Chapter 5
A Splitting Formula for Lower Algebraic K-Theory

Let Γ be a three-dimensional crystallographic group with lattice L and point group H. (We do not assume that Γ is a split crystallographic group.) In this chapter, we describe a simple construction of $E_{\mathcal{VC}}(\Gamma)$ and derive a splitting formula for the lower algebraic K-theory of any three-dimensional crystallographic group.

5.1 A Construction of $E_{\mathcal{FIN}}(\Gamma)$ for Crystallographic Groups

We will need to have a specific model of $E_{\mathcal{FIN}}(\Gamma)$ for our crystallographic groups Γ.

Proposition 5.1. *If Γ is a three-dimensional crystallographic group, then there is an equivariant cell structure on $\mathbb{R}^3$ making it a model for $E_{\mathcal{FIN}}(\Gamma)$.*

Proof. For every crystallographic group Γ, there is a crystallographic group Γ' of the same dimension, called the *splitting group* of Γ ([Ra94, pp. 312–313]), and an embedding $\phi : \Gamma \to \Gamma'$. The group Γ' is a split crystallographic group in our sense, by Lemma 7 on page 313 of [Ra94]. It is therefore sufficient to prove the proposition for every split three-dimensional crystallographic group. Table 4.1 shows that all of the split crystallographic groups are subgroups of seven maximal ones (consider the pairing of the maximal point group with each of the seven lattices). We will show in Chap. 6 (without circularity) that each of these maximal groups has the required model. The proposition now follows easily.

5.2 A Construction of $E_{\mathcal{VC}}(\Gamma)$ for Crystallographic Groups

Let Γ be a three-dimensional crystallographic group. We begin with a copy of $E_{\mathcal{FIN}}(\Gamma)$, which we can identify with a suitably cellulated copy of $\mathbb{R}^3$ by Proposition 5.1. For each $\ell \in L$ such that ℓ generates a maximal cyclic subgroup of

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L, we define

$$\mathbb{R}^2_\ell = \{\hat{\ell} \subseteq \mathbb{R}^3 \mid \hat{\ell} \text{ is a line parallel to } \langle \ell \rangle\},$$

where $\langle \ell \rangle$ denotes the one-dimensional vector subspace spanned by ℓ. Consider $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$, where the disjoint union is over all maximal cyclic subgroups $\langle \ell \rangle$ of L. We define a metric on $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ as follows. If $\ell_1, \ell_2 \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$, we set $d(\ell_1, \ell_2) = \infty$ if ℓ_1 and ℓ_2 are not parallel, and $d(\ell_1, \ell_2) = K$ if ℓ_1 is parallel to ℓ_2 and $K = \min\{d_{\mathbb{R}^3}(x, y) \mid x \in \ell_1, y \in \ell_2\}$. One readily checks that d is a metric on $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$, and that each $\mathbb{R}^2_\ell$ is isometric to $\mathbb{R}^2$. We will therefore freely refer to the $\mathbb{R}^2_\ell$ as "planes" in what follows. Moreover, Γ acts by isometries on $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$.

Next we would like to introduce an equivariant cell structure on $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$. Choose a plane $\mathbb{R}^2_\ell$.

Definition 5.1. Let $\pi : \Gamma \to H$ be the usual projection into the point group. We let $H_{\langle \ell \rangle} = \{h \in H \mid h \cdot \langle \ell \rangle = \langle \ell \rangle\}$ and $\Gamma(\ell) = \pi^{-1}(H_{\langle \ell \rangle})$.

It is straightforward to check that $\Gamma(\ell)$ acts on $\mathbb{R}^2_\ell$. Since $\langle \ell \rangle$ is a maximal cyclic subgroup of L, we can choose a basis $\{\ell_1, \ell_2, \ell_3\}$ of L, with $\ell_3 = \ell$. Each of the ℓ_i can be written $\ell_i = \alpha_i \ell + \hat{\ell}_i$, where $\alpha_i \in \mathbb{R}$ and $\hat{\ell}_i$ is perpendicular to ℓ. Since ℓ_1, ℓ_2, and ℓ_3 are linearly independent over $\mathbb{R}$, the same must be true of $\hat{\ell}_1$ and $\hat{\ell}_2$. The translation ℓ acts trivially on $\mathbb{R}^2_\ell$, so the action of L on $\mathbb{R}^2_\ell$ is the same as the action of $\langle \hat{\ell}_1, \hat{\ell}_2 \rangle$. In particular, the action of L has discrete orbits, from which it follows readily that the action of $\Gamma(\ell)$ on $\mathbb{R}^2_\ell$ has discrete orbits. We can therefore find a $\Gamma(\ell)$-equivariant cell structure on $\mathbb{R}^2_\ell$ making it a $\Gamma(\ell)$-CW complex.

Now we choose a (finite) left transversal $T \subseteq H$ of $\Gamma(\ell)$ in Γ. For each $t \in T$, we cellulate $\mathbb{R}^2_{t \cdot \ell}$ using the equality $\mathbb{R}^2_\ell = t \cdot \mathbb{R}^2_\ell$ (that is, for each cell $\sigma \subseteq \mathbb{R}^2_\ell$, we let $t \cdot \sigma$ be a cell in the cellulation of $\mathbb{R}^2_{t \cdot \ell}$). The result is an equivariant cellulation of all of $\Gamma \cdot \mathbb{R}^2_\ell$, which is a disjoint union of finitely many planes. We can continue in the same way, choosing a new plane $\mathbb{R}^2_{\ell'}$ and applying the same procedure, until we have cellulated all of $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$. The space $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ is a Γ-CW complex with respect to the resulting cellulation.

Proposition 5.2. *Let $Y = E_{\mathcal{FIN}}(\Gamma)$ and $Z = \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$. The space $X = Y * Z$ is a model for $E_{\mathcal{VC}}(\Gamma)$.*

Proof. Since Y and Z are Γ-CW complexes, the join X inherits a natural Γ-CW complex structure. For $G \leq \Gamma$ and $W \in \{X, Y, Z\}$, we let $\mathrm{Fix}_W(G) = \{w \in W \mid g \cdot w = w \text{ for all } g \in G\}$. We note that $\mathrm{Fix}_W(G)$ is a subcomplex of W, and $\mathrm{Fix}_X(G) = \mathrm{Fix}_Y(G) * \mathrm{Fix}_Z(G)$, for all $G \leq \Gamma$.

Let $G \in \mathcal{VC}(\Gamma)$. There are two cases. Assume first that G is finite. In this case, $\mathrm{Fix}_X(G) = \mathrm{Fix}_Y(G) * \mathrm{Fix}_Z(G)$, where $\mathrm{Fix}_Y(G)$ is contractible by our assumptions. It follows that $\mathrm{Fix}_X(G)$ is contractible.

If G is infinite and virtually cyclic, then there is a cyclic subgroup $\langle g \rangle \leq L$ having finite index in G such that $\langle g \rangle \trianglelefteq G$. This group $\langle g \rangle$ is contained in a maximal cyclic subgroup $\langle \ell \rangle$ of L. The kernel of the action of G on $\mathbb{R}^2_\ell$ therefore contains $\langle g \rangle$. It follows that the fixed set of the action of G on $\mathbb{R}^2_\ell$ is the same as the fixed set of the action of $G/\langle g \rangle$ on $\mathbb{R}^2_\ell$. The latter group is a finite group acting by isometries, so the fixed set is contractible. In particular, $\mathrm{Fix}_X(G) \cap \mathbb{R}^2_\ell$ is contractible. Now we claim that $\mathrm{Fix}_X(G) = \mathrm{Fix}_X(G) \cap \mathbb{R}^2_\ell$ (i.e., that $\mathrm{Fix}_X(G) \subseteq \mathbb{R}^2_\ell$). Indeed, it is enough to check that $\mathrm{Fix}_X(G) \cap \mathbb{R}^2_{\ell'} = \emptyset$ for $\langle \ell' \rangle \neq \langle \ell \rangle$ and $\mathrm{Fix}_X(G) \cap Y = \emptyset$. The latter equality follows directly from the definition of Y. If $\langle \ell' \rangle \neq \langle \ell \rangle$, then g acts as $\hat{g}$ on $\mathbb{R}^2_{\ell'}$, where $\hat{g}$ is the component of g perpendicular to ℓ'. The claim follows directly.

Now suppose that $G \notin \mathcal{VC}(\Gamma)$. It follows that $rk(G \cap L) \geq 2$. One easily sees that $G \cap L$ cannot have any global fixed point in any $\mathbb{R}^2_\ell$ and $\mathrm{Fix}_Y(G \cap L) = \emptyset$ by definition. It follows that $\mathrm{Fix}_X(G) = \emptyset$, as required.

5.3 A Splitting Formula for the Lower Algebraic K-Theory

In the next chapters, we will use the following theorem to compute the lower algebraic K-theory of the integral group ring of all 73 split three-dimensional crystallographic groups. Our goal in this section is to provide a proof.

Theorem 5.1. *Let Γ be a three-dimensional crystallographic group. For $n \leq 1$, we have a splitting*

$$\begin{aligned} &H_n^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty}) \\ &\quad \cong H_n^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{\hat{\ell} \in \mathcal{T}''} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *;\ \mathbb{KZ}^{-\infty}). \end{aligned}$$

The indexing set $\mathcal{T}''$ consists of a selection of one vertex $v \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ from each Γ-orbit of such non-negligible vertices.

We refer the reader to Definition 5.3 for a definition of negligible. We will eventually see that $\mathcal{T}''$ is finite. Note that Theorem 5.1 used the following definition.

Definition 5.2. Let G be a group acting on a set X. If $A \subseteq X$, we let $G_A = \{g \in G \mid g \cdot A = A\}$.

For each maximal cyclic subgroup $\langle \ell \rangle \leq L$, we set

$$\mathcal{VC}_{\langle \ell \rangle} = \mathcal{VC} \cap \{G \leq \Gamma(\ell) \mid |G \cap \langle \ell \rangle| = \infty \text{ if } |G| = \infty\}.$$

In words, $\mathcal{VC}_{\langle \ell \rangle}$ is the collection consisting of finite subgroups of $\Gamma(\ell)$ and the infinite virtually cyclic subgroups of $\Gamma(\ell)$ that contain some non-zero multiple of the translation ℓ. It is easy to check that $\mathcal{VC}_{\langle \ell \rangle}$ is a family of subgroups in $\Gamma(\ell)$.

The point group H acts on the set of maximal cyclic subgroups in L by the rule $h \cdot \langle \ell \rangle = \langle h(\ell) \rangle$. We choose a single maximal cyclic subgroup from each orbit and call the resulting collection $\mathcal{T}$.

Proposition 5.3. *Assume that Γ is given the discrete topology. We continue to write $Y = E_{\mathcal{FIN}}(\Gamma)$ and $Z = \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ when convenient.*

1. *$\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ is homeomorphic to $\coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} \mathbb{R}^2_\ell$ by a homeomorphism that is compatible with the Γ-action. (Here the action on the latter space is given by the rule $\gamma \cdot (\gamma', \ell') = (\gamma\gamma', \ell')$.) Each $\mathbb{R}^2_\ell$ is a model for $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$.*
2. *$\coprod_{\langle \ell \rangle} Y \times \mathbb{R}^2_\ell$ is homeomorphic to $\coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} (Y \times \mathbb{R}^2_\ell)$ by a homeomorphism that is compatible with the Γ-action. Each $Y \times \mathbb{R}^2_\ell$ is a model for $E_{\mathcal{FIN}}(\Gamma(\ell))$.*

The space $\coprod_{\langle \ell \rangle} Y \times \mathbb{R}^2_\ell$ can be identified with $Y \times Z \times \{1/2\} \subseteq Y * Z$ and $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ can be identified with the bottom of the join $Y * Z$.

Proof. We prove (1), the proof of (2) being similar. Consider the Γ-space $\Gamma \times \coprod_{\langle \ell \rangle \in \mathcal{T}} \mathbb{R}^2_\ell$, where Γ acts by left multiplication on the first coordinate and trivially on the second coordinate. We let $\coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} \mathbb{R}^2_\ell$ be the usual Borel construction (so that Γ acts only on the first coordinate). We regard $\coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ as a Γ-space with respect to its usual action.

Define maps $\pi_1 : \Gamma \times \coprod_{\langle \ell \rangle \in \mathcal{T}} \mathbb{R}^2_\ell \to \coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} \mathbb{R}^2_\ell$ and $\pi_2 : \Gamma \times \coprod_{\langle \ell \rangle \in \mathcal{T}} \mathbb{R}^2_\ell \to \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ by the rules $\pi_1(\gamma, x) = (\gamma, x)$ and $\pi_2(\gamma, x) = \gamma \cdot x$. Both of these are quotient maps and commute with the Γ-action.

We claim that π_1 is constant on point inverses of π_2 and π_2 is constant on point inverses of π_1. It will then follow from a well-known principle (see [Mu00, Theorem 22.2]) that there is a Γ-homeomorphism $f : \coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} \mathbb{R}^2_\ell \to \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$ such that $f \circ \pi_1 = \pi_2$.

$$\begin{tikzcd}
\Gamma \times \coprod_{\langle \ell \rangle \in \mathcal{T}} \mathbb{R}^2_\ell \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & \\
\coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} \mathbb{R}^2_\ell \arrow[r, "f"] & \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell
\end{tikzcd}$$

Let $\gamma \in \Gamma$ and $x \in \mathbb{R}^2_\ell$ for some $\langle \ell \rangle \in \mathcal{T}$. One easily checks that

$$\pi_1^{-1}(\gamma, x) = \{(\gamma\tilde{\gamma}, \tilde{\gamma}^{-1} \cdot x) \mid \tilde{\gamma} \in \Gamma(\ell)\}.$$

It follows directly that $\pi_2(\pi_1^{-1}(\gamma, x)) = \{\gamma \cdot x\}$ (a singleton), as required.

If $x \in \coprod_{\langle \ell \rangle} \mathbb{R}^2_\ell$, then

$$\pi_2^{-1}(x) = \{(\hat{\gamma}, \hat{x}) \mid \hat{\gamma} \cdot \hat{x} = x\}.$$

Now we suppose that (γ_1, x_1) and (γ_2, x_2) are in $\pi_2^{-1}(x)$. It follows that $\gamma_1 \cdot x_1 = \gamma_2 \cdot x_2$, so $\gamma_2^{-1}\gamma_1 \cdot x_1 = x_2$. Since x_1 and x_2 are in the same Γ-orbit, it must be that both are in $\mathbb{R}^2_\ell$, for some $\langle \ell \rangle \in \mathcal{T}$. It then follows that $\gamma_2^{-1}\gamma_1 \in \Gamma(\ell)$. Now we apply π_1:

$$\begin{aligned}\pi_1(\gamma_1, x_1) &= (\gamma_1, x_1)\\ &= (\gamma_2(\gamma_2^{-1}\gamma_1), x_1)\\ &\sim (\gamma_2, \gamma_2^{-1}\gamma_1 \cdot x_1)\\ &= (\gamma_2, x_2)\\ &= \pi_1(\gamma_2, x_2)\end{aligned}$$

It follows that $\pi_1(\pi_2^{-1}(x))$ is a singleton, as required.

We have now demonstrated the existence of f. The remaining statements are straightforward to check.

Proposition 5.4. *Let Γ be a three-dimensional crystallographic group. For all $n \in \mathbb{Z}$, we have a splitting*

$$\begin{aligned}&H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty})\\ &\quad\cong H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))\\ &\quad\to E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty})),\end{aligned}$$

where $H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)) \to E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty})$ denotes the cokernel of the relative assembly map

$$H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}) \longrightarrow H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}).$$

The proof of this proposition resembles others that have appeared in [J-PL06] and [LO09].

Proof. Let us work with the explicit model X for $E_{\mathcal{VC}}(\Gamma)$ constructed in Propositions 5.2 and 5.3. Since X is obtained as a join, there exists an obvious map $\rho : X \to [0, 1]$, which further has the property that every point pre-image is Γ-invariant. In particular, corresponding to the splitting of $[0, 1]$ into $[0, 2/3) \cup (1/3, 1]$, we get a Γ-invariant splitting of X. If we let $A = \rho^{-1}[0, 2/3)$, $B = \rho^{-1}(1/3, 1]$, then from the Mayer–Vietoris sequence in equivariant homology (and omitting coefficients in order to simplify our notation), we have that:

$$\ldots \to H_n^{\Gamma}(A \cap B) \to H_n^{\Gamma}(A) \oplus H_n^{\Gamma}(B) \to H_n^{\Gamma}(X) \to \ldots$$

Next, observe that we have obvious Γ-equivariant homotopy equivalences:

- $A = \rho^{-1}[0, 2/3) \simeq \rho^{-1}(0) = E_{\mathcal{FIN}}(\Gamma)$
- $B = \rho^{-1}(1/3, 1] \simeq \rho^{-1}(1) = \coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$
- $A \cap B = \rho^{-1}(1/3, 2/3) \simeq \rho^{-1}(1/2) = \coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} E_{\mathcal{FIN}}(\Gamma(\ell))$

Now, using the induction structure and the fact that our equivariant generalized homology theory turns disjoint unions into direct sums, we can evaluate the terms in the Mayer–Vietoris sequence as follows:

$$\begin{aligned} \ldots \to \bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \to H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma)) \oplus \bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)) \\ \to H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma)) \to \ldots \end{aligned}$$

Next, we study the relative assembly map

$$\Phi_{\langle \ell \rangle} : H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \to H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)).$$

We claim $\Phi_{\langle \ell \rangle}$ is split injective. This can be seen as follows. Consider the following commutative diagram:

$$\begin{tikzcd} H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \arrow[r, "\Phi_{\langle \ell \rangle}"] \arrow[dr, "\beta"'] & H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)) \arrow[d, "\alpha"] \\ & H_n^{\Gamma(\ell)}(E_{\mathcal{VC}}(\Gamma(\ell)) \end{tikzcd}$$

where α and β are the relative assembly maps induced by the inclusions $\mathcal{VC}_{\langle \ell \rangle} \subset \mathcal{VC}$ and $\mathcal{FIN} \subset \mathcal{VC}$. Recall that Bartels [Bar03] has established that for *any* group G, the relative assembly map:

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbb{KZ}^{-\infty}) \to H_n^G(E_{\mathcal{VC}}(G); \mathbb{KZ}^{-\infty})$$

is split injective for all n. Using this result from Bartels, it follows that

$$\beta : H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \longrightarrow H_n^{\Gamma(\ell)}(E_{\mathcal{VC}}(\Gamma(\ell))$$

is split injective. Therefore $\Phi_{\langle \ell \rangle}$ is also split injective.

Now, for each integer n, the above portion of the Mayer–Vietoris long exact sequence breaks off as a short exact sequence (since the initial term injects). Since the map from the $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma)) \to H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma))$ is also split injective (from the Bartels result), we obtain an identification of the cokernel of the latter map with the cokernel of the map

$$\bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \longrightarrow \bigoplus_{\langle \ell \rangle \in \mathcal{T}} H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))) \tag{5.1}$$

Next, since the inclusion map

$$\coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} E_{\mathcal{FIN}}(\Gamma(\ell)) \longrightarrow \coprod_{\langle \ell \rangle \in \mathcal{T}} \Gamma \times_{\Gamma(\ell)} E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$$

is the disjoint union of cellular $\Gamma(\ell)$-maps (for all $\langle \ell \rangle \in \mathcal{T}$), we see that the maps given in (1) split as a direct sum (over $\langle \ell \rangle \in \mathcal{T}$) of the relative assembly maps $H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell))) \to H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)))$. This immediately yields a corresponding splitting of the cokernel, completing the proof of the proposition.

The next step is to analyze the summands

$$H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)) \to E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty})$$

from Proposition 5.4.

We fix a maximal cyclic subgroup $\langle \ell \rangle \leq L$ for the remainder of this chapter. We note that the space $\mathbb{R}^3 * \mathbb{R}^2_\ell$ is a model for $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$, where both factors are given $\Gamma(\ell)$-equivariant cell structures and the action of $\Gamma(\ell)$ is the usual one.

Next we will need to describe the class of negligible groups.

Remark 5.1. If $G \in \mathcal{VC}$, then G has one of three possible forms:

1. G is finite, or
2. G is *infinite virtually cyclic of type I*; that is, G admits a surjective homomorphism onto $\mathbb{Z}$ with finite kernel. Such a group will necessarily have the form $G \cong F \rtimes \mathbb{Z}$, where F is the kernel of the surjection onto $\mathbb{Z}$, or
3. G is *infinite virtually cyclic of type II*; that is, G admits a surjective homomorphism onto D_∞ with finite kernel. In this case, $G \cong A *_F B$, where A, B, and F are finite groups, and F has index two in both A and B.

Definition 5.3. A group $G \in \mathcal{VC}$ is *negligible* if:

1. for each finite subgroup $H \leq G$, H is isomorphic to a subgroup of S_4 (the symmetric group on four symbols), and
2. if $G \in \mathcal{VC}_\infty$, then the finite group F from Remark 5.1 has square-free order.

(Thus, a finite group G is negligible if it is isomorphic to a subgroup of S_4. An infinite virtually cyclic group of type I is negligible if F is of square-free order and isomorphic to a subgroup of S_4. An infinite virtually cyclic group of type II is negligible if the factors A and B are isomorphic to subgroups of S_4, and F has square-free order.)

We will also say that a cell σ is negligible if its stabilizer group is negligible.

Remark 5.2. This is the first of two different definitions of "negligible" that we will use. We will need a different definition in Chaps. 8 and 9.

Definition 5.3 allows us to describe classes of cells that make no contribution to K-theory (see Lemma 5.1 below), which will let us ignore them in our work.

Lemma 5.1. *Let G be a negligible group. The groups $Wh_q(G)$ are trivial for $q \leq 1$, and the same is true for all subgroups of G.*

Proof. We first note that subgroups of negligible groups are negligible.

If G is finite and negligible, then G is isomorphic to a subgroup of S_4; i.e., $G \cong \{1\}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, D_2, D_3, D_4, A_4, or S_4. The lemma then follows from Table 7.1 and the accompanying discussion. (See also [LO09].)

Now we assume that G is negligible and infinite virtually cyclic of type I. Therefore, $G \cong F \rtimes_\alpha \mathbb{Z}$, where F is a subgroup of S_4 with square-free order. By results of Farrell and Hsiang [FH68] and Farrell and Jones [FJ95],

$$Wh_q(F \rtimes_\alpha \mathbb{Z}) \cong C \oplus NK_q(\mathbb{Z}F, \alpha) \oplus NK_q(\mathbb{Z}F, \alpha^{-1}),$$

where C is a suitable quotient of the group $Wh_{q-1}(F) \oplus Wh_q(F)$ and $q \leq 1$. Since F is finite and negligible, C is trivial. Therefore,

$$Wh_q(F \rtimes_\alpha \mathbb{Z}) \cong 2NK_q(\mathbb{Z}F, \alpha),$$

since Farrell and Hsiang also show that $NK_q(\mathbb{Z}F, \alpha) \cong NK_q(\mathbb{Z}F, \alpha^{-1})$. Since F has square-free order, $NK_q(\mathbb{Z}F, \alpha)$ is trivial for $q \leq 1$ by results of [Ha87] and [J-PR09]. (The case in which $\alpha = \mathrm{id}$ was established by Harmon [Ha87], and the general case is due to [J-PR09].) This proves the lemma in the case that G is negligible and infinite virtually cyclic of type I.

Finally, we assume that G is negligible and infinite virtually cyclic of type II. Therefore, we can write $G \cong G_1 *_F G_2$, where F has square-free order and index two in both factors, and both G_1 and G_2 are isomorphic to subgroups of S_4. By results of [Wal78] (see also [CP02]),

$$Wh_q(G) \cong X \oplus NK_q(\mathbb{Z}F; \mathbb{Z}[G_1 - F], \mathbb{Z}[G_2 - F]),$$

for all $q \leq 1$, where X is a suitable quotient of $Wh_q(G_1) \oplus Wh_q(G_2)$. Since G_1 and G_2 are negligible, both factors in the latter direct sum are trivial, so X is trivial. It follows that

$$Wh_q(G) \cong NK_q(\mathbb{Z}F; \mathbb{Z}[G_1 - F], \mathbb{Z}[G_2 - F]).$$

Let $F \rtimes_\alpha \mathbb{Z}$ be the canonical index two subgroup of G. Since $NK_q(\mathbb{Z}F, \alpha)$ is trivial for $q \leq 1$ by the previous case, it follows that $NK_q(\mathbb{Z}F; \mathbb{Z}[G_1 - F], \mathbb{Z}[G_2 - F])$ is also trivial for $q \leq 1$, by results of Lafont and Ortiz [LO08] (see also [DQR11] and [DKR11]). It follows that $Wh_q(G)$ is trivial for $q \leq 1$ in this case as well.

Lemma 5.2. *Let $G \leq \Gamma(\ell)$.*

1. *If there is a line $\hat{\ell}$ and a point $p \notin \hat{\ell}$ such that G fixes p and leaves $\hat{\ell}$ invariant, then G is negligible.*

2. *If G leaves a line $\hat{\ell}$ invariant and $\hat{\ell} \cap c \neq \emptyset$ for some open 2-cell $c \subseteq \mathbb{R}^3$, then G is negligible.*
3. *If G fixes two points $p_1, p_2 \in \mathbb{R}^3$ and the line $\overleftrightarrow{p_1 p_2}$ is not parallel to the line ℓ, then G is negligible.*
4. *If G leaves a strip $\hat{\ell} \times [0, K]$ invariant and acts trivially on the second factor, then G is negligible.*

Proof. 1. Let $\hat{p}$ be the point on $\hat{\ell}$ that is closest to p. We let v_1 be the vector originating at $\hat{p}$ and terminating at p. Let v_2 be a tangent vector to $\hat{\ell}$ at $\hat{p}$, and let v_3 be a vector that is perpendicular to v_1 and v_2. We note that the vectors v_1, v_2, and v_3 are pairwise orthogonal.

The point $\hat{p}$ must be fixed by G, and so G must act by orthogonal matrices with respect to the ordered basis $[v_1, v_2, v_3]$. By our assumptions, G fixes v_1. The inclusion $G \cdot v_2 \subseteq \{v_2, -v_2\}$ holds, since $\hat{\ell}$ is G-invariant. It follows from orthogonality that $G \cdot v_3 \subseteq \{v_3, -v_3\}$ as well. We conclude that G is isomorphic to a subgroup of $(\mathbb{Z}/2)^2$, and therefore negligible.

2. Suppose that G leaves $\hat{\ell}$ invariant, and $\hat{\ell} \cap c \neq \emptyset$ for some open 2-cell $c \subseteq \mathbb{R}^3$. We consider the restriction homomorphism $r : G \to \mathrm{Isom}(\hat{\ell})$. The kernel of this map is a subgroup of the stabilizer group of c. It follows that $|\ker r| = 1$ or 2. Thus, G maps into $\mathbb{Z}$ or D_∞ with 1 or $\mathbb{Z}/2$ as kernel, so G is negligible.
3. Let p_1, p_2 be fixed by G. We consider the line $\hat{\ell}$ through p_2 that is parallel to ℓ. Since $\overleftrightarrow{p_1 p_2}$ is not parallel to ℓ, $p_1 \notin \hat{\ell}$. Since $G \subseteq \Gamma(\ell)$, G leaves $\hat{\ell}$ invariant. As a result, G is negligible by (1).
4. If G leaves a strip $\hat{\ell} \times [0, K]$ invariant and acts trivially on the second coordinate, then G acts on each line $\hat{\ell} \times \{k\} \subseteq \hat{\ell} \times [0, K]$. At least one of these lines must meet an open 2-cell in $\mathbb{R}^3$, so G is negligible by (2).

Corollary 5.1. *Let $\hat{\ell} \in \mathbb{R}^2_\ell$ have a non-negligible stabilizer group. The point $\hat{\ell} \in \mathbb{R}^2_\ell$ must be a vertex, and the line $\hat{\ell} \subseteq \mathbb{R}^3$ occurs as a cellulated subcomplex in $E_{\mathcal{FIN}}(\Gamma(\ell))$.*

Proof. If $\hat{\ell}$ is not a vertex of $\mathbb{R}^2_\ell$, then the stabilizer group of $\hat{\ell}$ leaves a strip invariant in $\mathbb{R}^3$ and acts trivially on the bounded factor, so the stabilizer group of $\hat{\ell}$ is negligible by Lemma 5.2(4).

The second statement follows from the fact that $\hat{\ell} \subseteq (\mathbb{R}^3)^1$, by Lemma 5.2(2).

Definition 5.4. For each vertex $\hat{\ell} \in \mathbb{R}^2_\ell$ with non-negligible stabilizer, set

$$F(\hat{\ell}) = \hat{\ell} \subseteq \mathbb{R}^3.$$

Note that $\hat{\ell}$ is a cellulated line in $\mathbb{R}^3$ by Corollary 5.1, so $F(\hat{\ell})$ is a subcomplex of our model for $E_{\mathcal{FIN}}(\Gamma(\ell))$.

Under the same assumptions on $\hat{\ell}$, we also set

$$E(\hat{\ell}) = F(\hat{\ell}) * \hat{\ell}.$$

We note that $E(\hat{\ell})$ is a subcomplex of $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)) = \mathbb{R}^3 * \mathbb{R}^2_{\ell}$.

Proposition 5.5. *The subcomplexes*

$$F = \coprod_{\hat{\ell}} F(\hat{\ell}), \quad \textit{and} \quad E = \coprod_{\hat{\ell}} E(\hat{\ell})$$

of $E_{\mathcal{FIN}}(\Gamma(\ell))$ and $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$ (respectively) are $\Gamma(\ell)$-equivariant. (The disjoint unions are indexed over all lines $\hat{\ell} \in \mathbb{R}^2_{\ell}$ with non-negligible stabilizers.)

These subcomplexes also contain the only non-negligible cells in $E_{\mathcal{FIN}}(\Gamma(\ell))$ and $E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell))$ (respectively). In particular, the natural inclusions induce isomorphisms

$$H_n^{\Gamma(\ell)}(F; \mathbb{KZ}^{-\infty}) \cong H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}),$$
$$H_n^{\Gamma(\ell)}(E; \mathbb{KZ}^{-\infty}) \cong H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}).$$

Proof. The statement that the given subcomplexes are $\Gamma(\ell)$-equivariant follows from the $\Gamma(\ell)$-equivariance of the indexing sets.

Now we would like to show that the given subcomplexes contain the only non-negligible cells. It is good enough to do this for the subcomplex $E = \coprod_{\hat{\ell}} E(\hat{\ell})$, since

$$F = \coprod_{\hat{\ell}} F(\hat{\ell}) = E_{\mathcal{FIN}}(\Gamma(\ell)) \cap \coprod_{\hat{\ell}} E(\hat{\ell}).$$

We will consider cells in the top of the join $\mathbb{R}^3 * \mathbb{R}^2_{\ell}$ (i.e., in $\mathbb{R}^3$), then cells in $\mathbb{R}^2_{\ell}$, and finally the cells that can be described as joins of cells from $\mathbb{R}^3$ and $\mathbb{R}^2_{\ell}$.

We first describe the collection of all cells $c \subseteq \mathbb{R}^3$ with non-negligible stabilizers. It is clear that all such cells are 0- or 1-cells, since the stabilizer of a two-dimensional cell $\sigma \subseteq \mathbb{R}^3$ has order at most 2, and the stabilizer of a three-dimensional cell is necessarily trivial.

Suppose c is a 0-cell in $\mathbb{R}^3$ and c has non-negligible stabilizer. We consider the line $\hat{\ell}$ that is parallel to ℓ and passes through c. Thus $\hat{\ell} \in \mathbb{R}^2_{\ell}$ has a non-negligible stabilizer (since the stabilizer of c is contained in the stabilizer of $\hat{\ell}$), so it must be a vertex. It now follows that $c \in E(\hat{\ell})$.

Now suppose that $c \subseteq \mathbb{R}^3$ is an open 1-cell with non-negligible stabilizer. We choose two points $p_1, p_2 \in c$. The stabilizer group of c fixes both p_1 and p_2. Since the latter group is non-negligible, it must be that $\overleftrightarrow{p_1 p_2} = \hat{\ell}$ is parallel to ℓ by Lemma 5.2(3), so $\hat{\ell} \in \mathbb{R}^2_{\ell}$. The stabilizer group of $\hat{\ell}$ is non-negligible since it contains the stabilizer group of c. Thus, $\hat{\ell}$ is a vertex in $\mathbb{R}^2_\ell$, and $c \subseteq E(\hat{\ell})$. This concludes our analysis of cells in $\mathbb{R}^3$; we have shown that all non-negligible cells in $\mathbb{R}^3$ are contained in E.

We next consider the cells of $\mathbb{R}^2_\ell$. Corollary 5.1 shows that each open cell $c \subseteq \mathbb{R}^2_\ell$ of dimension greater than 0 has negligible stabilizer. Thus, if $c \subseteq \mathbb{R}^2_\ell$ has non-negligible stabilizer, then c is a vertex, so $c \in E(c)$.

Finally, we consider the cells $c = c_1 * c_2$ having non-negligible stabilizer, where $c_1 \subseteq \mathbb{R}^3$, $c_2 \subseteq \mathbb{R}^2_\ell$ are open cells. Since $G_c = G_{c_1} \cap G_{c_2}$, both c_1 and c_2 have non-negligible stabilizer groups. It follows from Corollary 5.1 that c_2 is a vertex (in $\mathbb{R}^2_\ell$) and a line in $\mathbb{R}^3$. Since G_c is non-negligible, we must have $c_1 \subseteq c_2$ by Lemma 5.2(1). Thus, $c \subseteq E(c_2)$. We have now shown that all of the non-negligible cells are in E.

The final statement now follows from Lemma 5.1. Indeed, consider the inclusion of F into $E_{\mathcal{FIN}}(\Gamma(\ell))$. The only cells to have non-zero K-groups are contained in the image (by the above argument and Lemma 5.1), proving that the inclusion induces an isomorphism. The other case is similar.

Proposition 5.6. *We have a $\Gamma(\ell)$-homeomorphism*

$$h : \coprod_{\hat{\ell} \in \mathcal{T}'} \Gamma(\ell) \times_{\Gamma(\ell)_{\hat{\ell}}} X(\hat{\ell}) \to \coprod_{\hat{\ell}} X(\hat{\ell}),$$

where $X \in \{E, F\}$ and $\mathcal{T}'$ is a selection of one vertex $\hat{\ell}$ from each $\Gamma(\ell)$-orbit of non-negligible vertices in $\mathbb{R}^2_\ell$. The domain is a $\Gamma(\ell)$-space relative to the action that is trivial on the second coordinate, and left multiplication on the first.

Moreover, each $F(\hat{\ell})$ is a model for $E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}})$, and each $E(\hat{\ell})$ is a model for $E_{\mathcal{VC}}(\Gamma_{\hat{\ell}})$.

Proof. The argument is similar to the proof of Proposition 5.3. We will prove the proposition in the case $X = E$, the other case being similar. Consider the commutative diagram:

$$\begin{tikzcd}
\Gamma(\ell) \times \coprod_{\widehat{\ell} \in \mathcal{T}'} E(\widehat{\ell}) \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & \\
\coprod_{\widehat{\ell} \in \mathcal{T}'} \Gamma(\ell) \times_{\Gamma(\ell)_{\widehat{\ell}}} E(\widehat{\ell}) \arrow[r, "h"] & \coprod_{\widehat{\ell}} E(\widehat{\ell})
\end{tikzcd}$$

The space on top is a $\Gamma(\ell)$-space relative to the action $\gamma' \cdot (\gamma, x) = (\gamma'\gamma, x)$. We set $\pi_1(\gamma, x) = (\gamma, x)$ and $\pi_2(\gamma, x) = \gamma \cdot x$. As in the proof of Proposition 5.3, we will show that π_1 is constant on point inverses of π_2, and π_2 is constant on point inverses of π_1. This will establish the existence of the desired $\Gamma(\ell)$-homeomorphism h, since both π_1 and π_2 are quotient maps that commute with the $\Gamma(\ell)$-action.

Choose $(\gamma, x) \in \coprod_{\hat{\ell} \in \mathcal{T}'} \Gamma(\ell) \times_{\Gamma(\ell)_{\hat{\ell}}} E(\hat{\ell})$. We have the equality

$$\pi_1^{-1}(\gamma, x) = \{(\gamma\gamma_1, \gamma_1^{-1}x) \mid \gamma_1 \in \Gamma(\ell)_{\hat{\ell}}\}.$$

It follows directly that $\pi_2(\pi_1^{-1}(\gamma, x)) = \{\gamma \cdot x\}$, as required.

Now we choose an arbitrary $x \in \coprod_{\hat{\ell}} E(\hat{\ell})$. We have

$$\pi_2^{-1}(x) = \{(\gamma, z) \mid \gamma \cdot z = x\}.$$

We choose two elements of the latter set, (γ_1, z_1) and (γ_2, z_2). It follows directly that $\gamma_2^{-1}\gamma_1 \cdot z_1 = z_2$, so z_1 and z_2 are in the same $\Gamma(\ell)$-orbit. Given the nature of the indexing set $\mathcal{T}'$, it must be that both z_1 and z_2 are in $E(\hat{\ell})$, for some non-negligible vertex $\hat{\ell} \in \mathbb{R}^2_{\ell}$. It now follows that $\gamma_2^{-1}\gamma_1 \in \Gamma(\ell)_{\hat{\ell}}$. Thus,

$$\begin{aligned}
\pi_1(\gamma_1, z_1) &= (\gamma_2\gamma_2^{-1}\gamma_1, z_1) \\
&\sim (\gamma_2, \gamma_2^{-1}\gamma_1 z_1) \\
&= (\gamma_2, z_2) \\
&= \pi_1(\gamma_2, z_2)
\end{aligned}$$

It follows that π_1 is constant on point inverses of π_2, as required. The existence of the homeomorphism h follows directly.

Finally, we note that $\Gamma(\ell)_{\hat{\ell}} = \Gamma_{\hat{\ell}}$ is an infinite virtually cyclic group. Since $F(\hat{\ell})$ is simply a cellulated line, and $E(\hat{\ell})$ is the join of $F(\hat{\ell})$ with a point, both are well-known models for $E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}})$ and $E_{\mathcal{VC}}(\Gamma_{\hat{\ell}})$, respectively.

Remark 5.3. We note that $\Gamma_{\hat{\ell}}$ denotes the same subgroup of Γ no matter whether we view $\hat{\ell}$ as a vertex in $\mathbb{R}^2_{\ell}$ or as a line in $\mathbb{R}^3$.

Proof (Proof of Theorem 5.1). Combining Propositions 5.5, 5.6, and the fact that our equivariant generalized homology theory turns disjoint unions into direct sums, we obtain the following isomorphisms

$$H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}) \cong \bigoplus_{\hat{\ell} \in \mathcal{T}'} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}); \mathbb{KZ}^{-\infty}), \quad \text{and}$$

$$H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}) \cong \bigoplus_{\hat{\ell} \in \mathcal{T}'} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{VC}}(\Gamma_{\hat{\ell}}); \mathbb{KZ}^{-\infty}).$$

We immediately get an identification of the cokernel of the relative assembly map $H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty}) \to H_n^{\Gamma(\ell)}(E_{\mathcal{VC}_{\langle \ell \rangle}}(\Gamma(\ell)); \mathbb{KZ}^{-\infty})$ with the direct sum of the cokernels of the relative assembly maps

$$H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}); \mathbb{KZ}^{-\infty}) \to H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{VC}}(\Gamma_{\hat{\ell}}); \mathbb{KZ}^{-\infty}).$$

Since $\Gamma_{\hat{\ell}} \in \mathcal{VC}$, then $E_{\mathcal{VC}}(\Gamma_{\hat{\ell}}) = \{*\}$, and the summands in Proposition 5.4 are

$$H_n^{\Gamma(\ell)}(E_{\mathcal{FIN}}(\Gamma(\ell)) \to E_{\mathcal{VC}_{\langle\ell\rangle}}(\Gamma(\ell))) \cong \bigoplus_{\hat{\ell} \in \mathcal{T}'} H_n^{\Gamma_{\hat{\ell}}}(E_{\mathcal{FIN}}(\Gamma_{\hat{\ell}}) \to *).$$

Finally, combining these observations with Proposition 5.4 completes the proof.