbb{R}} \left| \hat{g}(\gamma)\overline{\hat{\eta}}(a\gamma) \right|^2 |a|^{-1} dbda \\
&= \int_{\mathbb{R}} |\hat{g}(\gamma)|^2 \left(\int_{\mathbb{R}'} |\hat{\eta}(a\gamma)|^2 |a|^{-1} da \right) d\gamma \\
&= \left(\int_{\mathbb{R}} |\hat{g}(\gamma)|^2 d\gamma \right) \cdot \left(\int_{\mathbb{R}'} |\hat{\eta}(a\gamma)|^2 |a|^{-1} da \right) \\
&= c_\eta^2 \|g\|^2 \quad ,
\end{aligned}$$

where we used the fact that the measure $a^{-1}da$ is Haar measure of the multiplicative group $\mathbb{R}'$. Hence we have derived

$$\|V_\eta g\|_2^2 = c_\eta^2 \|g\|^2 \quad , \tag{2.20}$$

where

$$c_\eta^2 = \int_{\mathbb{R}} \frac{|\widehat{\eta}(\omega)|^2}{|\omega|} d\omega \quad . \tag{2.21}$$

Note that our calculations also include the case $c_\eta = \infty$, where (2.20) means that $V_\eta f \notin \mathrm{L}^2(G)$. For this additional observation we need the following extended version of the Plancherel theorem:

$$\forall h \in \mathrm{L}^1(\mathbb{R}) \quad : \quad \left(h \in \mathrm{L}^2(\mathbb{R}) \Longleftrightarrow \widehat{h} \in \mathrm{L}^2(\mathbb{R}) \right) \quad . \tag{2.22}$$

Now "$\Longrightarrow$" is due to Plancherel's theorem, but the other direction is not. In order to show it, let $g \in \mathrm{L}^2(\mathbb{R})$ denote the inverse Plancherel transform of $\widehat{h}$, we have to show $g = h$. But this follows from the injectivity of the Fourier transform on the space of tempered distributions, since restriction to $\mathrm{L}^1(G)$ resp. $\mathrm{L}^2(G)$ yields the Fourier- resp. Plancherel transform.

As in the case of the windowed Fourier transform, (2.20) implies that the representation is irreducible. This time, the admissibility condition reads as:

$$\eta \in \mathrm{L}^2(\mathbb{R}) \text{ is admissible } \Leftrightarrow c_\eta = 1 \quad . \tag{2.23}$$

Comparing our findings to Theorem 2.25, we see that we have a discrete series representation of a nonunimodular group. Accordingly, the admissibility condition is more restrictive, requiring not just the right normalization. As a matter of fact, it is straightforward to check the semi-invariance relation (2.16) to show that the Duflo-Moore operator is given by

$$(C_\pi f)^\wedge(\omega) = |\omega|^{-1/2}\widehat{f}(\omega) \quad ,$$

as (2.21) suggests.

Example 2.29. As observed in Remark (2.20) above, V_f need not be densely defined for arbitrary $f \in \mathrm{L}^2(G)$, when G is nonunimodular. Here we construct such an example for the case that G is the $ax+b$-group. For this purpose consider the quasi-regular representation π from Example 2.28. Pick a $\psi \in \mathrm{L}^2(\mathbb{R})$ which is not in the domain of the Duflo-Moore operator, and an admissible vector η. Defining $f = V_\eta\psi$ and $\mathcal{H} = V_\eta(\mathrm{L}^2(\mathbb{R})) \subset \mathrm{L}^2(G)$, we see that $V_f g = 0$ for $g \in \mathcal{H}^\perp$, whereas for $g = V_\eta\varphi \in \mathcal{H}$,

$$V_f g(x) = \langle V_\eta\varphi, \lambda_G(x)V_\eta\psi\rangle = \langle\varphi, \pi(x)\psi\rangle = V_\psi\phi(x) \quad ,$$

and the latter function is not in $\mathrm{L}^2(G)$ by 2.25 (b) and the choice of ψ. Hence $\mathrm{dom}(V_f) = \mathcal{H}^\perp$, and $V_f = 0$ on this domain.

Example 2.30. **2-D CWT.** This construction was first introduced by Murenzi [93], as a natural generalization of the continuous transform in one dimension. We consider the **similitude group** $G = \mathbb{R}^2 \rtimes (SO(2) \times \mathbb{R}^+)$. Hence G is the set $\mathbb{R}^2 \times SO(2) \times \mathbb{R}^+$ with the group law

$$(x, h, r)(x', h', r') = (x + rhx', hh', rr') \quad .$$

The group can be identified with the subgroup of the full affine group of the plane generated the translations, the rotations and the dilations. It thus acts naturally on $\mathbb{R}^2$, which gives rise to the quasi-regular representation π acting on $\mathrm{L}^2(\mathbb{R}^2)$ via

$$(\pi(x, h, r)f)(y) = |r|^{-1} f(r^{-1}h^{-1}(y - x)) \quad .$$

An adaptation of the argument for the 1D CWT yields

$$\|V_\eta f\|_2^2 = c_\eta^2 \|f\|_2 \quad , \tag{2.24}$$

where this time

$$c_\eta^2 = \int_{\mathbb{R}^2} \frac{|\widehat{f}(\omega)|^2}{|\omega|^2} d\omega \quad .$$

Therefore the admissibility condition reads

$$\eta \text{ is admissible } \iff \int_{\mathbb{R}^2} \frac{|\widehat{f}(\omega)|^2}{|\omega|^2} d\omega = 1 \quad .$$

As in the case of the 1D-CWT, we obtain from the norm equality that π is again irreducible. The Duflo-Moore operator is computed as

$$(C_\pi f)^\wedge(\omega) = |\omega|^{-1}\widehat{f}(\omega) \quad .$$

Next let us consider direct sums of discrete series representations. The following theorem describes how far the discrete series arguments carry. Recall that Proposition 2.23 gives criteria for direct sum representations, and the orthogonality relations for the discrete series case allow to derive admissibility criteria for multiplicities greater than one.

Theorem 2.31. *Let $\pi = \bigoplus_{i\in I} \pi_i$, where each π_i is a discrete series representation. Denote by P_i the projection onto the representation space $\mathcal{H}_{\pi_i}$, and by C_{π_i} the associated Duflo-Moore operators. Since the π_i are irreducible, there exist (up to normalization) unique intertwining operators $S_{i,j} : \mathcal{H}_{\pi_i} \to \mathcal{H}_{\pi_j}$. Then the following are equivalent:*

(a) η is admissible.
(b) $\eta_i \in \mathrm{dom}(C_{\pi_i})$, with $\|C_{\pi_i}\eta_i\| = 1$. Moreover, for all i,j with $\pi_i \simeq \pi_j$,

$$\langle C_{\pi_j} S_{i,j}\eta_i, C_{\pi_j}\eta_j\rangle = 0 \tag{2.25}$$

Proof. We apply Proposition 2.23. By Theorem 2.25(c), $\|C_{\pi_i}\eta_i\| = 1$ is the admissibility condition on η_i. Moreover, the orthogonality relation (2.15) shows that whenever $\pi_i \simeq \pi_j$,

$$\mathrm{Im}(V_{\eta_i}) \perp \mathrm{Im}(V_{\eta_i}) \iff \langle C_{\pi_j} S_{i,j}\eta_i, C_{\pi_j}\eta_j\rangle = 0 \quad .$$

Since any two irreducible representations are either equivalent or disjoint, Lemma 2.24 yields $\mathrm{Im}(V_{\eta_i}) \perp \mathrm{Im}(V_{\eta_i})$ for arbitrary vectors η_i and η_j, whenever $\pi_i \not\simeq \pi_j$. Hence the proof is finished.

The following remark is a preliminary version of one of the main results contained in this book: The existence criterion for admissible vectors given in Theorem 4.22. Here we only consider the case of direct sums of discrete series representations. Some of the phenomena encountered in the general case can be already examined in this simpler setting, in particular the striking difference between unimodular and nonunimodular groups and the role of the formal dimension operators in this context.

Remark 2.32. Let $\pi = \bigoplus_{i\in I} \pi_i$, where each π_i is a discrete series representation. We associate a *multiplicity function* $m_\pi : \widehat{G} \to \mathbb{N}_0 \bigcup \{\infty\}$ to π, by letting

$$m_\pi(\sigma) = |\{i \in I : \sigma \simeq \pi_i\}| \quad ,$$

where $|\cdot|$ denotes cardinality. m_π simply counts the multiplicity with which the representation π is contained in π. Note that $\mathcal{H}_\pi$ is assumed to be separable, hence the cardinalities are countably infinite at most, and only countably many σ have a nonzero multiplicity. Fix unitary intertwining operators $T_i : \mathcal{H}_{\pi_i} \to \mathcal{H}_\sigma$, for the unique $\sigma \in \widehat{G}$ with $\sigma \simeq \pi_i$. The uniqueness property of the Duflo-Moore operators entails that $C_\sigma = T_i C_{\pi_i} T_i^*$.

Using the operators T_i, the admissibility conditions from Theorem 2.31 can also be written as

$$\langle C_\sigma T_i \eta_i, C_\sigma T_j \eta_j \rangle = 0 \quad (\pi_i \simeq \pi_j) \tag{2.26}$$

$$\|C_{\pi_i} \eta_i\| = 1 \quad (\forall i \in I) \tag{2.27}$$

Both relations can be used to derive necessary conditions for the multiplicity function: (2.26) clearly implies that

$$m_\pi(\sigma) \leq \dim(\mathcal{H}_\sigma) \quad . \tag{2.28}$$

Moreover, the orthogonal decomposition of $\mathcal{H}_\pi$ yields in particular that

$$\infty > \|\eta\|^2 = \sum_{i \in I} \|P_i \eta\|^2 = \sum_{i \in I} \|\eta_i\|^2 \quad . \tag{2.29}$$

Now assume that G is unimodular. Then $C_\sigma = c_\sigma \times \mathrm{Id}_{\mathcal{H}_\sigma}$, with positive scalars c_σ. Hence (2.27) entails the necessary condition

$$\sum_{\sigma \in \widehat{G}} m_\pi(\sigma) c_\sigma^{-2} < \infty \quad . \tag{2.30}$$

Conversely, it is easily seen that vectors fulfilling (2.26) and (2.27) exist once (2.28) and (2.30) hold, therefore we have found a characterization of direct sums of discrete series representations with admissible vectors. Note that (2.30) implies $m_\pi(\sigma) < \infty$, which can be seen as a sharpening of (2.28).

In the nonunimodular case the situation is much less transparent. However, it turns out that the restrictions actually vanish! To begin with, $\dim(\mathcal{H}_\sigma) = \infty$ follows from the existence of an unbounded operator C_σ on $\mathcal{H}_\sigma$. In addition, while (2.29) still holds, implying in particular that (at least for I infinite) the norms of the η_i become arbitrarily small, it is no contradiction to (2.27). Here the fact that the C_{π_i} are unbounded makes it conceivable that there exist vectors that actually fulfill both conditions. Note that we still need to ensure (2.26), which requires more knowledge of the formal dimension operators than we have currently at our disposal. In any case the existence of an unbounded operator on $\mathcal{H}_\sigma$ entails $\dim(\mathcal{H}_\sigma) = \infty$, i.e., (2.28) holds trivially.

We will next study the space $\mathrm{L}^2_\pi(G)$ spanned by all coefficient functions associated to a fixed discrete series representation π. Most of the following is due to Duflo and Moore. The results can be seen as precursors of the Plancherel formula, or more precisely, as the contribution of the discrete series to the Plancherel formula. They also provide further insight into the role of Hilbert-Schmidt operators and the quasi-invariance relation (2.16).

Theorem 2.33. *Let π be a discrete series representation, and define*

$$\mathrm{L}^2_\pi(G) = \overline{\mathrm{span}\{\mathcal{H} : \lambda_G|_{\mathcal{H}} \simeq \pi\}} \quad .$$

(a) $\mathrm{L}^2_\pi(G) = \overline{\mathrm{span}\{V_\eta\varphi : \varphi, \eta \in \mathcal{H}_\pi \text{ such that } V_\eta\varphi \in \mathrm{L}^2(G)\}}$.
(b) $\mathrm{L}^2_\pi(G)$ *is* $\lambda_G \times \varrho_G$*-invariant, with* $\lambda_G \times \varrho_G|_{\mathrm{L}^2_\pi(G)} \simeq \pi \otimes \overline{\pi}$*. In particular,* $\lambda_G \times \varrho_G|_{\mathrm{L}^2_\pi(G)}$ *is irreducible.*
(c) Let $(\eta_i)_{i\in I}$ *denote an ONB of* $\mathcal{H}_\pi$ *contained in* $\mathrm{dom}(C_\pi^{-1})$*. Then* $\left(V_{C_\pi^{-1}\eta_i}\eta_j\right)_{i,j\in I}$ *is an ONB of* $\mathrm{L}^2_\pi(G)$.
(d) If σ *is another discrete series representation with* $\sigma \not\simeq \pi$*, then* $\mathrm{L}^2_\sigma(G) \perp \mathrm{L}^2_\pi(G)$.

Proof. For part (a) we let $\mathcal{H}_0 = \mathrm{span}\{V_\eta\varphi : \varphi, \eta \in \mathcal{H}_\pi, \eta \text{ is admissible}\}$. Then for every $f = V_\eta\varphi \in \mathcal{H}_0$, the leftinvariant space spanned by f is just the image of $\mathcal{H}_\pi$ under V_η, which is a unitary equivalence. Hence $\mathcal{H}_0 \subset \mathrm{L}^2_\pi(G)$, which extends to the closure of $\mathcal{H}_0$.

For the other direction we argue indirectly. Assume that there exists $g \in \mathrm{L}^2(G)$ such that the restriction of λ_G to the leftinvariant subspace generated by g is equivalent to π, yet g is not contained in the closed span of $\mathcal{H}_0$. W.l.o.g. we may assume that $g \perp \mathcal{H}_0$. Observe that $\mathcal{H}_0$ is rightinvariant also, since

$$V_{\pi(x)\eta}\varphi(y) = \langle \varphi, \pi(yx)\eta\rangle = \Delta_G(x)^{-1/2}\left(\rho_G(x) V_\eta\varphi\right)(y) \quad .$$

Let Q denote the projection onto the leftinvariant space generated by g. Pick $h \in C_c(G)$ such that $Qh \neq 0$. Then we have

$$V_{Qh}g = V_h g = g * h^* \in \mathrm{L}^2(G)$$

by choice of h and 2.19 (b). Moreover, $V_{Qh}g$ is nonzero since Qh and g are nonzero and π is irreducible. By definition of $\mathcal{H}_0$ we have $V_{Qh}g \in calH_0$. On the other hand, rightinvariance of $\mathcal{H}_0$ yields that if $g \perp \mathcal{H}_0$, then $g * h^* \perp \mathcal{H}_0$, and we have the desired contradiction.

For part (b), consider the mapping

$$T : \varphi \otimes \eta \mapsto V_{C_\pi^{-1}\eta}\varphi \quad ,$$

defined for all elementary tensors $\varphi \otimes \eta$ satisfying $\eta \in \mathrm{dom}(C_\pi^{-1})$. Since C_π^{-1} is densely defined, these tensors span a dense subspace of $\mathcal{H}_\pi \otimes \mathcal{H}_\pi$. Moreover, by the orthogonality relation (2.15), T is isometric. Hence there exists a unique linear isometry, also denoted by $T : \mathcal{H}_\pi \otimes \mathcal{H}_\pi \to \mathrm{L}^2_\pi(G)$. By part (a), it has dense image, hence T is in fact unitary. We will next show that T is an intertwining operator. For this purpose observe that (2.16) gives rise to

$$\pi(x) C_\pi^{-1} \pi(x)^* = \Delta_G(x)^{-1/2} C_\pi^{-1} \quad .$$

Then we compute

$$\begin{aligned}
T(\varphi \otimes \pi(x)\eta)(y) &= \langle \varphi, \pi(y) C_\pi^{-1} \pi(x)\eta \rangle \\
&= \langle \varphi, \Delta_G^{1/2} \pi(y)\pi(x) C_\pi^{-1} \eta \rangle \\
&= \Delta_G(x)^{1/2} T(\varphi \otimes \pi)(yx) \\
&= \left(\varrho(T(\varphi \otimes \pi)) \right)(y) \quad .
\end{aligned}$$

This shows that T intertwines $1 \otimes \overline{\pi}$ with ϱ_G; the left half is obvious.

Now part (c) is obtained by applying T to the ONB $(\eta_i \otimes \eta_j)_{i,j \in I}$. Part (d) follows from (a) and 2.24.

Observe that an ONB as in part (c) of the theorem always exists, since $\mathrm{dom}(C_\pi^{-1})$ is dense; simply apply Gram-Schmidt orthonormalization.

We will next show the announced contribution of π to the Plancherel formula. For this purpose we need the following definition: For $f \in \mathrm{L}^1(G)$ and a representation π, let

$$\pi(f) = \int_G f(x)\pi(x)dx \quad ,$$

where convergence is in the weak sense, which for $f \in \mathrm{L}^1(G)$ is guaranteed. This construction will be seen to yield the **operator-valued Fourier transform**, which is discussed in more detail in Chapter 3. We postpone a more complete discussion of the Fourier transform to that chapter, and only show the following result.

Theorem 2.34. *Let π be a discrete series representation. Denote by P_π the projection onto $\mathrm{L}^2_\pi(G)$. Then, for all $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$, $\pi(f)C_\pi^{-1}$ extends to a Hilbert-Schmidt operator, with*

$$\| \pi(f) C_\pi^{-1} \| = \| P_\pi(f) \| \quad .$$

Proof. Let an ONB $(\eta_i)_{i \in I} \subset \mathrm{dom}(C_\pi^{-1})$ of $\mathcal{H}_\pi$ be given. Then by part (c) of the previous theorem, we can compute the norm of $P_\pi(f)$ as

$$\begin{aligned}
\| P_\pi(f) \|^2 &= \sum_{i,j \in I} \left| \langle f, V_{C_\pi^{-1}\eta_j} \eta_i \rangle \right|^2 \\
&= \sum_{i,j \in I} \left| \int_G f(x) \overline{\langle \eta_i, \pi(x) C_\pi^{-1} \eta_j \rangle} dx \right|^2 \\
&= \sum_{i,j \in I} | \langle \pi(f) C_\pi^{-1} \eta_j, \eta_i \rangle |^2 \quad ,
\end{aligned}$$

where the last equation used the definition of the weak operator integral. But the last term is just the Hilbert-Schmidt norm of $\pi(f)C_\pi^{-1}$.

Let us now give a few examples for which the discrete series approach cannot work. Clearly, if the underlying group is compact, then every irreducible representation is in the discrete series: Wavelet coefficients are bounded functions and the Haar measure is finite, hence every wavelet coefficient is trivially in L^2. At the other end of the scale we have the reals: Every irreducible representation is a character, i.e. a group homomorphism $\mathbb{R} \to \mathbb{T}$. Matrix coefficients are constant multiples of that character, hence never square-integrable. The following theorem extends this observation to a larger class. The result is probably folklore, though I am not aware of a reference.

Theorem 2.35. *Let G be a* **SIN-group***, i.e., every neighborhood of unity contains a conjugation-invariant neighborhood. If G has a discrete series representation, then G is compact.*
In particular, if G is discrete and has a discrete series representation, then G is finite. If G is abelian and has a discrete series representation, then G is compact.

Proof. Note that SIN-groups are unimodular: For any conjugation-invariant neighborhood U of unity, and any $x \in G$ we have $|Ux| = |x^{-1}Ux| = |U|$.

Now let π be a discrete series representation. The first step consists in showing that $\dim(\mathcal{H}_\pi)$ is finite. For this purpose pick a conjugation-invariant neighborhood of unity such that $\pi(\mathbf{1}_U) \neq 0$. The existence of such a neighborhood is seen as follows: Since the characteristic functions of a neighborhood base at unity span a dense subspace of $\mathrm{L}^1(G)$, we would otherwise obtain $\pi(f) = 0$ for all $f \in \mathrm{L}^1(G)$. This would contradict [35, 13.3.1], hence U exists.

We next show that $\pi(\mathbf{1}_U)$ is an intertwining operator. Using conjugation-invariance of U and rightinvariance of Haar measure, we find

$$\begin{aligned}
\langle \phi, \pi\left(\mathbf{1}_U\right)\pi(y)\eta\rangle &= \int_G \mathbf{1}_U(x)\langle \phi, \pi(x)\pi(y)\eta\rangle d\mu_G(x) \\
&= \int_U \langle \phi, \pi(xy)\eta\rangle d\mu_G(x) \\
&= \int_{Uy} \langle \phi, \pi(x)\eta\rangle d\mu_G(x) \\
&= \int_{yU} \langle \phi, \pi(x)\eta\rangle d\mu_G(x) \\
&= \int_U \langle \phi, \pi(yx)\eta\rangle d\mu_G(x) \\
&= \langle \phi, \pi(y)\pi\left(\mathbf{1}_U\right)\eta\rangle .
\end{aligned}$$

Hence, by Schur's lemma, $\pi(\mathbf{1}_U)$ is a scalar, which is nonzero by choice of $\mathbf{1}_U$. On the other hand, $\pi(\mathbf{1}_U)$ is Hilbert-Schmidt. Hence $\dim(\mathcal{H}_\pi) < \infty$.

Now assume that G is not compact. Since G is σ-compact and locally compact, there is a sequence $(C_n)_{n\in\mathbb{N}}$ of compact sets in G with the property that $A \subset G$ is compact iff there exists $n \in \mathbb{N}$ such that $A \subset C_n$. Pick a sequence $(x_n)_{n\in\mathbb{N}} \subset G$ with $x_n \in G \setminus C_n$. Then for every compact set $A \subset G$ there exists $n_A \in \mathbb{N}$ with $x_k \notin C$ for all $k \geq n_A$, and this property is inherited by subsequences. Since $\dim(\mathcal{H}_\pi) < \infty$, we may assume that $\mathcal{H}_\pi = \mathbb{C}^n$ and $\pi(G) \subset U(n)$. Since $U(n)$ is compact, passing to a subsequence allows the assumption that $\pi(x_n) \to S \in U(n)$. Picking any unit vector η, we thus arrive at

$$V_\eta(S\eta)(x_n) = \langle S\eta, \pi(x_n)\eta\rangle \to \|S\eta\|^2 = 1 \quad .$$

On the other hand, $V_\eta(S\eta)$ vanishes at infinity, by 2.19, which yields the desired contradiction.

Note that the somewhat complicated choice of the sequence x_n is only made to avoid that any subsequence is relatively compact.

Let us close with an example that cannot be covered by the results in this section.

Example 2.36. **Dyadic wavelet transform.** This construction was first considered by Mallat and Zhang [92], though without referring to any group structure. We consider the group $H = \mathbb{R} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts by powers of 2. Hence H is the subgroup of the $ax+b$-group generated by $\mathbb{R} \times \{0\}$ and $(0, 2)$; let's call it the $2^k x + b$-group. We are interested in admissible vectors for the restriction of the quasiregular representation from Example 2.28 to H. An easy adaptation of the calculations there yields

$$\|V_\eta f\|_2^2 = \int_{\mathbb{R}} |\widehat{f}(\omega)|^2 \Phi_\eta(\omega) d\omega \quad , \tag{2.31}$$

where the function Φ_η is given by

$$\Phi_\eta(\omega) = \sum_{n\in\mathbb{Z}} |\widehat{\eta}(2^n\omega)|^2 \quad .$$

For the proof of a more general result we refer the reader to Theorem 5.8 below.

Unlike the previous examples, this representation is not irreducible: Consider a function f such that $\widehat{f}$ is supported in $[1, 1.5]$, and η with $\mathrm{supp}(\widehat{\eta}) \subset [1.5, 2]$. Then $\Phi_\eta = 0$ on the support of f, and thus (2.31) implies $V_\eta f = 0$.

On the other hand, (2.31) yields the admissibility criterion

$$\eta \text{ is admissible} \Leftrightarrow \Phi_\eta \equiv 1 \quad ,$$

and it is easy to construct such functions, say $\widehat{\eta} = \mathbf{1}_{[-2,-1]} + \mathbf{1}_{[1,2]}$. Hence we have found a representation which is not covered by the discrete series case. As a matter of fact, π does not contain irreducible subrepresentation: Suppose that $\mathcal{H} \subset \mathrm{L}^2(\mathbb{R})$ is an irreducible subspace. Since π has admissible vectors, the subrepresentation also does, by Proposition 2.14. Let $\eta \in \mathcal{H}$ be admissible. Then $\pi(G)\eta$ spans $\mathcal{H}$, therefore relation (2.31) yields that

$$\begin{aligned} \mathcal{H}^\perp &= \{\varphi \in \mathrm{L}^2(\mathbb{R}) : V_\eta\varphi = 0\} \\ &= \{\varphi \in \mathrm{L}^2(\mathbb{R}) : |\mathrm{supp}(\widehat{\varphi}) \cap \mathrm{supp}(\Phi_\eta|) = 0\} \quad . \end{aligned}$$

But the orthogonal complement of the latter space is easily computed, yielding

$$\mathcal{H} = \{\varphi \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(\widehat{\varphi}) \subset \mathrm{supp}(\Phi_\eta)\}$$

Recall that supp denotes the measure-theoretic support, and inclusion is understood up to sets of measure zero. Now it is easy to construct two nonzero vectors ξ_1 and ξ_2 such that Φ_{ξ_1} and Φ_{ξ_2} have disjoint supports, both contained in $\mathrm{supp}(\Phi_\eta)$. To see this observe that $\Phi_\eta(2^n\omega) = \Phi_\eta(\omega)$ implies that $\mathrm{supp}(\Phi_\eta) \subset \cup_{n\in\mathbb{Z}} 2^n A$, where $A = [1, 2[\cap \mathrm{supp}(\Phi_\eta)$. In particular, A has positive measure. Hence, if we pick $B_1, B_2 \subset A$ disjoint with positive measure, and let $\widehat{\xi_i} = \mathbf{1}_{B_i}$, we obtain two nonzero functions such that $\mathrm{supp}(\Phi_{\xi_1})$ and $\mathrm{supp}(\Phi_{\xi_2})$ are disjoint. But then (2.31) implies $V_{\xi_1}\xi_2 = 0$, in particular ξ_1 is not cyclic for $\mathcal{H}$.

Thus π has no irreducible subrepresentation, in particular Theorem 2.31 has significance either.

2.5 Selfadjoint Convolution Idempotents and Support Properties

We now continue the discussion of the subspaces of $\mathrm{L}^2(G)$ which arise as image spaces of wavelet transforms. The following notion describes the associated reproducing kernels. After proving this observation, we will draw several consequences from the properties of the reproducing kernel spaces. In particular, we study support properties of wavelet transforms, as well as the existence of admissible vectors for λ_G.

Definition 2.37. $S \in \mathrm{L}^2(G)$ *is called* **(right selfadjoint) convolution idempotent** *if* $S = S * S^* = S^*$.

Convolution idempotents in $\mathrm{L}^1(G)$ have been studied for instance in [59], and generally the existence of such idempotents is a strong restriction on the group. By contrast, we will see that L^2-convolution idempotents exist in abundance. But first the connection between convolution idempotents and generalized wavelet transforms.

Proposition 2.38. *(a) Let $S \in \mathrm{L}^2(G)$ be a convolution idempotent, and denote by $\mathcal{H}$ the closed leftinvariant subspace generated by S, i.e., $\mathcal{H} = \overline{\mathrm{span}(\lambda_G(G)S)}$. Then the projection onto $\mathcal{H}$ is given by right convolution with S, i.e. $\mathcal{H} = \mathrm{L}^2(G) * S = \{g * S : g \in \mathrm{L}^2(G)$. Moreover, if T is another convolution idempotent in $\mathcal{H}$ with $\mathcal{H} = \mathrm{L}^2(G) * T$, then $T = S$.*

(b) S is a selfadjoint convolution idempotent iff there exists a representation π and an admissible $\eta \in \mathcal{H}_\pi$ such that $S = V_\eta \eta$. Consequently, the image spaces of continuous wavelet transforms are precisely the spaces of the form $\mathrm{L}^2(G) * S$.

Proof. For (a) observe that clearly $f = f * S$ holds for all $f \in \mathrm{span}(\lambda_G(G)S)$, as well as $f * S = 0$ for all $f \perp \mathcal{H}$. Hence on a dense subspace $V_S = P_{\mathcal{H}}$, the latter being the projection onto $\mathcal{H}$. Since V_S is closed, the result follows. The uniqueness statement follows from $T = T * S$ and $T = T^*$, hence

$$T = S^* * T^* = S * T = S \quad .$$

The "if"-part of (b) is due to 2.16 (c). For the other direction let π be the restriction of λ_G to $\mathcal{H} = \mathrm{L}^2(G) * S$. Then $f = f * S = V_S f$ for $f \in \mathcal{H}$ shows that the inclusion map is a continuous wavelet transform. The last statement is obvious by now.

The next property will be relevant for sampling theorems, allowing to conclude uniform convergence from L^2-convergence. Note that this observation holds for a larger class of reproducing kernel Hilbert spaces.

Proposition 2.39. *Let $S \in \mathrm{L}^2(G)$ be a selfadjoint convolution idempotent, then for all $f \in \mathcal{H} = \mathrm{L}^2(G) * S$ we have $\|f\|_\infty \le \|f\|_2 \|S\|_2$.*

Proof. This follows from the Cauchy-Schwarz inequality:

$$|f(x)| = |(f * S^*)(x)| = |\langle f, \lambda_G(x) S \rangle| \le \|f\|_2 \|S\|_2 \quad .$$

The following proposition gives rise to a somewhat subtle distinction between unimodular and nonunimodular groups: In the unimodular case, any invariant subspace of $\mathrm{L}^2(G)$ which has admissible vectors possesses one in the form of a convolution idempotent. This will not be the case for nonunimodular groups, as will be clarified in Remark 2.43 below.

Proposition 2.40. *Suppose that $\mathcal{H} \subset \mathrm{L}^2(G)$ is closed and leftinvariant. Assume that $\mathcal{H}$ has an admissible vector η with $\eta^* \in \mathrm{L}^2(G)$. Then there exists a right convolution idempotent $S \in \mathcal{H}$ such that $\mathcal{H} = \mathrm{L}^2(G) * S$. In particular, in such a case $\mathcal{H} \subset C_0(G)$.*

Proof. Suppose that an admissible vector $\eta \in \mathcal{H}$ exists, then the projection onto $\mathcal{H}$ is given by $V_{\eta^*} V_\eta$. Since V_η is bounded, 2.19 (d) implies that $S = \eta^* * \eta = V_\eta^* \eta^* \in \mathcal{H}$. Hence, using associativity of convolution, $f = (f * \eta^*) * \eta = f * S$, for all $f \in \mathcal{H}$, whereas $f * (\eta^* * \eta) = 0$, for $f \perp \mathcal{H}$. Therefore $\mathcal{H} = \mathrm{L}^2(G) * S$, and S is the desired selfadjoint convolution idempotent.

We use the proposition to prove the following result due to Rieffel [102].

Proposition 2.41. *Let G be a unimodular group and $\mathcal{H} \subset \mathrm{L}^2(G)$ a closed, leftinvariant subspace. Then $\mathcal{H}$ contains a nonzero selfadjoint convolution idempotent.*

Proof. We start by choosing a nonzero bounded vector $\phi \in \mathcal{H}$: Pick $\phi_0 \in C_c(G)$ with nontrivial projection $P_{\mathcal{H}}\phi$ in $\mathcal{H}$. Then ϕ_0 is a bounded vector by 2.19(b), and Proposition 2.14 implies that $\phi = P_{\mathcal{H}}\phi_0$ is bounded as well. Pick a nonzero spectral projection Q of the selfadjoint operator $U = V_\phi^* V_\phi$ corresponding to a subset in $\mathbb{R}^+$ bounded away from zero. Then U restricted to $\mathcal{K} := Q(\mathcal{H})$ is a topological mapping. It follows that as a mapping $\mathcal{K} \to V_\phi(\mathcal{K})$ the operator $V_\phi = V_{Q\phi}$ is topological. Now Proposition 2.16 (b) ensures the existence of an admissible vector η in $\mathcal{K}$, and then Proposition 2.40 entails that $\mathcal{K}$ is generated by a convolution idempotent. (Note that the last step was the only instance where we used that G is unimodular; otherwise we have no way of checking $\eta^* \in \mathrm{L}^2(G)$.)

Our next aim is to decide for which unimodular groups G the regular representation itself allows admissible vectors. Note that in view of Proposition 2.14, the existence of admissible vectors for λ_G provides admissible vectors for all subrepresentations as well. Hence for these groups the necessary condition $\pi < \lambda_G$ is also sufficient, which yields a complete answer at least to the existence part of our problem. For unimodular groups, this approach turns out to be too bold, except for the somewhat trivial case of discrete groups. The following theorem first appeared in [53], with a somewhat sketchy proof.

Theorem 2.42. *Let G be unimodular. Then λ_G has an admissible vector iff G is discrete.*

Proof. First, if G is discrete, then the indicator function of $\{e_G\}$, where e_G is the neutral element of G, is admissible: The associated wavelet transform is the identity operator. Now assume that λ_G has an admissible vector, then by Proposition 2.40 $\mathrm{L}^2(G)$ consists of bounded continuous functions, in particular $\mathrm{L}^2(G) \subset \mathrm{L}^\infty(G)$. In order to show that this implies discreteness of G, we first show that for G nondiscrete there exist measurable sets of arbitrarily small, positive measure. For suppose otherwise, i.e.,

$$\epsilon := \inf\{|A| : A \subset G \text{ Borel }, |A| > 0\} > 0 \quad .$$

Then the infimum is actually attained: For $n \in \mathbb{N}$ there exists U_n such that $|U_n| < \epsilon + 1/n$. Using regularity, we find $V_n \supset U_n$ open with $|V_n \setminus U_n| < 1/n$. Pick $x_n \in V_n$, then $x_n^{-1}V_n$ is an open neighborhood of unity in G. It follows that letting for arbitrary $N \in \mathbb{N}$,

$$W_N = \bigcap_{n=1}^{N} x_n^{-1} V_n$$

defines a decreasing series of open neighborhoods of unity satisfying

$$\epsilon \leq |W_N| \leq |V_N| < \epsilon + 2/N \quad .$$

But then $U = \bigcap_{n \in \mathbb{N}} U_n$ has measure ϵ.

Next pick $C \subset U \subset V$, C compact, V open with $\mu(V \setminus C) < \epsilon$. C and V exist by regularity of Haar measure. Then $V \setminus C$ is open and has zero measure (by minimality of ϵ), hence $U = V$ is open. If U contains two distinct points, they can be divided by two disjoint open sets contained in U, which contradicts the minimality of $\mu(U)$. Hence U is an open singleton, and G is discrete, contrary to our assumptions.

Now suppose G is nondiscrete. Pick a sequence of Borel sets $U_n \subset G$ with the $0 < |U_n| < n^{-6}$, and define $f = \sum_{n \in \mathbb{N}} n \cdot \mathbf{1}_{U_n}$. Since $\|n\mathbf{1}_{U_n}\|_2 \le n^{-2}$, the sum converges in $\mathrm{L}^2(G)$, but clearly the limit is not in $\mathrm{L}^\infty(G)$. Hence $\mathrm{L}^2(G) \not\subset \mathrm{L}^\infty(G)$.

Remark 2.43. We note that if G is nonunimodular and type I, Theorem 4.22 provides the existence of an admissible vector for λ_G, even though G is obviously nondiscrete. Hence there is a sharp contrast between the unimodular and the nonunimodular setting.

The admissible vector for λ_G also shows that the assumption $\eta^* \in \mathrm{L}^2(G)$ in Proposition 2.40 cannot be dispensed with: If η is admissible for λ_G and such that $\eta^* \in \mathrm{L}^2(G)$, the Proposition implies $\mathrm{L}^2(G) \subset C_0(G)$. But then the proof of 2.42 can be adapted to show that G is discrete, which contradicts the fact that G is nonunimodular.

As one application of the connection between wavelets and convolution idempotents, we want to prove that wavelet coefficients have noncompact supports, at least for a large class of groups. We will see later on that in the abelian setting these results are related to the qualitative uncertainty property, stating that any L^2-function having support of finite Lebesgue measure both in time and frequency domains must be zero. The analog of that theorem for nonabelian groups will be given in Corollary 4.28 below.

But now let us show the result concerning the supports of wavelet coefficients. It was established by Wilczok both for the one-dimensional continuous wavelet transform and for the windowed Fourier transform [115], but the reasoning can be extended to a much larger class of groups. This was done by Arnal and Ludwig [14], who showed the unimodular version of 2.45, by an adaptation of the proof by Amrein and Berthier [5]. The only contribution of the author is realizing that with minor adjustments the proof goes through in the general case as well. First a small lemma is needed. Recall for this lemma that the **connected component** of a locally compact group is by definition the connected component of the unit element. It is a closed subgroup.

Lemma 2.44. *Let G be a locally compact group with noncompact connected component. Let $V \subset W$ be two measurable subsets of G such that $0 \neq |V|, |W| < \infty$. Then, whenever $|V| > \epsilon > 0$, there exists $x \in G$ such that*

$$|V| - 2\epsilon < |xV \cap W| < |V| - \epsilon \quad . \tag{2.32}$$

Proof. The function $\phi : x \mapsto |xV \cap W| = \langle \mathbf{1}_W, \lambda_G(x)\mathbf{1}_V \rangle$ vanishes at infinity, being a matrix coefficient associated to two L^2-functions (confer Proposition 2.5). Hence, if G_0 is the connected component of G, $\phi(G_0) \subset \mathbb{R}^+$ is a connected set, and since G_0 is noncompact, the closure of $\phi(G_0)$ contains 0. On the other hand, $\phi(e) = |V|$, hence $\phi(G)$ contains the half-open interval $]0, |V|]$. Hence there exists $x \in G_0$ such that $|V| - 2\epsilon < \phi(x) < |V| - \epsilon$, which is (2.32).

Theorem 2.45. *Let G be a locally compact group with noncompact connected component. Let $f \in \mathrm{L}^2(G)$ and suppose that there exists $S \in \mathrm{L}^2(G)$ such that $f = f * S$ and $S^* \in \mathrm{L}^2(G)$. If f is supported in a set of finite Haar measure, then $f = 0$.*

In particular, if $\eta \in \mathcal{H}_\pi$ is admissible for the representation π, and $V_\eta \varphi$ is supported in a set of finite Haar measure, for some $\varphi \in \mathcal{H}_\pi$, then $\varphi = 0$.

Proof. Suppose that $f \neq 0$ fulfills $f = f * S$, and in addition $C = f^{-1}(\mathbb{C} \setminus \{0\})$ has finite Haar measure. We pick $x_0 = e$, and apply the last lemma to recursively pick $x_1, x_2, \ldots \in G$ satisfying

$$|C| - \frac{1}{2^{k-1}} < |x_k C \cap C_{k-1}| < |C| - \frac{1}{2^k}, \quad k \in \mathbb{N}, \tag{2.33}$$

where $C_{k-1} = \bigcup_{i=0}^{k-1} x_i C \supset C$. Then, if we define $C_\infty = \bigcup_{i \in \mathbb{N}} C_i$, we find that $|C_\infty| < \infty$. Indeed, by (2.33), we have

$$\begin{aligned} |C_{k+1}| &= |C_k \cup x_{k+1} C| = |C_k| + |x_{k+1} C \setminus C_k| \\ &= |C_k| + |x_{k+1} C| - |C_{k+1} \cap x_k C_k| \le |C_k| + \frac{1}{2^k} \quad , \end{aligned}$$

which entails the desired finiteness.

Now define $\varphi = \mathbf{1}_{C_\infty}$, and consider the operator $K : g \mapsto \varphi \cdot (g * S)$. Writing

$$K(g)(x) = \varphi(x) \int_G g(y) S(y^{-1}x) dy = \int_G g(y) \varphi(x) S(y^{-1}x) dy$$

shows that K is an integral operator with kernel $(x, y) \mapsto \varphi(x) S(y^{-1}x)$. Since

$$\int_G \int_G |\varphi(x) S(y^{-1}x)|^2 dy dx = \mu_G(C_\infty) \|S^*\|_2^2 < \infty \quad ,$$

K is a Hilbert-Schmidt operator [101, VI.23], hence compact.

On the other hand, $(\lambda_G(x_k) f) * S = \lambda_G(x_k) f$ and $\mathrm{supp}(\lambda_G(x_k) f) = x_k C \subset C_\infty$ show that $\lambda_G(x_k) f$ is an eigenvector of K for the eigenvalue 1. In addition,

$$\left| \mathrm{supp}(\lambda_G(x_k) f) \setminus \bigcup_{i=0}^{k-1} \mathrm{supp}(\lambda_G(x_i) f) \right| > 0$$

by the lower inequality of (2.33), hence the $\lambda_G(x_k) f$ are linearly independent. But this means that the eigenspace of K for the eigenvalue 1 is infinite-dimensional, which contradicts the compactness.

The condition concerning the connected component is clearly necessary: If G is a compact group, the supports of the wavelet transforms are trivially of finite measure, yet there are many nontrivial convolution idempotents in $\mathrm{L}^2(G)$, arising for instance from irreducible representations.

Using Theorem 2.45 and Proposition 2.40, we can formulate the following sharpening of Theorem 2.42:

Corollary 2.46. *Let G be a locally compact unimodular group with noncompact connected component. Suppose that $\mathcal{H} \subset \mathrm{L}^2(G)$ is closed and leftinvariant. If $\mathcal{H}$ contains a nonzero function whose support has finite Haar measure, there is no admissible vector for $\mathcal{H}$.*

This concludes the discussion of the relations between continuous wavelet transforms and λ_G. Let us summarize the main results:

- A necessary condition for π to have admissible vectors is that $\pi < \lambda_G$. For nondiscrete unimodular groups, it is not sufficient.
- Embedding π into λ_G and making suitable identifications, we may assume that $\mathcal{H}_\pi = \mathrm{L}^2(G) * S$, with S a selfadjoint convolution idempotent.
- Admissible vectors in $\mathcal{H}_\pi$ are those η for which $f \mapsto f * \eta^*$ defines an isometry on $\mathcal{H}_\pi$. For $\mathcal{H}_\pi = \mathrm{L}^2(G) * S$, these vectors are characterized by $\eta^* * \eta = S$.

Therefore, in order to give a complete classification of representations with admissible vectors, we are faced with the following list of tasks:

T1 Give a concrete description of the closed, leftinvariant subspaces of $\mathrm{L}^2(G)$. In terms of the commuting algebra: Characterize the projections in $\mathrm{VN}_r(G)$.
T2 Given a leftinvariant subspace $\mathcal{H}$, give **admissibility criteria**, i.e. criteria for a right convolution operators $g \mapsto g * f^*$, with $f \in \mathcal{H}$, to be isometric.
T3 Characterize the subspaces $\mathcal{H}$ for which the admissibility conditions can be fulfilled. Equivalently, characterize the right convolution idempotents S.
T4 Given a concrete representation π, decide whether $\pi < \lambda_G$; if yes, make the criteria for **T1 - T3** explicit.

Remark 2.47. Item **T4** accounts for the fact that the discussion of the problem in $\mathrm{L}^2(G)$, while it makes perfect sense from a representation-theoretic point of view, limits the scope of the characterizations for concrete cases, where the realization of the representation is usually **not** given by left action on some suitable subspace of $\mathrm{L}^2(G)$. Indeed, in the case of the original wavelets arising from the $ax+b$-group, the focus of interest is on the action of that group on the real line by affine transformations, and the corresponding quasiregular representation. First finding an appropriate embedding into $\mathrm{L}^2(G)$ hence turns out to be a serious obstacle which must be overcome before the results presented here can be applied. However, for type I groups direct integral decompositions provide a systematic way of translating questions of containment of representations to the problem of absolute continuity of measures on the dual, and Chapter 5 contains a large class of examples to which this scheme is applicable.

2.6 Discretized Transforms and Sampling

In this section we want to embed the discretization problem into the L^2-setting, in a way which is complementary to the treatment of continuous transforms. In effect, we will only be able to do this in a satisfactory manner for unimodular groups.

Definition 2.48. *A family $(\eta_x)_{x \in X}$ of vectors in a Hilbert space $\mathcal{H}$ is called a* **frame** *if the associated coefficient operator is a topological embedding into $\ell^2(X)$, i.e., if there exist constants $0 < A \le B$ (called* **frame constants***) such that*

$$A \sum_{x \in X} |\langle \phi, \eta_x \rangle|^2 \le \|\phi\|^2 \le B \sum_{x \in X} |\langle \phi, \eta_x \rangle|^2 \quad .$$

A frame is **tight** *if $A = B$, and* **normalized tight** *if $A = B = 1$.*

In the terminology established in Section 1.1., a normalized tight frame is an admissible coherent state system based on a discrete space X with counting measure. We next formalize the notion of discretization.

Definition 2.49. *Let π be a representation and $\eta \in \mathcal{H}_\pi$ an admissible vector. Given a discrete subset $\Gamma \subset G$, the* **associated discretization of** *V_η is the coefficient operator $V_{\eta,\Gamma} : \mathcal{H}_\pi \to \ell^2(\Gamma)$ associated to the coherent state system $(\pi(\Gamma)\eta)$.*

Remark 2.50. (1) By 2.11 a discretization of V_η gives rise to the discrete reconstruction formula

$$f = \frac{1}{c_\eta} \sum_{\gamma \in \Gamma} V_\eta f(\gamma) \; \pi(\gamma)\eta \quad ,$$

which may be viewed as a Riemann sum version of the continuous reconstruction formula (2.10).

(2) Not all frames of the form $\pi(\Gamma)\eta$ arise as discretizations of continuous transforms, i.e., η need not be admissible. For instance, there exist frames associated to representations which are only square-integrable on a suitable quotient of the group [9]; these representations do not even possess admissible vectors in the sense discussed here.

On the other hand, the admissibility of functions giving rise to wavelet frames has been established in various settings, e.g., [33, 48, 8], which seems

to indicate that under certain topological conditions on the sampling set there is a strong connection between discrete and continuous transforms. See also Proposition 2.60 for the case that the sampling set is a lattice.
(3) We deal with discretization in a rather restrictive way, since only isometries are admitted. By now there exists extensive literature concerning the construction of wavelet frames and related constructions such as Gabor frames, see the monograph [28] and the references therein. We have refrained from discussing the discretization problem in full depth, since our focus is on Plancherel theory and its possible uses in connection with discretization.

The same remark applies to the structure of the sampling set: As the example of multiresolution ONB's of $\mathrm{L}^2(\mathbb{R})$ shows, the sampling set need not be a subgroup, i.e. it is not required to be regular. However, we will mostly concentrate on **regular sampling**, i.e., the sampling set will be a subgroup. It is obvious that the scope of purely group-theoretic techniques for dealing with irregular sampling will be limited, although examples like multiresolution ONB's are intriguing. A possible approach to obtain more general group-theoretic results, even in the irregular sampling case, could consist in adapting the techniques developed in [43] for certain discrete series representations (so-called *integrable representations*) to a more general setting.

Clearly discretization is closely connected to sampling the continuous transform. Hence the following notion arises quite naturally:

Definition 2.51. *Let G be a locally compact group, $\Gamma \subset G$. Let $\mathcal{H} \subset \mathrm{L}^2(G)$ be a leftinvariant closed subspace of $\mathrm{L}^2(G)$ consisting of continuous functions. We call $\mathcal{H}$ a* **sampling space (with respect to Γ)** *if it has the following two properties:*

(S_1) *There exists a constant $c_{\mathcal{H}} > 0$, such that for all $f \in \mathcal{H}$,*

$$\sum_{\gamma \in \Gamma} |f(\gamma)|^2 = c_{\mathcal{H}} \|f\|_2^2 \quad .$$

In other words, the restriction mapping $R_\Gamma : \mathcal{H} \ni f \mapsto (f|_\Gamma) \in \ell^2(\Gamma)$ is a scalar multiple of an isometry.

(S_2) *There exists $S \in \mathcal{H}$ such that every $f \in \mathcal{H}$ has the expansion*

$$f(x) = \sum_{\gamma \in \Gamma} f(\gamma) S(\gamma^{-1}x) \quad , \tag{2.34}$$

with convergence both in L^2 and uniformly.

The function S from condition (S_2) is called **sinc-type function**. *Furthermore, we say that a sampling space* **has the interpolation property** *if R_Γ maps onto all of $\ell^2(\Gamma)$, i.e. any element in $\ell^2(\Gamma)$ can be interpolated by a function in $\mathcal{H}$.*

It will become apparent below that the Heisenberg group allows a variety of sampling spaces associated to lattices, but none that has the interpolation property.

The definition is modelled after the following, prominent example:

Example 2.52 (**Whittaker, Shannon, Kotel'nikov).** Let $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$ and

$$\mathcal{H} = \{ f \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(\widehat{f}) \subset [-0.5, 0.5] \} \ .$$

Then $\mathcal{H}$ is a sampling subspace with the interpolation property, with associated sinc-type function

$$S(x) = \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \ .$$

A short proof of this fact, which uses the notions developed here, can be found in Remark 2.55 (1) below.

Our further discussion requires some basic and widely known facts about tight frames.

Proposition 2.53. *Let $(\eta_i)_{i \in I} \subset \mathcal{H}$ be a tight frame with frame constant c.*

(a) If $\mathcal{H}' \subset \mathcal{H}$ is a closed subspace and $P : \mathcal{H} \to \mathcal{H}'$ is the projection onto $\mathcal{H}'$, then $(P\eta_i)_{i \in I}$ is a tight frame of $\mathcal{H}'$ with frame constant c.
(b) Suppose that $c = 1$. Then $(\eta_i)_{i \in I}$ is an ONB iff $\|\eta_i\| = 1$ for all $i \in I$.
(c) If $\|\eta_i\| = \|\eta_j\|$, for all $i, j \in I$, then $\|\eta_i\|^2 \le c$.
(d) $(\eta_i)_{i \in I}$ is an orthonormal basis iff $c = 1$ and the coefficient operator is onto.

Proof. Part (a) follows from the fact that on $\mathcal{H}'$ the coefficient operator associated to $(P\eta_i)_{i \in I}$ coincides with the coefficient operator associated to $(\eta_i)_{i \in I}$. The "only-if"-part of (b) is clear. The "if"-part follows from

$$1 = \|\eta_i\|^2 = \sum_{i \in I} |\langle \eta_i, \eta_j \rangle|^2 = 1 + \sum_{i \neq j} |\langle \eta_i, \eta_j \rangle|^2 \ ,$$

whence $\langle \eta_i, \eta_j \rangle$ vanishes for $i \neq j$. Part (c) follows from a similar argument. The "only if" part of (d) is obvious. For the converse let $\delta_i \in \ell^2(I)$ be the Kronecker-delta at i, and let $T : \mathcal{H} \to \ell^2(I)$ denote the coefficient operator. Then $\langle T^* \delta_i, \varphi \rangle = \langle \delta_i, T\varphi \rangle = \langle \eta_i, \varphi \rangle$ for all $\varphi \in H$ implies $T^* \delta_i = \eta_i$, or $T\eta_i = \delta_i$ (T is by assumption unitary), which is the desired orthonormality relation.

The following proposition notes an elementary connection between sampling and discretization.

Proposition 2.54. *Let $\eta \in \mathcal{H}_\pi$ be admissible, and such that $\pi(\Gamma)\eta$ a tight frame with frame constant c_η. Then $\mathcal{H} = V_\eta(\mathcal{H}_\pi)$ is a sampling space, and $S = \frac{1}{c_\eta} V_\eta \eta$ is the associated sinc-type function for $\mathcal{H}$.*

Proof. Clearly $V_\eta(\mathcal{H}_\pi)$ consists of continuous functions. Using the isometry property of V_η together with the tight frame property of $\pi(\Gamma)\eta$, we obtain for all $f = V_\eta\phi \in \mathcal{H}$

$$\begin{aligned} f = V_\eta\phi &= V_\eta\left(\frac{1}{c_\eta}\sum_{\gamma\in\Gamma}\langle\phi,\pi(\gamma)\eta\rangle\pi(\gamma)\eta\right) \\ &= \sum_{\gamma\in\Gamma}\frac{1}{c_\eta}V_\eta\phi(\gamma)V_\eta(\pi(\gamma)\eta) = \sum_{\gamma\in\Gamma} f(\gamma)S(\gamma^{-1}\cdot) \quad , \end{aligned}$$

with convergence in $\|\cdot\|_2$. Uniform convergence follows from this by Proposition 2.39, since $V_\eta(\mathcal{H}_\pi) = \mathrm{L}^2(G) * V_\eta\eta$.

Remark 2.55. (1) The original sampling theorem in 2.52 can be seen to fit into this setting. If we pick η to be the sinc-function, we find that $V_\eta : \mathcal{H} \to \mathrm{L}^2(\mathbb{R})$ is just the inclusion map, hence η is admissible. Moreover, the Fourier transform of $(\lambda_{\mathbb{R}}(n)\eta)_{n\in\mathbb{Z}}$ yields precisely the Fourier basis of $\mathrm{L}^2([-1/2,1/2])$. Hence Proposition 2.54 applies.
(2) The proposition shows that various results on the relation between discrete wavelet or Weyl-Heisenberg systems and continuous ones give rise to sampling theorems: For the wavelet case, the underlying group is the $ax+b$-group. A result by Daubechies [33] ensures that every wavelet giving rise to a tight frame is in fact an admissible vector (up to normalization), hence we are precisely in the setting of the proposition. Similarly for discrete Weyl-Heisenberg system, where the underlying group is the reduced Heisenberg group we encountered in Example 2.27. Here admissibility of the window function is trivial. Again the expansion coefficients are sampled values of the windowed Fourier transform, which is the underlying continuous wavelet transform.

The following theorem serves various purposes. First of all it shows that, at least for a unimodular groups, the definition of a sampling space is redundant: Property (S_2) follows from (S_1). Moreover it shows that every sampling space can be obtained from the construction in Proposition 2.54, hence the construction of sampling subspaces and the discretization problem are (in a somewhat abstract sense) equivalent.

Theorem 2.56. *Assume that G is unimodular. Let $\mathcal{H} \subset \mathrm{L}^2(G)$ be a leftinvariant closed space consisting of continuous functions, and assume that it has property (S_1). Then $\mathcal{H}$ is a sampling subspace. More precisely, there exists a unique selfadjoint convolution idempotent S, such that $\frac{1}{c_{\mathcal{H}}}S$ is the associated sinc-type function, and in addition $\mathcal{H} = \mathrm{L}^2(G) * S$. In particular,*

$$\forall f \in \mathcal{H} \ , \ \forall \gamma \in \Gamma \quad : \quad f(\gamma) = \langle f, \lambda_G(\gamma)S\rangle \quad ,$$

and thus $\lambda_G(\Gamma)S$ is a tight frame for $\mathcal{H}$. $\mathcal{H}$ has arbitrary interpolation iff $\lambda_G(\Gamma)\frac{1}{\sqrt{c_{\mathcal{H}}}}S$ is an ONB of $\mathcal{H}$.

Proof. Define $S_\gamma = R_\Gamma^*(\delta_\gamma)$, where $\delta_\gamma \in \ell^2(\Gamma)$ is the Kronecker delta at γ. Then $\frac{1}{c_\eta} R_\Gamma^* R_\Gamma = \mathrm{Id}_\mathcal{H}$ shows that

$$f = \sum_{\gamma \in \Gamma} f(\gamma) \frac{1}{c_\mathcal{H}} S_\gamma \quad , \tag{2.35}$$

with convergence in the norm. The orthogonal projection $P : \ell^2(\Gamma) \to R_\Gamma(\mathcal{H})$ is given by $P = \frac{1}{c_\mathcal{H}} R_\Gamma R_\Gamma^*$. Moreover, we compute

$$f(\gamma) = \langle R_\Gamma f, \delta_\gamma \rangle = \langle f, R_\Gamma^* \delta_\gamma \rangle = \langle f, S_\gamma \rangle \quad . \tag{2.36}$$

Next use Zorn's lemma to pick a maximal family $(\mathcal{H}_i)_{i \in I}$ of nontrivial pairwise orthogonal closed subspaces of the form $\mathcal{H}_i = \mathrm{L}^2(G) * S_i$, where the S_i are selfadjoint convolution idempotents in $\mathrm{L}^2(G)$. Then Proposition 2.41 implies that $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$; this is the only place where we need that G is unimodular. Since right convolution with S_i is the orthogonal projection onto $\mathcal{H}_i$, equation (2.36) implies for all $f \in \mathcal{H}_i$

$$\langle f, S_\gamma * S_i \rangle = \langle f * S_i, S_\gamma \rangle = \langle f, S_\gamma \rangle = f(\gamma) = \langle f, \lambda_G(\gamma) S_i \rangle \quad .$$

Here the first equality used 2.19(d) and $S_i = S_i^*$, and the second one used $f = f * S_i$. As a consequence, $S_\gamma * S_i = \lambda_G(\gamma) S_i$. For all $\gamma \in \Gamma$,

$$S_\gamma = \sum_{i \in I} S_\gamma * S_i = \sum_{i \in I} \lambda_G(\gamma) S_i \quad , \tag{2.37}$$

with unconditionally converging sums. Since $\lambda_G(\gamma)$ is unitary, we can thus define

$$S = \sum_{i \in I} S_i$$

and conclude from (2.37) that

$$S_\gamma = \lambda(\gamma) S \quad .$$

Moreover, $S_i = S_i^*$ for all $i \in I$ implies $S = S^*$. Finally, for all $f \in \mathcal{H}$,

$$(f * S^*)(x) = \langle f, \lambda_G(x) S \rangle = \langle f, \sum_{i \in I} \lambda_G(x) S_i \rangle = \left(\sum_{i \in I} f * S_i \right)(x) = f(x) \quad .$$

Hence $\mathcal{H} = \mathrm{L}^2(G) * S$, and uniqueness of S was noted in Proposition 2.38. Now (2.35) and (2.37) shows that for $\frac{1}{c_\mathcal{H}} S$ to be the associated sinc-type function, only the uniform convergence of the sampling expansion remains to be shown, which follows from the normconvergence by Proposition 2.39. The statement concerning the tight frame property of $\lambda_G(\Gamma) S$ is now obvious. The last statement follows from Proposition 2.53 (d).

We next collect some additional observations which concerning regular sampling.

Definition 2.57. *A discrete subgroup $\Gamma < G$ is called a* **lattice** *if the quotient G/Γ carries a finite invariant measure. If a lattice exists, G is unimodular. If $A \subset G$ is any Borel transversal mod Γ, which exists by 3.4, we let*

$$\mathrm{covol}(\Gamma) = |A| \quad ,$$

which is independent of the choice of A.

The well-definedness of $\mathrm{covol}(\Gamma)$ is immediate from Weil's integral formula (2.2). The existence of a lattice implies that G is unimodular.

Proposition 2.58. *Let $\Gamma < G$, and suppose that there exists a frame of the form $\pi(\Gamma)\varphi$, with $\varphi \in \mathcal{H}_\pi$. Then there exist $\eta \in \mathcal{H}_\pi$ such that $\pi(\Gamma)\eta$ is a tight frame.*

Proof. First note that up to normalization the tight frame property is precisely admissibility for the restriction of π to Γ. Hence the statement is immediate from 2.16 (b).

Proposition 2.59. *Let G be unimodular and $\Gamma < G$ a discrete subgroup . Assume that $\mathcal{H} \subset \mathrm{L}^2(G)$ is a sampling subspace for Γ. Then Γ is a lattice, with* $\mathrm{covol}(\Gamma) = \frac{1}{c_{\mathcal{H}}}$.

Proof. If $f \in \mathcal{H}$ is any nonzero vector, and A is any measurable transversal, we compute

$$\begin{aligned}\|f\|^2 &= \int_A \sum_{\gamma\in\Gamma} |f(x\gamma)|^2 d\mu_G(x) = \int_A c_{\mathcal{H}} \|\lambda_G(x^{-1})f\|^2 d\mu_G(x) \\ &= \|f\|^2 c_{\mathcal{H}} \ \mathrm{covol}(\Gamma) \quad .\end{aligned}$$

The following general observation was pointed out to the author by K. Gröchenig:

Proposition 2.60. *Let $\Gamma < G$ be a lattice and assume that $\pi(\Gamma)\eta$ is a normalized tight frame. Then $\frac{1}{\mathrm{covol}(\Gamma)}\eta$ is admissible, i.e., the frame is a discretization of a continuous wavelet transform.*

Proof. For arbitrary $\phi \in \mathcal{H}_\pi$ and any measurable transversal A mod Γ

$$\begin{aligned}\|V_\eta \phi\|_2^2 &= \int_A \sum_{\gamma\in\Gamma} |\langle \phi, \pi(x\gamma)\eta\rangle|^2 dx \\ &= \int_A \sum_{\gamma\in\Gamma} |\langle \pi(x)^*\phi, \pi(\gamma)\eta\rangle|^2 dx \\ &= \int_A \|\lambda_G(x^{-1})\phi\|^2 dx \\ &= |A| \|\phi\|^2 \quad .\end{aligned}$$

The next proposition gives a representation-theoretic criterion for sampling spaces with the interpolation property.

Proposition 2.61. *Let $\Gamma < G$ be a lattice. There exists a sampling space $\mathcal{H} \subset \mathrm{L}^2(G)$ with interpolation property with respect to Γ iff there exists a representation π of G such that $\pi|_\Gamma \simeq \lambda_\Gamma$.*

Proof. For the "only-if" part pick $\pi = \lambda_G|_{\mathcal{H}}$. For the "if"-part, if $T : \mathcal{H} \to \ell^2(\Gamma)$ is a unitary equivalence, then $\eta = T^{-1}(\delta_0)$ is such that $\pi(\Gamma)\eta$ is an ONB of $\mathcal{H}$. Now the previous proposition implies that $V_\eta : \mathcal{H}_\pi \to \mathrm{L}^2(G)$ is an isometric embedding, and $\mathcal{H} = V_\eta(\mathcal{H}_\pi)$ has the interpolation property.

At least for unimodular groups, Theorem 2.56 implies that the discretization problem fits quite well into the framework developed for the continuous transforms. In particular the construction of sampling spaces and the discretization of continuous transforms are equivalent problems. We thus find one more task for our list:

T5 Characterize those convolution idempotents $S \in \mathrm{L}^2(G)$ such that in addition $\lambda_G(\Gamma)S$ is a tight frame of $\mathcal{H} = \mathrm{L}^2(G)*S$. For the interpolation property, decide which of these frames are in fact ONB's.

2.7 The Toy Example

In this section, we solve **T1** through **T5** for the group $G = \mathbb{R}$. This example will provide orientation for the further development, since in this setting the solutions turn out to be fairly simple exercises in real Fourier analysis; maybe with the exception of **T4**, which requires more sophisticated arguments.

As we saw in Section 2.3 , every representation of interest can be realized on some translationinvariant subspace on $\mathrm{L}^2(\mathbb{R})$. Moreover, in this setting wavelet transforms are convolution operators, hence it is quite natural to expect that the convolution theorem plays a role. Usually the convolution theorem is given on L^1, however for our purposes the following L^2-version will be more useful:

Theorem 2.62. *Let $f, g \in \mathrm{L}^2(\mathbb{R})$. Then $f * g^* \in \mathrm{L}^2(\mathbb{R})$ iff $\widehat{f}\overline{\widehat{g}} \in \mathrm{L}^2(\widehat{\mathbb{R}})$. In that case, $(f * g^*)^\wedge = \widehat{f}\overline{\widehat{g}}$.*

Proof. The computation

$$
\begin{aligned}
(f * g^*)(x) &= \langle f, \lambda_{\mathbb{R}}(x) g \rangle \\
&= \langle \widehat{f}, \mathrm{e}^{-2\pi \mathrm{i} x \cdot} \widehat{g} \rangle \\
&= \int_{\mathbb{R}} \widehat{f}(\omega) \overline{\widehat{g}(\omega)} \mathrm{e}^{2\pi \mathrm{i} x \omega} dx
\end{aligned}
$$

shows that the convolution theorem boils down to the "extended Plancherel formula" (2.22) proved in Example 2.28.

Solution to T1

Given a measurable subset $U \subset \widehat{\mathbb{R}}$, define

$$\mathcal{H}_U = \{f \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(\widehat{f}) \subset U\} \quad ,$$

where as usual inclusion is understood up to sets of measure zero. Then it is straightforward to show that $\mathcal{H}_U$ is a closed, translationinvariant subspace of $\mathrm{L}^2(\mathbb{R})$. The following theorem, which is essentially [104, 9.16], shows that this construction of translationinvariant subspaces is exhaustive.

Theorem 2.63. *The mapping*

$$\begin{array}{c} \{U \subset \widehat{\mathbb{R}} \textit{ measurable }\}/\ \ \text{nullsets} \longrightarrow \{\mathcal{H} \subset \mathrm{L}^2(G) \textit{ closed, leftinvariant}\} \quad , \\ U \ \mapsto \ \mathcal{H}_U \end{array}$$

is a bijection.

Solution to T2

Theorem 2.64. *Let $f \in \mathcal{H}_U$, for some measurable $U \subset \widehat{\mathbb{R}}$.*

$$f \textit{ is admissible} \Longleftrightarrow |\widehat{f}| = 1 \quad \textit{a.e. on } U \quad , \tag{2.38}$$

$$f \textit{ is a bounded vector} \Longleftrightarrow \widehat{f} \in \mathrm{L}^\infty(U) \quad , \tag{2.39}$$

$$f \textit{ is cyclic} \Longleftrightarrow \widehat{f} \neq 0 \quad (\textit{almost everywhere on } U) \quad . \tag{2.40}$$

Proof. Given any $f \in \mathrm{L}^2(\mathbb{R})$, denote by $M_{\widehat{f}}$ the multiplication operator with $\widehat{f}$, with the natural domain $\{g \in \mathrm{L}^2(\mathbb{R}) : \widehat{f}g \in \mathrm{L}^2(\mathbb{R})\}$. Then the L^2-convolution theorem implies that V_f and $M_{\widehat{f}}$ are conjugate under the Plancherel transform, including the domains. Now the equivalences follow immediately.

Solution to T3

Theorem 2.65. $\mathcal{H}_U \subset \mathrm{L}^2(\mathbb{R})$ *has admissible vectors iff $|U| < \infty$. $S \in \mathrm{L}^2(\mathbb{R})$ is a convolution idempotent iff $S = S_U := \mathbf{1}_U^\vee$, for $U \subset \widehat{\mathbb{R}}$ with $|U| < \infty$.*

Proof. Any admissible vector $f \in \mathcal{H}_U$ has to fulfill $|\widehat{f}| = 1$ on U, and of course $\widehat{f} \in \mathrm{L}^2(U)$. Thus follows the first condition. The characterization of convolution idempotents is immediate from the convolution theorem and $\widehat{f^*} = \overline{\widehat{f}}$.

Remark 2.66. The arguments for **T1** through **T3** generalize directly to locally compact abelian groups G. Simply replace $\widehat{\mathbb{R}}$ by the character group $\widehat{G}$ and Lebesgue measure by Haar measure on that group. This applies in particular to the cases $G = \mathbb{T}, \mathbb{Z}$.

Support Properties and the Qualitative Uncertainty Principle

Combining Theorems 2.65 and 2.45 yields the *qualitative uncertainty principle* over the reals:

Corollary 2.67. *If* $f \in \mathrm{L}^2(\mathbb{R})$ *fulfills* $|\mathrm{supp}(f)| < \infty$ *and* $|\mathrm{supp}(\widehat{f})| < \infty$, *then* $f = 0$.

Proof. $|\mathrm{supp}(\widehat{f})| < \infty$ implies $f = f * S = V_S f$ for a suitable convolution idempotent in $\mathrm{L}^2(\mathbb{R})$. Hence 2.45 applies.

Solution to T4

The arguments in this subsection were developed together with Keith Taylor. Suppose that $(\pi, \mathcal{H}_\pi)$ is an arbitrary representation of $\mathbb{R}$. A detailed description of such representation is obtainable by a combination of Stone's theorem and the spectral theorem for (possibly unbounded) operators. More precisely, Stone's theorem [101, VIII.8] implies the existence of an *infinitesimal generator*, i.e., a densely defined selfadjoint operator A on $\mathcal{H}_\pi$, such that

$$\pi(t) = \mathrm{e}^{-2\pi \mathrm{i} t A} \quad .$$

In order to understand this formula, we need to recall the spectral theorem [101, Chapter VIII]. Let Π denote the *spectral measure* of A. Then Π is a map from the Borel σ-algebra of $\mathbb{R}$ to the set of orthogonal projections on $\mathcal{H}$ mapping disjoint sets to projections with orthogonal ranges, satisfying $\Pi(A \cap B) = \Pi(A) \circ \Pi(B)$ as well as $\Pi(\mathbb{R}) = \mathrm{Id}_{\mathcal{H}_\pi}$. Π assigns to each pair of vectors $x, y \in \mathrm{dom}(A)$ a complex measure $\Pi_{x,y}$ on $\mathbb{R}$ by letting $\Pi_{x,y}(E) = \langle \Pi(E)x, y\rangle$. The spectral measure describes A via

$$\langle Ax, y\rangle = \int_{\mathbb{R}}^{\oplus} s d\Pi_{x,y}(s) \quad .$$

Th shorthand for this formula we use

$$A = \int_{\mathbb{R}} s d\Pi(s) \quad .$$

The spectral theorem can be viewed as a diagonalization of the selfadjoint operator. In particular, exponentiating amounts to exponentiating the diagonal elements, hence

$$\langle \mathrm{e}^{-2\pi \mathrm{i} t A} x, y\rangle = \int_{\mathbb{R}}^{\oplus} \mathrm{e}^{-2\pi \mathrm{i} s t} d\Pi_{x,y}(s) \quad ,$$

defines the unitary operator $\mathrm{e}^{-2\pi \mathrm{i} t A}$.

We want to decide in terms of the spectral measure whether admissible vectors exist. Recall that admissible vectors are in particular cyclic. The following lemma translates cyclicity into a property of the spectral measure, for which the construction of the representation π takes a somewhat more concrete form. The result is well-known, see for instance [95, Chapter I, Proposition 7.3]. For the proof of the Lemma we need one more ingredient, namely the *Fourier-Stieltjes* transform on $\mathbb{R}$. For this purpose let $M(\mathbb{R})$ denote the space of complex (finite) measures on $\mathbb{R}$. For every $\mu \in M(\mathbb{R})$, the exponentials are absolutely integrable with respect to $|\mu|$, hence the Fourier-Stieltjes-transform $\widehat{\mu} : \mathbb{R} \to \mathbb{C}$ of μ, given by

$$\widehat{\mu}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} \omega x} d\mu(x)$$

is well-defined. The crucial property of the Fourier-Stieltjes transform is that it is injective on $M(\mathbb{R})$ [45, 4.17]. As a consequence, to be used repeatedly in the next proof, we conclude for all positive $\nu \in M(\mathbb{R})$ that the exponentials are total in $\mathrm{L}^2(\mathbb{R},\nu)$: If $0 \neq f \in \mathrm{L}^2(\mathbb{R},\nu) \subset \mathrm{L}^1(\mathbb{R},\nu)$, then $f\nu \in M(\mathbb{R})$, and thus the Fourier-Stieltjes transform of $f\nu$ is nonzero. But the Fourier-Stieltjes transform of $f\nu$ is just the family of scalar products of f with the exponentials, taken in $\mathrm{L}^2(\mathbb{R},\nu)$.

Lemma 2.68. *The following are equivalent:*

(i) π *is cyclic.*
(ii) There exists a positive, finite Borel-measure μ *on* $\mathbb{R}$ *and a unitary map* $T : \mathcal{H}_\pi \to \mathrm{L}^2(\mathbb{R},\mu)$ *such that, for all* $B \in \mathcal{B}(\mathbb{R})$*, we have*

$$\mathit{\Pi}(B) = T^{-1} P_B T \quad , \tag{2.41}$$

where $P_B : \mathrm{L}^2(\mathbb{R},\mu) \to \mathrm{L}^2(\mathbb{R},\mu)$ *denotes multiplication with* $\mathbf{1}_B$*, as well as*

$$\pi(t) = T^{-1} M_t T \quad , \tag{2.42}$$

where M_t *is multiplication with* $\mathrm{e}^{-2\pi \mathrm{i} t \cdot}$.

The measure μ *is unique up to equivalence.*

Proof. "*(i)* $\Rightarrow$ *(ii)*": Let η be a cyclic vector, and define $\mu = \mathit{\Pi}_{\eta,\eta}$. Since μ is finite, all the characters $\mathrm{e}^{-2\pi \mathrm{i} t \cdot}$ are in $\mathrm{L}^2(\mathbb{R},\mu)$, and the equality

$$\langle \pi(t)\eta, \pi(s)\eta \rangle = \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}(t-s)\omega} d\mathit{\Pi}_{\eta,\eta} = \langle \mathrm{e}^{-2\pi \mathrm{i} t \cdot}, \mathrm{e}^{-2\pi \mathrm{i} s \cdot} \rangle_{\mathrm{L}^2(\mathbb{R},\mu)}$$

implies that the mapping $\pi(t)\eta \mapsto \mathrm{e}^{-2\pi \mathrm{i} t \cdot}$ may be extended linearly to an isometry $T : \mathcal{H}_\pi \to \mathrm{L}^2(\mathbb{R},\mu)$; here we used that $\pi(\mathbb{R})\eta$ spans a dense subspace of $\mathcal{H}_\pi$. It is onto, since $T(\mathcal{H}_\pi$ is closed and contains the exponentials, which are total in $\mathrm{L}^2(\mathbb{R},\mu)$.

For equation (2.41), we compute

$$\begin{aligned}
\langle T\Pi(B)\pi(s)\eta, \mathrm{e}^{-2\pi \mathrm{i} t\cdot}\rangle &= \langle \Pi(B)\pi(s)\eta, \pi(t)\eta\rangle \\
&= \int_B \mathrm{e}^{-2\pi\mathrm{i}(s-t)\lambda} d\Pi_{\eta,\eta}(\lambda) \\
&= \int_{\mathbb{R}} \mathbf{1}_B(\lambda)\mathrm{e}^{-2\pi\mathrm{i}s\lambda}\mathrm{e}^{-\mathrm{i}t\lambda} d\mu(\lambda) \\
&= \langle P_B T(\pi(s)\eta), \mathrm{e}^{-2\pi\mathrm{i}t\cdot}\rangle \quad ,
\end{aligned}$$

and thus $T\Pi(B) = P_B T$: The exponentials are total in $\mathrm{L}^2(\mathbb{R},\mu)$, and η is cyclic. Equation (2.42) is immediate on $\pi(\mathbb{R})\eta$, and extends to all of $\mathcal{H}$ in the same way.
"$(ii) \Rightarrow (i)$": The totality of the exponentials in $\mathrm{L}^2(\mathbb{R},\mu)$ implies that the constant function $\mathbf{1}$ is a cyclic vector with respect to the representation $t \mapsto M_t$. Hence $\eta = T^{-1}(\mathbf{1})$ is cyclic for π.

For the uniqueness result we observe that (2.41) clearly implies $\mu(B) = 0$ iff $\Pi(B) = 0$.

Now we can characterize arbitrary representations with admissible vectors. The argument is obtained by sharpening the proof of the lemma.

Theorem 2.69. *π has admissible vectors iff it is cyclic, and in addition the real-valued measure μ associated to its spectral measure by the previous lemma is absolutely continuous with respect to Lebesgue measure, with support in a set of finite Lebesgue measure.*

Proof. Suppose that η is an admissible vector for π. The proof consists essentially in repeating the construction proving the previous lemma and seeing that for admissible vectors η the measure μ is as desired. For this purpose we calculate

$$\begin{aligned}
V_\eta\phi(t) &= \langle \phi, \pi(t)\eta\rangle \\
&= \langle \phi, \mathrm{e}^{-2\pi\mathrm{i}tA}\eta\rangle \\
&= \int_{\mathbb{R}} \mathrm{e}^{2\pi\mathrm{i}t\omega} d\Pi_{\eta,\phi}(\omega) \quad ,
\end{aligned}$$

which exhibits $V_\eta\phi$ as the Fourier-Stieltjes transform of the measure $\Pi_{\eta,\phi}$. On the other hand, $V_\eta\phi$ is an L^2-function, hence $\Pi_{\eta,\phi}$ turns out to be absolutely continuous with respect to Lebesgue-measure λ. We let

$$T_\eta(\phi) = \frac{d\Pi_{\phi,\eta}}{d\lambda} \quad . \tag{2.43}$$

This sets up an isometry between $\mathcal{H}$ and some subspace of $\mathrm{L}^2(\mathbb{R})$. Let us next show that the projection onto $T(\mathcal{H})$ is given by restriction to an appropriately chosen subset Σ. For this purpose we compute $T_\eta(\Pi(B)\eta)$, for an arbitrary measurable subset B. On the one hand,

$$\langle \Pi(B)\eta,\eta\rangle = \int_{\mathbb{R}} d\mu_{\Pi(B)\eta,\eta}(\lambda) = \int_B d\Pi_{\eta,\eta}(\lambda) = \int_B T(\eta)(\lambda)d\lambda \quad .$$

On the other hand, the isometry property of T_η gives that

$$\begin{aligned}\langle \Pi(B)\eta,\eta\rangle &= \langle T_\eta(\Pi(B)\eta), T_\eta(\eta)\rangle = \int_{\mathbb{R}} T_\eta(\Pi(B))(\lambda)\overline{T_\eta(\eta)(\lambda)}d\lambda \\ &= \int_B T_\eta(\eta)(\lambda)\overline{T_\eta(\eta)(\lambda)}d\lambda \quad .\end{aligned}$$

Since this holds for all subsets B, we obtain that $T_\eta(\eta)(\lambda) = T_\eta(\eta)(\lambda)\overline{T_\eta(\eta)(\lambda)}$ a.e., which entails $T_\eta(\eta)(\lambda) \in \{0,1\}$. On the other hand, $T_\eta(\eta)$ is square-integrable, hence it is the characteristic function of a set Σ of finite Lebesgue measure. To show that $T_\eta(\mathcal{H}) = \mathrm{L}^2(\Sigma, dx)$, we note that $T_\eta(\Pi(B)\eta) = \mathbf{1}_B$, and the characteristic functions span a dense subspace of L^2.

Replacing $\Pi(B)\eta$ by $\mu(B)\phi$ in the above argument gives $T_\eta(\Pi(B)\phi) = \mathbf{1}_B T_\eta(\phi)$. Similarly we obtain $T_\eta(\pi(t)\phi) = \mathrm{e}^{-2\pi \mathrm{i} t\cdot}T_\eta(\phi)$, and we have shown the "only-if"-direction.

For the other direction, we construct an admissible vector for the equivalent representation acting on $\mathrm{L}^2(\Sigma)$, by picking $\eta = \mathbf{1}_\Sigma$. Then for every $\phi \in \mathrm{L}^2(\Sigma) \subset \mathrm{L}^1(\Sigma)$

$$V_\eta\phi(t) = \int_\Sigma \mathrm{e}^{2\pi \mathrm{i} t\lambda}\phi(\lambda)d\lambda = \widehat{\phi}(-t) \quad ,$$

which immediately implies $\|V_\eta\phi\|_{\mathrm{L}^2} = \|\phi\|$.

Remark 2.70. Given a representation π with admissible vector, we have now found two different ways to arrive at an equivalent representation $\widehat{\pi_U}$ acting on $\mathrm{L}^2(U,dx) \subset \mathrm{L}^2(\mathbb{R})$ for some measurable $U \subset \mathbb{R}$ by

$$\left(\widehat{\pi}(t)f\right)(\omega) = \mathrm{e}^{-2\pi \mathrm{i} t\omega}f(\omega) \quad .$$

The first one consists in embedding $\mathcal{H}_\pi$ in $\mathrm{L}^2(G)$ via V_η, and then applying Theorem 2.63 to see that $\pi \simeq \widehat{\pi}_U$ for a suitable U.

A shortcut is described by the mapping T_η constructed in the proof of 2.69. In fact, it is not hard to see that we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}_\pi & \overset{V_\eta}{\longrightarrow} & \mathcal{H}_U \\ & T_\eta \searrow & \downarrow \mathcal{F} \\ & & \mathrm{L}^2(U,dx) \end{array} \quad .$$

Indeed, observing that $T_\eta(\varphi) \in \mathrm{L}^2(\Sigma) \subset \mathrm{L}^1(\Sigma)$, we can apply the Fourier inversion formula to (2.43), obtaining

$$\begin{aligned} T(\varphi)^\vee(s) &= \int_{\mathbb{R}} \mathrm{e}^{2\pi \mathrm{i} st}T(\varphi)(t)dt = \int_{\mathbb{R}} \mathrm{e}^{2\pi \mathrm{i} st}d\Pi_{\varphi,\eta}(t) = \langle \varphi, \mathrm{e}^{-2\pi \mathrm{i} sA}\eta\rangle \\ &= V_\eta\varphi(s) \quad .\end{aligned}$$

Note also that the admissibility condition obtained in 2.64 coincides with the admissibility condition which is derived in the proof of Theorem 2.69. This is owed to the fact that both are just the admissibility conditions for the representation $\widehat{\pi_U}$, translated to the respective settings by the associated intertwining operators.

Solution to T5

Here we focus on regular sampling, i.e., $\Gamma = \alpha\mathbb{Z}$ is assumed to be a lattice, with $\alpha > 0$. The following theorem can be seen as a refinement of Shannon's sampling theorem. It can be regarded as folklore. Similar results for arbitrary locally compact groups were obtained for instance by Kluvánek [76].

Theorem 2.71. *Let $\mathcal{H}_U$ with associated idempotent $S = \mathbf{1}_U^\vee$. The following are equivalent:*

(a) $\lambda_\mathbb{R}(\Gamma)S$ is a tight frame, i.e., $\mathcal{H}_U$ is a sampling space. The constant $c_\mathcal{H}$ associated to $\mathcal{H}_U$ is $1/\alpha$.
(a') There exists $f \in \mathcal{H}_U$ such that $\lambda_\mathbb{R}(\Gamma)f$ is a frame.
(a") There exists $f \in \mathcal{H}_U$ such that $\lambda_\mathbb{R}(\Gamma)f$ is total.
(b) $|U \cap \frac{1}{\alpha}k + U| = 0$, for all $0 \neq k \in \mathbb{Z}$

Regarding the interpolation property, we have the following equivalent conditions:

(i) $\mathcal{H}_U$ is a sampling space with interpolation property.
(ii) $|U \cap \frac{k}{\alpha} + U| = 0$, for all $0 \neq k \in \mathbb{Z}$, and $|U| = \frac{1}{\alpha}$.

Proof. (a) $\Rightarrow$ (a') $\Rightarrow$ (a") is obvious. Now assume that (a") holds. If $\lambda_\mathbb{R}(\Gamma)f$ is total in $\mathcal{H}_U$ then f is a cyclic vector for the translation action of $\lambda_\mathbb{R}$ on $\mathcal{H}_U$. Hence $\widehat{f} \neq 0$ almost everywhere on U, by condition (2.40). Suppose that (b) is violated, i.e., there exists $A \subset U$ measurable with $|A| > 0$ and $\frac{k}{\alpha} + A \subset U$, for a suitable nonzero k. Possibly after passing to a smaller set A we may assume that $|\widehat{f}(\omega + \alpha k)| \geq \epsilon$ for some fixed $\epsilon > 0$ and all $\omega \in A$. Then letting

$$\widehat{g}(\omega) = \begin{cases} 1 & \text{for } \omega \in A \\ -\frac{\widehat{f}(\omega - k/\alpha)}{\widehat{f}(\omega)} & \text{for } \omega \in \frac{k}{\alpha} + A \end{cases}$$

defines an L^2-function supported on $A \cup \frac{k}{\alpha} + A$ satisfying

$$\widehat{f}(\omega)\widehat{g}(\omega) = -\widehat{f}(\omega + \frac{k}{\alpha})\widehat{g}(\omega + \frac{k}{\alpha})$$

on A. Given $\ell \in \mathbb{Z}$, the $1/\alpha$-periodicity of $\exp(2\pi\mathrm{i}\alpha\ell\cdot)$ then implies that

$$\langle g, \lambda_\mathbb{R}(\alpha\ell)f\rangle = \langle \widehat{g}, \mathrm{e}^{-2\pi\mathrm{i}\alpha\ell\cdot}\widehat{f}\rangle = 0 \quad .$$

Hence $\lambda_\mathbb{R}(\alpha\mathbb{Z})f$ is not complete, which gives the desired contradiction.

Now assume (b). Pick a Borel set $V \supset U$ fulfilling (b) and such that in addition, $|B| = 1/\alpha$. In other words, V is a measurable transversal containing U. Then the $1/\alpha$-periodicity of the exponentials and the fact that the exponentials (suitably normalized) form an ONB of $\mathrm{L}^2([0, 1/\alpha])$ implies that they also form an ONB of $\mathrm{L}^2(V)$. Then the restriction of the exponentials to U are the image of an ONB under a projection operator, i.e., still a normalized tight frame with frame constant $1/\alpha$, by 2.53(a). Pulling this back to $\mathcal{H}_U$ gives the desired statement. Moreover, we have also shown (ii) $\Rightarrow$ (i).

For (i) $\Rightarrow$ (ii) we note that the first condition in (ii) follows by (a) $\Rightarrow$ (b), whereas the second one follows from the requirement that $\alpha^{1/2}\|S\|_2 = 1$.

We close the section with the observation that not every continuous transform can be regularly sampled to give a discrete transform.

Example 2.72. There exists a space $\mathcal{H}_U$ which has admissible vectors but does not admit frames of the form $\lambda_{\mathbb{R}}(\alpha\mathbb{Z})f$. For this purpose, pick $U \subset \widehat{\mathbb{R}}$ open, dense and of finite measure, say a union of suitably small open balls around the rationals. Then there exist admissible vectors by Theorem 2.63, but since for $t \in \mathbb{R}$ arbitrary $U \cap t + U$ is open and nonempty, condition (b) of Theorem 2.71 is always violated.

Question: Does there exist a discrete set $\Gamma \subset \mathbb{R}$ and $\eta \in \mathcal{H}_U$ such that $\lambda_{\mathbb{R}}(\Gamma)\eta$ is a frame?