Exponential Decay Parameters Associated with Excessive Measures

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Summary. Let X be a Markov process with semigroup (P_t) and m an excessive measure of X. With m we associate the spectral radius $\lambda_r^{(p)}(m)$ of (P_t) on $L^p(m)$ $(1 \leqslant p \leqslant \infty)$ and the exit parameter $\lambda_e^{\mathcal{C}}(m)$ defined for an m-nest $\mathcal{C} = (C_n)$ in terms of the corresponding first exit times (τ_n). We discuss the impact of these parameters as well as their connection with other parameters of interest for the process.

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1 Introduction

When X is a transient Markov process it is sometimes possible to associate with it a new process $\widetilde{X}$ endowed with a conservative measure $\widetilde{m}$. Limit theorems obtained for $\widetilde{X}$ relative to $\widetilde{m}$ provide information on the long term behaviour of the initial process X. The process $\widetilde{X}$ and the measure $\widetilde{m}$ are obtained by means of a γ-subinvariant function and a γ-subinvariant measure (to be precisely defined in the sequel), whence the interest in the classes of γ-subinvariant functions and γ-subinvariant measures and the parameters associated with them.

In case of Harris irreducible processes this is a well established theory, sometimes called λ-*theory* (see [Ber97], [NN86], [TT79]). Related results for processes that are not necessarily irreducible are given in [Glo88] and [Str82].

The present paper is concerned with this kind of problems and they are considered in the context of the theory of excessive measures ([DMM92] and [Get90]). Unless otherwise mentioned the process X is assumed to be Borel right with state space $(E, \mathcal{E})$, semigroup (P_t), resolvent (U^q) and lifetime ζ.

M. Émery, M. Ledoux, and M. Yor (Eds.): LNM 1857, pp. 135–144, 2005.

For any $\gamma \geqslant 0$ we consider:

$$\mathbf{M}^{\gamma}(X) := \{\eta : \sigma\text{-finite measure on } (E,\mathcal{E}) \text{ such that } \mathrm{e}^{\gamma t}\eta P_t \leqslant \eta,\ \forall\, t \geqslant 0\},$$
$$\mathbf{F}^{\gamma}(X) := \{f \in \mathcal{E}^* : f \geqslant 0 \text{ on } E \text{ such that } \mathrm{e}^{\gamma t} P_t f \leqslant f,\ \forall\, t \geqslant 0\}.$$

An element of $\mathbf{M}^{\gamma}(X)$ (resp. $\mathbf{F}^{\gamma}(X)$) is called a γ-*subinvariant mesure* (resp. a γ-*subinvariant function*). They are called γ-invariant measures (resp. γ-invariant functions) when all the corresponding inequalities become equalities. Standard candidates as members of $\mathbf{F}^{\gamma}(X)$ are

$$\Phi^{\gamma} f(x) := \int_0^{\infty} \mathrm{e}^{\gamma t} P_t f(x)\,\mathrm{d}t, \qquad f \in \mathcal{E}^*,\ f \geqslant 0,$$

and as members of $\mathbf{M}^{\gamma}(X)$ the measures $\mu\Phi^{\gamma}$ provided they are σ-finite.

The following global parameters are of interest and were considered in various contexts ([Glo88], [NN86], [Str82], [TT79]):

$$\lambda_{\pi} := \sup\{\gamma \geqslant 0 : \mathbf{M}^{\gamma}(X) \neq \{0\}\};$$
$$\lambda_{\varphi} := \sup\{\gamma \geqslant 0 : \exists\, f \in \mathbf{F}^{\gamma}(X),\ f > 0 \text{ on } E,\ f \text{ not identically } \infty\}.$$

When $\gamma = 0$ instead of $\mathbf{M}^{\gamma}(X)$ we write as usual $\mathrm{Exc}(X)$ and consider its well known important subclasses:

$$\mathrm{Pur}(X) := \{m \in \mathrm{Exc}(X) : mP_t(h) \to 0 \text{ when } t \to \infty,\ \forall\, h > 0,\ m(h) < \infty\};$$
$$\mathrm{Inv}(X) := \{m \in \mathrm{Exc}(X) : mP_t = m,\ \forall\, t \geqslant 0\};$$
$$\mathrm{Con}(X) := \{m \in \mathrm{Exc}(X) : m(Uh < \infty) = 0,\ \forall\, h > 0,\ m(h) < \infty\};$$
$$\mathrm{Dis}(X) := \{m \in \mathrm{Exc}(X) : m(Uh = \infty) = 0,\ \forall\, h > 0,\ m(h) < \infty\}.$$

Whenever $m \in \mathrm{Exc}(X)$, each P_t, $t > 0$, and each qU^q, $q > 0$, may be thought as a contraction from $L^p(m)$ to $L^p(m)$ for $1 \leqslant p \leqslant \infty$; also, the semigroup (P_t) is strongly continuous on $L^p(m)$, $1 \leqslant p \leqslant \infty$ (these facts are discussed in a more general setting in [Get99]). As usual let $\lambda_r^{(p)}(m)$, the *spectral radius* of (P_t) on $L^p(m)$, $1 \leqslant p \leqslant \infty$, be defined as

$$\lambda_r^{(p)}(m) := \lim_{t \to \infty} \bigl\{-t^{-1} \ln \|P_t\|_{L^p(m)}\bigr\}.$$

The second section is devoted to the impact of $\lambda_r^{(p)}(m)$, $1 \leqslant p \leqslant \infty$, as decay parameters. First the connection of $\lambda_r^{(2)}(m)$ with $\lambda_{\pi}$is discussed. Then under certain restrictions on the process (imposed by the application of a very powerful result of Takeda [Tak00]) one gets that $\lambda_r^{(p)}(m)$ are independent of p and one identifies this common value with the decay parameter associated with X as irreducible process. The remaining part of section 2 is concerned exclusively with properties of $\lambda_r^{(1)}(m)$ having in view especially the connection with λ_{π}, λ_{φ}. An expression of $\lambda_r^{(1)}(m)$ in terms of the Kuznetsov measure Q_m associated with m is given. This allows to distinguish those measures m

for which $\lambda_r^{(1)}(m) > 0$ among the purely excessive ones. Finally, the underlying construction in the classical context of quasi-stationary distributions for irreducible processes is retrieved in the general case emphasizing the role of $\lambda_r^{(1)}(m)$.

Section 3 introduces the exit parameter associated with an m-nest, when m is a dissipative measure. These are similar to some parameters considered in [Str82] in a more specific context.

Notation. As is standard, we denote by $\mathcal{E}^e$ the σ-algebra generated by excessive functions and for any $B \in \mathcal{E}^e$, $T_B := \inf\{t > 0 : X_t \in B\}$, $\tau_B := T_{E\setminus B}$, $P_B f(x) := P^x\{f(X_{T_B}); T_B < \infty\}$.

2 Spectral radius as decay parameter

We start by revealing the special role played by the spectral radius in case of some irreducible processes. Recall that the process X is said to be μ-*irreducible*, with μ a σ-finite measure, if:

$$\mu\text{-(I)} \qquad \mu(B) > 0 \implies \forall\, x \in E, \quad U^1(x, B) > 0.$$

The following theorem introduces the *decay parameter* associated with the irreducible process X.

Theorem 1 ([TT79]). *For any Markov process X satisfying μ-(I) there exist a μ-polar set Γ, an increasing sequence of sets $(B_n) \subseteq \mathcal{E}^e$ with $E = \bigcup_n B_n$ and a parameter $\lambda \in [0, \infty[$ such that:*

(i) *For any $\gamma < \lambda$ we have $\Phi^\gamma(x, B_n) < \infty$, $\forall\, x \notin \Gamma$, $n \in \mathbb{N}$.*
(ii) *For any $\gamma > \lambda$ we have $\Phi^\gamma(x, B) \equiv \infty$, $\forall\, B \in \mathcal{E}$, $\mu(B) > 0$.*
(iii) *The process is either λ-transient, i.e.* (i) *holds for $\gamma = \lambda$, or it is λ-recurrent, i.e.* (ii) *holds for $\gamma = \lambda$.*
(iv) $\lambda = \lambda_\pi = \lambda_\varphi$.

We recall also that the whole theory of irreducible processes is based on the existence of a remarkable class of sets, namely:

$$\mathcal{L}(\mu) := \{B \in \mathcal{E}^e : \mu(B) > 0 \text{ for which there exists a measure } \nu_B \neq 0 \\ \text{such that } U^1(x, \,.\,) \geqslant \nu_B(\,.\,), \forall\, x \in B\}.$$

The impact of $\mathcal{L}(\mu)$ comes from the fact that whenever B is in $\mathcal{L}(\mu)$ and $\Phi^\gamma(x, B) = \infty$ for some $x \in E$, one has $\Phi^\gamma(x, A) = \infty$ for any $A \in \mathcal{E}$ such that $\mu(A) > 0$.

The next result is concerned with the connection between $\lambda_r^{(2)}(m)$ and λ_π.

Proposition 1. (i) *For any excessive measure* m *we have* $\lambda_r^{(2)}(m) \leqslant \lambda_\pi$.

(ii) *Suppose that the state space is locally compact with countable base and that* X *is a Feller process,* m*-symmetric with respect to the Radon measure* m*. Suppose further that* m-(I) *holds and that the support of* m *has non-empty interior. Then* $\lambda_r^{(2)}(m) = \lambda$.

Proof. (i) We shall actually show that given $h \in \mathcal{E}$ such that $0 < h \leqslant 1$ and $m(h) < \infty$, the parameter

$$\lambda(m;h) := \liminf_{t\to\infty}\left\{-t^{-1}\ln\frac{(h,P_t h)_m}{\|h\|_{L^{(2)}(m)}}\right\}$$

satisfies $\lambda(m;h) \leqslant \lambda_\pi$. This will be enough to prove (i) since $\lambda_r^{(2)}(m) \leqslant \lambda(m;h)$. To this end consider $\gamma < \lambda(m;h)$, $\gamma' \in \,]\gamma, \lambda(m;h)[$ and let $t_{\gamma'}$ be such that $(h, P_t h)_m \leqslant \mathrm{e}^{-\gamma' t}\|h\|_{L^{(2)}(m)}$, $\forall\, t \geqslant t_{\gamma'}$. Then the measure $\nu_\gamma(g) := (\gamma' - \gamma)\int_{t_{\gamma'}}^{\infty} \mathrm{e}^{t\gamma} m(hP_t g)\,\mathrm{d}t$, $g \in \mathcal{E}$, $g \geqslant 0$, is in $\mathbf{M}^\gamma$.

(ii) By (i) and m-(I) we have $\lambda_r^{(2)}(m) \leqslant \lambda_\pi = \lambda$. We will next prove that for any $\gamma > \lambda_r^{(2)}(m)$ there exists a compact K such that $m(K) > 0$ and $\Phi^\gamma 1_K \equiv \infty$. According to Proposition 6.3.8 (ii) in [MT96], whose hypotheses are the ones in (ii) but for the m-symmetry, any compact K such that $m(K) > 0$ is in $\mathcal{L}(m)$. This will be enough to ensure that $\gamma \geqslant \lambda$ and thus $\lambda \leqslant \lambda_r^{(2)}(m)$. Let $(E_z)_{z\in\mathbb{R}}$ be the resolution of the generator of $\{P_t;\, t > 0\}$ on $L^2(m)$ and let $\gamma' \in \,]\lambda_r^{(2)}(m), \gamma[$. There exists a function φ, continuous and with compact support such that $E_{-\gamma'}\varphi \neq E_{-\lambda_r^{(2)}(m)}\varphi$. Then φ may be written as $\varphi = \int_{-\gamma'}^{-\lambda_r^{(2)}(m)} \mathrm{d}E_z\varphi$ and

$$\big(|\varphi|, P_t|\varphi|\big)_m \geqslant \big(\varphi, P_t\varphi\big)_m = \int_{-\gamma'}^{-\lambda_r^{(2)}(m)} \mathrm{e}^{zt}\,\mathrm{d}(\varphi, E_z\varphi) \geqslant \mathrm{e}^{-\gamma' t}\Big(\varphi, E_{-\lambda_r^{(2)}(m)}\varphi\Big)_m.$$

Whence $(|\varphi|, \Phi^\gamma|\varphi|)_m = \infty$, which in turn implies $(1_{K_\varphi}, \Phi^\gamma 1_{K_\varphi})_m = \infty$, K_φ being the compact support of φ.

By m-(I) we have only two possibilities: either $\Phi^\gamma 1_{K_\varphi}$ is finite up to an m-polar set, or $\Phi^\gamma 1_{K_\varphi} \equiv \infty$. Assuming the first possibility we get an $M > 0$ such that $m(\Phi^\gamma 1_{K_\varphi} \leqslant M) > 0$ and therefore there exists a compact K such that $K \subseteq \{\Phi^\gamma 1_{K_\varphi} \leqslant M\}$. This implies $m(1_K \Phi^\gamma 1_{K_\varphi}) \leqslant Mm(K) < \infty$. On the other hand from $K_\varphi \in \mathcal{L}(m)$ and from the classical formula $\Phi^\gamma 1_K = \sum_{n=1}^{\infty}(1+\gamma)^{n-1}U^{1(n)}1_K$ we have $\Phi^\gamma 1_K \geqslant \nu_{K_\varphi}(K)(1+\gamma)\Phi^\gamma 1_{K_\varphi}$. Using this and m-symmetry we get

$$\big(1_K, \Phi^\gamma 1_{K_\varphi}\big)_m = \big(1_{K_\varphi}, \Phi^\gamma 1_K\big)_m \geqslant \nu_{K_\varphi}(K)(1+\gamma)\big(1_{K_\varphi}, \Phi^\gamma 1_{K_\varphi}\big)_m = \infty.$$

The obtained contradiction rules out the possibility that $\Phi^\gamma 1_{K_\varphi}$ is finite up to an m-polar set and thus $\Phi^\gamma 1_{K_\varphi} \equiv \infty$, the property which was to be proved. □

We now briefly discuss two other forms of irreducibility that will be involved in the sequel. We begin with

(I) for any finely open, non-empty set Γ we have $P_\Gamma 1(x) > 0, \forall x \in E$.

Condition (I) amounts to the property that all states communicate in the sense of [ADR66]. Immediate consequences of it are : the fact that all excessive measures are equivalent and the property that any $\xi \in \mathrm{Exc}(X)$ satisfies ξ-(I).

The next condition of irreducibility is the one imposed in [Tak00]. Let m be a σ-finite measure on $(E, \mathcal{E})$ and $\mathcal{I}(m) := \{A \in \mathcal{E} : 1_A P_t f = P_t 1_A f$ a.e.-m for any bounded $f \in L^2(m)\}$. The condition is the following:

m-(I)$'$ for any $A \in \mathcal{I}(m)$ we have either $m(A) = 0$ or $m(E \backslash A) = 0$.

In what follows we consider conditions under which it is possible to relate the three forms of irreducibility.

Lemma 1. *Let m belong to* $\mathrm{Exc}(X)$.

(i) *If m is Radon and m-*(I) *holds, then m-*(I)$'$ *also holds.*
(ii) *If the process is m-symmetric, if m is a reference measure and if m-*(I)$'$ *holds, then* (I) *also holds.*

Proof. To get (i) let $A \in \mathcal{I}(m)$ be such that $m(A) > 0$. For any compact $K \subseteq E \backslash A$ we have $m(1_A U^1 1_K) = 0$, implying by m-(I) that $m(K) = 0$, whence $m(E \backslash A) = 0$.

For (ii) let us note that by m-symmetry any absorbing set A is in $\mathcal{I}(m)$ and then apply this to the set $A = \{U^1 1_\Gamma = 0\}$ with Γ finely open, non-empty. Taking into account that m is a reference measure, the possibility that $m(E \backslash A) = 0$ is ruled out and from $m(A) = 0$ we get $P_\Gamma 1(x) > 0, \forall x \in E$. □

We now turn to the very special case, indicated in the introduction, when the p-independence of $\lambda_r^{(p)}(m)$, $1 \leqslant p \leqslant \infty$, occurs.

Theorem 2. *Assume that the state space $(E, \mathcal{E})$ is a locally compact metric space with countable base and that X is an m-symmetric Markov process, with m a Radon measure. Assume also that the following conditions are satisfied:*

(i) *m-*(I)$'$ *holds.*
(ii) *For each $t > 0$ and $x \in E$, $P_t(x, .) \ll m$.*
(iii) *For any $t > 0$, $P_t f \in \mathcal{C}_0(E)$ whenever $f \in \mathcal{C}_0(E)$, where $\mathcal{C}_0(E)$ denotes the space of all continuous functions vanishing at infinity.*
(iv) $U^1 1 \in \mathcal{C}_0(E)$.

Then $\lambda_r^{(p)}(m)$, $1 \leqslant p \leqslant \infty$, is independent of $p \in [1, \infty]$ and the common value $\lambda_r(m)$ coincides with the decay parameter λ associated with X as irreducible process.

Proof. The imposed conditions are precisely those in Theorem 2.3 in [Tak00] which provides the p-independence of $\lambda_r^{(p)}(m)$, $1 \leqslant p \leqslant \infty$.

The process is subject to (I) by Lemma 1 (ii) and the equality $\lambda_r(m) = \lambda$ follows from Proposition 1 (ii), since under the present assumptions the support of m is E. ☐

Remarks. 1. The conditions in Theorem 2 are met by the process (X, τ_D) equal to the d-dimensional Brownian motion killed at τ_D, where D is a regular domain satisfying $m(D) < \infty$. In this remarkable particular case we actually have a much stronger property, namely the corresponding transition operator P_t^D, $t > 0$, is a compact operator and has the same eigenvalues $\{\exp(-\lambda_k);\ k = 1, 2, \dots\}$ with $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots < \infty$, in all *appropriate spaces* $L^{(p)}(m_D)$, $1 \leqslant p < \infty$, and $\mathcal{C}_0(D)$ ([CZ95], Theorem 2.7). The decay parameter λ coincides in this case with λ_1.

2. In [Sat85], Sato studies the impact of λ_r^∞ as decay parameter assuming (iii), a condition weaker than (v) in Theorem 2 and the following condition of irreducibility: for any open set $G \in \mathcal{E}$ one has $P_G 1(x) > 0$, $\forall\, x \in E$. Symmetry and absolute continuity are not assumed.

3. In [Str82], Stroock proposed for a Radon measure m

$$\lambda_\sigma(m) := \sup_{\substack{\varphi \in C_c(E) \\ \varphi \geqslant 0}} \left\{ \limsup_{t \to \infty} \{ t^{-1} \ln (\varphi, P_t \varphi)_m \} \right\}$$

as decay parameter and shows that—under his conditions and m-symmetry—it coincides with the right end point of the spectrum of the generator on $L^2(m)$ (actually $-\lambda_r^{(2)}(m)$ in that case). Here $\mathcal{C}_c(E)$ denotes the space of all real valued continuous functions on E with compact support.

In what follows we turn to specific properties of $\lambda_r^{(1)}(m), m \in \mathrm{Exc}(X)$.

Let Q_m and Y be the Kuznetsov measure and process associated with m. Let also α be the birthtime of Y and $\mathcal{H}(m) := \{h \in \mathcal{E} : h \geqslant 0,\ 0 < m(h) < \infty\}$.

Proposition 2. *For any* $m \in \mathrm{Exc}(X)$ *we have*

$$\lambda_r^{(1)}(m) = \sup\Big\{\gamma \geqslant 0 : \lim_{t \to \infty} \mathrm{e}^{\gamma t} Q_m\big(h(Y_0);\ \alpha < -t\big) = 0,\ \forall\, h \in \mathcal{H}(m)\Big\}.$$

If $\lambda_r^{(1)}(m) > 0$*, then* $m \in \mathrm{Pur}(X)$.

Proof. First we note that by the Markov property of Y under Q_m and the stationarity of Q_m we have for any $f \in \mathcal{E}$, $f \geqslant 0$

$$\begin{aligned} m(P_t f) &= Q_m\big(P^{Y_0}\big(f(X_t)\big);\ \alpha < 0\big) = Q_m\big(f(Y_t);\ \alpha < 0\big) \\ &= Q_m\big(f(Y_0);\ \alpha < -t\big). \end{aligned}$$

Recall now the following well known property of the spectral radius: for any $a < \lambda_r^{(1)}(m)$ there exists $M_a \geqslant 1$ such that $\mathrm{e}^{at}\|P_t\|_{L^1(m)} \leqslant M_a$. Then put $\widetilde{\lambda}(m) := \sup\{\gamma \geqslant 0 : \lim_{t\to\infty} \mathrm{e}^{\gamma t} mP_t(h) = 0, \forall\, h \in \mathcal{H}(m)\}$. To show that $\lambda_r^{(1)}(m) \leqslant \widetilde{\lambda}(m)$, let $\gamma < \lambda_r^{(1)}(m)$ and $\gamma' \in \left]\gamma, \lambda_r^{(1)}(m)\right[$; then there exists $M_{\gamma'} \geqslant 1$ such that $\mathrm{e}^{\gamma' t} mP_t(h) \leqslant M_{\gamma'} m(h)$, $\forall\, h \in \mathcal{H}(m)$, and thus $\mathrm{e}^{\gamma t} mP_t(h) \to 0$ as $t \to \infty$, $\forall\, h \in \mathcal{H}(m)$. The converse inequality follows observing that for any $\gamma < \widetilde{\lambda}(m)$, the family of bounded operators $T_t := \mathrm{e}^{\gamma t} P_t$, $t \geqslant 0$, is such that $\sup_t \|T_t f\| < \infty$, $\forall\, f \in L^1(m)$. By the uniform boundedness principle we have $\sup_t \|T_t\| < \infty$, implying $\gamma \leqslant \lambda_r^{(1)}(m)$.

Finally from the obtained formula we get $Q_m(\alpha = -\infty) = 0$ when $\lambda_r^{(1)}(m) > 0$, which is well known to be equivalent to $m \in \mathrm{Pur}(X)$. □

We give now further results on $\lambda_r^{(1)}(m)$, $m \in \mathrm{Exc}(X)$, that will be of interest in connection with $\lambda_\pi, \lambda_\varphi$.

Proposition 3. (i) *If $m \in \mathbf{M}^\gamma(X)$, then $\gamma \leqslant \lambda_r^{(1)}(m)$. If m is γ-invariant then $\lambda_r^{(1)}(m) = \gamma$.*

(ii) *For any $\gamma < \lambda_r^{(1)}(m)$ there exist $\nu_\gamma \in \mathbf{M}^\gamma(X)$ and $f_\gamma \in \mathbf{F}^\gamma(X)$, $f_\gamma > 0$, such that $\nu_\gamma(f_\gamma) < \infty$.*

(iii) *For any $t > 0$, $\mathrm{e}^{-\lambda_r^{(1)}(m)t}$ is in the spectrum of P_t on $L^1(m)$.*

Proof. Property (i) is checked by direct verification.

For (ii) let $\gamma < \lambda_r^{(1)}(m)$, $\gamma' \in \left]\gamma, \lambda_r^{(1)}(m)\right[$ and $h \in \mathcal{E}$, $h > 0$, $m(h) < \infty$. Let also $M_{\gamma'} \geqslant 1$ be such that $\mathrm{e}^{\gamma t} m(P_t f) \leqslant M_{\gamma'} m(f)$, $\forall\, t > 0$, $\forall\, f \in \mathcal{E}$, $0 < m(f) < \infty$. We then set:

$$\nu_\gamma := m\Phi^\gamma \qquad \text{and} \qquad f_\gamma := \Phi^\gamma h$$

and these are the required elements in $\mathbf{M}^\gamma(X)$, respectively $\mathbf{F}^\gamma(X)$ because we can successively check that $m(f_\gamma) \leqslant M_{\gamma'}(\gamma' - \gamma)^{-1} m(h)$ and $\nu_\gamma(f_\gamma) \leqslant [M_{\gamma'}(\gamma' - \gamma)^{-1}]^2 m(h)$.

Property (iii) is a consequence of a very powerful result (Theorem 7.7 in [Dav81]) applied to the positive (in the sense that it applies non-negative functions from $L^1(m)$ into functions of the same kind) operator P_t, $t > 0$. (Unfortunately this does not ensure that any of the corresponding eigenfunctions is non-negative). □

Corollary 1. (i) $\lambda_\pi = \sup_{m \in \mathrm{Exc}(X)} \{\lambda_r^{(1)}(m)\} = \sup_{m \in \mathrm{Pur}(X)} \{\lambda_r^{(1)}(m)\}$.
(ii) $\lambda_\pi = \lambda_\varphi = \lambda_{\pi,\varphi}$ *where*

$$\lambda_{\pi,\varphi} := \sup\{\gamma \geqslant 0 : \exists\, f \in \mathbf{F}^\gamma(X),\ f > 0 \text{ on } E \text{ and } \nu \in \mathbf{M}^\gamma(X) \text{ such that } \nu(f) < \infty\}.$$

To further emphasize the role played by $\lambda_r^{(1)}(m)$ we end up this section with the construction of the process $\widetilde{X}$ and the measure $\widetilde{m}$ alluded to in the introduction. This amounts to considering $m \in \mathbf{M}^\gamma(X)$ and $h \in \mathbf{F}^\gamma(X)$ such that $m(E \backslash E_h) = 0$, where $E_h := \{0 < h < \infty\}$. With h we associate

$\widetilde{h} := \uparrow\lim_{t\downarrow 0} \mathrm{e}^{\gamma t} P_t h$ so that $\widetilde{h} \in \mathbf{F}^{\gamma}$, $\widetilde{h} \leqslant h$, $\{\widetilde{h} < h\}$ is a set of zero potential and $\widetilde{h}$ is an excessive function.

The supermartingale multiplicative functional

$$M_t := \mathrm{e}^{\gamma t}\Big[\widetilde{h}(X_0)\Big]^{-1}\widetilde{h}(X_t)1_{\{0<\widetilde{h}(X_0)<\infty\}}1_{\{t<T_{\{h=0\}}\}}$$

defines a subprocess $\widetilde{X}$ with state space $\widetilde{E} := \{0 < \widetilde{h} < \infty\}$ and semigroup

$$\widetilde{P}_t f(x) := \mathrm{e}^{\gamma t}\Big[\widetilde{h}(x)\Big]^{-1} P_t\Big(\widetilde{h}f\Big)(x), \qquad x \in \widetilde{E},\ f \in \mathcal{E}_{|\widetilde{E}},\ f \geqslant 0.$$

The process $\widetilde{X}$ is in turn a right Markov process ([Sha88], § 62) and $\widetilde{m} := \widetilde{h}m$ belongs to $\mathrm{Exc}(\widetilde{X})$; $\widetilde{m} \in \mathrm{Inv}(\widetilde{X})$ when m is γ-invariant.

A necessary and sufficient condition for $\widetilde{m}$ to be in $\mathrm{Con}(\widetilde{X})$ (which is a precondition for developing an ergodic theory with respect to $\widetilde{m}$) is the following: for any $f \in \mathcal{E}$, $f > 0$ such that $m(f1_{\widetilde{E}}) < \infty$ we have $m(\Phi^{\gamma} f 1_{\widetilde{E}} < \infty) = 0$. Note that when this condition is fulfilled we necessarily have $\gamma \geqslant \lambda_r^{(1)}(m)$; since m was taken from $\mathbf{M}^{\gamma}(X)$ we must have in fact in this case $\lambda_r^{(1)}(m) = \gamma$.

3 Exit parameters

Theorem 1 suggests that λ_π is (at least in the irreducible case) related to the amount of time spent by the process in *small sets*. The parameter λ_π may be also characterized in terms of escape from such sets and we are going to provide conditions for this. An alternative set of conditions are imposed in [Str82] in order to obtain Radon instead of σ-finite measures.

Recall from [FG96] that an *m-nest* associated with $m \in \mathrm{Dis}(X)$ is defined as an increasing sequence of finely open sets $\mathcal{C} = (C_n) \subseteq \mathcal{E}^e$ such that $P^m(\lim_n \tau_n < \infty) = 0$, where $\tau_n := \tau_{C_n}$.

For each $n \in \mathbb{N}$ let $(P_{t,n})$, (U_n^q) denote the semigroup and resolvent associated with the killed process (X, τ_n) and

$$\Phi_n^{\gamma} := \int_0^{\infty} \mathrm{e}^{\gamma t} P_{t,n}\,\mathrm{d}t = \sum_{p=1}^{\infty} (1+\gamma)^{p-1} U_n^{1(p)} f.$$

The m-nest (C_n) of interest for our problem will be assumed to have the following additional property:

(∗) there exists $D \in \mathcal{E}^e$, $D \subseteq C_1$, such that $U(x,B) > 0$, $\forall\, x \in E$ and $U_1^1(x,\,.\,) \geqslant \nu(\,.\,)$, $\forall\, x \in D$, where $\nu(\Gamma) := m_D(\Gamma)[m(D)]^{-1}$.

With the m-nest $\mathcal{C}$ having property (∗) we associate

$$\lambda_e^{\mathcal{C}}(m) := \sup\{\gamma \geqslant 0 : P^{\nu}(\mathrm{e}^{\gamma\tau_n}) < \infty,\ \forall\, n \in \mathbb{N}\}.$$

Proposition 4. *Let $\mathcal{C}$ be an m-nest satisfying condition* $(*)$. *Then* $\lambda_e^{\mathcal{C}}(m) \leqslant \lambda_\pi$.

Proof. Let $\gamma < \lambda_e^{\mathcal{C}}(m)$. In order to construct a measure in $\mathbf{M}^\gamma(X)$ we start by considering the measures $\nu_n := \nu \Phi_n^\gamma$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $\nu_n \in \mathbf{M}^\gamma(X, \tau_n)$ and it is finite because $\nu_n(P^\cdot(e^{\gamma' \tau_n})) < \infty$ for $\gamma' \in \left]\gamma, \lambda_e^{\mathcal{C}}(m)\right[$ as is easily checked. Also, for each $n \in \mathbb{N}$, $\nu_n(D) > 0$, $P^{\nu_n}(\lim \tau_n < \zeta) = 0$ and, due to condition $(*)$, $\nu_n \geqslant (1+\gamma)\nu_n(D)\nu$.

Next let $\mu_n := [\nu_n(D)]^{-1}\nu_n$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ the measure μ_n is in turn finite and such that $\mu_n \in \mathbf{M}^\gamma(X, \tau_n)$, $\mu_n(D) = 1$, $\mu_n \geqslant (1+\gamma)\nu$ and $P^{\mu_n}(\lim \tau_n < \zeta) = 0$.

Let further $\eta_n := \inf_{p \geqslant n} \mu_p$, $n \in \mathbb{N}$, define an increasing sequence of measures such that $\eta_n \in \mathbf{M}^\gamma(X, \tau_n)$, $\forall\, n \in \mathbb{N}$. For each $n \geqslant 1$, one has $\eta_n(D) \leqslant 1$, $\eta_n \geqslant (1+\gamma)$ and

$$\begin{aligned} \eta_n(U^1 1_D) &= \lim_{k\to\infty} P^{\eta_n}\left(\int_0^\infty e^{-u} 1_D(X_u)\, du\right) \\ &\leqslant \lim_{k\to\infty} \eta_k\big(U_k^1(D)\big) \leqslant \lim_{k\to\infty} \eta_k(D) \leqslant 1. \end{aligned}$$

Finally, let $\eta := \uparrow\!\lim_{n\to\infty} \eta_n$. Obviously $\eta \geqslant (1+\gamma)\nu$ and it is σ-finite because $\eta(U^1 1_D) \leqslant 1$ and $U^1 1_D(x) > 0$, $\forall\, x \in E$. It remains to show that $\eta \in \mathbf{M}^\gamma(X)$; this follows from the fact that for any $n \in \mathbb{N}$

$$e^{\gamma t}\eta_n(P_t f) = e^{\gamma t} \lim_{k\to\infty} P^{\eta_n}\big(f(X_t);\, t < \tau_k\big) \leqslant \lim_{k\to\infty} \eta_n(f) = \eta(f)$$

for each $f \in \mathcal{E}$, $f \geqslant 0$. □

It is perhaps worth mentioning that while there exist a number of m-nests associated with $m \in \mathrm{Dis}(X)$ (see [FG96] in his respect), condition $(*)$ is quite restrictive. Among other things the very existence of a set D with the properties involved in $(*)$ entails the ν-irreducibility of the process X.

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References

[ADR66] Azéma, J., Kaplan-Duflo, M., Revuz, D.: Récurrence fine des processus de Markov. *Ann. Inst. Henri Poincaré* **2**, 185–220 (1966).

[Ber97] Bertoin, J.: Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. *Ann. Appl. Probab.* **7**, 156–169 (1997).

[CZ95] Chung, K.L., Zhao, Z.: *From Brownian Motion to Schrödinger's Equation.* Springer, Berlin Heidelberg New York (1995).

[Dav81] Davies, E.B.: *One parameter semigroups.* Oxford University Press, Oxford (1981).

[DMM92] Dellacherie, C., Maisonneuve, B., Meyer, P.A.: *Probabilités et Potentiel.* Ch. XVII–XXIV, Hermann, Paris (1992).

[FG96] Fitzsimmons, P.J., Getoor, R.K.: Smooth measures and continuous additive functionals of right Markov processes. In: *Ito's Stochastic Calculus and Probability Theory.* Springer, Berlin Heidelberg New York (1996).

[Get90] Getoor, R.K.: *Excessive Measures.* Birkhäuser, Boston (1990).

[Get99] Getoor, R.K.: Measure perturbations of Markovian semigroups. *Potential Analysis* **11**, 101–133 (1999).

[Glo88] Glover, J.: Quasi-stationary distributions, eigenmeasures and eigenfunctions of Markov processes. In: *Seminar on Stochastic Processes 1984.* Birkhäuser, Boston (1986).

[MT96] Meyn, S.P., Tweedie, R.L.: *Markov Chains and Sochastic Stability.* Springer, Berlin Heidelberg New York (1996).

[NN86] Niemi, S., Nummelin, E.: On non-singular renewal kernels with applications to a semigroup of transition kernels. *Stochastic Process. Appl.* **22**, 177–202 (1986).

[Sat85] Sato, S.: An inequality for the spectral radius of Markov processes. *Kodai Math. J.* **8**, 5–13 (1985).

[Sha88] Sharpe, M.J.: *General Theory of Markov Processes.* Academic Press, San Diego (1988).

[Str82] Stroock, D.: On the spectrum of Markov semigoups and the existence of invariant measures. In: *Functionl Analysis in Markov Processes.* Springer, Berlin Heidelberg New York (1982).

[Tak00] Takeda, M.: L^p-independence of the spectral radius of symmetric Markov semigroups. *Canadian Math. Soc., Conference Proceedings* **20**, 613–623 (2000).

[TT79] Tuominen, P., Tweedie, R.J.: Exponential decay of general Markov processes and their discrete skeletons. *Adv. Appl. Probab.* **11**, 784–802 (1979).

[Yos80] Yosida, K.: *Functional Analysis, 6th ed.* Springer, Berlin Heidelberg New York (1980).