9

Brownian bridge asymptotics for random mappings

This chapter reviews Brownian bridge asymptotics for random mappings, first described in 1994 by Aldous and Pitman. The limit distributions as $n \to \infty$, of various functionals of a uniformly distributed random mapping from an n element set to itself, are those of corresponding functionals of a Brownian bridge. Similar results known to hold for various non-uniform models of random mappings, according to a kind of invariance principle. A mapping $M_n : [n] \to [n]$ can be identified with its digraph $\{i \to M_n(i),\ i \in [n]\}$, as in Figure 1.

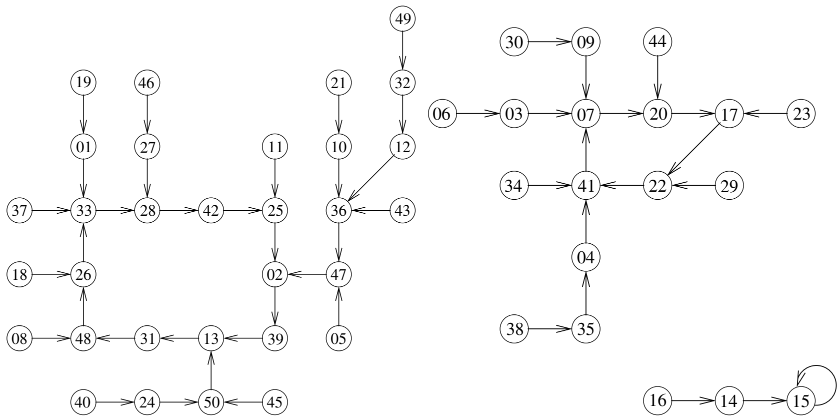

Figure 9.1: Digraph of a mapping $M_{50} : [50] \to [50]$.

Note how the mapping digraph encodes various features of iterates of the mapping. A mapping digraph can be decomposed as a collection of *rooted trees* together with some extra structure (*cycles, basins of attraction*). If each rooted tree is regarded as a plane tree and encoded by its Harris walk, defined by depth-first search following Harris [193], then given some ordering of tree-components, one can concatenate these Harris walks to define a *mapping-walk* which encodes numerous features of M_n.

From now on, we shall be interested in a uniformly distributed random mapping M_n. The connection between random mappings and Brownian bridge, first developed in [17], can be summarized as follows.

- For a uniform random mapping, the induced distribution on tree-components is such that the tree-walks, suitably normalized, converge to Brownian excursion as the tree size increases to infinity. So it is to be expected that the mapping-walks, suitably normalized, should converge to a limit process defined by some concatenation of Brownian excursions.
- With an appropriate choice of ordering of tree components, the weak limit of normalized mapping walks is reflecting Brownian bridge.

The subtle issue is how to order the tree components so that both

a) the mapping-walk encodes structure of cycles and basins of the mapping, and

b) the limit in distribution of the normalized mapping-walk can be explicitly identified.

How this can be done is discussed in some detail in the following sections, which are organized as follows:

9.1. Basins and trees deals with the definitions of basins and trees.
9.2. Mapping walks introduces two variants of the mapping walk.
9.3. Brownian asymptotics contains the main result: the scaled mapping-walk derived from a uniform random mapping M_n, with $2n$ steps of size $\pm 1/\sqrt{n}$ per unit time, converges in distribution to $2|B^{\rm br}|$ where $B^{\rm br}$ is a standard Brownian bridge.
9.4. The diameter As an application of the main result, the diameter of the digraph of M_n, normalized by $\sqrt{n}$, is shown to converge in distribution to an unusual functional of $B^{\rm br}$.
9.5. The height profile The normalized height profile of the forest derived from M_n converges weakly to the process of local times of $|B^{\rm br}|$.
9.6. Non-uniform random mappings This section collects references to extensions of these asymptotics to various kinds of non-uniform random mappings.

9.1. Basins and trees

Fix a mapping M_n. It has a set of *cyclic points*

$$\mathcal{C}_n := \{i \in [n] : M_n^k(i) = i \text{ for some } k \geq 1\},$$

where M_n^k is the kth iterate of M_n. Let $\mathcal{T}_{n,c}$ be the set of vertices of the tree component of the digraph with root $c \in \mathcal{C}_n$. Note that $\mathcal{T}_{n,c}$ might be a trivial tree with just a single root vertex. The tree components are bundled by the disjoint cycles $\mathcal{C}_{n,j} \subseteq \mathcal{C}_n$ to form the *basins of attraction* (connected components) of the mapping digraph, say

$$\mathcal{B}_{n,j} := \bigcup_{c \in \mathcal{C}_{n,j}} \mathcal{T}_{n,c} \quad \supseteq \mathcal{C}_{n,j} \text{ with } \bigcup_j \mathcal{B}_{n,j} = [n] \text{ and } \bigcup_j \mathcal{C}_{n,j} = \mathcal{C}_n \tag{9.1}$$

where all three unions are disjoint unions, and the $\mathcal{B}_{n,j}$ and $\mathcal{C}_{n,j}$ are indexed in some way by $j = 1, \ldots, |\mathcal{C}_n|$. Note that each tree component $\mathcal{T}_{n,c}$ is regarded here just as a subset of $[n]$, which is given the structure of a rooted tree by the action of M_n. The precise meaning of $\mathcal{B}_{n,j}$ and $\mathcal{C}_{n,j}$ now depends on the convention for ordering the cycles, which turns out to be of some importance. Two possible conventions are the *cycles-first ordering*, meaning the $\mathcal{C}_{n,j}$ are put in order of their least elements, and the *basins-first ordering* meaning the $\mathcal{B}_{n,j}$ are put in order of their least elements. Rather than introduce two separate notations for the two orderings, the same notation may be used for either ordering, with an indication of which is meant. Whichever ordering, the definitions of $\mathcal{B}_{n,j}$ and $\mathcal{C}_{n,j}$ are always linked by $\mathcal{B}_{n,j} \supseteq \mathcal{C}_{n,j}$, and (9.1) holds.

The following basic facts are easily deduced from these definitions, and results of Sections Section 2.4 and Section 4.5.

Structure of the basin partition Let Π_n^{basins} be the random partition of $[n]$ whose blocks are the basins of attraction of uniform random mapping M_n. Then Π_n^{basins} is a $\text{Gibbs}_{[n]}(1^\bullet, w_\bullet)$ partition, for w_j the number of mappings of $[j]$ whose digraph is connected. As remarked below Theorem 2.4, that implies the result of Aldous [14] that

$$\Pi_n^{\text{basins}} \xrightarrow{d} \Pi_\infty^{(0,\frac{1}{2})} \tag{9.2}$$

where the limit is a $(0, \frac{1}{2})$ partition of positive integers.

Structure of the tree partition Let Π_n^{trees} be the random partition of $[n]$ whose blocks are the tree components of the uniform random mapping M_n. So Π_n^{trees} is a refinement of Π_n^{basins}, with each basin split into its tree components. Note that the number of components of Π_n^{trees} equals the the number of cyclic points of M_n: $|\Pi_n^{\text{trees}}| = |\mathcal{C}_n|$. From the structure of a mapping digraph, Π_n^{trees} is a $\text{Gibbs}_{[n]}(v_\bullet, w_\bullet)$ partition for $v_k = k!$, the number of different ways that the restriction of M_n can act as a permutation of a given set of k cyclic points, and $w_j = j^{j-1}$ the number of rooted trees labeled by a set of size j. Let $q_j := e^{-j} j^{j-1}/j!$, so $(q_j, j = 0, 1, \ldots)$ is the distribution of total size of a critical Galton-Watson tree with Poisson offspring distribution. Since $q_j \sim (2\pi)^{-1/2} j^{-3/2}$, Theorem 2.5 gives for each $\ell > 0$, as $n \to \infty$

$$(\Pi_n^{\text{trees}} \text{ given } |\Pi_n^{\text{trees}}| = [\ell\sqrt{n}]) \xrightarrow{d} \Pi_\infty^{(\frac{1}{2}|\sqrt{2}\ell)} \tag{9.3}$$

where the limit is the partition of positive integers generated by lengths of excursions of a standard Brownian bridge B^{br} conditioned on $L_1^{\text{br}} = \ell$, where $L_1^{\text{br}} := L_1^0(B^{\text{br}})$. It is well known that L_1^{br} has the *Rayleigh density*

$$\mathbb{P}(L_1^{\text{br}} \in d\ell) = \ell \exp(-\tfrac{1}{2}\ell^2) d\ell \qquad (\ell > 0). \tag{9.4}$$

As a consequence of Cayley's result that kn^{n-k-1} is the number of forests labeled by $[n]$ with a specified set of k roots,

$$\mathbb{P}(|\mathcal{C}_n| = k) = \frac{k}{n} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \tag{9.5}$$

and hence that

$$|\mathcal{C}_n|/\sqrt{n} \xrightarrow{d} L_1^{\mathrm{br}} \tag{9.6}$$

jointly with

$$\Pi_n^{\mathrm{trees}} \xrightarrow{d} \Pi_\infty^{(\frac{1}{2},\frac{1}{2})} \tag{9.7}$$

where $\Pi_\infty^{(\frac{1}{2},\frac{1}{2})}$ is the random partition of positive integers generated by sampling from the interval partition defined by excursions of the standard Brownian bridge B^{br}, whose distribution is defined by the $(\frac{1}{2},\frac{1}{2})$ prediction rule. Recall from (4.45) that L_1^{br} is encoded in $\Pi_\infty^{(\frac{1}{2},\frac{1}{2})}$ as the almost sure limit as $n \to \infty$ of $|\Pi_n(\frac{1}{2},\frac{1}{2})|/\sqrt{2n}$, where $|\Pi_n(\frac{1}{2},\frac{1}{2})|$ is the number of distinct excursions of B^{br} discovered by n independent uniform points on $[0,1]$.

Joint distribution of trees and basins As a check on (9.2) and (9.7), and to understand the joint structure of tree and basin partitions generated by a uniform random mapping M_n, it is instructive to compute the joint law of the random variables

$$\#\mathcal{T}_n(1) := \text{ size of the tree containing 1 in the digraph of } M_n \tag{9.8}$$

and

$$\#\mathcal{B}_n(1) := \text{ size of the basin containing 1 in the digraph of } M_n. \tag{9.9}$$

Note that $\#\mathcal{T}_n(1)$ and $\#\mathcal{B}_n(1)$ are size-biased picks from the block-sizes of Π_n^{trees} and Π_n^{basins} respectively. So their limit distributions as $n \to \infty$, with normalization by n, are the structural distributions of the weak limits of Π_n^{trees} and Π_n^{basins} respectively.

To expose the combinatorial structure underlying the joint law of $\#\mathcal{T}_n(1)$ and $\#\mathcal{B}_n(1)$, introduce new variables

$$N_{n,1} := \#\mathcal{T}_n(1) - 1; \quad N_{n,2} := \#\mathcal{B}_n(1) - \#\mathcal{T}_n(1); \quad N_{n,3} := n - \#\mathcal{B}_n(1). \tag{9.10}$$

Then for each possible vector of integers

$$(n_1, n_2, n_3) \text{ with } n_i \geq 0 \text{ and } n_1 + n_2 + n_3 = n - 1, \tag{9.11}$$

there is the formula

$$\mathbb{P}(N_{n,i} = n_i, i = 1,2,3) = \binom{n-1}{n_1, n_2, n_3} \frac{(n_1+1)^{n_1} n_2^{n_2} n_3^{n_3}}{n^n}. \tag{9.12}$$

The multinomial coefficient appears here for obvious reasons. For each particular choice $n_1 + 1$ possible elements of the set $\mathcal{T}_n(1)$, the factor $(n_1+1)^{n_1}$ is the number of possible rooted trees induced by the action of M_n on this set, by Cayley's formula (6.24). For each choice n_3 possible elements of $[n]\backslash\mathcal{B}_n(1)$, the factor $n_3^{n_3}$ is the number of possible actions of M_n restricted to this set. This reflects part (i) of the following lemma. Part (ii) of the lemma explains the symmetry of formula (9.12) in (n_2, n_3) for fixed n_1. See also [359] for similar joint distributions derived from random mappings, known as *Abel multinomial distributions.*

Lemma 9.1. *For a uniform random mapping M_n,*

(i) *Conditionally given the restriction of M_n to $\mathcal{B}_n(1)$ with $\mathcal{B}_n(1) = \mathcal{B}$, the restriction of M_n to $[n]-\mathcal{B}$ is a uniform random mapping from $[n]-\mathcal{B}$ to $[n]-\mathcal{B}$.*

(ii) *Conditionally given that $\mathcal{T}_n(1)$ is some subset $\mathcal{T}$ of $[n]$ with $1 \in \mathcal{T}$, the restriction of Π_n^{trees} to $\mathcal{B}_n(1)\backslash\mathcal{T}$ and the restriction of Π_n^{trees} to $[n] - \mathcal{B}_n(1)$ are exchangeable.*

Proof. The first statement is obvious. To clarify statement (ii), given $\mathcal{T}_n(1) = \mathcal{T}$, each restriction of Π_n^{trees} is regarded as a random partition of a random subset of $[n]$, with some notion of a trivial partition if the subset is empty. According to (i), given also $\mathcal{B}_n(1) = \mathcal{B}$, the restriction of Π_n^{trees} to $[n] - \mathcal{B}$ is the tree-partition generated by a uniform random mapping from $\mathcal{B}$ to $\mathcal{B}$. On the other hand, the restriction of Π_n^{trees} to $\mathcal{B} - \mathcal{T}$ is the tree partition generated by a uniformly chosen composite structure on $\mathcal{B} - \mathcal{T}$, whereby $\mathcal{B} - \mathcal{T}$ is partitioned into tree components, and the roots of these components are assigned a linear order. But this is bijectively equivalent to a mapping from $\mathcal{B} - \mathcal{T}$ to $\mathcal{B} - \mathcal{T}$, hence the conclusion. □

By Stirling's formula, the probability in (9.12) is asymptotically equivalent to

$$\frac{1}{n^2}\frac{1}{2\pi}\frac{1}{\sqrt{n_1/n}\sqrt{n_2/n}\sqrt{n_3/n}} \text{ as } n_i \to \infty, i = 1, 2, 3, \tag{9.13}$$

hence as $n \to \infty$

$$(N_{n,1}, N_{n,2}, N_{n,3})/n \xrightarrow{d} \text{Dirichlet}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}). \tag{9.14}$$

Recalling the definitions (9.10) of the $N_{n,i}$, this implies

$$\frac{\#\mathcal{B}_n(1)}{n} \xrightarrow{d} \beta_{1,\frac{1}{2}} \text{ and } \frac{\#\mathcal{T}_n(1)}{n} \xrightarrow{d} \beta_{\frac{1}{2},1} \tag{9.15}$$

where $\beta_{a,b}$ has beta(a, b) distribution. As a check, according to Theorem 3.2, $\beta_{1,\frac{1}{2}}$ and $\beta_{\frac{1}{2},1}$ are the structural distributions of $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$ partitions respectively. So (9.15) agrees with (9.2) and (9.7). As indicated by Aldous [14], Lemma 9.1 (i) allows recursive application of the second convergence in (9.15) to show that the size-biased frequencies of Π_n^{basins} approach the GEM$(0, \frac{1}{2})$ frequencies (3.8), hence the convergence (9.2) of Π_n^{basins} to a $(0, \frac{1}{2})$ partition.

Exercises

9.1.1. Develop a variation of the above argument to show that the size-biased frequencies of Π_n^{trees} approach the GEM$(\frac{1}{2}, \frac{1}{2})$ frequencies, hence the convergence (9.7) of Π_n^{trees} to a $(\frac{1}{2}, \frac{1}{2})$ partition.

Notes and comments

This section is based on [17, 24]. The theory of random mappings has a long history. See [260, 17, 191] and papers cited there.

9.2. Mapping walks

The construction in [17] encodes the restriction of the digraph of M_n to each tree component $\mathcal{T}_{n,c}$ of size k by the Harris walk of $2k$ steps associated with this tree, which was defined in Section 6.3. This *tree-walk* derived from $\mathcal{T}_{n,c}$, with increments of ± 1 on the non-negative integers, makes an excursion which starts at 0 and returns to 0 for the first time after $2k$ steps, after reaching a maximum level $1 + h_n(c)$, where $h_n(c)$ is the maximal height above c of all vertices of the tree $\mathcal{T}_{n,c}$ with root c, that is

$$h_n(c) = \max\{h : \exists i \in [n] \text{ with } M_n^h(i) = c \text{ and } M_n^j(i) \notin \mathcal{C}_n \text{ for } 0 \leq j < h\}. \tag{9.16}$$

Given that c is a cyclic point such that the set of vertices $\mathcal{T}_{n,c}$ equals K for some subset K of $[n]$ with $c \in K$ and $|K| = k$, the restriction of the digraph of M_n to K has uniform distribution on the set of k^{k-1} trees labeled by K with root c. According to a basic result of Aldous, Theorem 6.4, as $k \to \infty$, the distribution of this tree-walk when scaled to have $2k$ steps of $\pm 1/\sqrt{k}$ per unit time, converges to the distribution of $2B^{\text{ex}}$, for B^{ex} a standard Brownian excursion.

We now define a mapping-walk (to code M_n) as a concatenation of its tree-walks, to make a walk of $2n$ steps starting and ending at 0 with exactly $|\mathcal{C}_n|$ returns to 0, one for each tree component of the mapping digraph. To concatenate the tree-walks, an order of tree-components must be specified. To retain useful information about M_n in the mapping-walk, we want the ordering of tree-walks to respect the cycle and basin structure of the mapping. Here are two orderings that do so.

Definition 9.2. (Cycles-first ordering) Fix a mapping M_n from $[n]$ to $[n]$. First put the cycles in increasing order of their least elements, say $c_{n,1} < c_{n,2} < \ldots < c_{n,|\mathcal{C}_n|}$. Let $\mathcal{C}_{n,j}$ be the cycle containing $c_{n,j}$, and let $\mathcal{B}_{n,j}$ be the basin containing $\mathcal{C}_{n,j}$. Within cycles, list the trees around the cycles, as follows. If the action of M_n takes $c_{n,j} \to c_{n,j,1} \to \cdots \to c_{n,j}$ for each $1 \leq j \leq |\mathcal{C}_n|$, the tree components $\mathcal{T}_{n,c}$ are listed with c in the order

$$(\overbrace{c_{n,1,1}, \ldots, c_{n,1}}^{\mathcal{C}_{n,1}}, \overbrace{c_{n,2,1}, \ldots, c_{n,2}}^{\mathcal{C}_{n,2}}, \ldots\ldots, \overbrace{c_{n,|\mathcal{C}_n|,1}, \ldots, c_{n,|\mathcal{C}_n|}}^{\mathcal{C}_{n,|\mathcal{C}_n|}}). \tag{9.17}$$

The *cycles-first mapping-walk* is obtained by concatenating the tree walks derived from M_n in this order. The *cycles-first search of* $[n]$ is the permutation $\sigma : [n] \to [n]$ where σ_j is the jth vertex of the digraph of M_n which is visited in the corresponding concatenation of tree searches.

Definition 9.3. (Basins-first ordering)[17] First put the basins $\mathcal{B}_{n,j}$ in increasing order of their least elements, say $1 = b_{n,1} < b_{n,2} < \dots b_{n,|\mathcal{C}_n|}$; let $c_{n,j} \in \mathcal{C}_{n,j}$ be the cyclic point at the root of the tree component containing $b_{n,j}$. Now list the trees around the cycles, just as in (9.17), but for the newly defined $c_{n,j}$ and $c_{n,j,i}$. Call the corresponding mapping-walk and search of $[n]$ the *basins-first mapping-walk* and *basins-first search*.

Let us briefly observe some similarities between the two mapping-walks. For each given basin B of M_n with say b elements, the restriction of M_n to B is encoded in a segment of each walk which equals at 0 at some time, and returns again to 0 after $2b$ more steps. If the basin contains exactly c cyclic points, this walk segment of $2b$ steps will be a concatenation of c excursions away from 0. Exactly where this segment of $2b$ steps appears in the mapping-walk depends on the ordering convention, as does the ordering of excursions away from 0 within the segment of $2b$ steps. However, many features of the action of M_n on the basin B are encoded in the same way in the two different stretches of length $2b$ in the two walks, despite the permutation of excursions. One example is the number of elements in the basin whose height above the cycles is h, which is encoded in either walk as the number of upcrossings from h to $h+1$ in the stretch of walk of length $2b$ corresponding to that basin.

9.3. Brownian asymptotics

The idea now is that for either of the mapping walks derived above from a uniform mapping M_n, a suitable rescaling converges weakly in $C[0,1]$ as $n \to \infty$ to the distribution of the reflecting Brownian bridge defined by the absolute value of a *standard Brownian bridge* B^{br} with $B^{\text{br}}_0 = B^{\text{br}}_1 = 0$ obtained by conditioning a standard Brownian motion B on $B_1 = 0$. Jointly with this convergence, results of [17] imply that for a uniform random mapping, the basin sizes rescaled by n, jointly with corresponding cycle sizes rescaled by $\sqrt{n}$, converge in distribution to a limiting bivariate sequence of random variables $(\lambda_{I_j}, L^{\text{br}}_{I_j})_{j=1,2,\dots}$ where $(I_j)_{j=1,2,\dots}$ is a random interval partition of $[0,1]$, with λ_{I_j} the length of I_j and $L^{\text{br}}_{I_j}$ the increment of local time of B^{br} at 0 over the interval I_j. For the basins-first walk, the limiting interval partition is $(I_j) = (I^D_j)$, according to the following definition. Here $U, U_1, U_2, \dots$ denotes a sequence of independent uniform $(0,1)$ variables, independent of B^{br}, and the local time process of B^{br} at 0 is assumed to be normalized as occupation density relative to Lebesgue measure.

Definition 9.4. (The D-partition [17]) Let $I^D_j := [D_{V_{j-1}}, D_{V_j}]$ where $V_0 = D_{V_0} = 0$ and V_j is defined inductively along with the D_{V_j} for $j \geq 1$ as follows: given that D_{V_i} and V_i have been defined for $0 \leq i < j$, let

$$V_j := D_{V_{j-1}} + U_j(1 - D_{V_{j-1}}),$$

so V_j is uniform on $[D_{V_{j-1}}, 1]$ given B^{br} and (V_i, D_{V_i}) for $0 \leq i < j$, and let

$$D_{V_j} := \inf\{t \geq V_j : B^{\text{br}}_t = 0\}.$$

On the other hand, for the cycles-first walk, the limits involve a different interval partition. This is the partition $(I_j) = (I_j^T)$ defined as follows using the local time process $(L_u^{\mathrm{br}}, 0 \le u \le 1)$ of B^{br} at 0:

Definition 9.5. (The T-partition) Let $I_j^T := [T_{j-1}, T_j]$ where $T_0 := 0$, $\hat{V}_0 :=$ 0, and for $j \ge 1$

$$\hat{V}_j := 1 - \textstyle\prod_{i=1}^{j}(1 - U_i), \tag{9.18}$$

so $\hat{V}_j$ is uniform on $[\hat{V}_{j-1}, 1]$ given B^{br} and $(\hat{V}_i, T_i)$ for $0 \le i < j$, and

$$T_j := \inf\{u : L_u^{\mathrm{br}}/L_1^{\mathrm{br}} > \hat{V}_j\}.$$

The main result can now be stated as follows:

Theorem 9.6. [24] *The scaled mapping-walk $(M_u^{[n]}, 0 \le u \le 1)$ derived from a uniform random mapping M_n, with $2n$ steps of $\pm 1/\sqrt{n}$ per unit time, for either the cycles-first or the basins-first ordering of excursions corresponding to tree components, converges in distribution to $2|B^{\mathrm{br}}|$ jointly with* (9.6) *and* (9.7), *where $(L_u^{\mathrm{br}}, 0 \le u \le 1)$ is the process of local time at 0 of B^{br}, and $\Pi_\infty^{(\frac{1}{2},\frac{1}{2})}$ is the random partition of positive integers generated by sampling from the interval partition defined by excursions B^{br}. Moreover,*
(i) *for the cycles-first ordering, with the cycles $\mathcal{B}_{n,j}$ in order of their least elements, these two limits in distribution hold jointly with*

$$\left(\frac{|\mathcal{B}_{n,j}|}{n}, \frac{|\mathcal{C}_{n,j}|}{\sqrt{n}}\right) \xrightarrow{d} (\lambda_{I_j}, L_{I_j}^{\mathrm{br}}) \tag{9.19}$$

as j varies, where the limits are the lengths and increments of local time of B^{br} at 0 associated with the interval partition $(I_j) := (I_j^T)$; whereas
(ii) [17] *for the basins-first ordering, with the basins $\mathcal{B}_{n,j}$ listed in order of their least elements, the same is true, provided the limiting interval partition is defined instead by $(I_j) := (I_j^D)$.*

The result for basins-first ordering is part of [17, Theorem 8]. The variant for cycles-first ordering can be established by a variation of the argument in [17], exploiting the exchangeability of the tree components in the cycles-first ordering. See also [58] and [15] for alternate approaches to the basic result of [17].

The random set of pairs $\{(|\mathcal{B}_{n,j}|/n, |\mathcal{C}_{n,j}|/\sqrt{n}), 1 \le j \le |\mathcal{C}_n|\}$ is the same, no matter what ordering convention is used. So Theorem 9.6 implies that the distribution of the random set of limit points, $\{(\lambda_{I_j}, L_{I_j}^{\mathrm{br}}), j \ge 1\}$, regarded as a point process on $\mathbb{R}^2_{>0}$, is the same for $(I_j) = (I_j^D)$ or $(I_j) = (I_j^T)$. This fact about Brownian bridge is not at all obvious, but can be verified by application of Brownian excursion theory. See [24] for further discussion.

To gain useful information about large random mappings from Theorem 9.6, it is necessary to understand well the joint law of B^{br} and one or other of the limiting interval partitions (I_j) whose definition depends on the path of B^{br}. To be definite, assume from now on that the ordering convention is *basins first.* One feature of natural interest is the maximal height above the cycle of the tallest tree in the basin. Let this maximal height be $H_{n,j}$ for the jth basin. Theorem 9.6 implies

$$\left(\frac{|\mathcal{B}_{n,j}|}{n}, \frac{|\mathcal{C}_{n,j}|}{\sqrt{n}}, \frac{H_{n,j}}{\sqrt{n}}\right) \xrightarrow{d} (\lambda_j, L_j, 2\overline{M}_j)_{j=1,2,\ldots} \tag{9.20}$$

where we abbreviate $\lambda_j := \lambda_{I_j}, L_j := L^{\mathrm{br}}_{I_j}$, and $\overline{M}_j := \overline{|B^{\mathrm{br}}|}(D_{V_{j-1}}, D_{V_j})$ is the maximal value of $|B^{\mathrm{br}}|$ on I_j. It follows easily from Definition 9.4, the strong Markov property of B^{br} at the times D_{V_j}, and Brownian scaling, that

$$\lambda_j = W_j \prod_{i=1}^{j-1}(1 - W_i) \tag{9.21}$$

for a sequence of independent random variables W_j with beta$(1, \frac{1}{2})$ distribution, and that

$$(L_j, \overline{M}_j) = \sqrt{\lambda_j}(\tilde{L}_j, \tilde{M}_j) \tag{9.22}$$

for a sequence of independent and identically distributed random pairs $(\tilde{L}_j, \tilde{M}_j)$, independent of (λ_j). The common distribution of $(\tilde{L}_j, \tilde{M}_j)$ is that of

$$(\tilde{L}_1, \tilde{M}_1) := \left(\frac{L^{\mathrm{br}}_{D_U}}{\sqrt{D_U}}, \frac{M^{\mathrm{br}}_{D_U}}{\sqrt{D_U}}\right) \tag{9.23}$$

where D_U is the time of the first zero of B^{br} after a uniform$[0,1]$ random time U which is independent of B^{br}, $L^{\mathrm{br}}_t := L^0_t(B^{\mathrm{br}})$, and and $M^{\mathrm{br}}_t := \max_{0 \le u \le t} |B^{\mathrm{br}}_u|$ for $0 \le t \le 1$. It is known [341] that for (λ_j) as in (9.21), assumed independent of B_1, the $B_1^2 \lambda_j$ are the points (in size-biased random order) of a Poisson process on $\mathbb{R}_{>0}$ with intensity measure $\frac{1}{2}t^{-1}e^{-t/2}dt$ which is the Lévy measure of the infinitely divisible gamma$(\frac{1}{2}, \frac{1}{2})$ distribution of B_1^2. Together with standard properties of Poisson processes, this observation and the previous formulae (9.21) to (9.23) yield the following lemma. See also [24] for related results.

Lemma 9.7. *If B_1 is a standard Gaussian variable independent of the sequence of triples $(\lambda_j, L_j, \overline{M}_j)_{j=1,2,\ldots}$ featured in* (9.20)*, then the random vectors*

$$(B_1^2 \lambda_j, |B_1| L_j, |B_1| \overline{M}_j)$$

are the points of a Poisson point process on $\mathbb{R}^3_{>0}$ with intensity measure μ defined by

$$\mu(dt\, d\ell\, dm) = \frac{e^{-t/2}\, dt}{2t} P(\sqrt{t}\tilde{L}_1 \in d\ell, \sqrt{t}\tilde{M}_1 \in dm) \tag{9.24}$$

for $t, \ell, m > 0$, where $(\tilde{L}_1, \tilde{M}_1)$ is the pair of random variables derived from a Brownian bridge by (9.23)*.*

For a process $X := (X_t, t \in J)$ parameterized by an interval J, and $I = [G_I, D_I]$ a subinterval of J with length $\lambda_I := D_I - G_I > 0$, we denote by $X[I]$ or $X[G_I, D_I]$ the *fragment of X on I*, that is the process

$$X[I]_u := X_{G_I + u} \qquad (0 \le u \le \lambda_I). \tag{9.25}$$

Denote by $X_*[I]$ or $X_*[G_I, D_I]$ the *standardized fragment of* X *on* I, defined by the *Brownian scaling operation*

$$X_*[I]_u := \frac{X_{G_I+u\lambda_I} - X_{G_I}}{\sqrt{\lambda_I}} \qquad (\ 0 \le u \le 1). \tag{9.26}$$

The process $\tilde{B}^{\mathrm{br}} := B_*[0, \tau_1]$, where τ_1 is an inverse local time at 0 for the unconditioned Brownian motion B, is known as a *Brownian pseudo-bridge*, and there is the following absolute continuity relation between the laws of $\tilde{B}^{\mathrm{br}}$ and B^{br} found in [59]: for each non-negative measurable function g on $C[0, 1]$,

$$\mathbb{E}[g(\tilde{B}^{\mathrm{br}})] = \sqrt{\frac{2}{\pi}} \mathbb{E}[g(B^{\mathrm{br}})/L_1^{\mathrm{br}}].$$

See Exercise 4.5.2 . It follows from [366, Theorem 1.3] and [17, Proposition 2] that the process $B_*^{\mathrm{br}}[0, D_U]$, obtained by rescaling the path of B^{br} on $[0, D_1]$ to have length 1 by Brownian scaling, has the same distribution as a rearrangement of the path of the pseudo-bridge $\tilde{B}^{\mathrm{br}}$. Neither the maximum nor the local time at 0 are affected by such a rearrangement, so there is the equality in distribution

$$(\tilde{L}_1, \tilde{M}_1) \stackrel{d}{=} (\tilde{L}_1^{\mathrm{br}}, \tilde{M}_1^{\mathrm{br}}) \tag{9.27}$$

where $\tilde{L}_1^{\mathrm{br}} := L_1^0(\tilde{B}^{\mathrm{br}})$ $\tilde{M}_1^{\mathrm{br}} := \max_{0 \le u \le 1} |\tilde{B}_u^{\mathrm{br}}|$. So (9.27) yields the formula

$$P(\sqrt{t}\tilde{L}_1 \in d\ell, \sqrt{t}\tilde{M}_1 \le y) = \sqrt{\frac{2}{\pi}} \frac{\sqrt{t}}{\ell} P(\sqrt{t} L_1^{\mathrm{br}} \in d\ell, \sqrt{t} M_1^{\mathrm{br}} \le y), \tag{9.28}$$

for $t, \ell, y > 0$. Now the joint law of L_1^{br} and M_1^{br} is characterized by the following identity: for all $\ell > 0$ and $y > 0$

$$\int_0^\infty \frac{e^{-t/2}}{\sqrt{2\pi t}} \, dt \, P(\sqrt{t} L_1^{\mathrm{br}} \in d\ell, \sqrt{t} M_1^{\mathrm{br}} \le y) = e^{-\ell} d\ell \, \exp\left(\frac{-2\ell}{e^{2y} - 1}\right) \tag{9.29}$$

which can be read from [374, Theorem 3, Lemma 4 and (36)], with the following interpretation. Let $(L_t, t \ge 0)$ be the local time process of the Brownian motion B at 0, let T be an exponential random variable with mean 2 independent of B, and let G_T be the time of the last 0 of B before time T. Then (9.29) provides two expressions for

$$P\left(L_T \in d\ell, \sup_{0 \le u \le G_T} |B_u| \le y\right),$$

on the left side by conditioning on G_T, and on the right side by conditioning on L_T. See also [384, Chapter XII, Exercise (4.24)].

Using (9.24), (9.28) and (9.29), we deduce that in the Poisson point process of Lemma 9.7,

$$\mathbb{E}[\text{number of points } (|B_1| L_j, |B_1| \overline{M}_j) \text{ with } |B_1| L_j \in d\ell \text{ and } |B_1| \overline{M}_j \le y] = \tag{9.30}$$

$$\int_0^\infty \frac{e^{-t/2}\,dt}{2t} P(\sqrt{t}\tilde{L}_1 \in d\ell, \sqrt{t}\tilde{M}_1 \le y) = \ell^{-1}e^{-\ell}d\ell \exp\left(\frac{-2\ell}{e^{2y}-1}\right). \tag{9.31}$$

A significant check on these calculations can be made as follows. By further integration, the expected number of points j with $|B_1|\overline{M}_j$ greater than y is

$$\eta(y) := \int_0^\infty \ell^{-1}e^{-\ell}d\ell \left[1 - \exp\left(\frac{-2\ell}{e^{2y}-1}\right)\right]. \tag{9.32}$$

Now the probability of no point greater than y is $e^{-\eta(y)}$, so

$$\mathbb{P}(|B_1| \max_j \overline{M}_j \le y) = e^{-\eta(y)}. \tag{9.33}$$

But the event $(|B_1| \max_j \overline{M}_j \le y)$ is identical to the event $(M_1^{\mathrm{br}} \le y)$, where $M_1^{\mathrm{br}} := \max_{0\le u\le 1} |B_u^{\mathrm{br}}|$. And $e^{-\eta(y)} = \frac{1}{1+2/(e^{2y}-1)} = \tanh y$ by application of the Lévy-Khintchine formula for the exponential distribution, that is

$$\frac{1}{1+\lambda} = \exp\left[-\int_0^\infty \ell^{-1}e^{-\ell}(1-e^{-\lambda\ell})d\ell\right],$$

for $\lambda = 2/(e^{2y}-1)$. Thus for B_1 standard Gaussian independent of B^{br} and $y > 0$, there is the remarkable formula

$$\mathbb{P}(|B_1|M_1^{\mathrm{br}} \le y) = \tanh y \qquad (y \ge 0) \tag{9.34}$$

which is a known equivalent of Kolmogorov's formula

$$\mathbb{P}(M_1^{\mathrm{br}} \le x) = \sum_{n=-\infty}^{\infty} (-1)^n e^{-2n^2x^2} \qquad (x \ge 0) \tag{9.35}$$

As observed in [60], formula (9.34) allows the Mellin transform of M_1^{br} to be expressed in terms of the Riemann zeta function. See also [339, 371, 373] for closely related Mellin transforms obtained by the technique of multiplication by a suitable independent random factor to introduce Poisson or Markovian structure.

Notes and comments

This section is based on [17] and [24].

9.4. The diameter

The *diameter* of M_n is the random variable

$$\Delta_n := \max_{i\in[n]} T_n(i)$$

where $T_n(i)$ is the number of iterations of M_n starting from i until some value is repeated:

$$T_n(i) := \min\{j \geq 1 : M_n^j(i) = M_n^k(i) \text{ for some } 0 \leq k < j\}$$

where $M_n^0(i) = i$ and $M_n^j(i) := M_n(M_n^{j-1}(i))$ is the image of i under j-fold iteration of M_n for $j \geq 1$. Since by definition $\Delta_n = \max_j(|\mathcal{C}_{n,j}| + H_{n,j})$, it follows from (9.20) that as $n \to \infty$

$$\frac{\Delta_n}{\sqrt{n}} \xrightarrow{d} \Delta := \max_j (L_j + 2\overline{M}_j). \tag{9.36}$$

So we obtain the following corollary of Theorem 9.6:

Corollary 9.8. [22] *Let B_1 be a standard Gaussian variable independent of Δ. Then the distribution of Δ in* (9.36) *is characterized by*

$$P(|B_1|\Delta \leq v) = e^{-E_1(v)-I(v)} \qquad (v \geq 0) \tag{9.37}$$

where

$$\begin{aligned} E_1(v) &:= \int_v^\infty u^{-1}e^{-u}du \\ I(v) &:= \int_0^v u^{-1}e^{-u}\left[1 - \exp\left(\frac{-2u}{e^{v-u}-1}\right)\right] du. \end{aligned}$$

Proof. From (9.36) and Lemma 9.7, the event $|B_1|\Delta \leq v$ is the event that there is no j with $|B_1|L_j + 2|B_1|\overline{M}_j > v$. But from (9.30) - (9.31), $E_1(v)$ is the expected number of j with $|B_1|L_j \geq v$, while $I(v)$ is the expected number of j with $|B_1|L_j < v$ and $|B_1|L_j + 2|B_1|\overline{M}_j > v$. □

Integration of (9.37) gives a formula for $\mathbb{E}(\Delta^p)$ for arbitrary $p > 0$, which is easily shown to be the limit as $n \to \infty$ of $\mathbb{E}((\Delta_n/\sqrt{n})^p)$. This formula was first found for $p = 1$ by Flajolet-Odlyzko [156, Theorem 7] using singularity analysis of generating functions. See also [408, 289, 97] for related asymptotic studies of the diameter of undirected random trees and graphs.

Exercises

9.4.1. (Problem: the diameter of a Brownian tree) Szekeres [408] found an explicit formula for the asymptotic distribution of the diameter of a uniform random tree labeled by $[n]$, with normalization by $\sqrt{n}$. Aldous [6, 3.4] observed that this is the distribution of the diameter of $\mathcal{T}(2B^{\text{ex}})$, and raised the following problem, which is still open: can this distribution be characterized directly in the Brownian world?

Notes and comments

This section is based on [22]. The technique of characterizing the law of some Brownian functional X by first considering the law of $|B_1|X$ for B_1 a standard Gaussian variable independent of X, and the related idea of *random Brownian scaling* have found numerous applications [439, 374, 372].

9.5. The height profile

We continue to suppose that M_n is a uniform random mapping from $[n]$ to $[n]$. For $v \in [n]$ let $h(v, M_n)$ be the least $m \geq 0$ such that $M_n^m(v) \in \mathcal{C}_n$. So $h(v, M_n)$ is the height of v in the forest derived from M_n whose set of roots is the random set $\mathcal{C}_n$ of cyclic points of M_n. For $h = 0, 1, 2, \ldots$ let $Z_{*,n}(h)$ be the number of $v \in [n]$ such that $h(v, M_n) = h$. Call this process $(Z_{*,n}(h), h \geq 0)$ the *height profile of the mapping forest.* Let $(Z_{k,n}(h), h \geq 0)$ be the height profile of the mapping forest conditioned on the event $(Z_{*,n}(0) = k)$ that M_n has exactly k cyclic points. Then $(Z_{k,n}(h), h \geq 0)$ has the same distribution as the height profile generated by a uniform random forest of k rooted trees labeled by $[n]$, to which the limit theorem (8.30) applies, by inspection of (8.34) and (4.9). To review:

Lemma 9.9. *If $(Z_{k,n}(h), h \geq 0)$ is either*
(i) *the height profile of a uniform random forest of k rooted trees labeled by $[n]$, or*
(ii) *the height profile of the forest derived from a random mapping from $[n]$ to $[n]$ conditioned to have k cyclic points,*
then the distribution of the sequence $(Z_{k,n}(h), h \geq 0)$ is that described by Lemma 8.6, and in the limit regime as $n \to \infty$ and $2k/\sqrt{n} \to \ell \geq 0$

$$\left(\frac{2}{\sqrt{n}} Z_{k,n}(2\sqrt{n}v), v \geq 0 \right) \xrightarrow{d} (Q_{\ell,1,v}, v \geq 0) \tag{9.38}$$

where the law of $(Q_{\ell,1,v}, v \geq 0)$ is defined by Theorem 8.4.

The following result is now obtained by mixing the result of the previous lemma with respect to the distribution of the number $Z_{*,n}(0) = |\mathcal{C}_n|$ of cyclic points of M_n. According to (9.6), $|\mathcal{C}_n|/\sqrt{n} \xrightarrow{d} L_1^0(B^{\mathrm{br}})$, so the result is:

Theorem 9.10. Drmota-Gittenberger [116] *The normalized height profile of the forest derived from a uniform random mapping M_n converges weakly to the process of local times of a reflecting Brownian bridge of length 1:*

$$\left(\frac{2}{\sqrt{n}} Z_{*,n}(2\sqrt{n}v), v \geq 0 \right) \xrightarrow{d} (L_1^v(|B^{\mathrm{br}}|, v \geq 0) \tag{9.39}$$

Notes and comments

This section is based on [358]. Presumably the convergence in distribution of height profiles (9.39) holds jointly with all the convergences in distribution described in Theorem 9.6. This must be true, but seems difficult to establish. Corresponding results of joint convergence in distribution of occupation time processes and unconditioned walk paths to their Brownian limits, for simple random walks, can be read from Knight [254]. Presumably corresponding results are known for simple random walks with bridges or excursions as limits, but I do not know a reference.

9.6. Non-uniform random mappings

Definition 9.11. *Let p be a probability distribution on $[n]$. Call M_n* p-mapping *from $[n]$ to $[n]$ if the images $M_n(i)$ of points $i \in [n]$ are independent and identically distributed according to p.*

Combinatorial properties of p-mappings, and some elementary asymptotics are reviewed in [359]. Further asymptotic features of p-mappings were studied in [327]. In [23, 16] it is shown that Brownian bridge asymptotics apply for models of random mappings more general than the uniform model, in particular for *p-mapping* model under suitable conditions. Proofs are simplified by use of Joyal's bijection between mappings and trees, discussed in Exercise 10.1.4 . Another important result on p-mappings is Burtin's formula which is presented in Exercise 10.1.5 . But these results for p-mappings are best considered in connection with *p-trees* and *p-forests*, which are the subject of Chapter 10.