Chapter 6
Iterative Method for Fixed Points of Nonexpansive Mappings

6.1 Introduction

We begin this chapter with the following well known definition and theorem.

Definition 6.1. Let (M, ρ) be a metric space. A mapping $T : M \to M$ is called a *contraction* if there exists $k \in [0, 1)$ such that $\rho(Tx, Ty) \leq k\rho(x, y)$ for all $x, y \in M$. If $k = 1$, then T is called *nonexpansive*.

Theorem 6.2. *(Banach Contraction Mapping Principle). Let (M, ρ) be a complete metric space and $T : M \to M$ be a contraction. Then T has a unique fixed point, i.e. there exists a unique $x^* \in M$ such that $Tx^* = x^*$. Moreover, for arbitrary $x_0 \in M$, the sequence $\{x_n\}$ defined iteratively by $x_{n+1} = Tx_n,\ n \geq 0$, converges to the unique fixed point of T.*

Apart from being an obvious generalization of the contraction mappings, nonexpansive maps are important, as has been observed by Bruck [59], mainly for the following two reasons:

- Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.
- Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusions of the form $0 \in \frac{du}{dt} + T(t)u$, where the operators $\{T(t)\}$ are, in general, set-valued and are *accretive* or *dissipative* and *minimally continuous*.

The following fixed point theorem has been proved for nonexpansive maps on uniformly convex spaces.

Theorem 6.3. *(Kirk, [283]) Let E be a reflexive Banach space and let K be a nonempty closed bounded and convex subset of E with normal structure.*

C. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations,*
Lecture Notes in Mathematics 1965,

Let $T : K \to K$ be a nonexpansive mapping of K into itself. Then T has a fixed point.

Unlike in the case of the Banach contraction mapping principle, trivial examples show that the sequence of successive approximations $x_{n+1} = Tx_n$, $x_0 \in K$, $n \geq 0$, (where K is a nonempty closed convex and bounded subset of a real Banach space E), for a nonexpansive mapping $T : K \to K$ even with a unique fixed point, may fail to converge to the fixed point. It suffices, for example, to take for T, a rotation of the unit ball in the plane around the origin of coordinates. More precisely, we have the following example.

Example 6.4. Let $B := \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$ and let T denote an anticlockwise rotation of $\frac{\pi}{4}$ about the origin of coordinates. Then T is nonexpansive with the origin as the only fixed point. Moreover, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n, x_0 = (1, 0) \in B, n \geq 0$, does not converge to zero.

Krasnoselskii [291], however, showed that in this example, one can obtain a convergent sequence of successive approximations if instead of T one takes the auxiliary nonexpansive mapping $\frac{1}{2}(I + T)$, where I denotes the identity transformation of the plane, i.e., if the sequence of successive approximations is defined by $x_0 \in K$,

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n), \quad n = 0, 1, ... \tag{6.1}$$

instead of by the usual so-called *Picard iterates*, $x_{n+1} = Tx_n$, $x_0 \in K$, $n \geq 0$. It is easy to see that the mappings T and $\frac{1}{2}(I + T)$ have the same set of fixed points, so that the limit of a convergent sequence defined by (6.1) is necessarily a fixed point of T.

More generally, if X is a normed linear space and K is a convex subset of X, a generalization of equation (6.1) which has proved successful in the approximation of fixed points of nonexpansive mappings $T : K \to K$ (when they exist), is the following scheme: $x_0 \in K$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n = 0, 1, 2, \ldots; \lambda \in (0, 1) \;, \tag{6.2}$$

λ constant (see, e.g., Schaefer [431]). However, the most general *Mann-type* iterative scheme now studied is the following: $x_0 \in K$,

$$x_{n+1} = (1 - C_n)x_n + C_nTx_n, \quad n = 0, 1, 2, ... \tag{6.3}$$

where $\{C_n\}_{n=1}^{\infty} \subset (0, 1)$ is a real sequence satisfying appropriate conditions (see, e.g., Chidume [87], Edelstein and O'Brian [218], Ishikawa [259]). Under the following additional assumptions (i) $\lim C_n = 0$; and (ii) $\sum_{n=0}^{\infty} C_n = \infty$, the sequence $\{x_n\}$ generated by (6.3) is generally referred to as the *Mann* sequence in the light of Mann [319]. The recursion formula (6.2) is consequently called the *Krasnoselskii-Mann (KM)* formula for finding fixed points of ne (*nonexpansive*) mappings. The following quotation further shows the importance of iterative methods for approximating fixed points of nonexpansive mappings.

- "Many well-known algorithms in signal processing and image reconstruction are iterative in nature .... A wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the KM iteration procedure, for particular choices of the ne operator...." (Charles Byrne , [63]).

Definition 6.5. Let K be a subset of a normed linear space E. Let $T : K \to E$ be a map such that $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. $F(T)$ is called the *fixed point set* of T. The map T is called *quasi-nonexpansive* if $\|Tx - Tx^*\| \leq \|x - x^*\|$ holds for all $x \in K$ and $x^* \in F(T)$.

It is clear that every nonexpansive map with a nonempty fixed point set, $F(T)$, is quasi-nonexpansive. In section 6.6, we will give an example of a quasi-nonexpansive map which is not nonexpansive.

6.2 Asymptotic Regularity

Let $T : K \to K$ be a nonexpansive self-mapping on a convex subset K of a normed linear space X. Let $S_\lambda := \lambda I + (1 - \lambda)T, \lambda \in (0, 1)$, where I denotes the identity map of K. Then for fixed $x_0 \in K, \{S_\lambda^n(x_0)\}$ is defined by $S_\lambda^n(x_0) = \lambda x_n + (1-\lambda)Tx_n$, where $x_n = S_\lambda^{n-1}(x_0)$. In [291], Krasnoselskii proved that if X is uniformly convex and K is compact then, for any $x_0 \in K$, the sequence $\{S_{\frac{1}{2}}^n(x_0)\}_{n=1}^\infty$, of iterates of x_0 under $S_{\frac{1}{2}} = \frac{1}{2}(I + T)$ converges to a fixed point of T. Schaefer [431] observed that the same holds for any $S_\lambda = \lambda I + (1 - \lambda)T$ with $0 < \lambda < 1$, and Edelstein [217] proved that strict convexity of X suffices. The important and natural question of whether or not strict convexity can be removed remained open for many years. In 1967, this question was resolved in the affirmative in the following theorem.

Theorem 6.6. *(Ishikawa, [259]) Let K be a subset of a Banach space X and let T be a nonexpansive mapping from K into X. For $x_0 \in K$, define the sequence $\{x_n\}_{n=1}^\infty$ by (6.3), where the real sequence $\{C_n\}_{n=0}^\infty$ satisfies: (a) $\sum_{n=0}^{\infty} C_n$ diverges, (b) $0 \leq C_n \leq b < 1$ for all positive integers n; and (c) $x_n \in K$ for all positive integers n. If $\{x_n\}_{n=1}^\infty$ is bounded, then $x_n - Tx_n \to 0$ as $n \to \infty$.*

One consequence of Theorem 6.6 is that if K is convex and compact, the sequence $\{x_n\}$ defined by (6.3) converges strongly to a fixed point of T (see Theorem 6.17 below). Another consequence of Theorem 6.6 is that for K convex and T mapping K into a bounded subset of X, the iterates of the map $S_\lambda = (1-\lambda)I + \lambda T$, $\lambda \in (0,1)$ are *asymptotically regular* at x, i.e., $\|S_\lambda^{n+1}x - S_\lambda^n x\| \to 0$ as $n \to \infty$. The concept of asymptotic regularity was introduced by Browder and Petryshyn [51] and, as a metric notion, a mapping $T : M \to M$ is said to be asymptotically regular on M if it is so at each x_0 in M. The relevance of asymptotic regularity to the existence of a fixed point for T can clearly be seen from the following theorem.

Theorem 6.7. *Suppose M is a metric space and $T : M \to M$ is continuous and asymptotically regular at x_0 in M. Then any cluster point of $\{T^n(x_0)\}_{n=1}^\infty$ is a fixed point of T.*

It follows that for *continuous* T, asymptotic regularity of S_λ at any x_0 in K implies $S_\lambda(p) = p$ for any cluster point p of $\{S_\lambda^n(x_0)\}_{n=1}^\infty$. Asymptotic regularity is not only useful in proving that fixed points exist but also in showing that in certain cases, the sequence of iterates at a point converges to a fixed point.

Proposition 6.8. *Let G be a linear mapping of a normed linear space E into itself and suppose G is power bounded (i.e., for some $k \geq 0$, $||G^n|| \leq k, (n = 1, 2, ...)$, and asymptotically regular. If, for some $x_0 \in E, \overline{co}\{G^n(x_0)\}$ contains a fixed point x^* of G, then $\{G^n(x_0)\}$ converges strongly to x^*.*

Proof. Let $\epsilon > 0$ be given and suppose that y is a point of $\overline{co}\{G^n(x_0)\}$ with $||x^* - y|| < \frac{\epsilon}{2(k+1)}$. Setting $y = \sum\limits_{j=1}^{m} \lambda_j G^j(x_0)$ we obtain, using the linearity of G,

$$\begin{aligned} G^n(x_0 - x^*) &= G^n(x_0 - y) + G^n(y - x^*) \\ &= G^n\Big(x_0 - \sum_{j=1}^{m} \lambda_j G^j(x_0)\Big) + G^n(y - x^*) \\ &= \sum_{j=1}^{m} \lambda_j [G^n(x_0) - G^{n+j}(x_0)] + G^n(y - x^*), \end{aligned}$$

since $\sum_{j=1}^m \lambda_j = 1$. Hence, $\Big|\Big|G^n(x_0 - x^*)\Big|\Big| \leq \Big|\Big|\sum_{j=1}^m \lambda_j [G^n(x_0) - G^{n+j}(x_0)]\Big|\Big| + \frac{k\epsilon}{2(k+1)}$ since $||G^n(y - x^*)|| \leq ||G^n||.||y - x^*|| \leq \frac{k\epsilon}{2(k+1)}$. Now, by asymptotic regularity, there exists an integer $N_0 > 0$ such that for all $n \geq N_0$, $||G^n(x_0) - G^{n+j}(x_0)|| \leq \frac{\epsilon}{2}, (j = 1, 2, ..., m)$. Hence $||G^n(x_0 - x^*)|| < \sum_{j=1}^m \lambda_j \left(\frac{\epsilon}{2}\right) + \frac{\epsilon}{2} = \epsilon \ \forall \ n \geq N_0$. This implies $G^n(x_0 - x^*) = G^n(x_0) - x^* \to 0$ as $n \to \infty$, proving Proposition 6.8. □

Remark 6.9. In connection with Theorem 6.7, we note that if E is a normed linear space and K is a subset of E which is only assumed to be weakly compact, then, in general, the sequence $\{S_\lambda^n(x_0)\}$ will not have any strong cluster point as is shown in the following example.

Example 6.10. There is a closed bounded and convex set K in the Hilbert space l_2, a nonexpansive self-map T of K and a point $x_0 \in K$ such that $\{S^n_{\frac{1}{2}}(x_0)\}$ does not converge in the norm topology.

For details, see Genel and Lindenstrauss, [228].

Definition 6.11. A Banach space X is called an *Opial space* (see, e.g. Opial, [366]) if for all sequences $\{x_n\}_{n=0}^{\infty}$ in X such that $\{x_n\}_{n=0}^{\infty}$ converges weakly to some x in X, the inequality

$$\liminf_{n\to\infty} \|x_n - y\| > \liminf_{n\to\infty} \|x_n - x\|$$

holds for all $y \neq x$.

Every Hilbert space is an Opial space (see, e.g., Edelstein and O'Brian, [218], Opial, [367]). In fact, for any normed linear space X, the existence of a weakly sequentially continuous duality map implies X is an Opial space but the converse implication does not hold (see, e.g. Edelstein and O'Brian, [218]). In particular, $\ell_p(1 < p < \infty)$ spaces are Opial spaces but L_p $(1 < p < \infty, p \neq 2)$ spaces are not. Suppose now K is a weakly compact convex subset of a real Opial space X and T is a nonexpansive mapping of K into itself. While example 6.10 shows that we cannot, in general, get strong convergence of the sequence defined by (6.3) to a fixed point of T, Theorem 6.20 (below) allows us to conclude that the sequence converges *weakly* to a fixed point of T *if E is an Opial space.*

6.3 Uniform Asymptotic Regularity

Definition 6.12. Let K be a subset of a real normed linear space X. A mapping $U : K \to X$ is called *uniformly asymptotically regular* if for any $\epsilon > 0$, there exists an integer $N > 0$ such that for any $x_0 \in K$ and for all $n \geq N$, $||U^{n+1}x_0 - U^n x_0|| < \epsilon$.

Definition 6.13. Given a set A and $x_0 \in A$, call a sequence $\{x_n\}_{n=0}^{\infty}$ *admissible* if there is a non-increasing sequence $\{C_n\}_{n=0}^{\infty}$ in $(0, 1)$ such that (6.3) holds.

We now prove the following theorems.

Theorem 6.14. *Let K be a subset of a real normed linear space X and let f be a nonexpansive mapping from K into X. Suppose for $x_0 \in K$ there exists an admissible sequence $\{x_n\}_{n=0}^{\infty} \subseteq K$ which is bounded. Then $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Moreover, if K is a bounded subset of X, then the above limit is uniform.*

Theorem 6.15. *With K, f and X as in Theorem 6.14 and $x_0 \in K$, suppose there exists an admissible sequence $\{x_n\}_{n=0}^{\infty} \subseteq K$ which is bounded and which is such that the non-increasing sequence $\{C_n\}_{n=0}^{\infty}$ also satisfies $0 < a \leq C_n < 1$ for all $n \geq 1$. Then $\lim_{n\to\infty} \|x_n - f(x_n)\| = 0$.*

The above theorems are easy consequences of the following more technical theorem.

Theorem 6.16. *Let K be a subset of a real normed linear space X and let f be a nonexpansive mapping from K into X. Suppose there exists a set $A \subseteq K$ such that for each $x_0 \in A$ there is an admissible sequence $\{x_n\}_{n=0}^{\infty} \subseteq A$, and suppose further that there exists some $\delta > 0$ such that for each positive integer N, and some admissible sequence $\{x_n\}_{n=0}^{\infty} \subseteq A$,*

$$\sup_{k\geq N} ||x_{k+1} - x_k|| > \delta. \tag{6.4}$$

Then, A is unbounded.

We prove Theorems 6.14 and 6.15 from Theorem 6.16.

Proof of Theorem 6.14. Both parts follow immediately from Theorem 6.16; the first by setting $\{x_n\}_{n=0}^{\infty} = A$ in the theorem and the second by setting $K = A$. □

Proof of Theorem 6.15. Since f is nonexpansive we obtain,

$$\begin{aligned} \|x_{n+1} - f(x_{n+1})\| &= \|(1 - C_n)(x_n - f(x_n)) + f(x_n) - f(x_{n+1})\| \\ &\leq (1 - C_n)\|x_n - f(x_n)\| \\ &\quad + \|x_n - ((1 - C_n)x_n + C_n f(x_n))\| \\ &= \|x_n - f(x_n)\|\cdot \end{aligned}$$

Thus the sequence $\{\|x_n - f(x_n)\|\}_{n=0}^{\infty}$ is non-increasing and bounded below, so $\lim_{n\to\infty} \|x_n - f(x_n)\|$ exists. But from $x_{n+1} = (1 - C_n)x_n + C_n f(x_n)$,

$$\lim_{n\to\infty} \|x_n - f(x_n)\| = \lim_{n\to\infty} \frac{1}{C_n}\|x_{n+1} - x_n\| \leq \frac{1}{a}\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

(by Theorem 6.16 since the admissible sequence $\{x_n\}_{n=0}^{\infty}$ is bounded), and this establishes Theorem 6.15. We next give the following proof.

Proof of Theorem 6.16. Assume by way of contradiction that A is bounded and let $\|x_n\| \leq \rho$ for each n. Let M be a fixed positive integer such that $(M - 1)\delta > 2\rho + 1$. Choose N, with $N > \max\{M, [(2\rho - \delta)M/(1 - C_1)^M C_1]\}$ (where here $[\cdot]$ denotes the greatest integer function) such that for some $\delta > 0$

and $x_0 \in A$, the corresponding admissible sequence $\{x_n\}_{n=0}^{\infty}$ in A satisfies $\|x_{N+1} - x_N\| > \delta$. Using the nonexpansiveness of f, we easily obtain the following:

$$\begin{aligned}\|x_{n+1} - x_n\| &= C_n\|(1 - C_{n-1})(x_{n-1} - f(x_{n-1})) + f(x_{n-1}) - f(x_n)\| \\ &\le C_n(1 - C_{n-1})\|x_{n-1} - f(x_{n-1})\| \\ &\quad +\|x_{n-1} - [(1 - C_{n-1})x_{n-1} + C_{n-1}f(x_{n-1})]\| \\ &= \tfrac{C_n}{C_{n-1}}\|x_n - x_{n-1}\| \le \|x_n - x_{n-1}\| ,\end{aligned}$$

the last equality following from (6.3) with n replaced by $(n-1)$ and T replaced by f while the last inequality follows since $\{C_n\}_{n=0}^{\infty}$ is a non-increasing sequence. Hence it follows that $\|x_{i+1}-x_i\| > \delta$ for all $i \le N$, and furthermore we obtain the following:

$$\delta < \|x_{N+1} - x_N\| \le \|x_N - x_{N-1}\| \le \cdots \le \|x_2 - x_1\| \le 2\rho; \tag{6.5}$$

$$\|f(x_{i+1}) - f(x_i)\| \le \|x_{i+1} - x_i\| \text{ for all } i = 0, 1, \ldots, N; \tag{6.6}$$

and $x_{i+1} = (1 - C_i)x_i + C_i f(x_i)$ so that

$$f(x_i) = \frac{x_{i+1}}{C_i} - \left(\frac{1 - C_i}{C_i}\right) x_i, \quad i = 1, 2, \ldots, N \;; \tag{6.7}$$

which implies,

$$\left\|\frac{1}{C_i}\{x_{i+1} - (1 - C_i)x_i\} - \frac{1}{C_{i-1}}\{x_i - (1 - C_{i-1})x_{i-1}\}\right\|$$

$$= \|f(x_i) - f(x_{i-1})\| \le \|x_i - x_{i-1}\|,$$

and this reduces to

$$\left\|\frac{1}{C_i}[x_{i+1} - x_i] - \left(\frac{1 - C_{i-1}}{C_{i-1}}\right)[x_i - x_{i-1}]\right\| \le \|x_i - x_{i-1}\| \tag{6.8}$$

for all $i = 1, 2, \ldots, N$. Now set $I = [(2\rho - \delta)/(1 - C_1)^M C_1]$ and consider the collection of I intervals $[s_k, s_{k+1}]$ where

$$s_k = \begin{cases} \delta + k(1 - C_1)^M C_1, & k = 0, 1, ..., I - 1, \\ 2\rho, & k = I. \end{cases}$$

We claim that some one of these intervals must contain at least M of the numbers $\{||x_i - x_{i+1}||\}_{i=0}^{N-1} \subseteq [\delta, 2\rho]$. If this is not the case, then $N < MI = M\left[\frac{2\rho-\delta}{(1-C_1)^M C_1}\right]$ contradicting our choice of N. Thus for some r, and some $s = s_k \in [\delta, 2\rho]$,

$$\|x_{r+i+1} - x_{r+i}\| \in [s, s + (1 - C_1)^M C_1] \tag{6.9}$$

for $i = 0, 1, \ldots, (M-1)$. Define $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$. Replacing i in (6.8) by $r + M - j - 1$, $(j = 0, 1, \ldots, M-1)$ we see that (6.8) and (6.9) imply

$$\begin{aligned}\left\| \frac{1}{C_{r+M-j-1}} \Delta x_{r+M-j} - \left(\frac{1 - C_{r+M-j-2}}{C_{r+M-j-2}} \right) \Delta x_{r+M-j-1} \right\| \\ \leq s + (1 - C_1)^M C_1.\end{aligned} \tag{6.10}$$

Choose $f^* \in X^*$ (the dual space of X) with $\|f^*\| = 1$ and $f^*(\Delta x_{r+M}) = \|\Delta x_{r+M}\|$. Then using (6.10) we obtain,

$$\begin{aligned}&\left| \frac{1}{C_{r+M-j-1}} f^*(\Delta x_{r+M-j}) - \left(\frac{1 - C_{r+M-j-2}}{C_{r+M-j-2}} \right) f^*(\Delta x_{r+M-j-1}) \right| \\ &\leq \|f^*\| \cdot \left\| \frac{1}{C_{r+M-j-1}} \Delta x_{r+M-j} - \left(\frac{1 - C_{r+M-j-2}}{C_{r+M-j-2}} \right) \Delta x_{r+M-j-1} \right\| \\ &\leq s + (1 - C_1)^M C_1,\end{aligned}$$

which yields

$$\begin{aligned}f^*(\Delta x_{r+M-j-1}) \geq &\left(\frac{C_{r+M-j-2}}{C_{r+M-j-1}} \right) \left(\frac{1}{1 - C_{r+M-j-2}} \right) f^*(\Delta x_{r+M-j}) \\ &- \left(\frac{C_{r+M-j-2}}{1 - C_{r+M-j-2}} \right) (s + (1 - C_1)^M C_1).\end{aligned} \tag{6.11}$$

Observe that since $\{C_i\}_{i=0}^{\infty}$ is non-increasing, for all $i \geq 1$, $(1 - C_i)^{-1} \leq (1 - C_1)^{-1}$ and $C_i(1 - C_i)^{-1} \leq C_1(1 - C_1)^{-1}$. Now for $j = 0$, using $f^*(\Delta x_{r+M}) = \|\Delta x_{r+M}\| \in [s, s + (1 - C_1)^M C_1]$ we obtain from (6.11),

$$\begin{aligned}f^*(\Delta x_{r+M-1}) &\geq \left(\frac{1}{1 - C_{r+M-2}} \right) s - \left(\frac{C_{r+M-2}}{1 - C_{r+M-2}} \right) \{s + (1 - C_1)^M C_1\} \\ &\geq s - C_1^2 (1 - C_1)^{M-1}.\end{aligned} \tag{6.12}$$

We will show that (6.12) implies

$$f^*(\Delta x_{r+M-j-1}) \geq s - (1 - C_1)^{M-1} C_1^2 \sum_{t=0}^{j} \left(\frac{1}{1 - C_1} \right)^t, \tag{6.13}$$

for $j = 1, 2, \ldots, M-1$. We establish this by induction. For $j = 0$, (6.13) reduces to (6.12). Suppose now (6.13) holds for $j \leq k$, for some $k \in \{1, 2, 3, \ldots, M-2\}$. Then from (6.11) and the inductive hypothesis we obtain,

$$
\begin{aligned}
f^*(\Delta x_{r+M-(k+1)-1}) &= f^*(\Delta x_{r+M-k-2}) \\
&\geq \left(\tfrac{C_{r+m-k-3}}{C_{r+M-k-2}}\right)\left(\tfrac{1}{1-C_{r+M-k-3}}\right) f^*\left(\Delta x_{r+M-k-1}\right) \\
&\quad - \left(\tfrac{C_{r+M-k-3}}{1-C_{r+M-k-3}}\right)\{s+(1-C_1)^M C_1\} \\
&\geq \left(\tfrac{1}{1-C_{r+M-k-3}}\right)\left[s-(1-C_1)^{M-1}C_1^2\sum_{t=0}^{k}\left(\tfrac{1}{1-C_1}\right)^t\right] \\
&\quad - \left(\tfrac{C_{r+M-k-3}}{1-C_{r+M-k-3}}\right)\ \{s+(1-C_1)^M C_1\} \\
&\geq s-\left(\tfrac{1}{1-C_1}\right)(1-C_1)^{M-1}C_1^2\sum_{t=0}^{k}\left(\tfrac{1}{1-C_1}\right)^t \\
&\quad - \left(\tfrac{C_1}{1-C_1}\right)(1-C_1)^M C_1 \\
&= s-(1-C_1)^{M-1}C_1^2\sum_{t=0}^{k+1}\left(\tfrac{1}{1-C_1}\right)^t\ ,
\end{aligned}
$$

which completes the induction. Recalling that f^* is linear and summing (6.13), by telescoping, from $j=0$ to $(M-2)$ yields:

$$
\begin{aligned}
f^*(x_{r+M-1}-x_r) &= f^*(x_{r+M-1})-f^*(x_r) \\
&\geq (M-1)s-(1-C_1)^{M-1}C_1^2\Big[1+\Big(1+\frac{1}{1-C_1}\Big) \\
&\quad +\cdots+\Big(1+\frac{1}{1-C_1}+\cdots+\Big(\frac{1}{1-C_1}\Big)^{M-2}\Big)\Big].
\end{aligned}
$$

Set $\lambda = 1-C_1$ so that,

$$
\begin{aligned}
f^*(x_{r+M-1}-x_r) &= (M-1)s-\lambda^{M-1}(1-\lambda)^2\Big[1+\Big(\frac{\lambda+1}{\lambda}\Big) \\
&\quad +\cdots+\Big(\frac{\lambda^{M-2}+\cdots+\lambda+1}{\lambda^{M-2}}\Big)\Big] \\
&= (M-1)s-\lambda(1-\lambda)\Big[\lambda^{M-1}\Big\{\Big(\frac{1-\lambda}{\lambda}\Big)+\Big(\frac{1-\lambda^2}{\lambda^2}\Big) \\
&\quad +\cdots+\Big(\frac{1-\lambda^{M-1}}{\lambda^{M-1}}\Big)\Big\}\Big] \geq (M-1)s-1,
\end{aligned}
$$

the last inequality following since

$$
\begin{aligned}
\lambda(1-\lambda)\left[\lambda^{M-1}\left\{\left(\frac{1-\lambda}{\lambda}\right)+\left(\frac{1-\lambda^2}{\lambda^2}\right)+\cdots+\left(\frac{1-\lambda^{M-1}}{\lambda^{M-1}}\right)\right\}\right] \\
< \lambda(1-\lambda)(\lambda^{M-2}+\cdots+\lambda+1) \leq 1\ .
\end{aligned}
$$

But $s \geq \delta$ implies $(M-1)s \geq (M-1)\delta > 2\rho+1$, so that $f^*(x_{r+M-1}-x_r) > \rho$. Also,

$$f^*(x_{r+M-1} - x_r) \le |f^*(x_{r+M-1} - x_r)| \le \|f^*\| \cdot \|x_{r+M-1} - x_r\|$$
$$= \|x_{r+M-1} - x_r\| \ .$$

Hence, $\|x_{r+M-1} - x_r\| > 2\rho$, contradicting the assumption that $\|x_n\| \le \rho$ for each n, and completing the proof of Theorem 6.16. □

6.4 Strong Convergence

Using the technique of Theorem 6.16 we are able to prove the following theorem.

Theorem 6.17. *With K, X and f as in Theorem 6.16, suppose for some $x_0 \in K$, the corresponding admissible sequence $\{x_n\}_{n=0}^{\infty} \subseteq K$ has a cluster point $q \in K$. Then $f(q) = q$ and $x_n \to q$. In particular, if the range of f is contained in a compact subset of K then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of f.*

Proof. In Edelstein [216], it was shown that q is also a cluster point of $\{f^n(q)\}$ and that $\|f^{n+1}(q) - f^n(q)\| = \|f(q) - q\|$ for all n. Thus if $f^i(q) := x_i$ and $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots$ then $\|\Delta x_{i+1}\| = \|\Delta x_i\|$ for all i. As in the proof of Theorem 6.16, we have, $\|\Delta x_{i+1}\| = \|x_{i+1} - x_i\| \le \frac{C_i}{C_{i-1}}\|x_i - x_{i-1}\| \le \|x_i - x_{i-1}\| = \|\Delta x_i\|$ and this implies $C_i = C_{i-1}$ for all i since $\|\Delta x_{i-1}\| = \|\Delta x_i\|$. Hence, from $x_{i+1} - x_i = (1 - C_i)x_i + C_i f(x_i) - (1 - C_{i-1})x_{i-1} - C_{i-1} f(x_{i-1})$ we obtain,

$$\|\Delta x_{i+1}\| \le (1 - C_i)\|\Delta x_i\| + C_i\|\Delta f(x_i)\|$$
$$\le (1 - C_i)\|\Delta x_i\| + C_i\|\Delta x_i\| = \|\Delta x_i\| \ ,$$

and this implies, $\|\Delta x_i\| = \|\Delta f(x_i)\|$. Assume for contradiction that,

$$\|\Delta x_i\| = \|\Delta f(x_i)\| = \beta > 0, \ i = 1, 2, \ldots. \tag{6.14}$$

Choose N, $K \in \mathbb{N}^+$ sufficiently large. From (6.14) we have, for $i = N + K$,

$$\|\Delta x_{N+K}\| = \|\Delta f(x_{N+K})\| = \beta > 0 \ . \tag{6.15}$$

Let $f^* \in X^*$ such that $\|f^*\| = 1$ and $f^*(\Delta x_{N+K}) = \|\Delta x_{N+K}\|$. Then for $j = 0, 1, 2, \ldots,$

$$f^*(\Delta f(x_{N+K-j})) \le \|f^*\| \cdot \|\Delta f(x_{N+K-j})\| = \|\Delta f(x_{N+K-j})\| = s\cdot \tag{6.16}$$

From $x_{N+K-j+1} = (1 - C_{N+K-j})x_{N+K-j} + C_{N+K-j} f(x_{N+K-j})$ we obtain, using $C_i = C_{i-1}$ for all i,

$$\Delta x_{N+K-j+1} = (1 - C_{N+K-j})\Delta x_{N+K-j} + C_{N+K-j}\Delta f(x_{N+K-j})\cdot \tag{6.17}$$

We will show that applying f^* to this equation yields:

$$f^*(\Delta x_{N+K-j}) \geq \beta \text{ for } j = 0, 1, \ldots. \tag{6.18}$$

We establish (6.18) by induction. Observe that $f^*(\Delta x_{N+K}) = \|\Delta x_{N+K}\| = \beta$ satisfies (6.18) with $j = 0$. Now if $j = 1$, applying f^* to (6.17) and using (6.16) yield:

$$\begin{aligned} f^*(\Delta x_{N+K-1}) &= \left(\frac{1}{1 - C_{N+K-1}}\right) f^*(\Delta x_{N+K}) - \left(\frac{C_{N+K-1}}{1 - C_{N+K-1}}\right) \times \\ &\quad f^*\left(\Delta f(x_{N+K-1})\right) \\ &\geq \left(\frac{1}{1 - C_{N+K-1}}\right) \beta - \left(\frac{C_{N+K-1}}{1 - C_{N+K-1}}\right) \beta = \beta \ , \end{aligned}$$

and (6.18) holds for $j = 1$. Assume it holds for $j = 0, 1, \ldots, t$. Then using (6.16), (6.17), and the inductive hypothesis we have,

$$\begin{aligned} f^*(\Delta x_{N+K-t-1}) &= \left(\tfrac{1}{1-C_{N+K-t-1}}\right) f^*\left(\Delta x_{N+K-t}\right) \\ &\quad - \left(\tfrac{C_{N+K-t-1}}{1-C_{N+K-t-1}}\right) f^*\left(\Delta f(x_{N+K-t-1})\right) \\ &\geq \left(\tfrac{1}{1-C_{N+K-t-1}}\right) \beta - \left(\tfrac{C_{N+K-t-1}}{1-C_{N+K-t-1}}\right) \beta = \beta \ , \end{aligned}$$

which completes the induction. Using the technique of the proof of Theorem 6.16 and summing (6.18) from $j = 0$ to $K - 1$ yields:

$$\|x_{N+K} - x_N\| \geq f^*(x_{N+K} - x_N) \geq K\beta \ , \tag{6.19}$$

and this implies that the sequence $\{x_i\}_{i=0}^{\infty}$ cannot have a convergent subsequence, a contradiction of the fact that $\{x_n\}_{n=0}^{\infty}$ has a cluster point. Hence $\beta = 0$ and $f(q) = q$. That $x_n \to q$ now follows readily from the nonexpansiveness of f. □

For our next result the following definition is needed.

Definition 6.18. (Petryshyn, [381]). Let C be a subset of a real normed linear space X. A mapping $f : C \to X$ is said to be *demicompact* at $h \in X$ if, for any bounded sequence $\{x_n\}_{n=0}^{\infty}$ in C such that $x_n - f(x_n) \to h$ as $n \to \infty$, there exist a subsequence $\{x_{n_j}\}_{j=0}^{\infty}$ and an $x \in C$ such that $x_{n_j} \to x$ as $j \to \infty$ and $x - f(x) = h$.

Corollary 6.19 *Suppose X is a real normed linear space, C is a closed bounded convex subset of X and f is a nonexpansive mapping of C into C. Suppose further that either, (i) f is demicompact at 0, or (ii) $(I - f)$ maps closed bounded subsets of X into closed subsets of X. For $x_0 \in C$ let $\{x_n\}_{n=0}^{\infty} \subseteq C$ be an admissible sequence where the real sequence $\{C_n\}_{n=0}^{\infty}$*

also satisfies $0 < a \leq C_n \leq b < 1$ *for all* $n \geq 1$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a fixed point of* f *in* C.

Proof. (i) From $x_{n+1} = (1 - C_n)x_n + C_n f(x_n)$ we obtain $x_n - f(x_n) = \frac{1}{C_n}\{x_n - x_{n-1}\}$. Since C is bounded, $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence and also $\{C_n\}_{n=1}^{\infty}$ bounded away from 0 implies (by Theorem 6.15) that $\{x_n - f(x_n)\}$ is convergent to 0 so that by the demicompactness of f at 0, $\{x_n\}_{n=0}^{\infty}$ has a cluster point in C. The result follows by Theorem 6.17.
(ii) If q is a fixed point of f, $\{\|x_n - q\|\}_{n=0}^{\infty}$ does not increase with n. It suffices, therefore, to show that there exists a subsequence of $\{x_n\}_{n=0}^{\infty}$ which converges strongly to a fixed point of f. For $x_0 \in C$, let K be the strong closure of the set $\{x_n\}_{n=0}^{\infty}$. By Theorem 6.15, $\{(I - f)(x_n)\}$ converges strongly to 0 as $n \to \infty$. Hence, 0 lies in the strong closure of $(I - f)(K)$ and since the latter is closed by hypothesis (since K is closed and bounded), 0 lies in $(I - f)(K)$. Hence, there is a subsequence $\{x_{n_j}\}_{j=0}^{\infty}$ such that $x_{n_j} \to \mu \in C$, where μ is a point such that $(I - f)\mu = 0$. Hence $x_n \to \mu$. □

6.5 Weak Convergence

Theorem 6.20. *Let* X *be an Opial space and* $f : K \to K$ *be a nonexpansive self-mapping of a weakly compact convex subset* K *of* X. *For any* x_0 *in* K, *let* $\{x_n\}_{n=0}^{\infty} \subseteq K$ *be the corresponding admissible sequence which is such that the non-increasing sequence* $\{C_n\}_{n=1}^{\infty}$ *also satisfies* $0 < a \leq C_n < 1$ *for all* $n \geq 1$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges weakly to a fixed point of* f.

We shall need the following definition.

Definition 6.21. A mapping $T : K \to X$ is called *demiclosed* at y if, for any sequence $\{x_n\}_{n=0}^{\infty} \subseteq K$ which converges weakly to an x in K, the strong convergence of the sequence $\{T(x_n)\}_{n=0}^{\infty}$ to y in K implies $Tx = y$.

The technique of Edelstein and O'Brian [218] together with Theorem 6.15 yields the following proof.

Proof of Theorem 6.20. Since X is an Opial space and f is nonexpansive, $(I - f)$ is demiclosed (see, e.g. Opial, [366]). Furthermore, by Theorem 6.15, f is asymptotically regular. Hence, by a result of Browder and Petryshyn [51], any weak cluster point of $\{x_n\}_{n=0}^{\infty} \subseteq K$ is a fixed point of f. We claim that $\{x_n\}_{n=0}^{\infty} \subseteq K$ has a unique weak cluster point. Suppose there exist two distinct weak cluster points of $\{x_n\}_{n=0}^{\infty}$, say q_1 and q_2, and two subsequences $\{x_{n_i}\}_{i=1}^{\infty}$ and $\{x_{n_j}\}_{j=1}^{\infty}$ such that $\{x_{n_i}\}_{i=1}^{\infty}$ converges weakly to q_1 and $\{x_{n_j}\}_{j=1}^{\infty}$ converges weakly to q_2. Let $p \in F(f)$ where $F(f)$ denotes the fixed point set of f. Then, it is easy to see that $\|x_{n+1} - p\| \leq \|x_n - p\|$ for each $n \geq 0$ so that $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for every $p \in F(f)$. Thus, since X is an Opial space, it follows that

$$\begin{aligned}\lim_{n\to\infty}\|x_n-q_1\| &= \lim_{i\to\infty}\|x_{n_i}-q_1\| < \lim_{i\to\infty}\|x_{n_i}-q_2\| \\ &= \lim_{n\to\infty}\|x_n-q_2\| \\ \lim_{n\to\infty}\|x_n-q_2\| &= \lim_{j\to\infty}\|x_{n_j}-q_2\| < \lim_{j\to\infty}\|x_{n_j}-q_1\| \\ &= \lim_{n\to\infty}\|x_n-q_1\|,\end{aligned}$$

this contradiction implies there exists exactly one weak cluster point q of $\{x_n\}_{n=0}^{\infty} \subseteq K$. By weak compactness of K, $\{x_n\}_{n=0}^{\infty}$ converges weakly to q.
□

Theorem 6.22. *Let K be a closed convex subset of a reflexive Banach space X, and T a continuous mapping of K into X such that (i) $F(T) \neq \emptyset$, where $F(T)$ denotes the fixed point set of T in K, (ii) If $Tp = p$, then $\|Tx - p\| \leq \|x - p\|$ for all x in K, (iii) There exist an x_0 in K and a corresponding admissible sequence $\{x_n\}_{n=0}^{\infty} \subseteq K$, (iv) T is asymptotically regular at x_0, (v) If $\{x_{n_j}\}_{j=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=0}^{\infty}$ such that $\{x_{n_j}\}_{j=1}^{\infty}$ converges weakly to $\tilde{x}$ in K and $\{x_{n_j} - Tx_{n_j}\}$ converges strongly to zero then $\tilde{x} - T\tilde{x} = 0$, (vi) X is an Opial space. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to a fixed point of T in K.*

Proof. Theorem 4.2 of Petryshyn and Williamson [382] implies the sequence $\{x_n\}_{n=0}^{\infty}$ contains a weakly convergent subsequence with its limit in $F(T)$ and, furthermore, that every weakly convergent subsequence of $\{x_n\}_{n=0}^{\infty}$ has a point q in $F(T)$ for its limit. As in the proof of Theorem 6.20, the sequence $\{x_n\}_{n=0}^{\infty}$ has a unique weak cluster point q in K. By weak compactness of K, $\{x_n\}_{n=0}^{\infty}$ converges weakly to q. □

6.6 Some Examples

Let K be a nonempty subset of a real normed linear space X. Recall that a mapping $T : K \to X$ is called *quasi-nonexpansive* provided T has a fixed point in K and that if $Tp = p$, p in K, then $\|Tx - p\| \leq \|x - p\|$ for all x in K. In this section, we shall exhibit large classes of quasi-nonexpansive mappings and these classes, in particular, properly contain the class of nonexpansive mappings with fixed points.

The concept of quasi-nonexpansive mappings was essentially introduced by Diaz and Metcalf [205]. A nonexpansive map $T : K \to K$ with at least one fixed point in K is quasi-nonexpansive. Also, a linear quasi-nonexpansive mapping on a subspace is nonexpansive on that subspace; but there exist continuous and discontinuous nonlinear quasi-nonexpansive mappings which are not nonexpansive (see, e.g., Example 6.23 below). We proved in Section 6.3 that if K is a nonempty closed convex bounded subset of X and T is a *nonexpansive* mapping of K into a bounded subset of X, the iterates of the map $S_\lambda = (1-\lambda)I+\lambda T$, $\lambda \in (0,1)$, are *uniformly asymptotically regular* on K. In this section, we shall show by means of an example that this result does not extend to the class of quasi-nonexpansive maps.

We start with the following example which shows that the class of quasi-nonexpansive mappings properly includes that of nonexpansive maps with fixed points.

Example 6.23. Let $X = \ell_\infty$ and $K := \{x \in \ell_\infty : \|x\|_\infty \leq 1\}$. Define $f : K \to K$ by $f(x) = (0, x_1^2, x_2^2, x_3^2, \dots)$ for $x = (x_1, x_2, x_3, \dots)$ in K. Then it is clear that f is continuous and maps K into K. Moreover $f(p) = p$ if and only if $p = 0$. Furthermore,

$$\begin{aligned}\|f(x) - p\|_\infty &= \|f(x)\|_\infty = \|(0, x_1^2, x_2^2, x_3^2, \dots)\|_\infty \\ &\leq \|(0, x_1, x_2, x_3, \dots)\|_\infty = \|x\|_\infty = \|x - p\|_\infty\end{aligned}$$

for all x in K. Therefore, f is quasi-nonexpansive. However, f is not nonexpansive.

For, if $x = (\frac{3}{4}, \frac{3}{4}, \dots)$ and $y = (\frac{1}{2}, \frac{1}{2}, \dots)$, it is clear that x and y belong to K. Furthermore, $\|x - y\|_\infty = \|(\frac{1}{4}, \frac{1}{4}, \dots)\| = \frac{1}{4}$, and $\|f(x) - f(y)\|_\infty = \|(0, \frac{5}{16}, \frac{5}{16}, \dots)\|_\infty = \frac{5}{16} > \frac{1}{4} = \|x - y\|_\infty$. □

Before we exhibit a large class of quasi-nonexpansive mappings we need the following preliminaries.

Suppose X is a Banach space and K is a bounded closed and convex subset of X. Within the past thirty years or so numerous papers have appeared concerning variants of the following contractive condition for mappings $T : K \to K$ introduced by Kannan [271]:

$$||Tx - Ty|| \leq \frac{1}{2}(||x - Tx|| + ||y - Ty||), x, y \in K \qquad (*)$$

(see e.g., Bianchini [31], Ciric [190], Hardy and Rogers [248], Ray [398], Reich [399]-[402], Rhoades [415]-[420], Shimi [448], Soardi [457]). These mappings are neither stronger nor weaker than the nonexpansive mappings. Nevertheless, it appears that most of the fixed point Theorems for nonexpansive mappings also hold for mappings which are continuous and satisfy $(*)$. A more general class of mappings was introduced in Hardy and Rogers [248] and the following result was proved.

Theorem 6.24. *(Hardy and Rogers, [248]) Let (M, d) be a complete metric space and $T : M \to M$ a continuous mapping satisfying for $x, y \in M$:*

$$\begin{aligned}d(Tx, Ty) \leq\ & a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) \\ & + a_4 d(x, Ty) + a_5 d(y, Tx),\end{aligned} \qquad (6.20)$$

where $a_i \geq 0$, $\sum_{i=1}^{5} a_i < 1$. Then T has a unique fixed point in M.

Condition (6.20), of course, implies T is a strict contraction if $a_i = 0$, $i = 2, \ldots, 5$; it reduces to the condition studied by Reich [399] if $a_4 = a_5 = 0$; and to a condition of Kannan [271] if $a_1 = a_4 = a_5 = 0$, $a_2 = a_3$.

In the case that M is replaced by a uniformly convex Banach space, inequality (6.20) has been weakened to allow $\sum_{i=1}^{5} a_i = 1$.

Theorem 6.25. *(Goebel, Kirk and Shimi, [233]) Let X be a uniformly convex Banach space, K a nonempty bounded closed and convex subset of X, $T : K \to K$ a continuous mapping satisfying for all x, y in K:*

$$\begin{aligned}\|Tx - Ty\| \leq a_1\|x - y\| + a_2\|x - Tx\| + a_3\|y - Ty\| \\ +a_4\|x - Ty\| + a_5\|y - Tx\| \end{aligned} \tag{6.21}$$

where $a_i \geq 0$, $\sum_{i=1}^{5} a_i \leq 1$. Then T has a fixed point in K.

Remark 6.26. Since the four points $\{x, y, Tx, Ty\}$ determine six distances in M, inequality (6.21) amounts to saying that the image distance $d(Tx, Ty)$ never exceeds a fixed convex combination of the remaining five distances. Geometrically, this type of condition is quite natural.

We now have the following proposition.

Proposition 6.27. *Let $T : K \to K$ be a map satisfying inequality (6.21). Then T is quasi-nonexpansive.*

Iterative methods for approximating fixed points of quasi-nonexpansive mappings have been studied by various authors (e.g., Chidume [93, 96, 101, 105], Dotson [211, 212, 213], Hardy and Rogers [248], Johnson [263], Outlaw [377], Outlaw and Groetsch [378], Reich [399]-[402], Rhoades [419], Senter and Dotson [441], Shimi [448]) and a host of other authors.

We now turn to the main example of this section which shows that a quasi-nonexpansive mapping in an arbitrary real Banach space need not be uniformly asymptotically regular.

Example 6.28. Let $X = \ell_\infty$ and $B(0,1) = \{x \in \ell_\infty : \|x\|_\infty \leq 1\}$. The example is the construction of an $f : \ell_\infty \to \ell_\infty$ such that, (i) f is continuous; (ii) $f : \ell_\infty \to B(0,1)$; (iii) $f(p) = p$ if and only if $p = 0$; (iv) $\|f(x) - p\|_\infty \leq \|x - p\|_\infty$ for all $x \in \ell_\infty$ and the fixed point p; (v) for all $n \in \mathbb{N}^+$, there exists $x \in B(0,1)$ such that

$$\|S_\lambda^{n+1}x - S_\lambda^n x\| > \lambda^2(1-\lambda)^2 ,$$

for arbitrary $\lambda \in (0,1)$ where $S_\lambda^n(x) = S_\lambda(S_\lambda^{n-1}x)$ and $S_\lambda x = \lambda x + (1-\lambda)f(x)$.

Define $f : \ell_\infty \to B(0,1) \subset \ell_\infty$ by

$$f(x) = \begin{cases} (0, x_1^2, x_2^2, x_3^2, \ldots, \ldots); & \text{if } \|x\|_\infty \leq 1 \\ \|x\|_\infty^{-2}(0, x_1^2, x_2^2, x_3^2, \ldots); & \text{if } \|x\|_\infty > 1 \ , \end{cases}$$

where $x = (x_1, x_2, x_3, \dots) \in \ell_\infty$. Then it is clear that f satisfies $(i) - (iii)$ above. For (iv) we have,

$$\|f(x) - p\|_\infty = \|f(x)\|_\infty \leq \begin{cases} \|x\|_\infty^2 \leq \|x\|_\infty; & \text{if } \|x\|_\infty \leq 1 \\ 1; & \text{if } \|x\|_\infty > 1 \end{cases},$$

i.e., $\|f(x) - p\|_\infty \leq \|x - p\|_\infty$ and (iv) is satisfied. For (v), we examine f and S_λ more closely. Now,

$$S_\lambda(x) = \{\lambda x_1, \lambda x_2 + (1-\lambda)x_1^2, \lambda x_3 + (1-\lambda)x_2^2, \dots\}.$$

Thus, if $x = (x_1, x_2, x_3, \dots)$ and $x_j = a$, where a is a constant for $j \geq k$, then

$$(S_\lambda(x))_j = \lambda a + (1-\lambda)a^2 \text{ for } j \geq k+1. \tag{6.22}$$

More generally, by induction, if $x_j = a$ for $j \geq k$ then $(S_\lambda^n(x))_j = a_n$, a constant for $j \geq k+n$, where the $\{a_n\}$ satisfies the recurrence relation,

$$a_n = \lambda a_{n-1} + (1-\lambda)a_{n-1}^2, \; n \geq 1, \; a_0 = a. \tag{6.23}$$

Suppose we have chosen $x = (a, a, a, \dots)$. Then

$$\begin{aligned} \|S_\lambda^{n+1}x - S_\lambda^n x\|_\infty &= \|\lambda(S_\lambda^n(x)) + (1-\lambda)f(S_\lambda^n(x)) - S_\lambda^n(x)\|_\infty \\ &= (1-\lambda)\|S_\lambda^n(x) - f(S_\lambda^n(x))\|_\infty \\ &= (1-\lambda)\sup_{j\geq 1}|(S_\lambda^n(x))_{j-1} - (S_\lambda^n(x))_{j-1}^2| \\ &\geq (1-\lambda)\sup_{j>n}|(S_\lambda^n(x))_{j-1} - (S_\lambda^n(x))_{j-1}^2| \\ &\geq (1-\lambda)\lambda|a_n - a_{(n-1)}^2|. \end{aligned}$$

If we could choose $0 < a < 1$ such that $a_n = \lambda$ then $x \in B(0,1)$ and condition (v) would be satisfied, completing the example. To do this, we note first that from (6.23), if $a < 1$, then $a_k < a$, inductively, and so

$$a_k = \lambda a_{k-1} + (1-\lambda)a_{k-1}^2 < \lambda a_{k-1} + (1-\lambda)a_{k-1} = a_{k-1}\;.$$

Thus, $1 > a > a_1 > a_2 > \cdots > a_k > \dots$. Also, if we know a_k then we can find a_{k-1} from (6.23) (i.e., solving) as

$$a_{k-1} = 2^{-1}(1-\lambda)^{-1}\{-\lambda + [\lambda^2 + 4(1-\lambda)a_k]^{\frac{1}{2}}\}. \tag{6.24}$$

Note that if $a_k < 1$, then $a_{k-1} < 1$ for, from (6.24), using $a_k < 1$ we have,

$$\begin{aligned} a_{k-1} &< 2^{-1}(1-\lambda)^{-1}\{-\lambda + [\lambda^2 + 4(1-\lambda)]^{\frac{1}{2}}\} \\ &= 2^{-1}(1-\lambda)^{-1}\{-\lambda + (2-\lambda)\} = 1\;. \end{aligned}$$

We can now show that (v) is satisfied. Choose $n \in N^+$, and put $a_n = \lambda$. Use (6.24) to compute $a_{n-1}, a_{n-2}, \ldots, a_1, a_0$; $a_0 < 1$. Then starting with $x = (a_0, a_0, a_0, \ldots)$ we have $x \in B(0,1)$, $a_n = \lambda$ and so, $\|S_\lambda^{n+1}x - S_\lambda^n x\| > \lambda^2(1-\lambda)^2$, as required. □

6.7 Halpern-type Iteration Method

Let E be a real Banach space, K a closed convex subset of E and $T : K \to K$ a nonexpansive mapping. For fixed $t \in (0,1)$ and arbitrary $u \in K$, let $z_t \in K$ denote the unique fixed point of T_t defined by $T_t x := tu + (1-t)Tx, x \in K$. Assume $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Browder [44] proved that if $E = H$, a Hilbert space, then $\lim_{t \to 0} z_t$ exists and is a fixed point of T. Reich [412] extended this result to uniformly smooth Banach spaces. Kirk [284] obtained the same result in arbitrary Banach spaces under the additional assumption that T has pre-compact range.

For a sequence $\{\alpha_n\}$ in $[0,1]$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} := \alpha_n u + (1-\alpha_n)Tx_n, n \geq 0. \tag{6.25}$$

Concerning this process, Reich [412] posed the following question.

Question. *Let E be a Banach space. Is there a sequence $\{\alpha_n\}$ such that whenever a weakly compact convex subset K of E has the fixed point property for nonexpansive mappings, then the sequence $\{x_n\}$ defined by (6.25) converges to a fixed point of T for arbitrary fixed $u \in K$ and all nonexpansive $T : K \to K$?*

Halpern [245] was the first to study the convergence of the algorithm (6.25) in the framework of Hilbert spaces. He proved the following Theorem.

Theorem H (Halpern, [245]) *Let K be a bounded closed convex subset of a Hilbert space H and $T : K \to K$ be a nonexpansive mapping. Let $u \in K$ be arbitrary. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}$, $\theta \in (0,1)$. Define a sequence $\{x_n\}$ in K by $x_1 \in K$, $x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n$, $n \geq 1$. Then, $\{x_n\}$ converges strongly to the element of $F(T) := \{x \in K : Tx = x\}$ nearest to u.*

An iteration method with recursion formula of the form (6.25) is now referred to as a *Halpern-type iteration method.*

Lions [313] improved Theorem H, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T if the real sequence $\{\alpha_n\}$

satisfies the following conditions: (i) $\lim\limits_{n\to\infty}\alpha_n = 0$; (ii) $\sum\limits_{n=1}^{\infty}\alpha_n = \infty$; and (iii) $\lim\limits_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n^2} = 0$.

Reich [412] gave an affirmative answer to the above question in the case when E is uniformly smooth and $\alpha_n = n^{-a}$ with $0 < a < 1$. It was observed that both Halpern's and Lions' conditions on the real sequence $\{\alpha_n\}$ excluded the natural choice, $\alpha_n := (n+1)^{-1}$. This was overcome by Wittmann [505] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies the following conditions:

$$(i)\lim_{n\to\infty}\alpha_n = 0;\ (ii)\sum_{n=1}^{\infty}\alpha_n = \infty;\ \text{and ; } (iii)\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty. \quad (6.26)$$

Reich [413] extended this result of Wittmann to the class of Banach spaces which are uniformly smooth and have weakly sequentially continuous duality maps (e.g., $l_p(1 < p < \infty)$), where the sequence $\{\alpha_n\}$ is required to satisfy conditions (i) and (ii) of (6.26) and to be decreasing (and hence also satisfies (iii) of (6.26)). Shioji and Takahashi [450] extended Wittmann's result to Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex bounded subset of K has the fixed point property for nonexpansive mappings (e.g., L_p spaces $(1 < p < \infty)$). They proved the following theorem.

Theorem ST. *Let E be a Banach space whose norm is uniformly Gâteaux differentiable and let K be a closed convex subset of E. Let T be a nonexpansive mapping from K into K such that the set $F(T)$ of fixed points of T is nonempty. Let $\{\alpha_n\}$ be a sequence which satisfies the following conditions: $0 \le \alpha_n \le 1,\ \lim\alpha_n = 0,\ \sum\alpha_n = \infty,\ \sum|\alpha_{n+1}-\alpha_n| < \infty$. Let $u \in K$ and let $\{x_n\}$ be the sequence defined by $x_0 \in K,\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, n \ge 0$. Assume that $\{z_t\}$ converges strongly to $z \in F(T)$ as $t \downarrow 0$, where for $0 < t < 1, z_t$ is the unique element of K which satisfies $z_t = tu + (1-t)Tz_t$. Then, $\{x_n\}$ converges strongly to z.*

A result of Reich [409] and that of Takahashi and Ueda [478] show that if K satisfies some additional assumption, then $\{z_t\}$ defined above converges strongly to a fixed point of T. In particular, the following is true.

Let E be a Banach space whose norm is uniformly Gâteaux differentiable and let K be a weakly compact convex subset of E. Let T be a nonexpansive mapping from K into K. Let $u \in K$ and let z_t be the unique element of K which satisfies $z_t = tu + (1-t)Tz_t$ for $0 < t < 1$. Assume that each nonempty $T-$invariant closed convex subset of K contains a fixed point of T. Then, $\{z_t\}$ converges strongly to a fixed point of T.

Morales and Jung [341] established the following result.

Theorem MJ (Morales and Jung, [341]) *Let K be a nonempty closed convex subset of a reflexive Banach space E which has uniformly Gâteaux differentiable norm and $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive mappings. Then there exists a continuous path $t \to z_t$, $0 < t < 1$ satisfying $z_t = tu + (1-t)Tz_t$, for arbitrary but fixed $u \in K$, which converges strongly to a fixed point of T.*

Xu [511] (see, also [510]) showed that the result of Halpern holds in uniformly smooth Banach spaces if condition (iii) of Lions is replaced with the condition $(iii)^*$ $\lim\limits_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$. He proved the following theorem.

Theorem 6.29. *(Xu, [511]) Let E be a uniformly smooth Banach space, K be a closed convex nonempty subset of E, $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $u, x_0 \in K$ be given and let $\{\alpha_n\} \subset [0,1]$ satisfy the conditions: (a) $\lim \alpha_n = 0$; (b) $\sum \alpha_n = \infty$; and (c) $\lim \frac{|\alpha_{n-1} - \alpha_n|}{\alpha_n} = 0$. Then the sequence $\{x_n\}$ generated by $x_0 \in K,\ x_{n+1} := (1-\alpha_n)Tx_n + \alpha_n u,\ n \geq 0$, converges strongly to some $x^* \in F(T)$.*

Remark 6.30. Wittman [505] had earlier proved Theorem 6.29 with condition (c) replaced by: $(c)^*$ $\sum |\alpha_{n+1} - \alpha_n| < \infty$. The conditions (c) and $(c)^*$ are not comparable. For instance, the sequence $\{\alpha_n\}$ defined by

$$\alpha_n := \begin{cases} n^{-\frac{1}{2}}, & \text{if } n \text{ is odd,} \\ (n^{-\frac{1}{2}} - 1)^{-1}, & \text{if } n \text{ is even,} \end{cases}$$

satisfies (c) but fails to satisfy $(c)^*$.

Remark 6.31. Halpern showed that the conditions (i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and (ii) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$ are necessary for the convergence of the sequence $\{x_n\}$ defined by (6.25). It is not known if generally they are sufficient. Some authors have established that if in the recursion formula (6.25), Tx_n is replaced with $T_n x_n := \left(\frac{1}{n}\right) \sum\limits_{k=0}^{n-1} T^k x_n$, then conditions (i) and (ii) are sufficient.

In order to prove the main theorems of this section, we shall make use of the following lemmas.

Lemma 6.32. *(Tan and Xu, [487]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq a_n + \sigma_n,\ n \geq 0,$$

such that $\sum\limits_{n=1}^{\infty} \sigma_n < \infty$. Then, $\lim a_n$ exists. If, in addition, $\{a_n\}$ has a subsequence that converges to 0, then a_n converges to 0 as $n \to \infty$.

Proof. From $0 \le a_{n+1} \le a_n + \sigma_n,\ n \ge 0$, we obtain that $0 \le a_{n+1} \le a_1 + \sum_1^n \sigma_n \le a_1 + \sum_1^\infty \sigma_n < \infty,\ n \ge 0$, and so $\{a_n\}$ is bounded. Furthermore, for fixed $m \in \mathbb{N}$, we have

$$\begin{aligned} 0 \le a_{n+m} &\le a_{n+m-1} + \sigma_{n+m-1} \\ &\le a_{n+m-2} + \sigma_{n+m-2} + \sigma_{n+m-1} \\ &\vdots \\ &\le \alpha_n + \sum_{i=n}^{n+m-1} \sigma_i. \end{aligned}$$

Taking "lim sup" as $m \to \infty$, we obtain that $\limsup_{m\to\infty} a_n \le a_n + \sum_{i=n}^\infty \sigma_i$. Now, taking "lim inf" as $n \to \infty$, we get $\limsup_{n\to\infty} a_n \le \liminf a_n$. Thus, $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$, and the limit exists. If, in addition, $\{a_n\}$ has a subsequence that converges to 0, since the limit of $\{a_n\}$ exists, then $\{a_n\}$ converges to 0 as $n \to \infty$. □

Aliter. We give another proof of the first part of Lemma 6.32.

Define $\rho_n := \sum_{k=1}^{n-1} \sigma_k$. Recall that $\lim_{n\to\infty} \rho_n := \sum_{k=1}^\infty \sigma_k < \infty$. Let $\lim_{n\to\infty} \rho_n = L < \infty$, for some $L \in \mathbb{R}$. Now, $a_{n+1} + \rho_n \le a_n + \sigma_n + \rho_n = a_n + \rho_{n+1}$ so that $a_{n+1} - \rho_{n+1} \le a_n - \rho_n \ \forall\ n \ge 1$. This implies that $\{a_n - \rho_n\}$ is non-increasing. Two cases arise.

Case 1: $\lim_{n\to\infty}(a_n - \rho_n) = -\infty$, or, **Case 2**: $\lim_{n\to\infty}(a_n - \rho_n) = M$, for some $M \in \mathbb{R}$. We show Case 1 is impossible. Suppose, for contradiction, case 1 holds. Then, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\Big(a_n - \rho_n + \rho_n\Big) = -\infty + L = -\infty$, contradicting the hypothesis that $a_n \ge 0$. So, case 2 holds. Hence, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\Big(a_n - \rho_n + \rho_n\Big) = M + L$, completing the proof.

Lemma 6.33. *(Suzuki, [463]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \le 0$. Then, $\lim_{n\to\infty} ||y_n - x_n|| = 0$.*

Lemma 6.34. *(Xu, [511]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \ge 0,$$

where,

(i) $\{\alpha_n\} \subset [0,1], \ \sum \alpha_n = \infty;$
(ii) $\limsup_{n\to\infty} \sigma_n \le 0;$
(iii) $\gamma_n \ge 0; \ (n \ge 0), \ \sum \gamma_n < \infty$. *Then,* $a_n \to 0$ *as* $n \to \infty$.

6.7.1 Convergence Theorems

In the sequel, $F(T) := \{x \in K : Tx = x\}$. In the next theorem, we shall assume that $\{z_t\}$ converges strongly to a fixed point z of T as $t \to 0$, where z_t is the unique element of K which satisfies $z_t = tu + (1-t)Tz_t$ for arbitrary $u \in K$.

Theorem 6.35. *Let K be a nonempty closed convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm and $T : K \to K$ be a nonexpansive mapping with $F(T) \ne \emptyset$. For a fixed $\delta \in (0,1)$, define $S : K \to K$ by $Sx := (1-\delta)x + \delta Tx \ \forall \ x \in K$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ which satisfies the following conditions: $C1 : \lim \alpha_n = 0$; $C2 : \sum \alpha_n = \infty$. For arbitrary $u, x_0 \in K$, let the sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Sx_n, n \ge 0. \tag{6.27}$$

Then, $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Observe first that S is nonexpansive and has the same set of fixed points as T. Define

$$\beta_n := (1-\delta)\alpha_n + \delta \ \forall \ n \ge 0; \ y_n := \frac{x_{n+1} - x_n + \beta_n x_n}{\beta_n}, \ n \ge 0. \tag{6.28}$$

Observe that $\beta_n \to \delta$ as $n \to \infty$, and that if $\{x_n\}$ is bounded, then $\{y_n\}$ is bounded. Let $x^* \in F(T) = F(S)$. One easily shows by induction that $||x_n - x^*|| \le \max\{||x_0 - x^*||, ||u - x^*||\}$ for all integers $n \ge 0$, and so, $\{x_n\}, \{y_n\}$, $\{Tx_n\}$ and $\{Sx_n\}$ are all bounded. Also,

$$||x_{n+1} - Sx_n|| = \alpha_n ||u - Sx_n|| \to 0, \tag{6.29}$$

as $n \to \infty$. Observe also that from the definitions of β_n and S, we obtain that $y_n = \frac{1}{\beta_n}(\alpha_n u + (1-\alpha_n)\delta Tx_n)$ so that

$$\begin{aligned} ||y_{n+1} - y_n|| - ||x_{n+1} - x_n|| &\le \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right|.||u|| \\ &+ \frac{(1-\alpha_{n+1})}{\beta_{n+1}} \ \delta \ ||Tx_{n+1} - Tx_n|| \\ &+ \left|\frac{1-\alpha_{n+1}}{\beta_{n+1}} - \frac{1-\alpha_n}{\beta_n}\right| \ \delta \ ||Tx_n|| - ||x_{n+1} - x_n||, \end{aligned}$$

so that, since $\{x_n\}$ and $\{Tx_n\}$ are bounded, we obtain that, for some constants $M_1 > 0$, and $M_2 > 0$,

$$\begin{aligned}\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \le \limsup_{n\to\infty}\Big\{\Big|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\Big|.||u|| \\ + \Big|\frac{(1-\alpha_{n+1})}{\beta_{n+1}}\delta - 1\Big|\ M_1 \\ + \Big|\frac{1-\alpha_{n+1}}{\beta_{n+1}} - \frac{1-\alpha_n}{\beta_n}\Big|\delta M_2\Big\} \le 0.\end{aligned}$$

Hence, by Lemma 6.33, $||y_n - x_n|| \to 0$ as $n \to \infty$. Consequently, $\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} \beta_n ||y_n - x_n|| = 0$. Combining this with (6.29) yields that

$$||x_n - Sx_n|| \to 0 \ as \ n \to \infty. \tag{6.30}$$

We now show that

$$\limsup_{n\to\infty}\langle u - z, j(x_n - z)\rangle \le 0. \tag{6.31}$$

For each integer $n \ge 1$, let $t_n \in (0,1)$ be such that

$$t_n \to 0, \text{ and } \frac{||x_n - Sx_n||}{t_n} \to 0, \ n \to \infty. \tag{6.32}$$

Let $z_{t_n} \in K$ be the unique fixed point of the contraction mapping S_{t_n} given by $S_{t_n}x := t_n u + (1-t_n)Sx, \ x \in K$. Then, $z_{t_n} - x_n = t_n(u - x_n) + (1 - t_n)(Sz_{t_n} - x_n)$. Using inequality (4.4), we compute as follows:

$$\begin{aligned}||z_{t_n} - x_n||^2 &\le (1-t_n)^2||Sz_{t_n} - x_n||^2 + 2t_n\langle u - x_n, j(z_{t_n} - x_n)\rangle \\ &\le (1-t_n)^2(||Sz_{t_n} - Sx_n|| + ||Sx_n - x_n||)^2 + 2t_n(||z_{t_n} - x_n||^2 \\ &+ \langle u - z_{t_n}, j(z_{t_n} - x_n)\rangle) \\ &\le (1+t_n^2)||z_{t_n} - x_n||^2 + ||Sx_n - x_n|| \times \\ &\quad (2||z_{t_n} - x_n|| + ||Sx_n - x_n||) \\ &+ 2t_n\langle u - z_{t_n}, j(z_{t_n} - x_n)\rangle,\end{aligned}$$

and hence,

$$\begin{aligned}\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \le \frac{t_n}{2}||z_{t_n} - x_n||^2 + \frac{||Sx_n - x_n||}{2t_n} \\ \times (2||z_{t_n} - x_n|| + ||Sx_n - x_n||).\end{aligned}$$

Since $\{x_n\}, \{z_{t_n}\}$ and $\{Sx_n\}$ are bounded and $\frac{||Sx_n - x_n||}{2t_n} \to 0, n \to \infty$, it follows from the last inequality that

$$\limsup_{n\to\infty}\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \le 0. \tag{6.33}$$

Moreover, we have that

$$\begin{aligned}\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle &= \langle u - z, j(x_n - z)\rangle \\ &\quad + \langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle \\ &\quad + \langle z - z_{t_n}, j(x_n - z_{t_n})\rangle. \qquad (6.34)\end{aligned}$$

But, by hypothesis, $z_{t_n} \to z \in F(S)$, $n \to \infty$. Thus, using the boundedness of $\{x_n\}$ we obtain that $\langle z - z_{t_n}, j(x_n - z_{t_n})\rangle \to 0$, $n \to \infty$. Also, $\langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle \to 0$, $n \to \infty$, since j is norm-to-weak* uniformly continuous on bounded subsets of E. Hence, we obtain from (6.33) and (6.34) that $\limsup_{n\to\infty}\langle u - z, j(x_n - z\rangle \le 0$. Furthermore, from (6.27) we get that $x_{n+1} - z = \alpha_n(u - z) + (1 - \alpha_n)(Sx_n - z)$. It then follows that

$$\begin{aligned}||x_{n+1} - z||^2 &\le (1 - \alpha_n)^2||Sx_n - z||^2 + 2\alpha_n\langle u - z, j(x_{n+1} - z)\rangle \\ &\le (1 - \alpha_n)||x_n - z||^2 + \alpha_n\sigma_n,\end{aligned}$$

where $\sigma_n := 2\langle u - z, j(x_{n+1} - z)\rangle$; $\gamma_n \equiv 0\ \forall\ n \ge 0$. Thus, by Lemma 6.34, $\{x_n\}$ converges strongly to a fixed point of T. □

Remark 6.36. We note that every uniformly smooth Banach space has a uniformly Gâteaux differentiable norm and is such that every nonempty closed convex and bounded subset of E has the fixed point property for nonexpansive maps (see e.g., [189]).

Let $S_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} S^k x$, where $S : K \to K$ is a nonexpansive map. With this definition, Xu proved the following theorem.

Theorem HKX (Xu, [511], Theorem 3.2) *Assume that E is a real uniformly convex and uniformly smooth Banach space. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm:*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)S_n x_n, n \ge 0. \qquad (6.35)$$

Assume that (i) $\lim \alpha_n = 0$; (i) $\sum \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a fixed point of S.

Remark 6.37. Theorem 6.35 is a significant improvement of Theorem HKX in the sense that the recursion formula (6.27) is simpler and requires less computation at each stage than the recursion formula (6.35). Moreover, the requirement that E be also uniformly convex imposed in Theorem HKX is dispensed with in Theorem 6.35. Furthermore, Theorem 6.35 is proved in the framework of the more general real Banach spaces with uniformly Gâteaux differentiable norms.

6.7.2 The Case of Non-self Mappings

Definition 6.38. Let K be a nonempty subset of a Banach space E. For $x \in K$, the *inward set* of x, $I_K x$, is defined by $I_K x := \{x + \alpha(u - x) : u \in K, \alpha \geq 1\}$. A mapping $T : K \to E$ is called *weakly inward* if $Tx \in cl[I_K(x)]$ for all $x \in K$, where $cl[I_K(x)]$ denotes the closure of the inward set. Every self-map is trivially weakly inward.

Definition 6.39. Let $K \subseteq E$ be closed convex and Q be a mapping of E onto K. Then Q is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Qz = z$ for every $z \in R(Q)$, *range of* Q. A subset K of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto K and it is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto K. *If $E = H$, the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset of H.*

Remark 6.40. We note that, if $T : K \to E$ is weakly inward, then $F(T) = F(QT)$, where Q is a sunny nonexpansive retraction of E onto K. In fact, clearly, $F(T) \subseteq F(QT)$. We show $F(QT) \subseteq F(T)$. Suppose this is not the case. Then there exists $x \in F(QT)$ such that $x \notin F(T)$. But, since T is weakly inward there exists $u \in K$, such that $Tx = x + \lambda(u - x)$ for some $\lambda > 1$ and $u \neq x$. Observe that if $u = x$ then $Tx = x$, a contradiction, since $x \notin F(T)$. As Q is sunny nonexpansive, we have $Q\big[QTx + t(Tx - QTx)\big] = x$ for all $t \geq 0$. But $QTx = x$ so that $Q\big[tTx + (1-t)x\big] = x$ for all $t \geq 0$. Since T is weakly inward, there exists $t_0 \in (0,1)$ such that $u := t_0 Tx + (1 - t_0)x$, and since $u \in K$, $Qu = u$. This implies $u = Qu = x$, a contradiction, since $x \neq u$. Therefore, $F(QT) \subseteq F(T)$, which implies that $F(QT) = F(T)$.

We now prove the following convergence theorem.

Theorem 6.41. *(Chidume et al., [184]) Let K be a nonempty closed convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm, and $T : K \to E$ be a nonexpansive mapping satisfying weakly inward condition with $F(T) \neq \emptyset$. Assume K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Assume that $\{z_t\}$ converges strongly to a fixed point z of QT as $t \to 0$, where for $0 < t < 1$, z_t is the unique element of K which satisfies $z_t = tx + (1-t)QTz_t$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ which satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and either (iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, or (iii)* $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$. For fixed $u, x_0 \in K$, let the sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) QTx_n, n \geq 0. \tag{6.36}$$

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.

Proof. Let $x^* \in F(T)$. One easily shows by induction that $||x_n - x^*|| \leq max\{||x_0 - x^*||, ||u - x^*||\}$ for all integers $n \geq 0$, and hence $\{x_n\}$ and $\{QTx_n\}$ are bounded. But this implies from (6.36) that

$$||x_{n+1} - QTx_n|| = \alpha_n ||u - QTx_n|| \to 0 \text{ as } n \to \infty. \qquad (6.37)$$

Furthermore, for some constant $M > 0$,

$$\begin{aligned} ||x_{n+1} - x_n|| &= ||(\alpha_n - \alpha_{n-1})(u - QTx_{n-1}) + (1 - \alpha_n)(QTx_n - QTx_{n-1})|| \\ &\leq M|\alpha_{n-1} - \alpha_n| + (1 - \alpha_n)||x_n - x_{n-1}||. \end{aligned}$$

We consider two cases.

Case 1. Condition $(iii)^*$ is satisfied. Then, $||x_{n+1} - x_n|| \leq (1 - \alpha_n)||x_n - x_{n-1}|| + \sigma_n$, where $\sigma_n := \alpha_n \beta_n$ and $\beta_n := (|\alpha_n - \alpha_{n-1}|M/\alpha_n)$ so that $\sigma_n = o(\alpha_n)$.

Case 2. Condition (iii) is satisfied. Then, $||x_{n+1} - x_n|| \leq (1 - \alpha_n)||x_n - x_{n-1}|| + \sigma_n$, where $\sigma_n := M|\alpha_n - \alpha_{n-1}|$ so that $\sum_{n=1}^{\infty} \sigma_n < \infty$. In either case, a lemma of [511] (see, Exercises 6.1, Problem 8) yields that $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$. Combining this with (6.36), we obtain that $||x_n - QTx_n|| \to 0$ as $n \to \infty$. For each integer $n \geq 0$, let $t_n \in (0, 1)$ be such that $t_n \to 0$ and $\frac{||x_n - QTx_n||}{t_n} \to 0$. Let $z_{t_n} \in K$ be the unique fixed point of the contraction mapping T_{t_n} given by $T_{t_n}x := t_n u + (1 - t_n)QTx, x \in K$. Then, $z_{t_n} - x_n = t_n(u - x_n.) + (1 - t_n)(QTz_{t_n} - x_n)$. Moreover, using inequality (4.4), we have

$$\begin{aligned} &||z_{t_n} - x_n||^2 \\ &\leq (1 - t_n)^2 ||QTz_{t_n} - x_n||^2 + 2t_n \langle u - x_n, j(z_{t_n} - x_n)\rangle \\ &\leq (1 - t_n)^2 (||QTz_{t_n} - QTx_n|| + ||QTx_n - x_n||)^2 + 2t_n(||z_{t_n} - x_n||^2 \\ &\quad + \langle u - z_{t_n}, j(z_{t_n} - x_n)\rangle) \\ &\leq (1 + t_n^2)||z_{t_n} - x_n||^2 + ||QTx_n - x_n||(2||z_{t_n} - x_n|| + ||QTx_n - x_n||) \\ &+ 2t_n \langle u - z_{t_n}, j(z_{t_n} - x_n)\rangle, \end{aligned}$$

and hence,

$$\begin{aligned} \langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leq \frac{t_n}{2}||z_{t_n} - x_n||^2 + \frac{||QTx_n - x_n||}{2t_n} \\ \times (2||z_{t_n} - x_n|| + ||QTx_n - x_n||). \end{aligned}$$

Since $\{x_n\}$, $\{z_{t_n}\}$ and $\{Tx_n\}$ are bounded and $\frac{||x_n - QTx_n||}{t_n} \to 0$ as $n \to \infty$, it follows from the last inequality that

$$\limsup_{n \to \infty} \langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leq 0. \qquad (6.38)$$

Moreover, we have that

$$\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle = \langle u - z, j(x_n - z)\rangle + \langle u - z, j(x_n - z_{t_n}) \\ -j(x_n - z)\rangle + \langle z - z_{t_n}, j(x_n - z_{t_n})\rangle. \quad (6.39)$$

But, by hypothesis, $z_{t_n} \to z \in F(QT)$ as $n \to 0$ and by Remark 6.40 we have that $QTz = z = Tz$. Thus, $\langle z - z_{t_n}, j(x_n - z_{t_n})\rangle \to 0$ as $n \to \infty$ (since $\{x_n\}$ is bounded). Also, $\langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle \to 0$ as $n \to \infty$ (since j is norm-to-w^* uniformly continuous on bounded subsets of E). Therefore, we obtain from (6.38) and (6.39) that $\limsup_{n\to\infty}\langle u - z, j(x_n - z)\rangle \leq 0$. Now from (6.36), we get $x_{n+1} - z = \alpha_n(u - z) + (1 - \alpha_n)(QTx_n - z)$. It follows that

$$||x_{n+1} - z||^2 \leq (1 - \alpha_n)^2||QTx_n - z||^2 + 2\alpha_n\langle u - z, j(x_{n+1} - z)\rangle \\ \leq (1 - \alpha_n)||x_n - z||^2 + \sigma_n,$$

where $\sigma_n := 2\alpha_n\langle u-z, j(x_{n+1}-z)\rangle$ so that $\limsup_{n\to\infty}\sigma_n \leq 0$. Thus, (see Exercise 6.1, Problem 8), $\{x_n\}$ converges strongly to a fixed point z of T. □

Remark 6.42. In [453], Shioji and Takahashi proved that if E is *a uniformly convex Banach space* whose norm is uniformly Gâteaux differentiable, and T is self-map of $K \subseteq E$ with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$, and T satisfies the following condition: $||T^nx - T^ny|| \leq k_n||x - y||\ \forall\ x, y \in K, n \in \mathbb{N}$ for some sequence $\{k_n\}, k_n \geq 1,\ \lim k_n = 1$, then the sequence $\{x_n\}$ defined iteratively by $u, x_0 \in K$ arbitrary,

$$x_n = \alpha_n u + (1 - \alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, n \geq N_0,$$

for N_0 sufficiently large, converges strongly to Pu, where P is the *sunny nonexpansive retract* from K onto $F(T)$ and $\{\alpha_n\}$ satisfies the following conditions: $0 \leq \alpha_n \leq 1,\ \lim \alpha_n = 0, \sum \alpha_n = \infty$ and

$$\sum \left(\left(1 - \alpha_n\right)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)^2 - 1\right) < \infty.$$

The two authors [450] had earlier established the same result in Hilbert space for the iterative scheme, $x_0, u \in K$ arbitrary,

$$y_n = \beta_n u + \frac{(1 - \beta_n)}{n+1}\sum_{j=0}^{n} T^j x_n;\ x_{n+1} = \alpha_n u + \frac{(1 - \alpha_n)}{n+1}\sum_{j=0}^{n} T^j y_n.$$

In these results, α_n and β_n are real sequences satisfying appropriate conditions.

Finally, we have the following Theorem which holds in *uniformly convex Banach spaces.*

Theorem 6.43. *(Xu, [511]). Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E, $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$. Assume that E has a Fréchet differentiable norm or satisfies Opial's condition. With an initial $x_0 \in K$, let $\{x_n\}$ be defined by $x_{n+1} := (1-\alpha_n)T_n x_n + \alpha_n u,\ n \geq 0$, where $\{\alpha_n\} \subset [0,1]$ satisfies the following conditions: (i) $\lim \alpha_n = 0$; (ii)$\sum \alpha_n = \infty$, and T_n is defined by $T_n x := \left(\frac{1}{n}\right)\sum\limits_{j=1}^{n-1} T^j x$. Then, $\{x_n\}$ converges weakly to some $x^* \in F(T)$.*

EXERCISES 6.1

1. Verify the assertions made in Example 6.4.
2. Prove Theorem 6.7, i.e., suppose M is a metric space and $T : M \to M$ is continuous and asymptotically regular at x_0 in M. Then, any cluster point of $\{T^n(x_0)\}$ is a fixed point of T.
3. Prove Proposition 6.27.
4. (a) Find an example of a complete metric space (E, ρ) and a mapping $f : E \to E$ such that $\rho(f(x), f(y)) < \rho(x, y)\ \ \forall\, x, y \in E$, and f has no fixed points.
   Hint: Consider $(E, \rho) \equiv (\mathbb{R}, \rho)$ where ρ is the usual metric and $f(x) = \ell n\ (1 + e^x)$.
   (b) Prove that if the Contraction Mapping Principle applies to f^n where n is a positive integer, then f has a unique fixed point.
   (c) Let (E, ρ) be a complete metric space and $f, g : E \to E$ be functions. Suppose f is a contraction and $f(g(x)) = g(f(x))\ \forall\, x \in E$. Prove that g has a fixed point but that such a fixed point need not be unique.
5. Let $E = C[0,1]$ with "sup" norm and let $K = \{f \in C[0,1] : f(0) = 0, f(1) = 1, 0 \leq f(x) \leq 1\}$. For each $f \in K$ define $\varphi : K \to C[0,1]$ by $(\varphi f)(x) = x f(x)$. Prove: (a) K is nonempty, closed, convex and bounded; (b) φ maps K into K; (c) φ is nonexpansive; (d) φ has no fixed points.
6. (a) State the Contraction Mapping Principle.
   (b) Let (E, ρ) be a complete metric space and $T : E \to E$ a contraction map with constant $k < 1$. Define the sequence $\{x_n\}$ inductively by $x_{n+1} = Tx_n,\ \ n = 1, 2, \ldots,\ \ \ x_0 \in E$. If x^* is the unique fixed point of T, prove: (i) $x_n \to x^*$ as $n \to \infty$; (ii) $\rho(x_n, x^*) \leq \dfrac{k^n}{1-k}\, \rho(x_1, x_0)$.
7. Let $C[0,1]$ be endowed with the "sup" metric. Define $T : C[0,1] \to C[0,1]$ *by* $(Tf)(t) = \int_0^t f(s)ds,\ \ f \in C[0,1],\ \ t \in [0,1]$. Prove:
   (a) T is *not* a contraction map;
   (b) T^2 is a contraction map.

(Note: "sup" metric ρ is given by $\rho(f,g) = \sup_{0\le t\le 1} |f(t) - g(t)|$).

(c) Does T have a fixed point?

8. Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:
$$a_{n+1} \le (1-\alpha_n)a_n + \sigma_n, n \ge 0,$$
where (i) $0 < \alpha_n < 1$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose, $\sum_{n=1}^{\infty} \sigma_n < \infty$. Prove that $a_n \to 0$ as $n \to \infty$.

6.8 Historical Remarks

Remark 6.44. If the constant vector u in the Halpern-type recursion formula (6.25) is replaced with $f(x_n)$, where $f : K \to K$ is a strict contraction, an iteration method involving the resulting formula is called the *viscosity method*. We make the following remarks concerning this method.

- The recursion formula with $f(x_n)$ involves more computation at each stage of the iteration than that with u and does not result in any improvement in the speed or rate of convergence of the scheme. Consequently, from the practical point of view, it is undesirable.
- When a Theorem has been proved using a Halpern-type recursion formula with a constant vector, say u, the proof of the same Theorem with u replaced by $f(x_n)$, the so-called viscosity method, generally does not involve any new ideas or method. Such a proof is generally an unnecessary repetition of the proof when the vector u is used.
- The so-called viscosity method may be useful in other iteration processes. But for the approximation of fixed points of nonexpansive and related operators, there seems to be no justification for studying it.

Let $I = [a,b]$ and let T be a self-map of I and suppose T has a unique fixed point in I. Mann [319] proved that the iteration process: $x_0 \in I$,
$$x_{n+1} = (1-c_n)x_n + c_nTx_n, \text{with } c_n = \frac{1}{n+1} \tag{6.40}$$
converges to the fixed point. Franks and Marzec [225] proved that the uniqueness assumption was unnecessary. Rhoades ([415], Theorem 1) extended the Franks and Marzec result to: $0 \le c_n \le 1, \sum c_n = \infty$. Outlaw and Groetcsh [378] obtained convergence for a nonexpansive mapping T of a convex compact subset of the complex plane. Groetsch [242] generalized the method for nonexpansive mappings on uniformly convex Banach spaces. Dotson [212] also used the method for quasi-nonexpansive mappings on strictly convex Banach spaces.

The concept of *uniform* asymptotic regularity was introduced by Edelstein and O'Brian in [218] where they also proved that, on any normed linear space E, and on any bounded convex subset $K \subset E$, S_λ is uniformly asymptotically regular. Results on the asymptotic regularity of S_λ were first obtained by Browder and Petryshyn [52]. They proved that if E is uniformly convex and $T : K \to K$ is a nonexpansive self-mapping of K, where K is a nonempty closed convex and bounded subset of E, then S_λ is asymptotically regular. As is easily seen, $S_\lambda(x^*) = x^*$ is equivalent to $Tx^* = x^*$, so that problems pertaining to the existence and location of fixed points for T reduce to similar problems for S_λ, where, by the result of Edelstein and O'Brian cited above, S_λ can be assumed to be uniformly asymptotically regular. Theorem 6.3 is the well known Browder-Göhde-Kirk theorem [42, 238, 283]. Theorem 6.7 and Proposition 6.8 are due to Edelstein and O'Brian [218]; Example 6.10 is due to Genel and Lindenstrauss [228]. Part (i) of Corollary 6.19 was originally proved by Petryshyn [381] for uniformly convex Banach spaces and part (ii) was first proved by Browder and Petryshyn [51] again, for uniformly convex Banach spaces. Edelstein and O'Brian [218] extended these results to arbitrary normed linear spaces for the sequence $\{S_\lambda^n\}$, defined by (6.2).

A consequence of a result of Browder and Petryshyn [51] shows that if T is asymptotically regular and $(I - T)$ is demiclosed, then any weak cluster point of $\{T^n(x_0)\}$ is a fixed point of T. It is also known that *in an Opial space*, $(I - T)$ is always demiclosed for any nonexpansive self-map T of a nonempty closed convex and bounded subset K. Edelstein and O'Brian [218] then proved that in an Opial space E, if $K \subset E$ is weakly compact and convex and T is a self-mapping of K, then for any $x_0 \in K$, the sequence $\{S_\lambda^n(x_0)\}$ converges weakly to a fixed point of T. This result is a generalization of an earlier result of Opial [366] who had proved the same result under the assumption that E is uniformly convex and has a weakly continuous duality map. It is pertinent to mention here that Gossez and Lami Dozo [241] have shown that for any normed linear space E, the existence of a weakly continuous duality map implies that E satisfies Opial's condition which in turn implies that E has *normal structure*, (see e.g., Brodskii and Mil'man, [38] for definition), but that none of the converse implications hold.

Theorems 6.14, 6.15 and 6.16 were proved by Edelstein and O'Brian [218] where the sequence $\{x_n\}$ is defined by (6.2). Theorem 6.15 was proved by Ishikawa for the more general sequence defined by (6.3). But then, while this result is somewhat stronger than the result of Edelstein and O'Brian in the sense that it involves the more general Mann iterates, the theorems of Edelstein and O'Brian are stronger in the sense that *uniform* asymptotic regularity is proved. Theorems 6.14, 6.15 and 6.16 unify these results of Ishikawa and those of Edelstein and O'Brian. The theorems are due to Chidume [87] who used a method which seems simpler and totally different from those of Ishikawa and, Edelstein and O'Brian. Finally, the results of Sections 6.3 to

6.5 together with Example 6.23 and Example 6.28 are also due to Chidume [87, 93]. Theorem 6.41 is due to Chidume *et al.* [184].

Strong convergence theorems for a generalization of nonexpansive mappings (relatively weak nonexpansive mappings) can be found in Zegeye and Shahzad [548].