Chapter 2
Foundations

The first section of this chapter introduces the complex plane, fixes notation, and discusses some useful concepts from real analysis. Some readers may initially choose to skim this section. The second section contains the definition and elementary properties of the class of holomorphic functions—the basic object of our study.

2.1 Introduction and Preliminaries

This section is a summary of basic notation, a description of some of the basic algebraic and geometric properties of the complex number system, and a disjoint collection of needed facts from real analysis (advanced calculus). We remind the reader of some of the formalities behind the standard notation which we usually approach quite informally. Not all concepts used as prerequisites are defined (among these are neighborhood, connected, path-connected, arc-wise connected, and compact sets); we assume that the reader has been exposed to them.[1]

We start with some standard **notation**:

$$\mathbb{Z}_{>0} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \widehat{\mathbb{C}}.$$

Here $\mathbb{Z}$ represents the integers, $\mathbb{Z}_{>0}$ the positive integers,[2] $\mathbb{Q}$ the rationals (the integer n is included in the rationals as the equivalence class of the quotient $\frac{n}{1}$), and $\mathbb{R}$ the reals. Whether one views the reals as the completion of the rationals or identifies

[1]The reader may want to consult J. R. Munkres *Topology (Second Edition)*, Dover, 2000, or J. L. Kelley, *General Topology*, Springer-Verlag, 1975 as well as definitions in Chap. 4.

[2]In general $X_{\text{condition}}$ and $\{x \in X; \text{condition}\}$ will describe the set of all x in X that satisfy the indicated condition.

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them with Dedekind cuts (we will not use these concepts explicitly), their most important property from the perspective of complex variables is the *least upper bound property*; that is, that every nonempty set of real numbers that has an upper bound has a least upper bound.

The inclusion of $\mathbb{R}$ into the complex numbers $\mathbb{C}$ needs a bit more explanation. It is specified as follows: for z in $\mathbb{C}$, we write $z = x + \imath\, y$ with x and y in $\mathbb{R}$, where the symbol $\imath$ represents a square root of -1; that is, $\imath^2 = -1$. With these conventions we can define addition and multiplication of complex numbers using the usual rules for these operations on the reals[3]: for all $x, y, \xi, \eta \in \mathbb{R}$,

$$(x + \imath\, y) + (\xi + \imath \eta) = (x + \xi) + \imath (y + \eta)$$

and

$$(x + \imath\, y)(\xi + \imath \eta) = (x\xi - y\eta) + \imath (x\eta + y\xi).$$

The *real numbers*, $\mathbb{R}$, are identified with the subset of $\mathbb{C}$ consisting of those numbers $z = x + \imath\, y$ with $y = 0$; the *imaginary* numbers, $\imath\mathbb{R}$, are those with $x = 0$. For $z = x + \imath\, y$ in $\mathbb{C}$ with x and y in $\mathbb{R}$ we write $x = \Re z$, the *real part* of z, and $y = \Im z$, the *imaginary part* of z. Geometrically, $\mathbb{R}$ and $\imath\mathbb{R}$ represent the *real* and *imaginary axes* of $\mathbb{C}$, viewed as the complex plane and identified with the cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (see Fig. 2.1).

The complex plane may be viewed as a subset of the *complex sphere* $\widehat{\mathbb{C}}$, which is $\mathbb{C}$ compactified by adjoining a point, known as *the point at infinity*, so that $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The space $\widehat{\mathbb{C}}$ is also called *the extended complex plane* or the *Riemann sphere*. This last name comes from identifying the points on the unit sphere in $\mathbb{R}^3$, with the exception of the *south pole*, the point $(0, 0, 1)$, with the points in the complex plane under what is known as stereographic projection; the point $(0, 0, 1)$ is identified with ∞. See Exercise 3.20 for the details.

We now describe some basic algebraic and geometric properties of the complex numbers.

For $z = x + \imath\, y$, with x and y real numbers, the complex number

$$\bar{z} = x - \imath\, y$$

is called the *complex conjugate* of z. Note that then

$$\Re z = \frac{z + \bar{z}}{2} \text{ and } \Im z = \frac{z - \bar{z}}{2\imath}.$$

One easily verifies the following basic

[3] With these operations $(\mathbb{C}, +, \cdot)$ is a field.

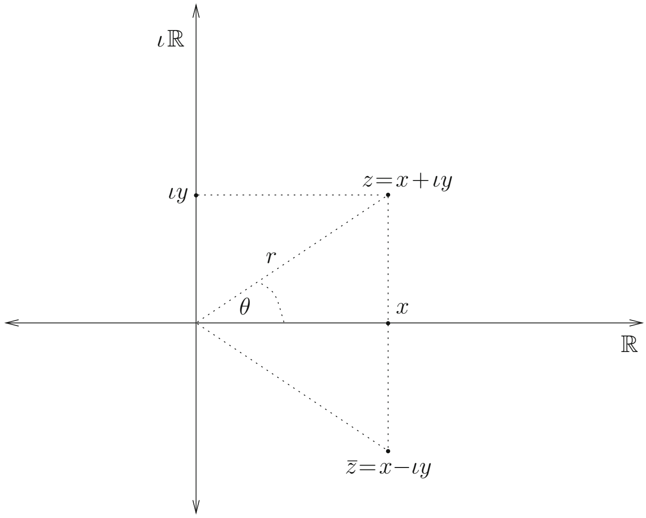

Fig. 2.1 The complex plane; rectangular and polar representations, conjugation

2.1.1 Properties of Conjugation

For$z$ and $w \in \mathbb{C}$,

(a) $\overline{z+w} = \bar{z} + \overline{w}$
(b) $\overline{zw} = \bar{z}\,\overline{w}$
(c) $\bar{z} = z$ if and only if $z \in \mathbb{R}$
(d) $\bar{\bar{z}} = z$

There is a simple and useful **geometric interpretation of conjugation**: it is represented by mirror reflection in the real axis; see Fig. 2.1.

From a slightly different point of view, conjugation may be seen as a self-map of $\mathbb{C}$, denoted by $^-$.

$$^- : \mathbb{C} \to \mathbb{C}.$$

Then its properties (a) through (d) may be restated as follows:

(a) $^-$ preserves the sum of complex numbers.
(b) $^-$ preserves the product of complex numbers.
(c) $^-$ fixes precisely the real numbers.
(d) $^-$ is an involution of $\mathbb{C}$; that is, when composed with itself, it gives the identity map on $\mathbb{C}$.

It is not hard to show that any self-map of $\mathbb{C}$ satisfying these properties coincides with complex conjugation; see Exercise 2.19.

Another important map, $z \mapsto |z|$ or

$$| \, | : \mathbb{C} \to \mathbb{R}_{\geq 0}$$

is defined by $r = |z| = (z\,\overline{z})^{\frac{1}{2}} = (x^2 + y^2)^{\frac{1}{2}}$, where $z = x + \imath\, y$. Note that $z\,\overline{z} = x^2 + y^2$ is always a real nonnegative number; we use the usual convention that unless otherwise specified the square root of a real nonnegative number is chosen to be nonnegative. The nonnegative real number r is called the *absolute value* or *norm* or *modulus* of the complex number z.

The following properties follow directly from the definition.

2.1.2 Properties of Absolute Value

For z and $w \in \mathbb{C}$,

(a) $|z| \geq 0$, and $|z| = 0$ if and only if $z = 0$
(b) $|zw| = |z|\,|w|$
(c) $|\overline{z}| = |z|$
(d) $|\Re z| \leq |z|$, and $\Re z = |z|$ if and only if $z = x \in \mathbb{R}_{\geq 0}$
(e) $|\Im z| \leq |z|$, and $\Im z = |z|$ if and only if $z = \imath\, y$ with $y \in \mathbb{R}_{\geq 0}$

2.1.3 Linear Representation of $\mathbb{C}$

As a vector space over $\mathbb{R}$, we can identify $\mathbb{C}$ with $\mathbb{R}^2$. Vector addition agrees with complex addition, and scalar multiplication by real numbers ($\mathbb{R} \times \mathbb{C} \to \mathbb{C}$) is the restriction of complex multiplication ($\mathbb{C} \times \mathbb{C} \to \mathbb{C}$).

This identification provides a powerful geometric interpretation for many results on complex numbers. One example is provided by conjugation, which can be viewed as the $\mathbb{R}$-linear map of $\mathbb{R}^2$ (with basis $1 = (1, 0)$ and $\imath = (0, 1)$) that sends 1 to 1 and $\imath$ to $-\imath$. Another instance is provided by the next geometric interpretation of the following:

2.1.4 Additional Properties of Absolute Value

From the Pythagorean equality, $r = |z|$ is the (Euclidean) distance in the plane from z to the origin; see Fig. 2.1.

Furthermore, for z and $w \in \mathbb{C}$, the following properties hold.

(f) $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.
(g) $|z + w| \leq |z| + |w|$. Equality holds whenever either z or w is equal to 0. If $z \neq 0$ and $w \neq 0$, then equality holds if and only if $w = az$ with $a \in \mathbb{R}_{>0}$.
(h) $||z| - |w|| \leq |z - w|$.

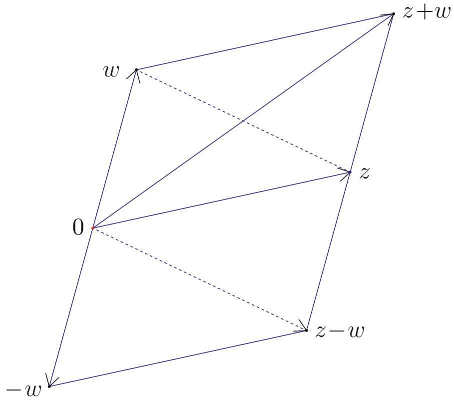

Fig. 2.2 Vector sums

Equality (f) is sometimes called the *parallelogram law*: the sum of the squares of the lengths of the diagonals in a parallelogram is equal to the sum of the squares of the lengths of its sides, see Fig. 2.2. This equality can be proven directly from the definition of absolute value and properties of the complex conjugation we have already stated:

$$|z \pm w|^2 = (z \pm w)(\overline{z} \pm \overline{w}) = |z|^2 \pm 2\,\Re(z\,\overline{w}) + |w|^2\,. \tag{2.1}$$

Inequality (g) is called the *triangle inequality*: the length of a side of a triangle is at most equal to the sum of the lengths of the other two sides, with equality if and only if the triangle is degenerate (z and w lie on the same ray); see Fig. 2.2.

The triangle inequality (or rather its squared version) follows from (2.1) and the previous properties of the absolute value, by observing that $\Re(z\,\overline{w}) \le |z\,\overline{w}| = |z|\;|w|$ and using the conditions for equality given in property (d) for the absolute value.

Through the identification of $\mathbb{R}^2$ with $\mathbb{C}$ given above, we can use real or complex notation to describe geometric shapes in the plane. As we show next, sometimes the use of complex notation simplifies the description of the objects under study.

2.1.5 Lines, Circles, and Half Planes

Any line in the plane $\mathbb{R}^2$ with orthogonal coordinates $x$and y is given by an equation of the form

$$a\,x + b\,y + c = 0, \tag{2.2}$$

with a, b, and c real numbers, and a and b not both equal to zero.

Similarly, any circle in the plane is given by an equation of the form

$$(x-d)^2+(y-f)^2-R^2=0, \tag{2.3}$$

with d, f, and R real numbers and $R>0$. In this case, we can read off from the equation that the center of the circle is at the point (d, f), and its radius is R.

We will now see an advantage of using complex notation: both of the above types of geometric figures may be described algebraically by a single equation, thus implying that there is a certain relation between lines and circles on the plane (this relation will be explained later: see Exercise 3.21).

Replacing x by $\dfrac{z+\bar{z}}{2}$ and y by $\dfrac{z-\bar{z}}{2\imath}$ first in (2.2) and then in (2.3), we obtain the following two equations:

$$B\,z+\overline{B}\,\bar{z}+c=0\,, \quad \text{with } B=\frac{a-\imath\, b}{2}\neq 0, \tag{2.4}$$

and

$$|z|^2+(-d+\imath\, f)\,z+(-d-\imath\, f)\,\bar{z}+d^2+f^2-R^2=0, \tag{2.5}$$

or, equivalently,

$$|z-E|=R \quad \text{with } E=d+\imath\, f. \tag{2.6}$$

We claim that both equations (2.4) and (2.5) are special cases of

$$A\,|z|^2+B\,z+\overline{B}\,\bar{z}+C=0\,, \tag{2.7}$$

with A and C real numbers, B complex, $A\geq 0$, and $|B|^2>AC$.

Indeed, if $A=0$ then (2.7) becomes (2.4), which is equivalent to (2.2), whereas if $A>0$ then (2.7) becomes (2.5), which is equivalent to (2.3) with center $E=-\dfrac{\overline{B}}{A}$ and radius $R=\dfrac{\sqrt{|B|^2-AC}}{A}$.

We have thus shown that any circle or line in the plane is given by Equation (2.7), depending on whether $A>0$ or $A=0$.

Similarly, half planes in $\mathbb{C}$ are given by equations of the form

$$\Re(B\,z)>C \quad \text{or, equivalently, } \Im(B\,z)>C,$$

with B in $\mathbb{C}_{\neq 0}$ and C real.

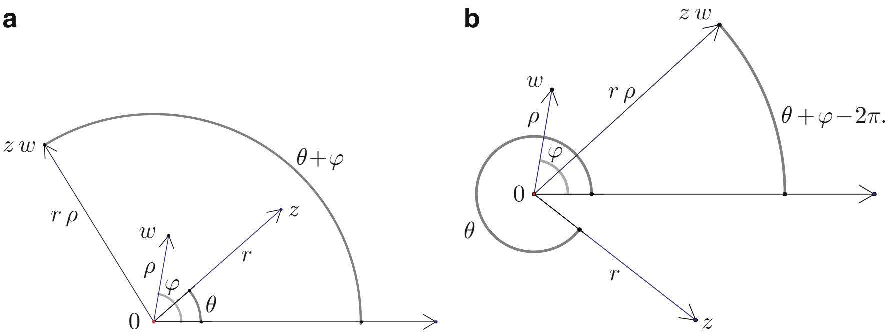

Fig. 2.3 Vector multiplication. (**a**) Sum of arguments smaller than 2π. (**b**) Sum of arguments larger than 2π

2.1.6 Polar Coordinates

A nonzero vector in $\mathbb{C}$ can be described by polar coordinates (r, θ) as well as by the rectangular coordinates (x, y) we have been using. If $z \in \mathbb{C}$ and $z \neq 0$, then we can write

$$z = x + \imath\, y = r\,(\cos\theta + \imath\,\sin\theta)\,,$$

where $r = |z|$ and $\theta = \arg z$ (an *argument* of z) $= \arcsin \dfrac{y}{r} = \arccos \dfrac{x}{r}$.

Note that the last two identities are needed to define the argument and that $\arg z$ is defined up to addition of an integral multiple of 2π. This is why we labeled θ *an* argument of z as opposed to *the* argument.[4]

If $w = \rho[\cos\varphi + \imath\,\sin\varphi]$ is another nonzero complex number, then, using the usual addition formulas for the sine and cosine functions, we have

$$zw = (r\rho)[\cos(\theta + \varphi) + \imath\,\sin(\theta + \varphi)].$$

This polar form of the multiplication formula shows that the complex multiplication of two (nonzero) complex numbers is equivalent to the real multiplication of their moduli and the addition of their arguments, giving a geometric interpretation of how the operation of multiplication acts on vectors represented in polar coordinates; see Fig. 2.3. It also shows (again) that $|z\,w| = |z|\,|w|$. Polar coordinates also provide another way to view Fig. 2.2.

In particular, it follows that if $n \in \mathbb{Z}$ and $z = r\,(\cos\theta + \imath\,\sin\theta)$ is a nonzero complex number, then

$$z^n = r^n[\cos n\theta + \imath\,\sin n\theta];$$

[4]The number π will be defined rigorously in Definition 3.34. Trigonometric functions will be introduced in the next chapter where some of their properties, including addition formulae, will be developed. For the moment, polar coordinates should not be used in proofs.

it also proves the famous *de Moivre's formula*:

$$(\cos\theta + \imath\ \sin\theta)^n = \cos(n\,\theta) + \imath\ \sin(n\,\theta)$$

for $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$.

Therefore, for each nonzero complex number $z = r\,(\cos\theta + \imath\ \sin\theta)$ and each n in $\mathbb{Z}_{>0}$, there exist precisely n complex numbers w such that $w^n = z$; they are the n n-th *roots* of z, and are given by

$$r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + \imath\ \sin\left(\frac{\theta + 2\pi k}{n}\right)\right],$$

with $k = 0, 1, \ldots, n-1$.

Note that these n complex numbers are the vertices of a regular n-gon in the plane.

2.1.7 Coordinates on $\mathbb{C}$

We have already seen that we can use three sets of coordinates on $\mathbb{C}$, as follows.

1. Rectangular (x, y): Each equation x = constant (respectively y = constant) yields a line parallel to the imaginary axis (respectively real axis), while the equation $x = y$ yields the line through the origin with slope equal to 1.
2. Complex $(z, \bar{z})$: Only one of these coordinates is needed to describe a point by the equation z = constant (or $\bar{z}$ = constant), while the equation $z = \bar{z}$ yields the real axis.
3. Polar (r, θ): The equation $r = a$ with $a > 0$ is a circle of radius a centered at the origin, whereas the equation θ = constant is a ray emanating from (but not including) the origin. The equation $r = \theta$ denotes a type of spiral ending at (but not passing through) the origin.

The choice of the appropriate one among the various possible coordinates on $\mathbb{C}$ may simplify a problem. As an example we solve the following one.

Let n be a positive integer, and suppose we want to find the set of points z in $\mathbb{C}$ that satisfy the equation

$$z^n = \bar{z}^n. \tag{2.8}$$

Using rectangular coordinates would lead us to solve

$$(x + \imath\ y)^n = (x - \imath\ y)^n,$$

which is doable but far from pleasant.

Instead, we first note that certainly $z = 0$ satisfies (2.8). For $z \neq 0$, we may use the polar coordinates: the equation we are trying to solve is then equivalent to

$$r^n[\cos n\theta + \imath \ \sin n\theta] = r^n[\cos n\theta - \imath \ \sin n\theta],$$

which implies that $n\,\theta = k\,\pi$ for some integer k. Thus we immediately see that the complete solution to (2.8) is the set of $2n$ rays $\theta = \theta_o$ from the origin (including the origin) with

$$\theta_o \in \left\{0\,, \frac{\pi}{n}\,, \frac{2\pi}{n}\,, \ldots, \frac{(2n-1)\,\pi}{n}\right\}.$$

2.2 More Preliminaries that Rely on Topology, Metrics, and Sequences

We collect some facts on sets of complex numbers and functions defined on them, that mostly follow from translating to the complex system the analogous results from real analysis.

The formula $d(z, w) = |z - w|$, for z and $w \in \mathbb{C}$, defines a **metric on** $\mathbb{C}$. Thus $(\mathbb{C}, d)$ is a metric space, with a metric that agrees with the Euclidean metric on $\mathbb{R}^2$ (under the linear representation of the complex plane described earlier).

Definition 2.1. We say that a sequence (indexed by $n \in \mathbb{Z}_{>0}$) $\{z_n\}$ of complex numbers *converges* to $\alpha \in \mathbb{C}$ if given $\epsilon > 0$, there exists an $N \in \mathbb{Z}_{>0}$ such that $|z_n - \alpha| < \epsilon$ for all $n > N$; in this case we write

$$\lim_{n\to\infty} z_n = \alpha.$$

A sequence $\{z_n\}$ of complex numbers is called *Cauchy* if given $\epsilon > 0$, there exists an $N \in \mathbb{Z}_{>0}$ such that $|z_n - z_m| < \epsilon$ for all $n, m > N$.

Theorem 2.2. *If $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences of complex numbers, then*

(a) *$\{z_n + \alpha\, w_n\}$ is Cauchy for all $\alpha \in \mathbb{C}$.*
(b) *$\{\bar{z}_n\}$ is Cauchy.*
(c) *$\{\,|z_n|\,\} \subset \mathbb{R}_{\geq 0}$ is Cauchy.*

Proof. (a) It suffices to assume that $\alpha \neq 0$. Given $\epsilon > 0$, choose N_1 such that $|z_n - z_m| < \frac{\epsilon}{2}$ for all $n, m > N_1$ and choose N_2 such that $|w_n - w_m| < \frac{\epsilon}{2|\alpha|}$ for all $n, m > N_2$. Choose $N = \max\{N_1, N_2\}$. Then, for all n and $m > N$, we have

$$|(z_n + \alpha\, w_n) - (z_m + \alpha\, w_m)| \leq |z_n - z_m| + |\alpha|\,|w_n - w_m| < \epsilon.$$

(b) It follows directly from $|\bar{z}_n - \bar{z}_m| = |\overline{z_n - z_m}| = |z_n - z_m|$.
(c) We know that for all z and w in $\mathbb{C}$ we have

$$||z| - |w|| \leq |z - w| .$$

Applying this inequality to z_n and z_m in the sequence, we obtain

$$||z_n| - |z_m|| \leq |z_n - z_m| ,$$

and the result follows.

□

Remark 2.3. The above arguments mimic arguments in real analysis needed to establish the corresponding results for real sequences. We will, in the sequel, leave such routine arguments as exercises for the reader.

Corollary 2.4. $\{z_n\}$ *is a Cauchy sequence of complex numbers if and only if* $\{\Re z_n\}$ *and* $\{\Im z_n\}$ *are Cauchy sequences of real numbers.*

Corollary 2.5. $(\mathbb{C}, d)$ *is a complete metric space; that is, every Cauchy sequence of complex numbers converges to a complex number.*

Proof. Observe that the metric on $\mathbb{C}$ restricts to the Euclidean metric on $\mathbb{R}$, which is complete, and applies the previous corollary. □

Definition 2.6. Let $A \subseteq \mathbb{C}$. We say that A is *bounded* if the set of nonnegative real numbers $\{\, |z| \,;\; z \in A\}$ is; that is, if there exists a positive real number M such that $|z| < M$ for all z in A.

Definition 2.7. Let $c \in \mathbb{C}$ and $\epsilon > 0$. The *ϵ-ball about c*, or the *open disc* with center c and radius ϵ, is the set

$$U_c(\epsilon) = U(c, \epsilon) = \{z \in \mathbb{C}; |z - c| < \epsilon\},$$

that is, the interior of the circle with center c and radius ϵ.

Proposition 2.8. *A subset A of $\mathbb{C}$ is bounded if and only if there exist a complex number c and a positive number R such that*

$$A \subset U(c, R).$$

Remark 2.9. A proof is omitted for one of three reasons (in addition to the reason described in Remark 2.3): either it is trivial or it follows directly from results in real analysis or it appears as an exercise at the end of the corresponding chapter.[5] The third possibility is always labeled as such; when standard results in real analysis are needed, there is some indication of what they are or where to find them. For example, the next two theorems are translations to $\mathbb{C}$ of standard metric results for $\mathbb{R}^2$. It

[5]Exercises can be found at the end of each chapter and are numbered by chapter, so that Exercise 2.7 is to be found at the end of Chap. 2.

should be clear from the context when the first possibility occurs. It is recommended that the reader ensures that he/she is able to supply an appropriate proof when none is given.

Theorem 2.10 (Bolzano–Weierstrass). *Every bounded infinite set S in $\mathbb{C}$ has at least one limit point; that is, there exists at least one $c \in \mathbb{C}$ such that, for each $\epsilon > 0$, the ball $U(c, \epsilon)$ contains a point $z \in S$ with $z \neq c$.*

Theorem 2.11. *A set $K \subset \mathbb{C}$ is compact if and only if it is closed and bounded.*

We will certainly be using a number of consequences of compactness not discussed in this chapter (e.g., in a compact metric space, every sequence has a convergent subsequence) and also of connectedness, which we will not define here.

Definition 2.12. Let f be a function defined on a set S in $\mathbb{C}$. We assume that f is complex-valued, unless otherwise stated. Thus f may be viewed as either a map from S into $\mathbb{R}^2$ or into $\mathbb{C}$ and also as two real-valued functions defined on the set S.

Let c be a limit point of S and let α be a complex number. We say that the *limit of f at c* is α, and we write

$$\lim_{z \to c} f(z) = \alpha$$

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - \alpha| < \epsilon \text{ whenever } z \in S \text{ and } 0 < |z - c| < \delta.$$

Remark 2.13. The condition that c is a limit point of S ensures that there are points z in S arbitrarily close to (but different from) c so that $f(z)$ is defined there. Note that it is not required that $f(c)$ be defined.

The above definition is again a translation of language from $\mathbb{R}^2$ to $\mathbb{C}$. Thus we will be able to adopt many results (the next three theorems, in particular) from real analysis. In addition to the usual algebraic operations on pairs of functions $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$ familiar from real analysis, such as $f + cg$ with $c \in \mathbb{C}$, fg, and $\dfrac{f}{g}$ (provided g does not *vanish* on S; that is, if $g(z) \neq 0$ for any $z \in S$ or, equivalently, if no $z \in S$ is a *zero* of g), we will consider other functions constructed from a single function f, that are usually not emphasized in real analysis. Among them are the following:

$$(\Re f)(z) = \Re f(z), \ (\Im f)(z) = \Im f(z), \ \overline{f}(z) = \overline{f(z)}, \ |f|(z) = |f(z)|,$$

also defined on S.

For instance, if $f(z) = z^2 = x^2 - y^2 + 2\imath\, x\, y$ for $z \in \mathbb{C}$, we have $(\Re f)(z) = x^2 - y^2$, $(\Im f)(z) = 2\,x\,y$, $\overline{f}(z) = \overline{z}^2 = x^2 - y^2 - 2\imath\, x\, y$, and $|f|(z) = |z|^2 = x^2 + y^2$ for $z \in \mathbb{C}$.

Theorem 2.14. *Let S be a subset of $\mathbb{C}$ and let f and g be functions defined on S. If c is a limit point of S, then:*

(a) $\lim_{z\to c}(f + a\, g)(z) = \lim_{z\to c} f(z) + a \lim_{z\to c} g(z)$ *for all* $a \in \mathbb{C}$
(b) $\lim_{z\to c}(fg)(z) = \lim_{z\to c} f(z) \lim_{z\to c} g(z)$
(c) $\lim_{z\to c} |f|(z) = \left|\lim_{z\to c} f(z)\right|$
(d) $\lim_{z\to c} \overline{f}(z) = \overline{\lim_{z\to c} f(z)}$

Remark 2.15. The usual interpretation of the above formulae is used here and in the rest of the book: the LHS[6] exists whenever the RHS exists, and then we have the stated equality.

Corollary 2.16. *Let S be a subset of $\mathbb{C}$, let f be a function defined on S, and $\alpha \in \mathbb{C}$. Set $u = \Re f$ and $v = \Im f$ (so that $f(z) = u(z) + \imath\, v(z)$). If c is a limit point of S, then*

$$\lim_{z\to c} f(z) = \alpha$$

if and only if

$$\lim_{z\to c} u(z) = \Re\alpha \quad \textit{and} \quad \lim_{z\to c} v(z) = \Im\alpha.$$

Definition 2.17. Let S be a subset of $\mathbb{C}$, $f : S \to \mathbb{C}$ be a function defined on S, and $c \in S$ be a point in S. We say that:

(a) f is *continuous at c* if $\lim_{z\to c} f(z) = f(c)$.
(b) f is *continuous on S* if it is continuous at each c in S.
(c) f is *uniformly continuous on S* if for all $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - f(w)| < \epsilon \text{ for all } z \text{ and } w \text{ in } S \text{ with } |z - w| < \delta.$$

Remark 2.18. A function f is (uniformly) continuous on S if and only if both $\Re f$ and $\Im f$ are.

Uniform continuity implies continuity, but the converse is not true in general.

Theorem 2.19. *Let f and g be functions defined in appropriate sets, that is, sets where the composition $g \circ f$ of these functions makes sense. Then the following properties hold:*

(*a*) *If f is continuous at c and $f(c) \neq 0$, then $\dfrac{1}{f}$ is defined in a neighborhood of c and is continuous at c.*
(*b*) *If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.*

Theorem 2.20. *Let $K \subset \mathbb{C}$ be a compact set and $f : K \to \mathbb{C}$ be a continuous function on K. Then f is uniformly continuous on K.*

[6] LHS (RHS) are standard abbreviations for left (right) hand side and will be used throughout this book.

Proof. A continuous mapping from a compact metric space to a metric space is uniformly continuous. □

Definition 2.21. Given a sequence of functions $\{f_n\}$, all defined on the same set S in $\mathbb{C}$, we say that $\{f_n\}$ *converges uniformly* to a function f on S if for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}_{>0}$ such that

$$|f(z) - f_n(z)| < \epsilon \text{ for all } z \in S \text{ and all } n > N.$$

Remark 2.22. $\{f_n\}$ converges uniformly on S (to some function f) if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}_{>0}$ such that

$$|f_n(z) - f_m(z)| < \epsilon \text{ for all } z \in S \text{ and all } n \text{ and } m > N.$$

Note that in this case the limit function f is uniquely determined; it is the pointwise limit $f(z) = \lim_{n\to\infty} f_n(z)$, for all $z \in S$.

Theorem 2.23. *Let $\{f_n\}$ be a sequence of functions defined on $S \subseteq \mathbb{C}$. If:*

(1) *$\{f_n\}$ converges uniformly on S.*
(2) *Each f_n is continuous on S.*

Then the function f defined by

$$f(z) = \lim_{n\to\infty} f_n(z), \; z \in S$$

is continuous on S.

Proof. Start with two points z and c in S. Then for each natural number n we have

$$|f(z) - f(c)| \le |f(z) - f_n(z)| + |f_n(z) - f_n(c)| + |f_n(c) - f(c)|.$$

Now fix $\epsilon > 0$. By (1), the first and third term on the right-hand side are less than $\frac{\epsilon}{3}$ for n large. If we now fix c and n, it follows from (2) that the second term is less than $\frac{\epsilon}{3}$ as soon as z is close enough to c. Thus f is continuous at c. □

Definition 2.24. A *domain* or *region* in $\mathbb{C}$ is a subset of $\mathbb{C}$ which is open and connected.

Remark 2.25. Note that a domain in $\mathbb{C}$ could also be defined as an open arcwise connected subset of $\mathbb{C}$. (See also Exercise 2.20.) Also note that each point in a domain D is a limit point of D, and therefore it makes sense to ask, at each point in D, about the limit of any function defined on D.

2.3 Differentiability and Holomorphic Mappings

Up to now, the complex numbers were used mainly to supply us with a convenient alternative notation. This is about to change. The definition of the derivative of a complex-valued function of a complex variable mimics that for the derivative of a real-valued function of a real variable. However, we shall see shortly that the properties of the two classes of functions are quite different.

Definition 2.26. Let f be a function defined in some disc about $c \in \mathbb{C}$. We say that f is *(complex) differentiable at* c provided

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \tag{2.9}$$

exists. In this case the limit is denoted by

$$f'(c), \ \frac{\mathrm{d}f}{\mathrm{d}z}(c), \ \left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_{z=c}, \text{ or } (Df)(c),$$

and is called the *derivative of* f *at* c.

Remark 2.27. (1) It is important that h be an arbitrary *complex* number (of small nonzero modulus) in the above definition.

(2) Note that

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{z \to c} \frac{f(z) - f(c)}{z - c}.$$

(3) If f is differentiable at c, then f is continuous at c. The converse is not true in general; see Example 2.32.4.

(4) We consider two identities for a function f defined in a neighborhood of $c \in \mathbb{C}$:

$$f(c+h) = a_0 + \epsilon(h) \text{ with } \lim_{h \to 0} \epsilon(h) = 0,$$

and

$$f(c+h) = a_0 + a_1 h + h\epsilon(h) \text{ with } \lim_{h \to 0} \epsilon(h) = 0.$$

As in real analysis, the first of these says that f is continuous at c if and only if $f(c) = a_0$; the second says that f is differentiable at c if and only if $f(c) = a_0$ and $f'(c) = a_1$. Whereas in the real case the second statement is sharp with regard to smoothness, we shall see that in the complex case, under appropriate conditions, it can be improved significantly.

Notation 2.28. If the function f is differentiable on a domain D (i.e., at each point of D), then it defines a function $f' : D \to \mathbb{C}$.

Thus for every $n \in \mathbb{Z}_{\geq 0}$ we can define inductively $f^{(n)}$, the n-th derivative of f, as follows:

$f^{(0)} = f$, and if $f^{(n)}$ is defined for $n \geq 0$, then we set $f^{(n+1)} = \left(f^{(n)}\right)'$ whenever the appropriate limits exist.

It is customary to abbreviate $f^{(2)}$ and $f^{(3)}$ by f'' and f''', respectively. Of course, $f^{(1)} = f'$.

Definition 2.29. Let f be a function defined in a neighborhood of $c \in \mathbb{C}$. Then f is *holomorphic* or *analytic at* c if it is differentiable in a neighborhood (perhaps smaller) of c. A function defined on an open set U is *holomorphic* or *analytic on* U if it is holomorphic (equivalently, differentiable) at each point of U. It should be emphasized that holomorphicity is always defined on open sets.

A function f is called *anti-holomorphic* if $\bar{f}$ is holomorphic.

The usual rules of differentiation hold. Let f and g be functions defined in a neighborhood of $c \in \mathbb{C}$, let F be a function defined in a neighborhood of $f(c)$, and let $a \in \mathbb{C}$. Then (recall Remark 2.15):

(a) $(f + ag)'(c) = f'(c) + ag'(c)$
(b) $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$
(c) $(F \circ f)'(c) = F'(f(c))f'(c)$ (the chain rule)
(d) $\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{f(c)^2}$ provided $f(c) \neq 0$
(e) if $f(z) = z^n$ with $n \in \mathbb{Z}$ (and $z \in \mathbb{C}_{\neq 0}$ if $n \leq 0$), then $f'(z) = n\,z^{n-1}$

Remark 2.30. About the chain rule (c): If $f(z) = w$ is a differentiable function of z and if $F(w) = \zeta$ is a differentiable function of w, then we often write the chain rule as

$$\frac{d\zeta}{dz} = \frac{d\zeta}{dw}\,\frac{dw}{dz}.$$

A "proof" follows. Let z_0 be arbitrary in the domain of f, and set $w_0 = f(z_0)$ and $\zeta_0 = F(w_0)$. Note that $w = f(z) \to w_0$ as $z \to z_0$. Now

$$\begin{aligned}(F \circ f)'(z_0) &= \frac{d\zeta}{dz}(z_0) = \lim_{z \to z_0} \frac{\zeta - \zeta_0}{z - z_0} \\ &= \lim_{z \to z_0} \frac{(\zeta - \zeta_0)(w - w_0)}{(w - w_0)(z - z_0)} = \lim_{w \to w_0} \frac{\zeta - \zeta_0}{w - w_0}\ \lim_{z \to z_0} \frac{w - w_0}{z - z_0} \\ &= \frac{d\zeta}{dw}(w_0)\,\frac{dw}{dz}(z_0)\ = F'(w_0)\,f'(z_0).\end{aligned}$$

This "proof" has an error in it, what is it?

Definition 2.31. A function defined on the complex plane is called *entire* if it is holomorphic on $\mathbb{C}$, that is, if its derivative exists at each point of $\mathbb{C}$.

Example 2.32. We illustrate some of the concepts introduced with more or less familiar examples.

1. Every polynomial (in one complex variable) is entire. These (apparently) simple objects have fairly complicated behavior, that is studied, for example, as part of complex dynamics.
2. A *rational* function is a function of the form $R = \dfrac{P}{Q}$, where P and Q are polynomials (in one complex variable), with Q not the zero polynomial. Note that the polynomial Q has only finitely many zeros (the number of zeros, properly counted, equals the *degree* of Q; see Exercise 3.19). The rational function R is holomorphic on $\mathbb{C} - \{\text{zeros of } Q\}$.
3. A special case of Example 2.32.2 is $R(z) = \dfrac{az+b}{cz+d}$ with a, b, c, and d fixed complex numbers satisfying $ad - bc \neq 0$. These rational functions are called *fractional linear transformations* or *Möbius transformations* and will be studied in detail in Sect. 8.1. They are the building blocks for much that will follow in this book—automorphisms of domains in the Riemann sphere, and Blaschke products, and as important ingredients for much current research in areas of complex analysis: Riemann surfaces, Fuchsian, and (the more general case of) Kleinian groups.
4. In real analysis it takes work to construct a continuous function on $\mathbb{R}$ that is nowhere differentiable. The situation with respect to complex differentiability is much simpler. The functions $z \mapsto \bar{z}$ and $z \mapsto |z|$ are both continuous on $\mathbb{C}$, but they are nowhere (complex) differentiable, since the corresponding limits (2.9) do not exist at any c in $\mathbb{C}$.

2.3.1 Convention

Whenever we write $z = x + \imath\, y$ for variables and $f = u + \imath\, v$ for functions, then we automatically mean that $x = \Re z$, $y = \Im z$, $u = \Re f$, and $v = \Im f$. We also write $u = u(x, y)$ and $v = v(x, y)$, as well as $u = u(z)$ and $v = v(z)$. We naturally use subscripts to denote partial derivatives with respect a given variable, so that notation such as u_x, u_y, v_x, or v_y has the obvious meaning.

2.3.2 The Cauchy-Riemann (CR) Equations

Theorem 2.33. *If $f = u + \imath\, v$ is differentiable at $c = a + \imath\, b$, then u and v have partial derivatives with respect to x and y at c, and they satisfy the* ***Cauchy–Riemann equations:***

$$u_x(a,b) = v_y(a,b), \quad u_y(a,b) = -v_x(a,b). \tag{CR}$$

Furthermore,

$$f'(c) = u_x(a,b) + \imath\, v_x(a,b) = -\imath\, u_y(a,b) + v_y(a,b).$$

Proof. First take $h = \alpha$, with α real, in the limit (2.9) appearing in the definition of differentiability and compute

$$f'(c) = u_x(a,b) + \imath\, v_x(a,b).$$

Then take $h = \imath\beta$, with β real, and compute

$$f'(c) = -\imath\, u_y(a,b) + v_y(a,b).$$

Comparing the two expressions we obtain the desired result. $\square$

Remark 2.34. Let $f = u + \imath\, v$ be a function defined in a neighborhood of c, such that the partial derivatives u_x, v_x, u_y, and v_y exist at c. Then we will use the obvious notation:

$$f_x = u_x + \imath\, v_x \text{ and } f_y = u_y + \imath\, v_y.$$

In this language the CR equations (CR) for the function f are written as follows:

$$f_x(c) = -\imath\, f_y(c), \tag{2.10}$$

and Theorem 2.33 may be stated as follows.
If f is differentiable at c, then f has partial derivatives with respect to x and y at c, and they satisfy the Cauchy–Riemann equation (2.10). *Furthermore,*

$$f'(c) = f_x(c) = -\imath\, f_y(c). \tag{2.11}$$

Remark 2.35. The CR equations are not sufficient for differentiability. To see this, define

$$f(z) = \begin{cases} z^5\,|z|^{-4} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

It is easy to verify that the function f is continuous on $\mathbb{C}$. Furthermore, for α real and nonzero we have $\dfrac{f(\alpha)}{\alpha} = 1$, and for β real and nonzero we have $\dfrac{f(\imath\beta)}{\imath\beta} = 1$. Therefore $f_x(0) = 1$ and $f_y(0) = \imath$, and f satisfies the Cauchy–Riemann equation (2.10) at $z = 0$. However, f is not differentiable at $z = 0$. Indeed, if it were, we would conclude from (2.11) that $f'(0) = 1$. Now take $h = (1+\imath)\gamma$ with γ real and nonzero and observe that $\dfrac{f(h)}{h} = -1$ so that $f'(0)$ would be equal to -1.

Remark 2.36. We may use the CR equations to try to manufacture an entire function with a given real (or imaginary) part. Let us start with the real-valued function $u(x, y) = x^2+y^2$. If this were to be the real part of some entire function $f = u+\imath\, v$, then the CR equations would help us to determine v. Since $u_x = 2x = v_y$ and $u_y = 2y = -v_x$ must be satisfied, by integrating $2x = v_y$ with respect to y we obtain that $v(x, y) = 2\,x\,y + h(x)$, for some function h of x alone; by integrating $2y = -v_x$ with respect to x we obtain that $v(x, y) = -2\,y\,x + g(y)$, for some function g of y. It is quite obvious that these two expressions for v are incompatible, and hence there is no such function f.

The situation changes dramatically for $u(x, y) = x^2 - y^2$, where similar calculations lead to $v(x, y) = 2\,x\,y + h(x) = 2\,y\,x + g(y)$, and we may choose $h(x) = g(y) = a$, for any real value of a. We have thus obtained a family of entire functions f with prescribed real part u; these are given by

$$f(z) = u(x, y) + \imath\, v(x, y) = x^2 - y^2 + \imath\,(2xy + a) = z^2 + \imath\, a,$$

with a any real number.

We will determine later the class of real-valued functions u for which the construction outlined above leads us to an entire function f.

Definition 2.37. For a complex-valued function f defined on a region in the complex plane, such that both f_x and f_y exist in this region, set

$$f_z = \frac{1}{2}\left(f_x - \imath\, f_y\right)$$

and

$$f_{\bar{z}} = \frac{1}{2}\left(f_x + \imath\, f_y\right).$$

Remark 2.38. The partial derivatives just defined are computed as if z and $\bar{z}$ were independent variables. For instance, if $f(z) = z^2 + 5z\bar{z}^3$, then it is easy to verify that $f_z = 2z + 5\bar{z}^3$ and $f_{\bar{z}} = 15z\bar{z}^2$.

These partials not only simplify the notation: for example, the two Cauchy–Riemann equations (CR) are written as the single equation

$$f_{\bar{z}} = 0, \qquad \text{(CR complex)}$$

or, if f is differentiable at c, then

$$f'(c) = f_z(c),$$

but they also allow us to produce more concise arguments (and, as we shall see later, prettier formulae), as illustrated in the proof of the lemma below.

We use the notation $\dfrac{\partial f}{\partial \bar{z}}$ (respectively $\dfrac{\partial f}{\partial z}$) interchangeably with $f_{\bar{z}}$ (respectively f_z).

Lemma 2.39. *If f is a $\mathbf{C}^1$-complex-valued function defined in a neighborhood of $c \in \mathbb{C}$, then*

$$f(z) - f(c) = (z-c)\, f_z(c) + \overline{(z-c)}\, f_{\bar{z}}(c) + |z-c|\ \varepsilon(z,c), \tag{2.12}$$

for all $z \in \mathbb{C}$ with $|z-c|$ small, where $\varepsilon(z,c)$ is a complex-valued function of z and c such that

$$\lim_{z \to c} \varepsilon(z,c) = 0.$$

Proof. As usual we write $z = x + \imath\, y$, $c = a + \imath\, b$, and $f = u + \imath\, v$ and abbreviate $\triangle u = u(z) - u(c)$, $\triangle x = x - a$, $\triangle y = y - b$, and $\triangle z = z - c = \triangle x + \imath\, \triangle y$. By hypothesis, the real-valued function u has continuous first partial derivatives defined in a neighborhood of c, and we can define ε_1 by

$$\varepsilon_1(z,c) = \frac{\triangle u - u_x(c)\triangle x - u_y(c)\triangle y}{|\triangle z|}$$

for $z \neq c$, and $\varepsilon_1(c,c) = 0$. Then it is clear that

$$u(z) - u(c) = (x-a)u_x(a,b) + (y-b)u_y(a,b) + |z-c|\, \varepsilon_1(z,c).$$

We now show that

$$\lim_{z \to c} \varepsilon_1(z,c) = 0. \tag{2.13}$$

If we rewrite $\triangle u$ as

$$\triangle u = [u(x,y) - u(x,b)] + [u(x,b) - u(a,b)]\,,$$

it follows from the (real) mean value theorem applied to the two summands on the RHS that

$$\triangle u = u_y(x, y_0)\triangle y + u_x(x_0, b)\triangle x,$$

where y_0 is between y and b and x_0 is between x and a. Thus

$$\varepsilon_1(z,c) = \frac{[u_y(x,y_0) - u_y(a,b)]\triangle y + [u_x(x_0,b) - u_x(a,b)]\triangle x}{|\triangle z|},$$

for $z \neq c$. Hence we see that

$$|\varepsilon_1(z,c)| \le \left|u_y(x,y_0) - u_y(a,b)\right| + |u_x(x_0,b) - u_x(a,b)|\,,$$

and the claim (2.13) follows.

Similarly,

$$v(z) - v(c) = (x-a)v_x(a,b) + (y-b)v_y(a,b) + |z-c|\, \varepsilon_2(z,c),$$

with

$$\lim_{z\to c} \varepsilon_2(z, c) = 0. \tag{2.14}$$

With obvious notational conventions, we compute that

$$\begin{aligned}
\triangle f &= \triangle u + \imath\, \triangle v \\
&= [u_x(a, b) + \imath\, v_x(a, b)]\, \triangle x + \left[u_y(a, b) + \imath\, v_y(a, b)\right] \triangle y + |\triangle z|\, \varepsilon(z, c) \\
&= \frac{\triangle z + \overline{\triangle z}}{2} f_x(c) + \imath\, \frac{\overline{\triangle z} - \triangle z}{2} f_y(c) + |\triangle z|\, \varepsilon(z, c) \\
&= \triangle z f_z(c) + \overline{\triangle z} f_{\bar z}(c) + |\triangle z|\, \varepsilon(z, c),
\end{aligned}$$

with $\varepsilon(z, c) = \varepsilon_1(z, c) + \imath\, \varepsilon_2(z, c)$. Now equalities (2.13) and (2.14) imply that

$$\lim_{z\to c} \varepsilon(z, c) = 0. \qquad \square$$

Theorem 2.40. *If the function f has continuous first partial derivatives in a neighborhood of c that satisfy the* CR *equations at c, then f is (complex) differentiable at c.*

Proof. The theorem is an immediate consequence of (2.12), since in this case $f_{\bar z}(c) = 0$ and hence $f'(c) = f_z(c)$. $\square$

Corollary 2.41. *If the function f has continuous first partial derivatives in an open neighborhood U of $c \in \mathbb{C}$ and the* CR *equations hold at each point of U, then f is holomorphic at c (in fact on U).*

Remark 2.42. The converse to this corollary is also true. It will take us some time to prove it.

Theorem 2.43. *If f is holomorphic and real-valued on a domain D, then f is constant.*

Proof. As usual we write $f = u + \imath\, v$; in this case $v = 0$. The CR equations say $u_x = v_y = 0$ and $u_y = -v_x = 0$. Thus u is constant, since D is connected. $\square$

Theorem 2.44. *If f is holomorphic and $f' = 0$ on a domain D, then f is constant.*

Proof. As above $f = u + \imath\, v$ and $f' = u_x + \imath\, v_x = 0$. The last equation together with the CR equations say $0 = u_x = v_y$ and ; $0 = v_x = -u_y$. Thus both u and v are constant, since D is connected. $\square$

Exercises

2.1. (a) Let $\{z_n\}$ be a sequence of complex numbers and assume

$$|z_n - z_m| < \frac{1}{1 + |n - m|}\,, \quad \text{for all } n \text{ and } m.$$

Show that the sequence converges.
Do you have enough information to evaluate $\lim_{n\to\infty} z_n$?
What else can you say about this sequence?

(b) Let $\{z_n\}$ be a sequence with $\lim_{n\to\infty} z_n = 0$ and let $\{w_n\}$ be a bounded sequence. Show that

$$\lim_{n\to\infty} w_n z_n = 0.$$

2.2. (a) Let z and c denote two complex numbers. Show that

$$|\overline{c}\, z - 1|^2 - |z - c|^2 = (1 - |z|^2)\,(1 - |c|^2).$$

(b) Use (a) to conclude that if c is any fixed complex number with $|c| < 1$, then

$$\{z \in \mathbb{C}; |z - c| < |\overline{c}\, z - 1|\} = \{z \in \mathbb{C}; |z| < 1\},$$
$$\{z \in \mathbb{C}; |z - c| = |\overline{c}\, z - 1|\} = \{z \in \mathbb{C}; |z| = 1\} \text{ and}$$
$$\{z \in \mathbb{C}; |z - c| > |\overline{c}\, z - 1|\} = \{z \in \mathbb{C}; |z| > 1\}.$$

2.3. Let a, b, and c be three distinct points on a straight line with b between a and c. Show that

$$\frac{a-b}{c-b} \in \mathbb{R}_{<0}.$$

2.4. (a) Given two points z_1, z_2 such that $|z_1| < 1$ and $|z_2| < 1$, show that for every point $z \neq 1$ in the closed triangle with vertices z_1, z_2, and 1,

$$\frac{|1-z|}{1-|z|} \leq K,$$

where K is a constant that depends only on z_1 and z_2.

(b) Determine the smallest value of K for $z_1 = \dfrac{1+\imath}{2}$ and $z_2 = \dfrac{1-\imath}{2}$.

2.5. Verify the Cauchy–Riemann equations for the function $f(z) = z^3$ by splitting f into its real and imaginary parts.

2.6. Suppose $z = x + \imath\, y$. Define

$$f(z) = \frac{xy^2\,(x + \imath\, y)}{x^2 + y^4},$$

for $z \neq 0$ and $f(0) = 0$. Show that

$$\lim \frac{f(z) - f(0)}{z} = 0$$

as $z \to 0$ along any straight line. Show that as $z \to 0$ along the curve $x = y^2$, the limit of the difference quotient is $\frac{1}{2}$, thus showing that $f'(0)$ does not exist.

2.7. Let $x = r\cos\theta$ and $y = r\sin\theta$. Show that the Cauchy–Riemann equations in polar coordinates for $F = U + \imath V$, where $U = U(r,\theta)$ and $V = V(r,\theta)$, are

$$r\frac{\partial U}{\partial r} = \frac{\partial V}{\partial \theta} \quad \text{and} \quad r\frac{\partial V}{\partial r} = -\frac{\partial U}{\partial \theta},$$

or, in alternate notation,

$$rU_r = V_\theta \quad \text{and} \quad rV_r = -U_\theta.$$

2.8. Let f be a complex-valued function defined on a region in the complex plane, and assume that both f_x and f_y exist in this region. Using the definitions of f_z and $f_{\bar{z}}$, show that for $\mathbf{C}^1$-functions f,

$$f \text{ is holomorphic if and only if } f_{\bar{z}} = 0$$

and that in this case $f_z = f'$.

2.9. Let R and Φ be two real-valued $\mathbf{C}^1$-functions of a complex variable z. Show that $f = Re^{\imath\Phi}$ is holomorphic if and only if

$$R_{\bar{z}} + \imath R\Phi_{\bar{z}} = 0.$$

2.10. Show that if f and g are $\mathbf{C}^1$-functions, then the (complex) chain rule is expressed as follows (here $w = f(z)$ and g is viewed as a function of w).

$$(g \circ f)_z = g_w f_z + g_{\overline{w}} \overline{f}_z$$

and

$$(g \circ f)_{\bar{z}} = g_w f_{\bar{z}} + g_{\overline{w}} \overline{f}_{\bar{z}}.$$

2.11. Let p be a complex-valued polynomial of two real variables:

$$p(z) = \sum a_{ij} x^i y^j.$$

Write

$$p(z) = \sum_{j \geq 0} P_j(z)\bar{z}^j,$$

where each P_j is of the form $P_j(z) = \sum b_{ij} z^i$ (a polynomial in z). Prove that p is an entire function if and only if

$$0 \equiv P_1 \equiv P_2 \equiv \ldots.$$

What can you conclude in this case for the matrix $[a_{ij}]$?

2.12. Deduce the analogues of the CR equations for anti-holomorphic functions, in rectangular, polar, and complex coordinates.

2.13. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, and set $g(z) = \overline{f}(\overline{z})$ and $h(z) = f(\overline{z})$, for z in $\mathbb{C}$. Show that g is holomorphic and h is anti-holomorphic on $\mathbb{C}$. Furthermore, h is holomorphic on $\mathbb{C}$ if and only if f is a constant function.

2.14. Let D be an arbitrary (nonempty) open connected set in $\mathbb{C}$. Describe the class of complex-valued functions on D that are both holomorphic and anti-holomorphic.

2.15. Does there exist a holomorphic function f on $\mathbb{C}$ whose real part is:

(a) $u(x, y) = \mathrm{e}^x$? Or
(b) $u(x, y) = \mathrm{e}^x(x \cos y - y \sin y)$?

Justify your answer; that is, if yes, exhibit the holomorphic function(s) and if not, prove it.

2.16. Prove the *fundamental theorem of algebra*: If $a_0, \ldots, a_{n-1}$ are complex numbers ($n \geq 1$) and $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, then there exists a number $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Hint: A standard method of attack:

(a) Show that there are an $M > 0$ and an $R > 0$ such that for all $|z| \geq R$, $|p(z)| \geq M$ holds.
(b) Show next that there is a $z_0 \in \mathbb{C}$ such that

$$|p(z_0)| = \min\{|p(z)|; z \in \mathbb{C}\}.$$

(c) By the change of variable $p(z + z_0) = g(z)$, it suffices to show that $g(0) = 0$.
(d) Write $g(z) = \alpha + z^m(\beta + c_1 z + \cdots + c_{n-m}z^{n-m})$ with $\beta \neq 0$. Choose γ such that

$$\gamma^m = -\frac{\alpha}{\beta}.$$

If $\alpha \neq 0$, obtain the contradiction $|g(\gamma z)| < |\alpha|$ for some z.

Note. We will later have several simpler proofs of this theorem using results from complex analysis, for instance, in Theorem 5.16 and Exercise 6.1. See also the April 2006 issue of *The American Mathematical Monthly* for yet other proofs of this fundamental result.

2.17. Conclude from the fundamental theorem of algebra that a nonconstant complex polynomial of degree n has n complex roots, counted with multiplicities.

Use this result to show that a nonconstant real polynomial that cannot be factored as a product of two nonconstant real polynomials of lower degree (i.e., a real irreducible nonconstant polynomial) has degree one or two.

2.18. Using the fundamental theorem of algebra stated in Exercise 2.16, prove the *Frobenius theorem*: If F is a field containing the reals whose dimension as a real vector space is finite, then either F is the reals or F is (isomorphic to) $\mathbb{C}$.

Hint: An outline of possible steps follows.

(a) Assume $\dim_{\mathbb{R}} F = n > 1$. Show that for θ in $F - \mathbb{R}$ there exists a nonzero *real* polynomial p with leading coefficient 1 and such that $p(\theta) = 0$.
(b) Show that there exist real numbers β and γ such that

$$\theta^2 - 2\beta\theta + \gamma = 0.$$

(c) Show that there exists a positive real number δ such that $(\theta - \beta)^2 = -\delta^2$, and therefore

$$\sigma = \frac{\theta - \beta}{\delta}$$

is an element of F satisfying $\sigma^2 = -1$.
(d) The field

$$G = \mathbb{R}(\sigma) = \{x + y\sigma : x, y \in \mathbb{R}\} \subseteq F$$

is isomorphic to $\mathbb{C}$, so without loss of generality assume $\sigma = \imath$ and $G = \mathbb{C}$.
Conclude by showing that any element of F is the root of a *complex* polynomial with leading coefficient 1 and is therefore a complex number.

2.19. Prove the following statements, where automorphism is a bijection preserving sums and products.

(a) Every automorphism of the real field is the identity.
(b) Every automorphism of the complex field fixing the reals is either the identity or conjugation.
(c) Every continuous automorphism of the complex field is either the identity or conjugation.

2.20. A domain is defined to be an open connected set. It was remarked that it could also be defined to be an open arcwise connected set. Can it be defined as an open path connected set? Justify your answer.