7

Sketches of Solutions for the Exercises

This chapter is devoted to giving solutions, hints and references for the exercises of the seven preceding chapters, from Chapter 0 to Chapter 6. For ease of the reader, we have indicated the page where the text of a given exercise may be found.

S_0 Preliminaries

• **Solution to Exercise 1** *(p.8)*
Using Itô's formula, we obtain

$$Z_t = Z_0 + \int_0^t c'(S_u) dB_u$$

The conclusion follows easily.

Comment 7.1 *This generalization of the balayage formula was introduced by [DÉY91] to study sets in which multidimensional martingales may have surprising behaviors.*

• **Solution to Exercise 2** *(p.8)*
For any $f \in \mathcal{C}^1$, $(F(W_t) - f(W_t)Y_t;\ t \geq 0)$ *is a local martingale if and only if*

$$(f(W_t) - f'(W_t)Y_t)dW_t - f(W_t)dV_t = 0$$

Taking $f = 1$, *we obtain* $W = V$. *Then, we are left with* $Y_t dV_t = 0$.
Hence, a necessary and sufficient condition is $W = V$ *and* dV_t *is carried by the zero set of* Y.

Remark 7.1 *Moreover, if* $Y_t \geq 0$, *then* $V_t = 2L_t^0(Y)$.

• **Solution to Exercise 3** *(p.8)*
For a "reasonable" function f:

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*, Lect. Notes Math. **1873**, 117–139 (2006)
www.springerlink.com

$$f(N_t) = f(0) + \sum_{s \le t} (f(N_s) - f(N_{s-}))$$
$$= f(0) + \sum_{s \le t} (f(N_{s-} + 1) - f(N_{s-}))\, \Delta N_s$$

which is compensated by $\int\limits_0^t (f(N_{s-}+1) - f(N_{s-}))\, \lambda ds$.

Thus, for $f_1(x) = x^2$, $f_2(x) = e^x$, *the compensators are* $\int_0^t (2N_s + 1)\lambda ds$ *and* $\int_0^t e^{N_s}(e-1)\lambda ds$.

• **Solution to Exercise 4** *(p.8)*
Let $s < t$, *then*

$$\mathbb{E}\left[{}^{(o)}A_t | \mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}\left[A_t|\mathcal{F}_t\right]|\mathcal{F}_s\right]$$
$$= \mathbb{E}\left[A_t|\mathcal{F}_s\right] \ge \mathbb{E}\left[A_s|\mathcal{F}_s\right] = {}^{(o)}A_s$$

Hence ${}^{(o)}A$ *is a submartingale.*
Moreover,

$$\mathbb{E}\left[{}^{(o)}A_t - {}^{(o)}A_s|\mathcal{F}_s\right] = \mathbb{E}\left[A_t - A_s|\mathcal{F}_s\right] = \mathbb{E}\left[A_t^{(p)} - A_s^{(p)}|\mathcal{F}_s\right]$$

Hence ${}^{(o)}A - A^{(p)}$ *is a martingale.*

• **Solution to Exercise 5** *(p.8)*

a) We compute in two different manners $\mathbb{E}_x\left[F(X_u;\ u \le s) f(X_t)\right]$ *for* $F(X_u;\ u \le s)$ *a generic, positive* $\mathcal{F}_s$*-measurable function, and* $f : \mathbb{R} \mapsto \mathbb{R}^+$ *a Borel function;*
- *on one hand, we use the definition of* $P^t_{x \to y}$
- *on the other hand, we replace* $f(X_t)$ *by*

$$P_{t-s}f(X_s) = \int dy \phi_{t-s}(y) f(X_s + y)$$

(For a rigorous definition of the bridges of Markov processes, see [FPY93]).

b) Let ν *denote the Lévy measure of* X.
Under P_x, $M^f_u := \sum\limits_{s \le u} f(\Delta X_s) - u < \nu, f >$ *is a martingale (for suitable* f*'s). We then use Girsanov's theorem to see how the martingale* $(M^f_u; u \ge 0)$ *is transformed when passing from* P_x *to* $P^t_{x \to y}$.

$$M^f_u = \widetilde{M}^f_u + \int_0^u \frac{d < M^f, \phi_{t-\cdot}(y - X_\cdot) >_s}{\phi_{t-s}(y - X_s)}$$

where $(\widetilde{M}^f_u;\ u \ge 0)$ *is a* $P^t_{x \to y}$*- local martingale.*
It remains to compute $< M^f, \phi_{t-\cdot}(y - X_\cdot) >_u$ *which is the compensator under* P_x *of*

$$\sum_{s\le u} f(\Delta X_s)\left(\phi_{t-s}(y-X_s)-\phi_{t-s}(y-X_{s-})\right) =$$

$$= \sum_{s\le u} f(\Delta X_s)\left(\phi_{t-s}(y-X_{s-}-\Delta X_s)-\phi_{t-s}(y-X_{s-})\right)$$

This compensator is precisely given by

$$\int_0^u ds \int \nu(dz) f(z)\left(\phi_{t-s}(y-X_s-z)-\phi_{t-s}(y-X_s)\right)$$

Finally the compensator we are looking for is:

$$u < \nu, f > + \int_0^u ds \int \nu(dz) f(z)\left(\frac{\phi_{t-s}(y-X_s-z)}{\phi_{t-s}(y-X_s)} - 1\right)$$

Comment 7.2 *The reader may find it interesting to extend such computations starting from a general Markov process, not necessarily a Lévy process. See Kunita [Kun69] and [Kun76], and Sato [Sat99] for such general discussions.*

• **Solution to Exercise 6** *(p.9)*

a) *It is a simple consequence of the integration by parts formula* (I_o) *for the conveniently stopped local martingale* $(M_t;\ t \ge 0)$.
b) *Idem with the integration by parts formula* (I_p).
c) *From integration by parts formula* (0.1), *we deduce*

$$[M, A]_t = \left(M_t A_t - \int_0^t M_{s-} dA_s\right) - \int_0^t A_s\, dM_s + \int_0^t \Delta A_s dM_s$$

We deduce that $[M, A]$ *is a local martingale. Since* $[M, A]$ *is purely discontinuous and has the same jumps as* $\int_0^{\cdot} \Delta A_s dM_s$, *the result follows.*

• **Solution to Exercise 7** *(p.9)*

a) *The martingale associated with the variable* B_1 *is* $(B_{t\wedge 1};\ t \ge 0)$*; hence, its increasing process is bounded, which implies the result. The result for* $(|B_t|;\ t \le 1)$ *follows from Tanaka's formula.*
b) *This result follows from the John-Nirenberg inequality, see [DM80] for a proof.*
c) *First note that, for any* $t \le 1$, $M_t := \mathbb{E}\left[B_1^2|\mathcal{F}_t\right] = B_t^2 + 1 - t$. *Consequently,*

$$\begin{aligned}\mathbb{E}\left[(M_1-M_t)^2|\mathcal{F}_t\right] &= \mathbb{E}\left[(B_1^2-B_t^2-1+t)^2|\mathcal{F}_t\right] = \mathbb{E}\left[(2\int_t^1 B_u dB_u)^2|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[4\int_t^1 B_u^2 du|\mathcal{F}_t\right] = 4\int_t^1 \mathbb{E}\left[B_u^2|\mathcal{F}_t\right] du \\ &\ge 4B_t^2(1-t)\end{aligned}$$

Therefore $(M_t;\ t \le 1)$ *is not in BMO.*

S_1 Chapter 1

• **Solution to Exercise 8** *(p.22)*

a) *Apply Lemma 0.3.*

b) *Let* $(T_n, n \in \mathbb{N})$ *denote an increasing sequence of stopping times which reduces the local martingale* $(\int_0^t k_{\gamma_u} dN_u;\ t \geq 0)$. *We take* k *predictable, bounded. Then:*

$$\mathbb{E}\left[k_{\gamma_{T_n}}(\overline{N}_{T_n} - N_{T_n})\right] = \mathbb{E}\left[\int_0^{T_n} k_{\gamma_u} d\overline{N}_u\right] = \mathbb{E}\left[\int_0^{T_n} k_u d\overline{N}_u\right]$$

In the preceding formula, we now replace $(k_u;\ u \geq 0)$ *by* $(k_u/(\alpha + \varepsilon\overline{N}_u),\ u \geq 0)$ *for two positive constants* α *and* ε. *Then, by dominated convergence, we get:*

$$\mathbb{E}\left[k_\Lambda \frac{\overline{N}_\infty}{\alpha + \varepsilon\overline{N}_\infty}\right] = \mathbb{E}\left[\int_0^\infty k_u \frac{d\overline{N}_u}{\alpha + \varepsilon\overline{N}_u}\right] \tag{7.1}$$

The result in b) is now obtained by letting $\varepsilon \downarrow 0$, *and applying Beppo-Levi.*

c) *Similarly, in* (7.1), *we let* α *decrease to 0, which yields to the result.*
The distribution of A_∞^Λ *follows from Lemma 0.1, since* $\overline{N}_\infty \overset{(law)}{=} 1/U$ *with* U *a uniform variable.*

d) *With the same care as before, we arrive to the formula*

$$\mathbb{E}\left[k_\Lambda(\overline{N}_\infty - N_\infty)\right] = \mathbb{E}\left[\int_0^\infty k_u d\overline{N}_u\right]$$

hence

$$\mathbb{E}\left[k_\Lambda(\overline{N}_\infty - n_\Lambda)\right] = \mathbb{E}\left[\int_0^\infty k_u d\overline{N}_u\right]$$

and finally, $A_t^\Lambda = \int_0^t \frac{d\overline{N}_u}{\overline{N}_u - n_u}$.

• **Solution to Exercise 9** *(p.23)*

a) *Using that* $Z_\Lambda = 1$ *and that* $(A_{\Lambda+t}^\Lambda;\ t \geq 0)$ *is constant, we find*

$$\begin{aligned} 1 - Z_{\Lambda+t} = \mu_\Lambda - \mu_{\Lambda+t} &= \hat{\mu}_t - \int_\Lambda^{\Lambda+t} \frac{d < \mu, 1 - Z >_s}{1 - Z_{s-}} \\ &= \hat{\mu}_t + \int_0^t \frac{d < \hat{\mu} >_s}{1 - Z_{\Lambda+s}} \end{aligned} \tag{7.2}$$

b) *From* (7.2), *we deduce that there exists a 3-dimensional Bessel process* $R^{(3)}$ *such that* $1 - Z_{\Lambda+t} = R^{(3)}_{<\hat{\mu}>_t}$.

• **Solution to Exercise 10** *(p.23)*

a) *Let* $(L^x,\ x \in \mathbb{R})$ *be the family of (Meyer) local times of the local martingale* $s(Y)$.
Using the Markov property of Y *and the local martingale property of* $s(Y)$, *we obtain that*

$$Z_t^{\mathcal{L}_a} = 1 \wedge \left(\frac{s(Y_t)}{s(a)}\right)$$

Applying Itô-Tanaka's formula to the RHS of this equality, we find

$$A_t^{\mathcal{L}_a} = -\frac{1}{2s(a)} L_t^{s(a)}$$

b) *Taking the expectation in the occupation time formula, we obtain* $\int_0^t p_u(y,a)\, du = \mathbb{E}_y\left[L_t^{s(a)}\right]$ *and the result follows by differentiation.*

c) *Applying the definition of the predictable compensator (Definition 0.1) with the predictable process* $k_u = 1_{0 \le u \le t}$, *we deduce*

$$P_y(0 \le \mathcal{L}_a \le t) = -\frac{1}{2s(a)} \mathbb{E}_y\left[L_t^{s(a)}\right]$$

and then we use b).

• **Solution to Exercise 11** *(p.24)*

The proof of (1.4) *follows easily from the Hint, which itself is a consequence of the Markov property;* $C = \sqrt{\pi/2}$.

• **Solution to Exercise 12** *(p.24)*

$C = 2\lambda$; *Lemma 0.1 yields* $S_\infty^{(-\lambda)} \overset{(law)}{=} -\frac{1}{2\lambda}\log U \overset{(law)}{=} \mathbf{e}_{2\lambda} \overset{(law)}{=} \frac{1}{2\lambda}\mathbf{e}$ *where* $\mathbf{e}_\alpha$, *resp.* $\mathbf{e}$, *denotes an exponential variable with parameter* α, *resp. 1 and* U *is a uniform variable on* $[0,1]$.

• **Solution to Exercise 13** *(p.24)*

a) *It easily follows from the Hint and Example 1.4:* $L_{T_a^*}$ *is* $\exp(a)$.

b) *It suffices to apply the strong Markov property at time* $\tau_l^{(\beta)}$. *One then obtains (thanks to a)):*

$$P_t(x, dy) = e^{-\frac{t}{x}}\varepsilon_x(dy) + e^{-\frac{t}{y}}\frac{t}{y^2}1_{\{y \ge x\}}dy$$

For more details about the process $(W_l;\ l \ge 0)$, *which has been introduced by S. Watanabe, see [RY05] Exercice 4.11 Chapter XII and references therein.*

• **Solution to Exercise 14** *(p.25)*

a) a.1) *From Itô's formula (or the balayage formula),* $(M_t^\varphi := \frac{B_t}{\varphi(L_t)},\ t \ge 0)$ *is a local martingale with quadratic variation* $(\int_0^t \frac{ds}{\varphi^2(L_s)},\ t \ge 0)$.

a.2) Using Tanaka's formula for B, we deduce that

$$\frac{1}{\varphi(L_t)}|B_t| = \int_0^{L_t} \frac{dx}{\varphi(x)} + \int_0^t \frac{sgn(B_s)dB_s}{\varphi(L_s)}$$

Hence, the local time of M^φ *is* $(\int_0^{L_t} \frac{dx}{\varphi(x)},\ t \geq 0)$, *and it is also the local time* L^* *of* β, *time-changed with* $\int_0^{\cdot} \frac{ds}{\varphi^2(L_s)}$.

a.3)

$$P(\exists t \leq \tau_l,\ |B_t| \geq \varphi(L_t)) = P(\sup_{v \leq \tau_A^{(\beta)}} |\beta_v| \geq 1)$$

with $A = \int_0^l \frac{dx}{\varphi(x)}$. *The result follows from Exercise 13 a).*

b) The event $\Gamma = \{\forall A > 0,\ \exists t \geq A,\ |B_t| \geq \varphi(L_t)\}$ *is in the tail σ-field of the process* $(|B|_\cdot, L_\cdot)$, *so its probability is 0 or 1.*

- *If* $\int_0^\infty \frac{dx}{\varphi(x)} < \infty$, *then* $P(\exists t \geq 0,\ |B_t| \geq \varphi(L_t)) < 1$ *and, consequently,* $P(\Gamma) = 0$.
- *If* $\int_0^\infty \frac{dx}{\varphi(x)} = \infty$, *then*

$$1 = P(\exists t \geq \tau_l,\ |B_t| \geq \varphi(L_t))(= P(\exists t \geq 0,\ |\hat{B}_t| \geq \varphi(l + \hat{L}_t)))$$

Finally, $P(\Gamma) = 1$ *since* Γ *is the decreasing limit of* $\{\exists t \geq \tau_l,\ |B_t| \geq \varphi(L_t)\}$ *when* l *tends to* ∞.

c) To compute $P(\exists t \geq 0,\ B_t \leq \psi(S_t))$, *we use both Lévy's theorem (Proposition 0.5) and question a) with* $\varphi(x) = x - \psi(x)$

Comment 7.3 *In [Kni78], Knight studied some more general quantities such as*

$$\mathbb{E}\left[e^{-\int_0^{\tau_l} ds f(B_s, L_s)}\right]$$

and the preceding exercise focuses on a particular case of this study (Corollary 1.3 p.435). Nevertheless, the methods developed here are more directly taken from [JY81].

• **Solution to Exercise 15** *(p.26)*

a) We consider Itô's excursion process $e_l(t) := B_{\tau_{l-}+t} 1_{t \leq \tau_l - \tau_{l-}}$, $l \geq 0$.
From excursion theory, we know that $N_l^\varphi := \sum_{\lambda < l} 1_{\{\max_u e_l(u) \geq \varphi(\lambda)\}}$ *is a Poisson variable with parameter* $\int_0^l d\lambda\ \mathbf{n}(\max_u \varepsilon_u \geq \varphi(\lambda))$.

b) Apply a) with constant φ *and use the martingale property of* $(R_t^{2\nu} - L_t;\ t \geq 0)$ *to compute the expectation of* L *at the first hitting time of* a *by* R.

c) Applying Doob's maximal identity to the martingale $(h(L_t)R_t^{2\nu} + 1 - H(L_t),\ t \geq 0)$, *we obtain, for any* $x \in [0, 1]$,

$$1 - x = P(\exists t \geq 0,\ h(L_t)R_t^{2\nu} + 1 - H(L_t) \geq x)$$
$$= P\left(\exists t \geq 0,\ R_t \geq \left(\frac{H(L_t) + x - 1}{h(L_t)}\right)_+^{1/2\nu}\right)$$

Then if we consider $\varphi(l) = \left(\frac{H(l)+x-1}{h(l)}\right)_+^{1/2\nu}$, *the desired result is obtained by finding* h, *the positive solution of the ODE* $\varphi(l)^{2\nu} y'(l) = y(l) + x - 1$.

- **Solution to Exercise 16** *(p.26)*✠

If f *is twice differentiable, Itô's formula yields to linear ODE for* f *and the result follows from the resolution of these equations.*
In the general case, the question turns to be more tricky; nevertheless Jan Obłoj [OY05] [Obł05] recently managed to solve this problem and to prove the announced result. The main arguments used in Obłoj's proof are Motoo's theorem and the fact that the scale functions of Brownian motion are the affine functions.

- **Solution to Exercise 17** *(p.27)*

a) Let ρ *be a process such that* $\exp\{B_t - \mu t\} = \rho_{A_t^{(-\mu)}}$. *After applying Itô's formula to* $\exp\{2(B_t - \mu t)\}$ *and time-changing with the inverse of* $A_t^{(-\mu)}$, ρ *is seen to satisfy the Bessel SDE with index* $-\mu$.
b) Easy since $\lim_{t\to\infty} \exp\{B_t - \mu t\} = 0$.
c) Using time-reversal at T_0 *for the Bessel process* $R^{(-\mu)}$, T_0 *may be seen as the last passage time at level 1 for a Bessel process with index* μ; *then, the result is a particular case of Exercise 10*
d) The computations of λ_t *and* $\dot{\lambda}_t$ *are simple consequences of the decomposition* $A_\infty^{(-\mu)} = A_t^{(-\mu)} + \hat{A}_\infty^{(-\mu)} \exp\{B_t - \mu t\}$ *where we use the distribution found in question c).*
e) From (1.10), *we deduce*

$$B_t^{(-\mu)} = \widetilde{B}_t^{(\mu)} - \int_0^t \frac{dA_s^{(-\mu)}}{A_\infty^{(-\mu)} - A_s^{(-\mu)}} = \widetilde{B}_t^{(\mu)} + \log\left(\frac{A_\infty^{(-\mu)} - A_s^{(-\mu)}}{A_\infty^{(-\mu)}}\right)$$

Replacing $\widetilde{B}$ *by this expression in the computation of* $\widetilde{A}_t^{(\mu)}$, *we obtain*

$$\widetilde{A}_t^{(\mu)} = \int_0^t \left(\frac{A_\infty^{(-\mu)}}{A_\infty^{(-\mu)} - A_s^{(-\mu)}}\right)^2 dA_s^{(-\mu)}$$

- **Solution to Exercise 18** *(p.28)*✠

- **Solution to Exercise 19** *(p.28)*

a) 1. See Table 2 p.34 line 3.

2. We deduce from the preceding question that

$$S_t - B_t =$$

$$= \widetilde{B}_t - \int_0^t ds \left(\frac{2S_T - B_T - B_s}{T-s} - \frac{1}{2S_T - B_T - B_s} \right) 1_{S_s < S_T} + S_t$$

$$- \int_0^t ds \frac{1}{T-s} \left(S_s - B_s - (S_T - B_T) \coth\left[\frac{(S_T - B_T)(S_s - B_s)}{T-s} \right] \right) 1_{S_T = S_s}$$

b) *1. See Table 2 p.34 line 5.*
2. Combined with Tanaka's formula, the preceding question yields to

$$|B_t| = \widetilde{B}_t + \int_0^t ds \left(\frac{1}{L_T - L_s + |B_T| + |B_s|} - \frac{L_T - L_s + |B_T| + |B_s|}{T-s} \right) 1_{s<\gamma_T}$$

$$+ \int_0^t ds \frac{1}{T-s} \left(|B_T| \coth\left[\frac{|B_T||B_s|}{T-s} \right] - |B_s| \right) 1_{T \geq s \geq \gamma_T} + L_t$$

c) These two formulae are identical thanks to Lévy's equivalence (Lemma 0.5).

- **Solution to Exercise 20** *(p.28)*

a) See Remark 1.4.

*b) The two formulae are identical; in [Jeu80] p58-59 or [JY85], Jeulin deals with a more general framework involving any honest time L which satisfies assumption (**A**). Then the coincidence between the formulae obtained by enlarging progressively with L (i.e. w.r.t. $\mathcal{F}^L$) and by enlarging initially with the terminal value of the predictable compensator A^L_∞ follows from the fact that the (**H**) hypothesis is satisfied between the filtration $\mathcal{F}^L$ and $\mathcal{F}^L \vee \sigma(A^L_\infty)$, i.e. the first one is immersed in the second one.*

Remark 7.2 *In fact, the coincidence observed in this exercise turns out to be the general case under (**CA**) hypothesis. Indeed, the key argument is the multiplicative representation of Z^L, the Azéma supermartingale of an honest time L, obtained in Proposition 1.3. With $Z_t = N_t/\overline{N}_t$, enlarging progressively with L (resp. initially with A^L_∞) is equivalent to enlarging progressively with the random time $\sup\{t \geq 0,\ N_t = \overline{N}_t\}$ (resp. initially with $\overline{N}_\infty$).*

- **Solution to Exercise 21** *(p.28)*

a) Consider the closed subspace $\mathbb{G}_t^{h,-}$ of the Gaussian space $\mathbb{G}_t$ generated by $(B_u,\ u \leq t)$:

$$\mathbb{G}_t^{h,-} = span \left\{ \int_0^t f(u) dB_u \text{ with } f \in L^2([0,t]) \text{ and } \int_0^t f(u)h(u)du = 0 \right\}$$

Then $\mathbb{G}_t$ *is the orthogonal sum of* $\mathbb{G}_t^{h,-}$ *and the one-dimensional space:* $\left\{\lambda \int_0^t h(u)dB_u;\ \lambda \in \mathbb{R}\right\}$. *Then, it suffices to consider* $\mathcal{F}_t^{h,-}$ *the* σ*-field generated by* $\mathbb{G}_t^{h,-}$.

The fact that $\mathcal{F}_t^{h,-} \subset \mathcal{F}_{t+t'}^{h,-}$ *follows from the remark that* $\mathbb{G}_t^{h,-} \subset \mathbb{G}_{t+t'}^{h,-}$.

b) It suffices to remark that $B^{h,-}$ *is the projection of* B *on the filtration* $\mathcal{F}^{h,-}$.

c) i) From question b), using integration by parts, we get that

$$B_t - \left(\frac{t^{\alpha+1}}{\alpha+1}\frac{2\alpha+1}{t^{2\alpha+1}}\right)\int_0^t u^\alpha dB_u = B_t - \left(\frac{2\alpha+1}{\alpha+1}\right) \times \frac{1}{t^\alpha}\left[t^\alpha B_t - \int_0^t \alpha B_s s^{\alpha-1} ds\right] = -\left(\frac{\alpha}{\alpha+1}\right) B_t + \left(\frac{\alpha}{\alpha+1}\right)(2\alpha+1)\frac{1}{t^\alpha}\int_0^t B_s s^{\alpha-1} ds$$

is a $(\mathcal{F}_t^{\alpha,-},\ t \geq 0)$*-martingale whose quadratic variation is* $\left(\frac{\alpha}{\alpha+1}\right)^2 t$.

- *For* $\alpha \neq 0$, *we deduce that:*

$$\left(B_t^{\alpha,-} := B_t - \frac{2\alpha+1}{t^\alpha}\int_0^t B_s s^{\alpha-1} ds,\ t \geq 0\right)$$

is a $(\mathcal{F}_t^{\alpha,-},\ t \geq 0)$*-Brownian motion.*

- *For* $\alpha = 0$, *by passage to the limit, we obtain that:*

$$\left(B_t - \int_0^t ds \frac{B_s}{s},\ t \geq 0\right)$$

is a $(\mathcal{F}_t^{0,-},\ t \geq 0)$*-Brownian motion.*

ii) In order to show $\left(B_t^{\alpha,-},\ t \geq 0\right)$ *generates* $(\mathcal{F}_t^{\alpha,-},\ t \geq 0)$, *it suffices to prove that the orthogonal in* $\mathbb{G}_t$, *the Gaussian space generated by* $(B_u,\ u \leq t)$, *of* $\mathbb{G}_t^{\alpha,-}$, *is precisely the one-dimensional space* $\left\{\lambda \int_0^t u^\alpha dB_u,\ \lambda \in \mathbb{R}\right\}$.

Indeed, in general, $\int_0^t k(u)dB_u$ *is orthogonal to* $\mathbb{G}_t^{h,-}$ *if and only if:*

$$\forall s \leq t, \qquad \mathbb{E}\left[\left(\int_0^t k(u)dB_u\right)\left\{B_s - \sigma_h(s)\int_0^s h(u)dB_u\right\}\right] = 0$$

where $\sigma_h(s) = \frac{\int_0^s h(u)du}{\int_0^s h(u)^2 du}$.

Hence:

$$\int_0^s k(u)du = \sigma_h(s)\int_0^s h(u)k(u)du \tag{7.3}$$

In the case $h(s) = s^\alpha$*, we get* $\sigma_h(s) = \frac{2\alpha+1}{\alpha+1}\frac{1}{s^\alpha}$*, hence (7.3) is equivalent to:*

$$s^\alpha \int_0^s k(u)du = \frac{2\alpha+1}{\alpha+1}\int_0^s u^\alpha k(u)du$$

i.e., for $\alpha \neq 0$*:*

$$-\frac{1}{\alpha+1}u^\alpha k(u) + u^{\alpha-1}\int_0^u k(t)dt = 0, \qquad du \text{ a.s.}$$

and $k(u) = \frac{\alpha+1}{u}\int_0^u k(t)dt$ *which yields:* $\int_0^u k(t)dt = Cu^{\alpha+1}$*, hence the result.*
A similar computation holds for $\alpha = 0$*, starting with*

$$B_t^{0,-} = B_t - \int_0^t ds \frac{B_s}{s}$$

Comment 7.4 *We note that, at least concerning the functions* $h_\alpha(s) = s^\alpha$*, one may recover the original Brownian filtration by enlarging adequately* $(\mathcal{F}_t^{h_\alpha,-},\ t \geq 0)$*.*

Indeed, note that, for $0 < s < t$*:*

$$\frac{B_t}{t} - \frac{B_s}{s} = \int_s^t \frac{dB_u}{u} - \int_s^t \frac{B_u}{u^2}du$$

which yields: $\frac{B_s}{s} = -\int_s^\infty \frac{dB_u^{0,-}}{u}$*;hence:* $\mathcal{F}_t = \mathcal{F}_t^{0,-} \vee \sigma\left\{\int_t^\infty \frac{dB_u^{0,-}}{u}\right\}$.

A similar computation may be done with h_α*,* $\alpha > -1/2$*.*

S_2 Chapter 2

- **Solution to Exercise 22** *(p.49)*

a) $\lambda = a^{\frac{1}{r}}$*.*
b) $(b_u := uB_{u/(1-u)},\ u \leq 1)$ *is a standard Brownian bridge.*
c) ✠

It may be worth noticing that

$$P(\sup_t\{|B_t| - t^{r/2}\} \geq x) = P(\exists t \geq 0,\ B_t \geq x + t^{r/2})$$
$$= P(\inf\{t \geq 0,\ B_t \geq x + t^{r/2}\} < \infty)$$

and that the density of the variable $\inf\{t \geq 0,\ B_t \geq x + t^{r/2}\}$ *is shown, in [dlPHDV04] (following [RSS84]), to satisfy a Volterra integral equation of second type.*
Note that this question has already been solved for $r = 4$ *by Groeneboom [Gro89] which relates first passage times for* $B_t - ct^2$ *to Bessel processes, and then find their densities explicitly in terms of Airy functions.*

Remark 7.3 *It follows from a well-known theorem due to Fernique about the supremum of any centered Gaussian process that both quantities in equation* (2.6) *have moments of all orders.*

• **Solution to Exercise 23** *(p.50)*

a) See Exercise 14.

b) From Example 2.1, we obtain

$$\mathbb{E}\left[\sup_t\{|B_t|^\alpha - L_t^{\alpha\beta}\}\right] = \mathbb{E}\left[(\mathbf{e}_{\alpha,\alpha\beta})^{\frac{\alpha\beta}{1-\beta}}\right]$$

with $\mathbf{e}_{a,b}$ *an exponential random variable with parameter* $c_{a,b}$ *defined in Example 2.1. Therefore, this quantity is finite if and only if* $\frac{\alpha\beta}{\beta-1} < 1$

c) For any $c > 0$*, we write:*

$$\begin{aligned}\mathbb{E}\left[|B_\Lambda|^\alpha\right] &= \mathbb{E}\left[|B_\Lambda|^\alpha - cL_\Lambda^{\alpha\beta}\right] + c\mathbb{E}\left[L_\Lambda^{\alpha\beta}\right]\\ &\leq \mathbb{E}\left[\sup_{t\in\mathbb{R}^+}\{|B_t|^\alpha - cL_t^{\alpha\beta}\}\right] + c\mathbb{E}\left[L_\Lambda^{\alpha\beta}\right]\\ &\leq c^{-\frac{1}{\beta-1}}\mathbb{E}\left[\sup_{t\in\mathbb{R}^+}\{|B_t|^\alpha - L_t^{\alpha\beta}\}\right] + c\mathbb{E}\left[L_\Lambda^{\alpha\beta}\right] \qquad \textit{(scaling)}\end{aligned}$$

Minimizing with respect to c, we obtain $\mathbb{E}\left[|B_\Lambda|^\alpha\right] \leq K.\mathbb{E}\left[L_\Lambda^{\alpha\beta}\right]^{\frac{1}{\beta}}$*, with*

$$K = \beta\left(\frac{\mathbb{E}\left[\sup_{t\in\mathbb{R}^+}\{|B_t|^\alpha - L_t^{\alpha\beta}\}\right]}{\beta - 1}\right)^{\frac{\beta-1}{\beta}}$$

d) We prove that the constant K found in c) is the best constant, simply by considering the random time

$$\inf\{u \geq 0,\ |B_u|^\alpha - L_u^{\alpha\beta}\} = \sup_t\{|B_t|^\alpha - L_t^{\alpha\beta}\}\ .$$

For more details, see Song-Yor [SY87].

• **Solution to Exercise 24** *(p.50)*

a) $A^{(2)}$ *and* $C^{(2)}$ *are not moment-equivalent, as may be seen by considering* $T = T_a := \inf\{t;\ B_t = a\}$*, and the moment-equivalence between* $A^{(1)}$ *and* $C^{(1)}$*.*

b) Take $T = \tau_l$*.* $C^{(3)}_{\tau_l}$ *is distributed as the reciprocal of an exponential variable (see Example 13); hence,* $A^{(3)}$ *and* $C^{(3)}$ *are not moment-equivalent.*

Remark 7.4 *Although* $A^{(i)}$ *and* $C^{(i)}$ *are not moment-equivalent for* $i = 2, 3$*, if we consider only small exponents p, that is* $p < 1$*, then*

$$\mathbb{E}\left[(C_T^{(2)})^p\right] \leq \gamma_p^{(2)}\mathbb{E}\left[(A_T^{(2)})^p\right] \qquad \mathbb{E}\left[(C_T^{(3)})^p\right] \leq \gamma_p^{(3)}\mathbb{E}\left[(A_T^{(3)})^p\right]$$

with universal constants $\gamma_p^{(2)}$ *and* $\gamma_p^{(3)}$. *These inequalities are particular consequences of the domination relation introduced by Lenglart [Len77] to whom we refer the reader.*

- **Solution to Exercise 25** *(p.51)*

a) From (2.10), *we deduce that*

$$\rho_t^{(n)} - t = \frac{2}{\sqrt{n}} \int_0^t \sqrt{\rho_s^{(n)}}\, d\beta_s$$

and we conclude thanks to some ersatz of BDG inequalities applied to the martingale $\left(\frac{2}{\sqrt{n}} \int_0^t \sqrt{\rho_s^{(n)}}\, d\beta_s;\ t \geq 0\right)$

b) Similar arguments yields to the result.

- **Solution to Exercise 26** *(p.51)*✠

It may be helpful to write these inequalities in an equivalent form, namely

a) $\mathbb{E}\left[|M_T|\right] \leq C_p^{(1)}\mathbb{E}\left[< M >_T^{p/2}\right]^{1/p}$ *for any martingale* $(M_t;\ t \geq 0)$

b) $\mathbb{E}\left[\varphi(L_T)|B_T|\right] \leq C_p^{(2)}\mathbb{E}\left[\Phi(L_T)^p\right]^{1/p}$ *for any function* φ *and* $\Phi(x) = \int_0^x \varphi(y)dy$

S_3 Chapter 3

- **Solution to Exercise 27** *(p.66)*

a) $B_u^{[0,\gamma]} = u\beta_{\frac{1}{u}-1}$ *where* $\beta_v = \frac{1}{\sqrt{\hat{\delta}}}(\hat{B}_{\hat{\delta}+\hat{\delta}v} - \hat{B}_{\hat{\delta}})$.

b) $m_1 = \sqrt{\hat{B}_1^2 + \frac{\hat{B}_1^2}{\hat{\delta}-1}}$; *moreover,* $\frac{\hat{\delta}-1}{\hat{B}_1^2}$ *is independent of* $\hat{B}_1^2$ *and is distributed as the first hitting time of 1 by a Brownian motion* $\widetilde{B}$; *finally:*

$$\frac{1}{\widetilde{T}_1} \stackrel{(law)}{=} \widetilde{B}_1^2$$

c) ✠

- **Solution to Exercise 28** *(p.67)*

We note the equalities between events:

$$\begin{aligned}(\gamma \leq u) = (1 \leq \delta_u) &= (1 \leq u + \hat{T}_{B_u}) \\ &= (1 \leq u + B_u^2 \hat{T}_1) = \left(1 \leq u\left(1 + \frac{B_1^2}{\hat{B}_1^2}\right)\right)\end{aligned}$$

with $\delta_u = \inf\{v \geq u,\ B_v = 0\}$ *and* $\hat{T}_{B_u}$ *(resp.* $\hat{T}_1$*) the first hitting time of* B_u *(resp. 1) by a Brownian motion independent of* $(B_s;\ s \leq u)$.

• **Solution to Exercise 29** *(p.67)*

a) Using the expression of Z^γ computed at the beginning of Chapter 3, we find

$$\lambda_t(f) = f(\gamma_t)\left(1 - \psi\left(\frac{|B_t|}{\sqrt{1-t}}\right)\right) + \mathbb{E}\left[f(\gamma)1_{\gamma>t}|\mathcal{F}_t\right]$$

To compute the conditional law of γ knowing $\mathcal{F}_t$, we note that

$$P(\gamma \in du|\mathcal{F}_t) = \sqrt{\frac{2}{\pi}}\,\frac{\mathbb{E}\left[dL_u|\mathcal{F}_t\right]}{\sqrt{1-u}} = \sqrt{\frac{2}{\pi}}\,\frac{\mathbb{E}_{B_t}\left[dL_{u-t}\right]}{\sqrt{1-u}}, \qquad t < u$$

Then, we use $\mathbb{E}_x\left[L_h^y\right] = \int_0^h ds p_s(x,y)$ to show that

$$\lambda_t(f) = f(\gamma_t)\left(1 - \psi\left(\frac{|B_t|}{\sqrt{1-t}}\right)\right) + \int_t^1 du \frac{f(u)e^{-\frac{B_t^2}{2(u-t)}}}{\pi\sqrt{(1-u)(u-t)}}$$

The two enlargement formulae are identical (see the solution to Exercise 20).

b) In order to compute $\mathbb{E}\left[f(\gamma_T,\delta_T)|\mathcal{F}_t\right]$ for any Borel function f, we compute successively:

$$\mathbb{E}\left[f(\gamma_T,\delta_T)1_{t>\delta_T}|\mathcal{F}_t\right] = f(\gamma_T,\delta_T)1_{t>\delta_T}$$

$$\mathbb{E}\left[f(\gamma_T,\delta_T)1_{\delta_T>t>T}|\mathcal{F}_t\right] = \frac{1}{\sqrt{2\pi}}\int_t^\infty f(\gamma_t,u)e^{-\frac{B_t^2}{2(u-t)}}\frac{|B_t|}{(u-t)^{3/2}}du1_{\gamma_t<T<t}$$

$$\mathbb{E}\left[f(\gamma_T,\delta_T)1_{T>t>\gamma_T}|\mathcal{F}_t\right] = \frac{1}{\sqrt{2\pi}}\int_T^\infty f(\gamma_t,u)e^{-\frac{B_t^2}{2(u-t)}}\frac{|B_t|}{(u-t)^{3/2}}du1_{t<T}$$

$$\mathbb{E}\left[f(\gamma_T,\delta_T)1_{\gamma_T>t}|\mathcal{F}_t\right] = \frac{1}{2\pi}\int_t^T du\int_t^\infty dv f(u,v)\frac{e^{-\frac{B_t^2}{2(u-t)}}}{\sqrt{(v-u)^3(u-t)}}$$

where we use the conditional distribution of γ_T knowing $\mathcal{F}_t$ already computed in question a) above and where the last equality is obtained by conditioning successively on $\mathcal{F}_T$ and $\mathcal{F}_{\gamma_T}$.
Then, the result follows easily.

• **Solution to Exercise 30** *(p.67)*

Thanks to Lévy's equivalence (Proposition 0.5), we may reduce our discussion to path decomposition after and before γ. More precisely, we deduce from Lévy's equivalence that $(m_u^{(1)} := \frac{B_\sigma - B_{\sigma+u(1-\sigma)}}{\sqrt{1-\sigma}},\ u \le 1)$ and $(m_u^{(2)} := \frac{B_\sigma - B_{\sigma-u\sigma}}{\sqrt{\sigma}},\ u \le 1)$ are independent and that $m^{(1)}$ is a meander.
$m^{(2)}$ is also a meander since Wiener measure is invariant by time-reversal at fixed time 1.

• **Solution to Exercise 31** *(p.67)*

The result follows easily from the Hint and the explicit law of $\hat{L}_{1-t}$ (obtained e.g. via Lévy's equivalence, Proposition 0.5).

• **Solution to Exercise 32** *(p.68)*

a) and b): The process in (3.12) *is the limit in* L^2*, uniformly in* t $(\leq T)$ *of*

$$z_{\gamma_t} B_t \exp\left(i \int_0^t f(s) 1_{|B_s| \geq \frac{1}{n}} d\hat{B}_s + \frac{1}{2} \int_0^t f(s)^2 ds\right)$$

and we use Itô's formula to obtain in the limit:

$$z_{\gamma_t} B_t \widehat{\mathcal{E}}_t = \int_0^t z_{\gamma_s} \widehat{\mathcal{E}}_s (1 + i f(s) B_s) dB_s$$

with $\widehat{\mathcal{E}}_t = \exp\left(i \int_0^t f(s) d\hat{B}_s + \frac{1}{2} \int_0^t f(s)^2 ds\right)$.

Therefore, $(z_{\gamma_t} B_t \widehat{\mathcal{E}}_t;\ t \geq 0)$ *is a martingale, which yields the desired result.*

c) • *We write:*

$$|B_{\gamma_t+u}| = sgn(B_t)\left(\hat{B}_{\gamma_t+u} - \hat{B}_{\gamma_t}\right) + \int_0^u \frac{dh}{|B_{\gamma_t+h}|}$$

The result now follows from Imhof's relation (3.1) *and the fact that the three dimensional Bessel process generates the same filtration as its driving Brownian motion.*

• *We have shown previously that* B_t *is measurable with respect to*

$$\sigma(\gamma_t) \vee \sigma(sgn(B_t)) \vee \hat{\mathcal{B}}_t$$

But, $\gamma_t = \inf\{s < t,\ sgn(B_u) \text{ is constant on } [s,t]\}$*, hence the result.*

d) We write:

$$f(B_t) = f(|B_t|) 1_{s_t=+1} + f(-|B_t|) 1_{s_t=-1},$$

where $s_t = sgn(B_t)$*, and the desired formula will follow from the computation of* $\sigma_t^+ := P(s_t = +1 | \widehat{\mathcal{B}}_t \vee \mathcal{F}_{\gamma_t})$.

For this purpose, we use: $\mathbb{E}[B_t | \widehat{\mathcal{B}}_t \vee \mathcal{F}_{\gamma_t}] = 0$

This implies: $\lambda_t^+ \sigma_t^+ - \lambda_t^- (1 - \sigma_t^+) = 0$*, hence:*

$$\sigma_t^+ = \frac{\lambda_t^-}{\lambda_t^+ + \lambda_t^-}$$

e) Under the signed measure Q_a, $\widehat{\mathcal{E}}_t := \exp\left(i \int_0^t f(s) d\hat{B}_s + \frac{1}{2} \int_0^t f(s)^2 ds\right)$ *defines a martingale since* $(\frac{B_t}{a} \widehat{\mathcal{E}}_t;\ t \geq 0)$ *is a* P_a*-martingale. Moreover, since* $\mathbb{E}_{P_a}\left[\frac{B_t}{a} \widehat{\mathcal{E}}_t\right] = 1$*, we deduce that* $\mathbb{E}_{Q_a}\left[\widehat{\mathcal{E}}_t\right] = 1$*, so that:*

$$\mathbb{E}_{Q_a}\left[\exp\left(i \int_0^t f(s) d\hat{B}_s\right)\right] = \exp\left(-\frac{1}{2} \int_0^t f(s)^2 ds\right)$$

The main differences with the preceding questions can be explained using T_0*, the first hitting time by* B *of the level 0. More precisely, in the following decomposition*

$$\frac{B_1}{a} = \frac{B_{1\wedge T_0}}{a} + \frac{B_1}{a} 1_{T_0<1}$$

the first part on the right corresponds to the density between the Wiener measure P_a and the law of the three dimensional Bessel process starting from a.

S_4 Chapter 4

• **Solution to Exercise 33** *(p.83)*

a) $(X_t^D;\ t \geq 0)$ *is a* $(Q, (\mathcal{F}_t;\ t \geq 0))$*-Brownian motion.*

b) *It is easy to show that every* $(Q, (\mathcal{F}_t;\ t \geq 0))$ *local martingale* $(M_t^D;\ t \geq 0)$ *may be obtained as the Girsanov transform of* $(M_t; t \geq 0)$*, a* $(W, (\mathcal{F}_t; t \geq 0))$ *local martingale, namely,*

$$M_t^D = M_t - \int_0^t \frac{d < D, M >_s}{D_s}$$

From the representation of $(M_t;\ t \geq 0)$ *as a stochastic integral with respect to* $(X_t;\ t \geq 0)$*, we deduce the representation of* $(M_t^D;\ t \geq 0)$ *as a stochastic integral with respect to* $(X_t^D;\ t \geq 0)$*.*

• **Solution to Exercise 34** *(p.84)*

a) *The first point follows from*

$$< \mathcal{A} >_t = \int_0^t |Z_s|^2 ds \qquad < \theta >_s = \int_0^t \frac{ds}{|Z_s|^2}$$
$$\mathcal{A}_t = \int_0^t |Z_s|^2 d\theta_s \qquad \theta_t = \int_0^t \frac{d\mathcal{A}_s}{|Z_s|^2}$$

The second point follows from the polar representation of $(Z_t;\ t \geq 0)$*, namely* $Z_t = |Z_t| \exp(i\theta_t)$*, also called skew product decomposition of* $(Z_t;\ t \geq 0)$*, see, e.g. [IM74] and [PR88].*

b) *We note that it is not possible to represent either* X *or* Y *as a stochastic integral with respect to* $\mathcal{A}$*(or* θ*); why?*
Then, use Theorem 4.4.

c) *When* $z_0 = 0$*,* $(\theta_t;\ t \geq 0)$ *cannot be defined via formula* (4.8) *(see, e.g. [IM74] Section 7.16 p276).*
On the other hand, the definition of $(\mathcal{A}_t;\ t \geq 0)$ *makes sense as before; furthermore,* $(|Z_t|,\ t \geq 0)$ *is adapted to the filtration of* $(\mathcal{A}_t;\ t \geq 0)$ *since*

$$|Z_t|^2 = \frac{d}{dt}(< \mathcal{A} >_t)$$

The natural filtration of $|Z|$ *is that of* $\beta_t = \int_0^t \frac{X_s dX_s + Y_s dY_s}{|Z_s|}$ *and* $\gamma_t = \int_0^t \frac{d\mathcal{A}_s}{|Z_s|}$ *is another Brownian motion independent from* β*. The filtration of*

(β, γ) is precisely that of $\mathcal{A}$.
Finally, for any fixed time t, $Z_t/|Z_t|$ is uniformly distributed on the unit circle, and independent from the process $(\mathcal{A}_s;\ s \geq 0)$. Indeed, for any positive, Borel function f and any positive, bounded functional Φ:

$$\mathbb{E}\left[\Phi(\mathcal{A}) f\left(\frac{Z_t}{|Z_t|}\right)\right] = \mathbb{E}\left[\Phi(\mathcal{A}) f\left(\frac{e^{i\theta} Z_t}{|Z_t|}\right)\right] = \mathbb{E}\left[\Phi(\mathcal{A}) \frac{1}{2\pi}\int_0^{2\pi} d\theta f\left(\frac{e^{i\theta} Z_t}{|Z_t|}\right)\right]$$
$$= \mathbb{E}\left[\Phi(\mathcal{A})\right] \frac{1}{2\pi}\int_0^{2\pi} d\mu f(e^{i\mu})$$

• **Solution to Exercise 35** *(p.84)*
We first detail the arguments suggested in the Hint:
for $f : (E, \mathcal{E}) \to (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, compute, for any $t, s > 0$, $\mathbb{E}\left[f(X_{t+s})|\mathcal{Y}_s\right]$ in two different manners, which shall yield

$$\Lambda(P_t f)(Y_s) = Q_t(\Lambda f)(Y_s) \qquad a.s.$$

We now consider points a) and b):

a) $X_t = B_t^{(n)}$, $Y_t = |B_t^{(n)}|$.
Then, under P_0, $X_t = Y_t \Theta_t$, with $\Theta_t = X_t/Y_t$ uniformly distributed on the unit sphere, and independent of $(Y_u;\ u \geq 0)$. Hence $\Lambda(x, dy)$ is the uniform law on the sphere of radius y.

b) It will be easier to work with the squared Bessel processes $(Z_t^{(m)};\ t \geq 0)$ and $(Z_t^{(n)};\ t \geq 0)$. From the additivity property of squared Bessel processes, we may construct the pair $(Z^{(m)}, Z^{(n)})$ as:

$$Z_t^{(m)} = Z_t^{(m-n)} + Z_t^{(n)}, \qquad t \geq 0$$

where, on the right hand side, the two processes $(Z_t^{(m-n)};\ t \geq 0)$ and $(Z_t^{(n)};\ t \geq 0)$ are independent.
We now take

$$\mathcal{X}_t = \sigma\{Z_u^{(m-n)},\ Z_u^{(n)},\ u \leq t\}$$
$$\mathcal{Y}_t = \sigma\{Z_u^{(m)},\ u \leq t\}$$

Then, we use the following properties:

- *$(Z_t^{(n)};\ t \geq 0)$ is a Markov process with respect to $(\mathcal{X}_t;\ t \geq 0)$.*
- *For any $t > 0$, the variable $Z_t^{(n)}/Z_t^{(m)}$ is independent from $\mathcal{Y}_t$ and $\beta(\frac{n}{2}, \frac{m-n}{2})$ distributed.*

The second property may be proven by using time inversion for the two-dimensional process $(Z^{(m)}, Z^{(n)})$, an elementary beta-gamma result which yields $Z_t^{(n)}/Z_t^{(m)}$ is $\beta(\frac{n}{2}, \frac{m-n}{2})$ distributed, and finally the Markov property of $(Z_t^{(m)};\ t \geq 0)$ with respect to $(\mathcal{X}_t;\ t \geq 0)$.
This argument is taken from [CPY98].

• **Solution to Exercise 36** *(p.85)*

a) $\mathcal{L}^{(\beta)} f = 0$ *with* $f(x) = x$.

b) *Noting that* $< X >_t = t$ *and that* $\Delta X_t = \beta X_{t-} 1_{\Delta X_t \neq 0}$ *and using Theorem 4.7, it suffices to prove that* X_t *is bounded; from the equations* $d(X_t^2) = 2X_{t-} dX_t + d[X,X]_t$ *and* $d[X,X]_t = \beta X_{t-} dX_t + dt$, *we deduce that*

$$(\beta + 2) d[X,X]_t - \beta d(X_t^2) = 2dt$$

therefore $X_t^2 \leq X_0^2 \leq \frac{2t}{-\beta}$.

c) *The fact that there exists a Markov kernel* M *such that* $\mathcal{L}^{(\beta)} M = M\left(\frac{1}{2} D^2\right)$ *follows from the representation of* $\mathcal{L}^{(\beta)}$ *as* $\mathcal{L}^{(\beta)} f(x) = \frac{1}{2} \mathbb{E}\left[f''(xV)\right]$ *where the law of the random variable* V *is given by*

$$\mathbb{P}(V \in dv) = \frac{2}{\beta^2} (v - 1 - \beta)^+ 1_{v \leq 1} dv$$

and $Mf(x) = \mathbb{E}[f(xV)]$. *The announced result follows easily.*

Remark 7.5 *In the particular case* $\beta = -2$, $(X_t;\ t \geq 0)$ *is called the parabolic martingale[Val95]; its paths belong to the parabola* $x^2 = t$. *More precisely,* $P(X_t = \sqrt{t}) = P(X_t = -\sqrt{t}) = 1/2$ *and its jumps, which occur when* X *changes signs, happen at times distributed according to a Poisson point process with intensity* $dt/(4t)$. $(X_t;\ t \geq 0)$ *may be realized as* $X_t = B_{T^*_{\sqrt{t}}}$ *where* B *is a Brownian motion and* $T^*_a = \inf\{u \geq 0;\ |B_u| = a\}$.

• **Solution to Exercise 37** *(p.85)*

a) $\mathcal{L}_k f_1 = 0$; $\mathcal{L}_k f_2(x) = (1 + 2k)$, *with* $f_1(x) = x$, $f_2(x) = x^2$.

b) $\mathcal{L}_k$ *restricted to even functions is the infinitesimal generator of the Bessel process.*
Since $|X^{(k)}|$ *is a continuous process, the jump process satisfies* $\Delta X_t^{(k)} = -2X_{t-}^{(k)} 1_{\Delta X_t^{(k)} \neq 0}$.

c) *Since* $(X^{(k)})^2$ *is a squared Bessel process with dimension* $(2k+1)$, *there exists a Brownian motion* B *such that*

$$(X_t^{(k)})^2 = x^2 + 2\int_0^t \sqrt{(X_s^{(k)})^2} dB_s + (2k+1)t$$

Identifying this formula with Itô's formula, we obtain:

$$\begin{cases} \int_0^t X_{s-}^{(k)} d(X^{(k)})_s^c & = \int_0^t |X_s^{(k)}| dB_s \\ 2\int_0^t X_{s-}^{(k)} d(X^{(k)})_s^d + \sum_{s \leq t} \left(\Delta X_s^{(k)}\right)^2 & = 2kt \end{cases}$$

Therefore $1_{X_s^{(k)} \neq 0} d < (X^{(k)})^c >_s = ds$ *and* $1_{X_s^{(k)} = 0} d < (X^{(k)})^c >_s = 0$, *but the zero set of the squared Bessel process* $(X^{(k)})^2$ *(hence of* $X^{(k)}$*)*

has 0-Lebesgue measure. Hence $(X^{(k)})^c$ is a Brownian motion (Lévy's characterization theorem). The result then follows.

d) *It suffices to show that the processes $(Y_u;\ u \geq 0)$ and $(X^{(k)}_{\tau_u},\ u \geq 0)$, where $\tau_u = \inf\{t \geq 0,\ A_t > u\}$, are both Markovian with the same infinitesimal generator, namely*

$$\widetilde{\mathcal{L}}f(x) = \frac{x^2}{2}f''(x) + k\left(xf'(x) + \frac{f(-x) - f(x)}{2}\right)$$

e) *The hypotheses made in Theorem 4.7 apply:*
   - *the totality property follows from the fact that each variable $(X^{(k)}_t)^2$ admits some exponential moments*
   - $\Delta X^{(k)}_t = -2X^{(k)}_{t-}1_{\Delta X^{(k)}_t \neq 0}$
   - *we have just seen previously that*

$$< (X^{(k)})^{(c)} >_t = t, \qquad < (X^{(k)})^{(d)} >_t = 2kt$$

S_5 Chapter 5

- **Solution to Exercise 38** *(p.99)*

a) *$(\mathcal{A}_t;\ t \geq 0)$ is immersed in $(\mathcal{B}_t;\ t \geq 0)$ if and only if every uniformly integrable $(\mathcal{A}_t;\ t \geq 0)$-martingale $(M^{\mathcal{A}}_t;\ t \geq 0)$ is a $(\mathcal{B}_t;\ t \geq 0)$-martingale; hence, it may be written as $M^{\mathcal{A}}_t = \mathbb{E}\left[M^{\mathcal{A}}_\infty|\mathcal{B}_t\right]$, that is*

$$\mathbb{E}\left[M^{\mathcal{A}}_\infty|\mathcal{A}_t\right] = \mathbb{E}\left[M^{\mathcal{A}}_\infty|\mathcal{B}_t\right]$$

Since $M^{\mathcal{A}}_\infty$ may be taken to be any variable in $L^1(\mathcal{A}_\infty)$, we get the desired result.

b) *Using both the PRP and the $(\mathcal{B}_t;\ t \geq 0)$-martingale properties of $(a_t;\ t \geq 0)$, the result follows from the fact that $(\mathcal{A}_t;\ t \geq 0)$-predictable processes remain $(\mathcal{B}_t;\ t \geq 0)$-predictable processes.*

- **Solution to Exercise 39** *(p.99)*

a) *We give two different arguments:*
   - *Since the filtration of $|B|$ is that of $\beta := \int_0^\cdot sgn(B_s)dB_s$, which is a $(\mathcal{F}_t;\ t \geq 0)$-Brownian motion, we can use the predictable representation property of β, and use Exercise 38.*
   - *$|B|$ is a $(\mathcal{F}_t;\ t \geq 0)$-Markov process (Dynkin's criteria, see e.g. [RP81]). Then, the martingale additive functionals of $|B|$ are also $(\mathcal{F}_t;\ t \geq 0)$-martingales; since they generate (in the sense of Kunita-Watanabe) all $(\sigma\{|B_u|,\ u \leq t\},\ t \geq 0)$-martingales, the latter are also $(\mathcal{F}_t;\ t \geq 0)$-martingales.*

b) *For general integers n, both arguments may be adequately extended.*

c) This is immediate since there exist discontinuous $(\mathcal{F}_t^+,\ t \geq 0)$*-martingales as proven in Theorem 5.5.*

• **Solution to Exercise 40** *(p.101)*

a) *1. Use Tanaka's formula.*

2. For any bounded functional F, the Cameron-Martin formula yields

$$\begin{aligned}\mathbb{E}\left[F(|B_s^{(\mu)}|,\ s \leq t)\right] &= \mathbb{E}\left[F(|B_s|,\ s \leq t)e^{\mu B_t - \frac{\mu^2}{2}t}\right] \\ &= \mathbb{E}\left[F(|B_s|,\ s \leq t)e^{-\mu B_t - \frac{\mu^2}{2}t}\right] \\ &= \mathbb{E}\left[F(|B_s|,\ s \leq t)\cosh(\mu B_t)e^{-\frac{\mu^2}{2}t}\right]\end{aligned}$$

Formula (5.9) *now follows from Girsanov's theorem.*
The equality of the two filtrations follows by considering $(B_t^{(\mu)})^2$ *which appears as a strong solution of a SDE driven by* β*.*

3. Comparing the two decomposition formulae (5.9) *and* (5.10)*, we see that* β *is not a* $(\mathcal{F}_t;\ t \geq 0)$*-martingale.*

Comment 7.5 *With the notation of Proposition 5.13, we have*

$$A_t^{\mathcal{F}} = \mu \int_0^t sgn(B_s^{(\mu)})ds + L_t^0(B^{(\mu)}), \quad (A^{\mathcal{F}})_t^{(p)} = \mu \int_0^t \tanh(\mu|B_s^{(\mu)}|)ds + L_t^0(B^{(\mu)})$$

b) *1. With the conditional law recalled in the exercise, we deduce*

$$\begin{aligned}\mathbb{E}\left[F(R_s^{(\mu)};\ s \leq t)\right] &= \mathbb{E}\left[F(R_s^{(0)};\ s \leq t)e^{\mu B_t - \frac{\mu^2}{2}t}\right] \\ &= \mathbb{E}\left[F(R_s^{(0)};\ s \leq t)\frac{1}{2R_t^{(0)}}\int_{-R_t^{(0)}}^{R_t^{(0)}} dx\ e^{\mu x}e^{-\frac{\mu^2}{2}t}\right] \\ &= \mathbb{E}\left[F(R_s^{(0)};\ s \leq t)\frac{\sinh(\mu R_t^{(0)})}{\mu R_t^{(0)}}e^{-\frac{\mu^2}{2}t}\right]\end{aligned}$$

Equation (5.11) *is obtained, once more, from Girsanov's theorem.*

2. This is a consequence of the non-canonical decomposition

$$R_t^{(\mu)} = -B_t + (2S_t^{(\mu)} - \mu t).$$

3. Using the Cameron-Martin formula together with the result recalled for processes with 0-drift, we obtain that the law of $B_t^{(\mu)}$ *conditionally on* $\sigma(R_s^{(\mu)};\ s \leq t)$ *and* $R_t^{(\mu)} = r$ *is*

$$\frac{\mu e^{\mu x}}{2\sinh(\mu r)} 1_{x \in (-r,r)} dx$$

• **Solution to Exercise 41** *(p.102)*

a) Recall that $A_1^+ \stackrel{(law)}{=} \gamma a_1^+ + \varepsilon(1-\gamma)$ *(see Subsection 3.1.2) and multiply both sides of this identity with* $2\mathbf{e}$, *where* $\mathbf{e}$ *denotes a standard exponential variable. From the beta-gamma algebra, we obtain*

$$\mathcal{N}^2 \stackrel{(law)}{=} \mathcal{N}^2 a_1^+ + \varepsilon \mathcal{N}'^2 \tag{7.4}$$

where on the RHS, the four variables are independent, and $\mathcal{N}$ *and* $\mathcal{N}'$ *are standard normal variables. We then easily deduce from* (7.4) *that*

$$\mathbb{E}\left[e^{-\lambda a_1^+ \mathcal{N}^2}\right] = \frac{\sqrt{2\lambda+1}-1}{\lambda}$$

with $\mathcal{N}$ *a standard Gaussian variable independent from* a_1^+. *Therefore,* $a_1^+\mathcal{N}^2 \stackrel{(law)}{=} U\mathcal{N}^2$ *with* U *a uniform variable. The result follows from the injectivity of the Gauss transform.*

b) For any bounded function f *on* $[0,1]$,

$$\begin{aligned}
\mathbb{E}(f(a_1^+)) &= \int_0^\infty dl\ le^{-\frac{l^2}{2}} \mathbb{E}\left[f(A_1^+)|L_1 = l, B_1 = 0\right] \\
&= \int_0^\infty dl\ le^{-\frac{l^2}{2}} \mathbb{E}\left[f(A_1^+)|\tau_l = 1\right] && \textit{(switching identity)} \\
&= \int_0^\infty dl\ le^{-\frac{l^2}{2}} \mathbb{E}\left[f(A_{\tau_l}^+)|\tau_l = 1\right] \\
&= \int_0^\infty dm\ e^{-m} \mathbb{E}\left[f(2mA_{\tau_1}^+)|2m\tau_1 = 1\right] && \textit{(scaling)} \\
&= \int_0^\infty \frac{dm}{\sqrt{m}} e^{-m} \mathbb{E}\left[\sqrt{m} f(\frac{A_{\tau_1}^+}{\tau_1})|\frac{1}{2\tau_1} = m\right] \\
&= \int_0^\infty \frac{dm}{\sqrt{m}} e^{-m} \mathbb{E}\left[\frac{1}{\sqrt{2\tau_1}} f(\frac{A_{\tau_1}^+}{\tau_1})|\frac{1}{2\tau_1} = m\right] \\
&= \sqrt{\frac{\pi}{2}} \mathbb{E}\left[\frac{1}{\sqrt{\tau_1}} f(\frac{A_{\tau_1}^+}{\tau_1})\right]
\end{aligned}$$

Now, the result follows from simple computations with the variables $\mathcal{N}^2 \stackrel{(law)}{=} \frac{1}{2A_{\tau_1}^+}$ *and* $\mathcal{N}'^2 \stackrel{(law)}{=} \frac{1}{2A_{\tau_1}^-}$, *where* $\mathcal{N}$ *and* $\mathcal{N}'$ *may be chosen to be independent standard Gaussian variables.*

Remark 7.6 *In [Yor95], using the absolute continuity between the standard Brownian bridge and the pseudo bridge*[1] $(\frac{1}{\sqrt{\tau_1}} B_{u\tau_1},\ u \le 1)$, *one also arrives at the identity*

[1] See [BLGY87] for more details about this process; in fact, this absolute continuity result may also be obtained as a consequence of the switching identity, see e.g. [PY92].

$$\mathbb{E}(f(a_1^+)) = \sqrt{\frac{\pi}{2}}\mathbb{E}\left[\frac{1}{\sqrt{\tau_1}} f\left(\frac{A_{\tau_1}^+}{\tau_1}\right)\right]$$

c) On one hand,
$\mathbb{E}\left[\int_0^\infty dL_t F(B_u;\ u \le t)\varphi(L_t)\psi(t)\right] =$

$$= \mathbb{E}\left[\int_0^\infty dL_t \mathbb{E}\left[F(B_u;\ u \le t)\varphi(L_t)|B_t = 0\right]\psi(t)\right]$$
$$= \int_0^\infty \frac{dt}{\sqrt{2\pi t}}\mathbb{E}\left[F(B_u;\ u \le t)\varphi(L_t)|B_t = 0\right]\psi(t)$$
$$= \int_0^\infty \frac{dt\psi(t)}{\sqrt{2\pi t}}\int_0^\infty P(L_t \in dl|B_t = 0)\varphi(l)\mathbb{E}\left[F(B_u;\ u \le t)|B_t = 0, L_t = l\right]$$

On the other hand,

$$\mathbb{E}\left[\int_0^\infty dL_t F(B_u;\ u \le t)\varphi(L_t)\psi(t)\right] = \mathbb{E}\left[\int_0^\infty dl F(B_u;\ u \le \tau_l)\varphi(l)\psi(\tau_l)\right]$$
$$= \int_0^\infty dl\varphi(l)\int_0^\infty P(\tau_l \in dt)\psi(t)\mathbb{E}\left[F(B_u;\ u \le \tau_l)|\tau_l = t\right]$$

We can identify these two quantities for any φ, ψ. Thus
$\frac{dt}{\sqrt{2\pi t}}P(L_t \in dl|B_t = 0)\mathbb{E}\left[F(B_u;\ u \le t)|B_t = 0, L_t = l\right] =$

$$dlP(\tau_l \in dt)\mathbb{E}\left[F(B_u;\ u \le \tau_l)|\tau_l = t\right]$$

Taking $F = 1$, we get $\frac{dt}{\sqrt{2\pi t}}P(L_t \in dl|B_t = 0) = dlP(\tau_l \in dt)$ (note that this entails (5.12) *since: $P(\tau_l \in dt) = \frac{le^{-l^2/2t}dt}{\sqrt{2\pi t^3}}$) and, therefore, the switching identity:*

$$\mathbb{E}\left[F(B_u;\ u \le t)|B_t = 0, L_t = l\right] = \mathbb{E}\left[F(B_u;\ u \le \tau_l)|\tau_l = t\right]$$

• **Solution to Exercise 42** *(p.102)*

a) This result is a simple consequence of Tanaka's formula for $(B_t - x)^+$.
b) Let $\alpha_{\cdot}^{x,-}$ denote the right-continuous inverse of $A^{x,-}$.

$$\mathbb{E}\left[f(B_\xi)1_{B_\xi<x}|\mathcal{E}_{\mathbb{W}}^x\right] = \lambda\mathbb{E}\left[\int_0^\infty f(B_t)1_{B_t<x}e^{-\lambda t}dt|\mathcal{E}_{\mathbb{W}}^x\right]$$
$$= \lambda\int_0^\infty duf(B_{\alpha_u^{x,-}})\mathbb{E}\left[e^{-\lambda\alpha_u^{x,-}}|\mathcal{E}_{\mathbb{W}}^x\right]$$
$$= \lambda\int_0^\infty duf(B_{\alpha_u^{x,-}})e^{-\lambda u}\mathbb{E}\left[e^{-\lambda A^{x,+}(\alpha_u^{x,-})}|\mathcal{E}_{\mathbb{W}}^x\right]$$
$$= \lambda\int_0^\infty duf(B_{\alpha_u^{x,-}})e^{-\lambda u}e^{-\sqrt{2\lambda}\frac{1}{2}L_{\alpha_u^{x-}}^x} \qquad \textit{(using a))}$$
$$= \lambda\int_0^\infty dt1_{B_t<x}f(B_t)e^{-\lambda A_t^{x,-}-\sqrt{\frac{\lambda}{2}}L_t^x}$$

Remark 7.7 *In fact, this exercise gives a glimpse of D. Williams' attempt to prove the continuity of martingales in the filtration* $\mathcal{E}_{\mathbb{W}}^{\cdot}$ *of Definition 5.2. More precisely, D. Williams considers the martingales* $\left(\mathbb{E}\left[\prod_{i=1}^{N} f_i(B_{\xi_i})|\mathcal{E}_{\mathbb{W}}^{x}\right];\ x \in \mathbb{R}\right)$ *and uses arguments similar to those in this exercise (or their excursion theory counterpart) to obtain the a.s. continuity.*

• **Solution to Exercise 43** *(p.102)*✠

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• **Solution to Exercise 44** *(p.105)*✠

• **Solution to Exercise 45** *(p.114)*
The main difficulty is to justify the exchange of the symbols $\vee$ *and* $\bigcap$ *in*

$$\mathcal{F}_{L+}\left(=\bigcap_{\varepsilon>0}\mathcal{F}_{L+\varepsilon}\right)=\bigcap_{\varepsilon>0}\left(\mathcal{F}_L \vee \sigma\{B_{L+u},\ u \le \varepsilon\}\right)$$

This exchange (which has been the cause of many errors! See some discussion in Chaumont-Yor [CY03], Exercise 2.5 p.29) is licit in the following cases because of the following independence property.

a) $L = \sup\{t \le \gamma_{T_1},\ B_t = S_t\}$ *or* $L = \gamma_{T_1}$*; then, the independence property between* $\mathcal{F}_L$ *and* $(B_{L+u};\ u \ge 0)$ *holds.*
b) $L = \gamma$*. then,* $\mathcal{F}_\gamma$ *and* $\bigcap_{\eta>0}\sigma\left\{B_u^{[\gamma,1]},\ u \le \eta\right\}$ *are independent and the second* σ*-field reduces to* $\sigma(A)$ *where* $A = \{B_1 > 0\}$.

• **Solution to Exercise 46** *(p.115)*
First note that, from the comparison theorem, the sticky Brownian motion starting from $x \ge 0$ *turns out to be a* $\mathbb{R}_+$*-valued diffusion.*

a) A simple application of Itô's formula yields the desired martingale property. Then, we deduce, for any $t \ge 0$,

$$\mathbb{E}\left[e^{-\sqrt{2\lambda}\ X_t-\lambda t}+(\sqrt{2\lambda}\ \theta+\lambda)\int_0^t e^{-\lambda s}1_{X_s=0}ds\right]=1$$

Since X *is a positive process, the right-hand side tends to* $(\sqrt{2\lambda}\ \theta + \lambda)\mathbb{E}\left[\int_0^\infty e^{-\lambda s}1_{X_s=0}ds\right]$ *as* t *tends to* $+\infty$.
b) Once again, the result is deduced from Itô's formula and from the positivity of $X \vee \widetilde{X}$.
c) It suffices to use the L^2*-convergence of* $1_{X_t^{(n)}=0} - 1_{\widetilde{X}_t^{(n)}=0}$ *towards 0.*

• **Solution to Exercise 47** *(p.116)*✠

• **Solution to Exercise 48** *(p.116)*
Here is an interesting example:
Consider $(f_i)_{i=1,..,n}$ *a set of* n *space-time harmonic functions for Brownian motion, i.e.*

$$\frac{\partial}{\partial t} f_i + \frac{1}{2} \frac{\partial^2}{\partial x^2} f_i = 0$$

such that

$$\begin{cases} f_i(x,t) \geq 0 & \textit{if } x \geq 0 \\ f_i(x,t) = 0 \textit{ if and only if } x = 0 \\ \frac{\partial}{\partial x} f_i(0,t) & \textit{is independent of } i \end{cases}$$

Then, if $(Z_t^i;\ t \geq 0)_{i=1,..,n}$ *denotes Walsh's Brownian motion, then* $(f_i(Z_t^i, t),\ t \geq 0)_{i=1,..,n}$ *is a spider-martingale.*
The example $f_i(x,t) = \sinh(\lambda x) e^{-\frac{\lambda^2 t}{2}}$ *(with the same* λ*, independent of* i*) is of particular interest in the following*

Application:
To prove formulae (6.7) *and* (6.8)*, we consider the previous spider martingale. There exists a constant* C*, independent of* i*, such that*

$$C = \sinh(\lambda z_i) \mathbb{E}\left[e^{-\frac{\lambda^2}{2} T_{\{z_1,\dots,z_n\}}} 1_{Z_{T_{\{z_1,\dots,z_n\}}} = z_i} \right]$$

Moreover, using the martingale $(\cosh(\lambda |Z_t|) e^{-\frac{\lambda^2 t}{2}};\ t \geq 0)$*, we obtain from Doob's stopping theorem*

$$1 = \sum_{i=1}^{n} \cosh(\lambda z_i) \mathbb{E}\left[e^{-\frac{\lambda^2}{2} T_{\{z_1,\dots,z_n\}}} 1_{Z_{T_{\{z_1,\dots,z_n\}}} = z_i} \right]$$

Thus, $$1 = \sum_{i=1}^{n} \cosh(\lambda z_i) \frac{C}{\sinh(\lambda z_i)}$$

The result (6.7) *follows.*