# Chapter 10
# An Example; Mann Iteration for Strictly Pseudo-contractive Mappings

## 10.1 Introduction and a Convergence Theorem

We have seen (Chapter 6) that the Mann iteration method has been successfully employed in approximating fixed points (when they exist) of nonexpansive mappings. This success has not carried over to the more general class of pseudo-contractions. If $K$ is a compact convex subset of a Hilbert space and $T : K \to K$ is Lipschitz, then, by Schauder fixed point theorem, $T$ has a fixed point in $K$. All efforts to approximate such a fixed point by means of the Mann sequence when $T$ is also assumed to be pseudo-contractive proved abortive. In 1974, Ishikawa introduced a new iteration scheme and proved the following theorem.

**Theorem 10.1.** *If $K$ is a compact convex subset of a Hilbert space $H$, $T : K \mapsto K$ is a Lipschitzian pseudo-contractive map and $x_0$ is any point of $K$, then the sequence $\{x_n\}_{n\geq 0}$ converges strongly to a fixed point of $T$, where $x_n$ is defined iteratively for each positive integer $n \geq 0$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n; \quad y_n = (1-\beta_n)x_n + \beta_n T x_n, \tag{10.1}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions $(i)$ $0 \leq \alpha_n \leq \beta_n < 1$; $(ii)$ $\lim\limits_{n\to\infty} \beta_n = 0$; $(iii)$ $\sum\limits_{n\geq 0} \alpha_n \beta_n = \infty$.*

The recursion formula (10.1) with conditions $(i), (ii)$ and $(iii)$ is generally referred to as the *Ishikawa iteration process.*

## 10.2 An Example

Since its publication in 1974, it had remained an open question (see e.g., Borwein and Borwein [33], Chidume and Moore [145], Hicks and Kubicek [251]) whether or not the Mann recursion formula defined by (6.3), which

C. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations,*
Lecture Notes in Mathematics 1965,

is certainly simpler than the Ishikawa recursion formula (10.1), converges under the setting of theorem 10.1 to a fixed point of T if the operator T is pseudo-contractive and Lipschitz. Hicks and Kubicek [251], gave an example of a *discontinuous* pseudo-contraction with a unique fixed point for which the Mann iteration does not always converge. Borwein and Borwein [33], (Proposition 8), gave an example of a Lipschitz map (which is not pseudo-contractive) with a unique fixed point for which the Mann sequence fails to converge. The problem for *Lipschitz* pseudo-contraction still remained open. This was eventually resolved in the negative by Chidume and Mutangadura [148] in the following example.

*Example 10.2.* Let $X$ be the real Hilbert space $\mathbb{R}^2$ under the usual Euclidean inner product. If $x = (a, b) \in X$ we define $x^\perp \in X$ to be $(b, -a)$. Trivially, we have $\langle x, x^\perp\rangle = 0$, $||x^\perp|| = ||x||$, $\langle x^\perp, y^\perp\rangle = \langle x, y\rangle$, $||x^\perp - y^\perp|| = ||x - y||$ and $\langle x^\perp, y\rangle + \langle x, y^\perp\rangle = 0$ for all $x$, $y \in X$. We take our closed and bounded convex set $K$ to be the closed unit ball in $X$ and put $K_1 = \{x \in X : ||x|| \leq \frac{1}{2}\}$, $K_2 = \{x \in X : \frac{1}{2} \leq ||x|| \leq 1\}$. We define the map $T : K \longrightarrow K$ as follows:

$$Tx = \begin{cases} x + x^\perp\,, & \text{if } x \in K_1. \\ \frac{x}{||x||} - x + x^\perp\,, & \text{if } x \in K_2. \end{cases}$$

Then, $T$ is a Lipschitz pseudo-contractive map of a compact convex set into itself with a unique fixed point for which no Mann sequence converges.

We notice that, for $x \in K_1 \cap K_2$, the two possible expressions for $Tx$ coincide and that $T$ is continuous on both of $K_1$ and $K_2$. Hence $T$ is continuous on all of $K$. We now show that $T$ is, in fact, Lipschitz. One easily shows that $||Tx - Ty|| = \sqrt{2}||x - y||$ for $x$, $y \in K_1$. For $x$, $y \in K_2$, we have

$$\begin{aligned}\left|\left|\frac{x}{||x||} - \frac{y}{||y||}\right|\right|^2 &= \frac{2}{||x||||y||}(||x||||y|| - \langle x, y\rangle) \\ &= \frac{1}{||x||||y||}\{||x - y||^2 - (||x|| - ||y||)^2\} \\ &\leq \frac{1}{||x||||y||}\,2||x - y||^2 \leq 8||x - y||^2\,.\end{aligned}$$

Hence, for $x$, $y \in K_2$, we have

$$||Tx - Ty|| \leq \left|\left|\frac{x}{||x||} - \frac{y}{||y||}\right|\right| + ||x - y|| + ||x^\perp - y^\perp|| \leq 5||x - y||\,,$$

so that $T$ is Lipschitz on $K_2$. Now let $x$ and $y$ be in the interiors of $K_1$ and $K_2$ respectively. Then there exist $\lambda \in (0, 1)$ and $z \in K_1 \cap K_2$ for which $z = \lambda x + (1 - \lambda)y$. Hence

$$\begin{aligned}||Tx - Ty|| &\leq ||Tx - Tz|| + ||Tz - Ty|| \\ &\leq \sqrt{2}||x - z|| + 5||z - y|| \\ &\leq 5||x - z|| + 5||z - y|| = 5||x - y||\,.\end{aligned}$$

Thus $||Tx - Ty|| \leq 5||x - y||$ for all $x$, $y \in K$, as required. The origin is clearly a fixed point of $T$. For $x \in K_1$, $||Tx||^2 = 2||x||^2$, and for $x \in K_2$, $||Tx||^2 = 1 + 2||x||^2 - 2||x||$. From these expressions and from the fact that $Tx = x^{\perp} \neq x$ if $||x|| = 1$, it is easy to show that the origin is the only fixed point of $T$. We now show that no Mann iteration sequence for $T$ is convergent for any nonzero starting point.

First, we show that no such Mann sequence converges to the fixed point. Let $x \in K$ be such that $x \neq 0$. Then, in case $x \in K_1$, any Mann iterate of $x$ is actually further away from the fixed point of $T$ than $x$ is. This is because $||(1-\lambda)x + \lambda Tx||^2 = (1+\lambda^2)||x||^2 > ||x||^2$ for $\lambda \in (0,1)$. If $x \in K_2$ then, for any $\lambda \in (0,1)$,

$$\begin{aligned} ||(1-\lambda)x + \lambda Tx||^2 &= ||(\tfrac{\lambda}{||x||} + 1 - 2\lambda)x + \lambda x^{\perp}||^2 \\ &= [(\tfrac{\lambda}{||x||} + 1 - 2\lambda)^2 + \lambda^2]||x||^2 > 0\cdot \end{aligned}$$

More generally, it is easy to see that for the recursion formula

$$x_0 \in K,\ x_{n+1} = (1 - c_n)x_n + c_n T x_n,\ n \geq 0, \tag{10.2}$$

if $x_0 \in K_1$ then $||x_{n+1}|| > ||x_n||$ for all integers $n \geq 0$, and if $x_0 \in K_2$, then $||x_{n+1}|| \geq \frac{\sqrt{2}}{2}||x_n||$ for all integers $n \geq 0$. We therefore conclude that, in addition, any Mann iterate of any non zero vector in $K$ is itself non zero. Thus any Mann sequence $\{x_n\}$, starting from a nonzero vector, must be infinite. For such a sequence to converge to the origin, $x_n$ would have to lie in the neighborhood $K_1$ of the origin for all $n > N_0$, for some real $N_0$. This is not possible because, as already established for $K_1$, $||x_n|| < ||x_{n+1}||$ for all $n > N_0$.

We now show that no Mann sequence converges to $x \neq 0$. We do this in the form of a general lemma.

**Lemma 10.3.** *Let $M$ be a nonempty, closed and convex subset of a real Banach space $E$ and let $S : M \to M$ be any continuous function. If a Mann sequence for $S$ is norm convergent, then the corresponding limit is a fixed point for $S$.*

*Proof.* Let $\{x_n\}$ be a Mann sequence in $M$ for $S$, as defined in the recursion formula (10.2). Assume, for proof by contradiction, that the sequence converges, in norm, to $x$ in $M$, where $Sx \neq x$. For each $n \in \mathbb{N}$, put $\epsilon_n = x_n - Sx_n - x + Sx$. Since $S$ is continuous, the sequence $\epsilon_n$ converges to 0. Pick $p \in \mathbb{N}$ such that, if $m \geq p$ and $n \geq p$, then $||\epsilon_n|| < \frac{1}{3}||x - Sx||$ and $||x_n - x_m|| < \frac{1}{3}||x - Sx||$. Pick any positive integer $q$ such that $\sum\limits_{n=p}^{p+q} c_n \geq 1$. We have that

$$
\begin{aligned}
||x_p - x_{p+q+1}|| &= \Big|\Big|\sum_{n=p}^{p+q}(x_n - x_{n+1})\Big|\Big| = \Big|\Big|\sum_{n=p}^{p+q} c_n(x - Sx + \epsilon_n)\Big|\Big| \\
&\geq \Big|\Big|\sum_{n=p}^{p+q} c_n(x - Sx)\Big|\Big| - \Big|\Big|\sum_{n=p}^{p+q} c_n\epsilon_n\Big|\Big| \\
&\geq \sum_{n=p}^{p+q} c_n\Big(||x - Sx|| - \tfrac{1}{3}||x - Sx||\Big) \geq \tfrac{2}{3}||x - Sx||\,.
\end{aligned}
$$

The contradiction proves the result. □

We now show that $T$ is a pseudo-contraction. First, we note that we may put $j(x) = x$, since $X$ is Hilbert. For $x$, $y \in K$, put $\Gamma(x; y) = ||x - y||^2 - \langle Tx - Ty, x - y\rangle$ and, if $x$ and $y$ are both non zero, put $\lambda(x; y) = \dfrac{\langle x, y\rangle}{||x||||y||}$. Hence, to show that $T$ is a pseudo-contraction, we need to prove that $\Gamma(x; y) \geq 0$ for all $x$, $y \in K$. We only need examine the following three cases:

1. $x$, $y \in K_1$: An easy computation shows that $\langle Tx - Ty, x - y\rangle = ||x - y||^2$ so that $\Gamma(x; y) = 0$; thus we are home and dry for this case.
2. $x$, $y \in K_2$: Again, a straightforward calculation shows that

$$
\begin{aligned}
\langle Tx - Ty, x - y\rangle &= ||x|| - ||x||^2 + ||y|| - ||y||^2 \\
&\quad + \langle x, y\rangle(2 - \frac{1}{||x||} - \frac{1}{||y||}) \\
&= ||x|| - ||x||^2 + ||y|| - ||y||^2 \\
&\quad + \lambda(x; y)(2||x||||y|| - ||x|| - ||y||).
\end{aligned}
$$

   Hence $\Gamma(x; y) = 2||x||^2 + 2||y||^2 - ||x|| - ||y|| - \lambda(x; y)(4||x||||y|| - ||x|| - ||y||)$. It is not hard to establish that $(4||x||||y|| - ||x|| - ||y||) \geq 0$ for all $x$, $y \in K_2$. Hence, for fixed $||x||$ and $||y||$, $\Gamma(x; y)$ has a minimum when $\lambda(x; y) = 1$. This minimum is therefore $2||x||^2 + 2||y||^2 - 4||x||||y|| = 2(||x|| - ||y||)^2$. Again, we have that $\Gamma(x; y) \geq 0$ for all $x$, $y \in K_2$ as required.
3. $x \in K_1$, $y \in K_2$: We have
   $\langle Tx - Ty, x - y\rangle = ||x||^2 + ||y|| - ||y||^2 - \lambda(x; y)||x||$.
   Hence $\Gamma(x; y) = 2||y||^2 - ||y|| + (||x|| - 2||x||||y||)\lambda(x; y)$. Since $||x|| - 2||x||||y|| \leq 0$ for $x \in K_1$ and $y \in K_2$, $\Gamma(x; y)$ has its minimum, for fixed $||x||$ and $||y||$ when $\lambda(x; y) = 1$. We conclude that

$$
\begin{aligned}
\Gamma(x; y) &\geq 2||y||^2 - ||y|| + ||x|| - 2||x||||y||| \\
&= (||y|| - ||x||)(2||y|| - 1) \\
&\geq 0 \text{ for all } x \in K_1,\ y \in K_2.
\end{aligned}
$$

This completes the proof. □

## 10.3 Mann Iteration for a Class of Lipschitz Pseudo-contractive Maps

A mapping $T : K \to E$ is said to be *strictly pseudo-contractive in the sense of Browder and Petryshyn* [52] if

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|x-y-(Tx-Ty)\|^2 \tag{10.3}$$

holds for $x, y \in K$ and for some $\lambda > 0$.

It is easy to see that such mappings are Lipschitz with Lipschitzian constant $L = \dfrac{1+\lambda}{\lambda}$.

This class of mappings was introduced in 1967 by Browder and Petryshyn [52] who actually defined it in a Hilbert space as follows: Let $K$ be a nonempty subset of real Hilbert space. A mapping $T : K \to K$ is said to be strictly pseudo-contractive if

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|x-y-(Tx-Ty)\|^2 \tag{10.4}$$

holds for all $x, y \in K$ and for some $0 < k < 1$. Clearly, nonexpansive mappings satisfy (10.4) and it is also easy to see that in real Hilbert spaces, inequalities (10.3) and (10.4) are equivalent. The class of strictly pseudo-contractive mappings is a subclass of the class of Lipshitz pesudo-contractive ones.

From now on, we shall denote the class of mappings which are *strictly* pseudo-contractive in the sense of Browder and Petryshyn by $\mathcal{M}$. For $\mathcal{M}$, Browder and Petryshyn proved the following theorem.

**Theorem 10.4.** *(Browder and Petryshyn, [52]) Let $K$ be a bounded closed convex subset of a real Hilbert space $H$ and let $T : K \to K$ be a mapping belonging to $\mathcal{M}$. Then, for any $x_0 \in K$ and any fixed $\gamma$ such that $1-k < \gamma < 1$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = \gamma Tx_n + (1-\gamma)x_n,\ n = 1, 2, ...,$$

*converges weakly to some fixed point of $T$ in $K$. If, in addition, $T$ is demicompact, then $\{x_n\}$ converges strongly to some fixed point of $T$ in $K$.*

Maruster [325] proved, for $T \in \mathcal{M},\ F(T) \neq \emptyset$, in a real Hilbert space, that a Halpern-type iteration process converges to a fixed point of $T$. Chidume [101] extended this result of Maruster to $L_p$ spaces, $p \geq 2$. Interest in iterative methods for approximating fixed points of mappings belonging to $\mathcal{M}$ has continued to grow (see, e.g., [255],[256],[120],[374] and the references contained therein).

While the example of Chidume and Mutangadura shows that the Mann sequence will not always converge to a fixed point of a Lipschitz pseudo-contractive mapping defined even on a compact convex subset of a Hilbert space, we show in this section that it can be used to approximate a fixed point of the *important subclass* of the class of Lipschitz pseudo-contractive maps consisting of mappings which are strictly pseudo-contractive in the sense of Browder and Petryshyn.

We begin with the following lemma.

**Lemma 10.5.** *Let $\{\sigma_n\}$ and $\{\beta_n\}$ be sequences of non-negative real numbers satisfying the following inequality $\beta_{n+1} \leq (1+\sigma_n)\beta_n,\ n \geq 0$. If $\sum\limits_{n\geq 0} \sigma_n < \infty$ then $\lim\limits_{n\to\infty} \beta_n$ exists and if there exists a subsequence of $\{\beta_n\}$ converging to 0, then $\lim\limits_{n\to\infty} \beta_n = 0$.*

We now prove the following results.

In what follows $L$ will denote the Lipschitz constant of $T$, and $\lambda$ is the constant appearing in inequality (10.3).

**Lemma 10.6.** *Let $E$ be a real Banach space. Let $K$ be a nonempty closed and convex subset of $E$. Let $T : K \to K$ be a strictly pseudo-contractive map in the sense of Browder and Petryshyn with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Let $x^* \in F(T)$. For a fixed $x_0 \in K$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \tag{10.5}$$

*where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying the following conditions: (i) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\sum\limits_{n=1}^{\infty} \alpha_n^2 < \infty$. Then, (a) $\{x_n\}$ is bounded; (b) $\lim ||x_n - x^*||$ exists for some $x^* \in F(T)$; (c) $\liminf\limits_{n\to\infty} \|x_n - Tx_n\| = 0$.*

*Proof.* Since $T$ is strictly pseudo-contractive in the sense of Browder and Petryshyn, we have $\langle Tx_n - Tx^*, j(x_n - x^*)\rangle \leq \|x_n - x^*\|^2 - \lambda\|x_n - x^* - (Tx_n - Tx^*)\|^2$. Using the recursion formula (10.5) and inequality (4.4), we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\alpha_n\langle x_{n+1} - x^* + x^* - Tx_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad - 2\alpha_n\langle x_n - x_{n+1} + Tx_{n+1} - Tx_n, j(x_{n+1} - x^*)\rangle \\
&\leq \|x_n - x^*\|^2 - 2\alpha_n\|x_{n+1} - x^*\|^2 \\
&\quad + 2\alpha_n\langle Tx_{n+1} - x^*, j(x_{n+1} - x^*)\rangle \\
&\quad + 2\alpha_n^2(1+L)\|x_n - Tx_n\|\ \|x_{n+1} - x^*\| \\
&\leq \|x_n - x^*\|^2 - 2\alpha_n\lambda\|x_{n+1} - Tx_{n+1}\|^2 \\
&\quad + 2\alpha_n^2(1+L)\|x_n - Tx_n\|\ \|x_{n+1} - x^*\| \\
&\leq \|x_n - x^*\|^2 + 2\alpha_n^2(1+L) \times \\
&\qquad \Big(\|x_n - x^*\| + L\|x_n - x^*\|\Big)\|x_{n+1} - x^*\|
\end{aligned}$$

$$
\begin{aligned}
&\le \|x_n - x^*\|^2 + 2\alpha_n^2(1+L)^2\|x_n - x^*\| \times \\
&\quad \Big[\big(1+\alpha_n(1+L)\big)\|x_n - x^*\|\Big] \\
&= \Big[1 + 2\alpha_n^2(1+L)^2\big(1+\alpha_n(1+L)\big)\Big]\|x_n - x^*\|^2 \\
&\le \Big(1 + C\alpha_n^2)\Big)\|x_n - x^*\|^2 \le \|x_1 - x^*\|^2 e^{C\sum\limits_{n=1}^{\infty}\alpha_n^2} < \infty, \qquad (*)
\end{aligned}
$$

for some constant $C > 0$. Thus, $\{x_n\}$ is bounded and from $(*)$, Lemma 10.5, and condition $(ii)$, we obtain that $\lim\limits_{n\to\infty}\|x_n - x^*\|$ exists.
Furthermore,

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 - 2\alpha_n\lambda\|x_{n+1} - Tx_{n+1}\|^2 \\
&\quad + 2\alpha_n^2(1+L)\|x_n - Tx_n\|\ \|x_{n+1} - x^*\| \\
&\le \|x_n - x^*\|^2 - 2\alpha_n\lambda\|x_{n+1} - Tx_{n+1}\|^2 + \alpha_n^2 M,
\end{aligned}
$$

where $M := 2(1+L)\sup(\|x_n - Tx_n\|\ \|x_{n+1} - x^*\|)$. The last inequality now implies that

$$
\sum \alpha_n\lambda\|x_{n+1} - Tx_{n+1}\|^2 \le \sum\Big(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\Big) + \sum \alpha_n^2 M < \infty.
$$

Condition $(i)$ now implies that $\liminf\|x_{n+1} - Tx_{n+1}\| = 0$, completing the proof. □

**Theorem 10.7.** *Let $E$ be a real Banach space. Let $K$ be a nonempty closed and convex subset of $E$. Let $T : K \to K$ be a strictly pseudo-contractive map in the sense of Browder and Petryshyn with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. For a fixed $x_0 \in K$, define a sequence $\{x_n\}$ by*

$$
x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n \qquad (10.6)
$$

*where $\{\alpha_n\}$ is a real sequence satisfying the following conditions: $(i)\sum\alpha_n = \infty$ and $(ii)\sum\alpha_n^2 < \infty$. If $T$ is demicompact, then $\{x_n\}$ converges strongly to some fixed point of $T$ in $K$.*

*Proof.* By lemma 10.6, $\liminf\limits_{n\to\infty}\|x_n - Tx_n\| = 0$. Hence, there exists a subsequence of $\{x_n\}$, say $\{x_{n_k}\}$, such that $\lim\limits_{k\to\infty}\|x_{n_k} - Tx_{n_k}\| = 0$. From the fact that $T$ is demicompact, the continuity of $T$ and passing to a subsequence which we still denote by $\{x_{n_k}\}$ we obtain that $\{x_{n_k}\}$ converges to some fixed point, say $q$, of $T$. Since $\lim\|x_n - q\|$ exists, we have that $\{x_n\}$ converges strongly to $q$. The proof is complete. □

**Corollary 10.8** *Let $E$ be a real Banach space. Let $K$ be a nonempty compact and convex subset of $E$. Let $T : K \to K$ be a strictly pseudo-contractive map in the sense of Browder and Petryshyn. For a fixed $x_0 \in K$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \tag{10.7}$$

*where* $\{\alpha_n\}$ *is a real sequence in* $(0,1)$ *satisfying the following conditions:* $(i) \sum \alpha_n = \infty$ *and* $(ii) \sum \alpha_n^2 < \infty$. *Then,* $\{x_n\}$ *converges strongly to some fixed point of* $T$ *in* $K$.

## EXERCISES 10.1

1. Let $H$ be the complex plane and $K := \{z \in H : |z| \leq 1\}$. Define $T : K \to K$ by
$$T(re^{i\theta}) = \begin{cases} 2re^{i(\theta + \frac{\pi}{3})}, & \text{for } 0 \leq r \leq \frac{1}{2}, \\ e^{i(\theta + \frac{2\pi}{3})}, & \text{for } \frac{1}{2} < r \leq 1. \end{cases}$$
Prove that

   $(i)$ Zero is the only fixed point of $T$.
   $(ii)$ $T$ is pseudo-contractive.
   Let $c_n := \frac{1}{n+1}$ for all integers $n \geq 0$. Define the sequence$\{z_n\}$ by $z_{n+1} := (1 - c_n)z_n + c_n T z_n, z_0 \in K$, $n \geq 0$.
   $(iii)$ Prove $\{z_n\}$ does not converge to zero.
   $(iv)$ Prove $T$ is not continuous.

2. Let $K$ be a compact convex subset of a real Hilbert space, $H$; $T : K \to K$ a continuous hemi-contractive map (i.e., $F(T) := \{x \in K : Tx = x\} \neq \emptyset$ and $||Tx - x^*||^2 \leq ||x - x^*||^2 + ||x - Tx||^2 \ \forall \ x \in H, x^* \in F(T)$). Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ be real sequences in [0,1] satisfying the following conditions:

   $(i)$ $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1 \forall n \geq 1$;
   $(ii)$ $\lim b_n = \lim b'_n = 0$;
   $(iii)$ $\sum c_n < \infty; \sum c'_n < \infty$;
   $(iv)$ $0 \leq \alpha_n \leq \beta_n < 1, \forall n \geq 1$, where $\alpha_n := b_n + c_n; \beta_n := b'_n + c'_n$.
   $(v)$ $\sum \alpha_n \beta_n = \infty; \sum \alpha_n \beta_n \delta_n < \infty$, where $\delta_n := ||Tx_n - Ty_n||^2$.

   For arbitrary $x_1, u_1, v_1 \in K$, define the sequence $\{x_n\}$ iteratively by $x_{n+1} = a_n x_n + b_n T y_n + c_n u_n$; $y_n := a'_n x_n + b'_n T x_n + c'_n v_n, n \geq 1$, where $\{u_n\}$ and $\{v_n\}$ are arbitrary bounded sequences in $K$. Prove that $\{x_n\}$ converges strongly to a fixed point of $T$.
   (Hint. Chidume and Moore [145], Theorem 1).
3. Prove Lemma 10.5.
4. Let $T : K \to E$ be strictly pseudocontractive in the sense of Browder and Petryshyn. Prove the statement made at the begining of section 10.3 that $T$ is Lipschitz with Lipschitz constant $L := \frac{1}{\lambda}(1 + \lambda)$.
5. Prove that in a real Hilbert space, inequalities (10.3) and (10.4) are equivalent.

## 10.4 Historical Remarks

Qihou [389] extended Theorem 10.1 to the slightly more general class of Lipschitz hemi-contractive maps. In [390] he proved that if K is a compact convex subset of *a Hilbert space* and $T : K \to K$ is a *continuous* pseudo-contractive map *with a finite number of fixed points* then the Ishikawa iteration sequence defined by (10.1) converges strongly to a fixed point of T. Consequently, while the Mann sequence does not converge to the fixed point of T in Example 10.2, the Ishikawa sequence does. Chidume and Moore [145] also extended Theorem 10.1 to continuous maps under additional assumption that $\sum \alpha_n \beta_n \delta_n < \infty$, where $\delta_n := ||Tx_n - Ty_n||^2$. Conditions similar to this had been imposed in the literature. Reich [408] imposed the condition $\sum\limits_{n=0}^{\infty} c_n^2 ||Tx_n||^2 < \infty$ (where $\{c_n\}$ is a real sequence in (0,1) satisfying appropriate conditions). Furthermore, if $T$ is Lipschitz, then $\sum \alpha_n \beta_n \delta_n < \infty$, for suitable choices of $\alpha_n,\ \beta_n$ (see Chidume and Moore, [145]).

While the example of Chidume and Mutangadura [148] shows that the Mann iteration method cannot always be used to approximate a fixed point of Lipschitz pseudo-contractive maps even in a compact convex domain, it has been shown in this section that for the important subclass $\mathcal{M}$ of the class of Lipschitz pseudo-contractions, Mann iteration method can always be used. Marino and Xu [321] recently proved *weak* convergence of the Mann process to a fixed point of $T \in \mathcal{M}$ in *a real Hilbert space.* Theorem 10.7 and Corollary 10.8 yield *strong* convergence in *arbitrary real Banach spaces.* All the results of section 10.3 are due to Chidume, Abbas and Ali [120].