7

Classification and Elementary Properties of Müntz Sequences

Again we consider $\Lambda = \{\lambda_k\}_{k=1}^\infty$ satisfying $0 < \lambda_1 < \lambda_2 < \dots$ and $\sum_{n=1}^\infty 1/\lambda_k < \infty$. At first we will be concerned with different classes of Λ which are distinguished by special properties. Then we study the underlying Müntz polynomials $\sum_{k=1}^n \alpha_k t^{\lambda_k}$. In particular we give estimates for the inclination of the elements t^{λ_k} as well as of the differences $t^{\lambda_k} - t^{\lambda_{k+1}}$. One of the main results of this chapter is Theorem 7.4.4 where we show that Λ is non-lacunary if and only if $M(\Lambda)$ is closing. For E we always consider C_0 or L_p.

7.1 Different Classes of Λ

We start with

Definition 7.1.1 *A sequence Λ satisfying $0 < \lambda_1 < \lambda_2 < \dots$ (as well as the corresponding sequence $M(\Lambda)$ and the Müntz space $[M(\Lambda)]_E$) will be called*
1. standard *, if*

$$\lim_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} = 1 ,$$

2. rational *, if all λ_k are non-negative rationals,*
3. integer*, if all λ_k are non-negative integers,*
4. sparse *or* non-dense *, if $\sum_k 1/\lambda_k < \infty$,*
5. dense *, if $\sum_k 1/\lambda_k = \infty$.*

In 6.3.1 we already introduced lacunary and quasilacunary sequences. Now we extend this notion.

Definition 7.1.2 *Λ will be called* block lacunary *if, for some increasing sequence of integers $\bar{n} = \{n_k\}_{k=1}^\infty$ and some $\beta > 1$, we have $\lambda_{n_k+1}/\lambda_{n_k} \geq \beta$, $k = 1, 2, \dots$*

If Λ is block lacunary with respect to $\bar{n}$ and β we also speak of a $(\bar{n}, \beta)$-block lacunary *sequence. The intervals*

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$$I_k = \{m : m \text{ an integer }, \; n_k + 1 \le m \le n_{k+1}\}$$

will be called block intervals.

Recall that, according to 6.3.1, Λ is quasilacunary if it is $(\bar{n}, \beta)$-block lacunary for some $\beta > 1$ and we have $\sup_k (n_{k+1} - n_k) < \infty$. Moreover, Λ is lacunary if it is $(\bar{n}, \beta)$-block lacunary with $n_{k+1} - n_k = 1$ for all k.

Proposition 7.1.3 *The following are equivalent*
(i) Λ is quasilacunary
(ii) There are lacunary $\Lambda_1, \dots, \Lambda_m$ such that $\Lambda = \cup_{i=1}^m \Lambda_i$
(iii) For arbitrary $\beta > 1$ there is N such that $\Lambda \cap [\beta^j, \beta^{j+1}]$, $j = 1, 2, \dots$, has at most N elements
(iv) There is an increasing sequence of integers $\bar{n} = \{n_k\}_{k=1}^\infty$ and some $\beta > 1$ such that

$$\lambda_{n_{k+1}} / \lambda_{n_k} \ge \beta, \;\; k = 1, 2, \dots, \quad \text{and} \quad \sup_k (n_{k+1} - n_k) < \infty \; .$$

Proof. $(i) \Rightarrow (ii)$: If we fix exactly one λ_j in each block we obtain finitely many lacunary Λ_i satisfying (ii).
$(ii) \Rightarrow (iii)$: Fix an arbitrary $\beta > 1$. Since all Λ_j are lacunary there are $N_j > 0$ such that $\Lambda_j \cap [\beta^k, \beta^{k+1}]$ has at most N_j elements for all k. Put $N = \sup_{j=1,\dots,m} N_j$.
$(iii) \Rightarrow (iv)$: Fix $\beta > 1$ and put $m_k = \sup\{i : \lambda_i \le \beta^k\}$. Then, by (iii), $\sup_k (m_{k+1} - m_k) < \infty$. We may assume $\Lambda \cap]\beta^k, \beta^{k+1}] \ne \emptyset$ for each k, otherwise enlarge Λ. Hence we have $m_k \ne m_{k+1}$ and $\beta^k \le \lambda_{m_k+1}, \dots, \lambda_{m_{k+1}} \le \beta^{k+1}$. Put $n_k = m_{2k}$. Then $\sup_k (n_{k+1} - n_k) < \infty$ and

$$\frac{\lambda_{n_{k+1}}}{\lambda_{n_k}} \ge \frac{\beta^{2k+1}}{\beta^{2k}} = \beta \; .$$

$(iv) \Rightarrow (i)$: By assumption we have

$$\prod_{j=n_k}^{n_{k+1}-1} \left(\frac{\lambda_{j+1}}{\lambda_j} \right) = \frac{\lambda_{n_{k+1}}}{\lambda_{n_k}} \ge \beta$$

and the numbers of the factors in the preceding products are uniformly bounded. Therefore we find $\delta > 1$ and, for each k, some index m_k such that $n_k \le m_k \le n_{k+1}$ and $\lambda_{m_k+1}/\lambda_{m_k} \ge \delta$. Since $\sup_k (n_{k+1} - n_k) < \infty$ we also have $\sup_k (m_{k+1} - m_k) < \infty$. Now, Λ is quasilacunary with respect to $\bar{m} = \{m_k\}_{k=1}^\infty$ and δ. □

We also note that block lacunary and standard are opposite properties.

Proposition 7.1.4 *The following are equivalent*
(i) Λ is not block lacunary
(ii) Λ is standard

Proof. $(i) \Rightarrow (ii)$: Otherwise find $\delta > 0$ and indices $n_1 < n_2 < \ldots$ with $\lambda_{n_k+1}/\lambda_{n_k} \geq 1 + \delta$ for all k.
$(ii) \Rightarrow (i)$ is obvious. □

We conclude this section with two more classes of Λ.

Definition 7.1.5 *1. Let, for some $a > 0a$, $\delta > 0$ and $s \geq 1$*

$$\lambda_k = a + \delta k^s, \qquad k = 1, 2, \ldots$$

*Then we call Λ an s-*arithmetic *sequence. If $s = 1$ then Λ is simply called an* arithmetic *sequence.*
2. Let $\bar{n} = \{n_k\}_{k=1}^{\infty}$ be an increasing sequence of indices and assume that there are numbers $a_k > 0$ and $\delta_k > 0$ with

$$\lambda_j = a_k + (j - n_k)\delta_k, \qquad n_k + 1 \leq j \leq n_{k+1}, \quad k = 1, 2, \ldots$$

*Put $\bar{a} = \{a_k\}_{k=1}^{\infty}$ and $\bar{\delta} = \{\delta_k\}_{k=1}^{\infty}$. Then Λ will be called a $(\bar{n}, \bar{a}, \bar{\delta})$-*block arithmetic *sequence.*

Of course, an s-arithmetic sequence is non-dense if and only if $s > 1$.

Virtually nothing is known about the Banach space $[M(\{k^s\}_{k=1}^{\infty})]_C$ if $s > 1$. On the other hand, in 9.3 we will give a complete Banach space characterization of $[M(\Lambda)]_E$ for $E = C$ and $E = L_p$, $1 \leq p < \infty$, if Λ is quasilacunary. (Then $[M(\Lambda)]_C \sim c_0$ and $[M(\Lambda)]_{L_p} \sim l_p$.) Moreover, in 10.2 we show that there is a block lacunary Λ where $[M(\Lambda)]_C$ is not isomorphic to c_0.

Definition 7.1.6 *1. Let, for some $a > 0$ and $q > 0$, $\lambda_k = aq^k$, $k = 1, 2, \ldots$ Then we call Λ a* geometric *(or (a, q)-*geometric*) sequence.*
2. Let $\bar{n} = \{n_k\}_{k=1}^{\infty}$ be an increasing sequence of indices and assume that there are numbers $a_k > 0$ and $q_k > 0$ with

$$\lambda_j = a_k q_k^{j-n_k}, \qquad n_k + 1 \leq j \leq n_{k+1}, \quad k = 1, 2, \ldots$$

*Put $\bar{a} = \{a_k\}_{k=1}^{\infty}$ and $\bar{q} = \{q_k\}_{k=1}^{\infty}$. Then Λ will be called a $(\bar{n}, \bar{a}, \bar{q})$-*block geometric *sequence.*

7.2 Iterated Differences

For the sequence Λ let $d\Lambda$ denote the differences $d\Lambda = \{\lambda_{k+1} - \lambda_k\}_{k=1}^{\infty}$. Then go on to define in the same fashion $d^2(\Lambda) = d(d\Lambda)$, $d^3(\Lambda) = d(d^2(\Lambda))$ etc. Put $d^0(\Lambda) = \Lambda$.

Definition 7.2.1 *The sequence Λ will be called*
1. k-regular *(or* strictly k-regular*) if, for some positive integer k, all sequences $d^j(\Lambda)$, $j = 0, 1, \ldots, k$, consist of non-negative (or strictly positive) numbers,*
2. absolutely monotone *(or* strictly absolutely monotone*), if Λ is k-regular (or strictly k-regular) for all positive integers k.*

For example, the sequence $\Lambda = \{q^k\}_{k=1}^{\infty}$ is strictly absolutely monotone for any $q > 1$. It will turn out that strictly absolutely monotone sequences are always block lacunary but not necessarily lacunary.

Proposition 7.2.2 *Let Λ be a strictly absolutely monotone sequence of integers. Then we have $\lambda_k \geq 2^{k-1}$ for all k. Moreover, Λ is block-lacunary.*

Proof. Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a strictly absolutely monotone sequence of integers. Define $a_{1,k} = \lambda_k$, $k = 1, 2, \ldots$ and, by induction, $a_{m,k} = a_{m-1,k} - a_{m-1,k-1}$, $k = m, m+1, \ldots$. Then all $a_{m,k}$ are positive integers. In particular, $a_{m,m} \geq 1$ for all m. Induction on $k - m$ yields $a_{m,k} \geq 2^{k-m}$ for all k and m.

Assume that Λ is not block lacunary. Then, according to 7.1.4, Λ is standard and we have $\lim_{m\to\infty} \lambda_{m+1}/\lambda_m = 1$. Fix $\epsilon \in]0, 1[$ and find m_0 such that $\lambda_{m+1}/\lambda_m \leq 1 + \epsilon$ for all $m \geq m_0$. With the first part of Proposition 7.2.2 we obtain

$$2^{m-1} \leq \lambda_m \leq (1 + \epsilon)^{m-m_0} \lambda_{m_0}$$

for all $m \geq m_0$ and hence

$$1 \leq \lim_{m\to\infty} \left(\frac{1+\epsilon}{2}\right)^{m-m_0} \left(\frac{1}{2}\right)^{m_0-1} \lambda_{m_0} = 0 \,,$$

a contradiction. □

Proposition 7.2.3 *There exists a strictly absolutely monotone non-lacunary sequence Λ of integers.*

Proof. At first we observe that, if $\{\alpha_k\}_{k=1}^{\infty}$ is strictly absolutely monotone and $\{\beta_k\}_{k=1}^{\infty}$ is absolutely monotone then $\{\alpha_k + \beta_k\}_{k=1}^{\infty}$ is strictly absolutely monotone. This is a straightforward consequence of the definitions.

Now we use induction to introduce strictly absolutely monotone sequences $\{a_{m,j}\}_{j=1}^{\infty}$, $m = 1, 2, \ldots$, and indices $j_1 = 1 < j_2 < \ldots$. Put $a_{1,j} = 2^j$ and $j_1 = 1$.

If we have already $\{a_{m,j}\}_{j=1}^{\infty}$ and j_m for some m then let $j_{m+1} > j_m$ be such that

$$\frac{j_{m+1} - j_m}{j_{m+1} - 1 - j_m} \leq 1 + \frac{1}{2(m+1)} \,.$$

Let b be a positive integer with $a_{m,j_{m+1}}/b \leq 2^{-1}(m+1)^{-1}$. Put

$$a_{m+1,j} = \begin{cases} a_{m,j}, & j \leq j_m \\ a_{m,j} + b(j - j_m), & j > j_m \end{cases} \,.$$

Since $\underbrace{0, \ldots, 0}_{j_m \text{ times}}, b, 2b, 3b, \ldots$ is absolutely monotone, $\{a_{m+1,j}\}_{j=1}^{\infty}$ is strictly absolutely monotone and we obtain

$$\frac{a_{m+1,j_{m+1}}}{a_{m+1,j_{m+1}-1}} \leq 1 + \frac{1}{m+1} \,.$$

Finally, put $\lambda_j = a_{m,j}$ if $j_{m-1} < j \le j_m$. It follows from the construction that $\Lambda = \{\lambda_j\}_{j=1}^{\infty}$ is a strictly absolutely monotone sequence of integers and we have

$$\lim_{m\to\infty} \frac{\lambda_{j_m}}{\lambda_{j_m-1}} = \lim_{m\to\infty} \frac{a_{m,j_m}}{a_{m,j_m-1}} = 1 \, .$$

Hence Λ is not lacunary. □

7.3 Elementary Properties of Müntz Sequences and Polynomials

Now we focus on $M(\Lambda)$ instead of Λ. We want to discuss elementary properties of $M(\Lambda)$ where Λ satisfies some of the preceding conditions. We start with a technical lemma.

Lemma 7.3.1 *Let $g(t) = t^\lambda - t^\mu$, where $0 < \lambda < \mu$, and put $\rho = \mu/\lambda$. Then we obtain*

$$||g||_C = \nu(\rho)\cdot\left(1 - \frac{1}{\rho}\right) \qquad \textit{with} \qquad \nu(\rho) = \rho^{\frac{1}{1-\rho}} \, .$$

$\nu(\rho)$ is a strictly increasing function on $]1,\infty[$ satisfying $1/e < \nu(\rho) < 1$ and $\lim_{\rho\to 1}\nu(\rho) = 1/e$. Moreover, $t_0 := (\lambda/\mu)^{1/(\mu-\lambda)}$ is the unique maximum point of $|g(t)|$.

Proof. It follows from simple calculus that g attains its unique maximum at t_0 and that $||g||_C = \nu(\rho)(1-1/\rho)$. We have

$$\frac{d\log\nu}{d\rho} = \frac{1/\rho - 1 + \log\rho}{(1-\rho)^2}$$

and

$$\frac{d(1/\rho - 1 + \log\rho)}{d\rho} = \frac{1}{\rho}\left(1 - \frac{1}{\rho}\right) > 0 \qquad \text{for } \rho > 1 \, .$$

Since $1/\rho - 1 + \log\rho = 0$ if $\rho = 1$ we obtain $\frac{d\log\nu}{d\rho} > 0$ and hence $\log\nu$ and ν are strictly increasing. □

Lemma 7.3.1 has a number of consequences. At first we note

Proposition 7.3.2 *The Müntz sequences $\{t^{\lambda_k}\}_{k=1}^N$ and $\{t^{\mu_k}\}_{k=1}^N$ are isometrically equivalent in $C[0,1]$ if and only if*

$$\frac{\lambda_{j+1}}{\lambda_j} = \frac{\mu_{j+1}}{\mu_j} \qquad \textit{for} \quad j = 1, 2, \ldots, N-1 \, .$$

Proof. The sufficiency of the condition for isometric equivalence follows easily by substituting $\tau = t^{\lambda_1}$ and $\tau = t^{\mu_1}$.

For the necessity assume that $||t^{\lambda_1} - t^{\lambda_j}||_C = ||t^{\mu_1} - t^{\mu_j}||_C$. With Lemma 7.3.1 we conclude $\lambda_j/\lambda_1 = \mu_j/\mu_1$. □

As a direct consequence of 2.7.2 we have

Proposition 7.3.3 *Let E be either L_p, for $1 \leq p < \infty$, or C. Then span $\{t^{\lambda_k}\}_{k=1}^{n} \subset E$ is a continuous function of $(\lambda_1, \ldots, \lambda_n)$ with respect to (the logarithm of) the Banach-Mazur distance and the ball opening Θ of subspaces in E.*

Using the last two propositions we obtain

Proposition 7.3.4 *Let Λ be a (a, q)-geometric sequence with $q > 1$. Then*

(a) $\{t^{\lambda_k}\}_{k=1}^{\infty}$ is isometrically equivalent in C to $\{t^{\lambda_k}\}_{k=l}^{\infty}$ for all $l = 1, 2, \ldots$,
(b) If Λ is finite, i.e. if $\Lambda = \{aq, aq^2, \ldots, aq^n\}$ for some n, then $M_{a,q} = [M(\Lambda)]_C$ is a continuous function of the parameters a and q with respect to the ball opening Θ as metric. Furthermore, for some $\epsilon > 0$ we find a Lipschitz constant $c(q, \epsilon)$ with

$$\Theta(M_{a,q}, M_{a,\tau}) \leq c(q, \epsilon) \cdot |q - \tau|, \qquad \tau \in [q - \epsilon, q + \epsilon] \,.$$

We finish this section with an estimate of the values of a Müntz polynomial which has only two summands.

Lemma 7.3.5 *There is a constant $\kappa > 0$ satisfying the following: Let $0 < \lambda < \mu$ and $g(t) = t^\lambda - t^\mu$. Then, for any $t \in [0, 1]$, we have*

$$|g(t)| \leq \kappa t^{\lambda/2} ||g||_C \,.$$

Proof. Put $\rho = \lambda/\mu$. Then, according to 7.3.1, we obtain

$$||g||_C = \rho^{\frac{\rho}{1-\rho}} (1 - \rho) \,.$$

Put $f(t) = t^{\lambda/2} - t^{\mu - \lambda/2}$. Then, with the preceding ρ, 7.3.1 implies

$$||f||_C = \left(\frac{\rho}{2 - \rho} \right)^{\frac{\rho}{2(1-\rho)}} \frac{2 - 2\rho}{2 - \rho} \,.$$

Put

$$\tau(s) = \left(\frac{1}{s(2 - s)} \right)^{\frac{s}{2(1-s)}} \frac{2 - 2s}{2 - s}, \quad s \in \,]0, 1[\,.$$

Then τ is continuous and we have

$$\lim_{s \to 0} \tau(s) = 1 \quad \text{and} \quad \lim_{s \to 1} \tau(s) = \lim_{x \to \infty} \tau\left(\frac{x}{1 + x} \right) = 0 \,.$$

Hence there is a constant $\kappa > 0$ with $\tau(s) \leq \kappa$ for all $s \in \,]0, 1[$. We obtain, for any $t \in [0, 1]$,

$$|g(t)| \leq t^{\lambda/2}||f||_C = t^{\lambda/2}\frac{||f||_C}{||g||_C}||g||_C$$
$$= t^{\lambda/2}\tau(\rho)||g||_C$$
$$\leq \kappa t^{\lambda/2}||g||_C$$

□

Now we turn to general Müntz polynomials with two summands.

Proposition 7.3.6 *Let $p(t) = at^\lambda + bt^\mu$ with $0 < \lambda < \mu$. Then for any $t \in [0,1]$ we have*

$$|p(t)| \leq (2\kappa + 1)t^{\lambda/2}||p||_C .$$

where κ is the constant of Lemma 7.3.5.

Proof. Assume $||p||_C = 1$. We have

$$p(t) = at^\lambda + bt^\mu = a(t^\lambda - t^\mu) + (a+b)t^\mu .$$

Hence $p(1) = a+b$ and $|a+b| \leq ||p||_C = 1$. Put $g(t) = t^\lambda - t^\mu$. Then we obtain $|a| \cdot ||g||_C = ||a(t^\lambda - t^\mu)||_C \leq 2$. Using 7.3.5 we see that, for any $t \in [0,1]$,

$$|p(t)| \leq \kappa t^{\lambda/2}|a| \cdot ||g||_C + t^\mu \leq (2\kappa + 1)t^{\lambda/2} .$$

□

Compare Proposition 7.3.6 with Corollary 6.1.3. There the constant depends on the given exponents λ_j while here κ is independent of λ and μ.

If t in the preceding proposition is small then $|p(t)|$ is small. In particular we obtain a lower estimate for

$$\min\{t_0 \in [0,1] \ : \ |p(t_0)| = ||p||_C\}$$

In the next chapter we extend Proposition 7.3.6 to general Müntz polynomials.

7.4 Differences of Müntz Sequences

Lemma 7.3.1 implies that the elements of a Müntz sequence $M(\Lambda)$ have, in general, "bad" mutual disposition. As we have noted already, even if Λ has "large gaps", i.e. if $\sum_{k=1}^\infty 1/\lambda_k < \infty$, in general $M(\Lambda)$ is not a basis or uniformly minimal. (Lateron in 9.2 we shall see that $M(\Lambda)$ is a basis if and only if Λ is lacunary. This is also equivalent to the condition that Λ is uniformly minimal or separated.) For standard Λ the normalized elements of $M(\Lambda)$ are even closing (see 7.4.4, for the definition of closing see 2.2.1).

The situation does not improve if we go over to the sequence of differences. Again, we obtain a closing sequence in general. However, the geometry of differences of a Müntz sequence helps to understand more complicated phenomena such as the geometry of octants which we discuss in 7.5

Lemma 7.4.1 *Consider $\mu > \lambda > 0$ and $\tau > 0$ such that*

$$2\mu \le \lambda + \tau \qquad \text{and} \qquad \tau\lambda \le \mu^2 .$$

Then for the functions $g_1(t) = t^\lambda - t^\mu$, $g_2(t) = t^\mu - t^\tau$ and $\Delta(t) = g_1(t) - g_2(t) = t^\lambda - 2t^\mu + t^\tau$ we have $||\Delta||_C \le 5e^2||g_1||_C^2$.

Proof. By substituting $t^\lambda = s$ we can assume that $\lambda = 1$ and hence $\mu \ge 1$. So we deal with $g_1(t) = t - t^\mu$, $g_2(t) = t^\mu - t^\tau$ where $2\mu - 1 \le \tau \le \mu^2$. Consider two cases.

1. $\tau = 2\mu - 1$. Here, $\tau - 1 = 2(\mu - 1) = 2\alpha$ with $\alpha = \mu - 1$. Moreover,

$$\Delta(t) = t - 2t^\mu + t^{2\mu-1} = t(1 - t^\alpha)^2 ,$$

$$\Delta'(t) = 1 - 2\mu t^\alpha + (2\mu - 1)t^{2\alpha} \text{ and } \mu^2 - (2\mu - 1) = \alpha^2 .$$

So we have a unique point t_0 of maximum for Δ with $t_0^\alpha = 1/(2\mu - 1)$. Thus,

$$||\Delta||_C = (2\mu - 1)^{-1/\alpha}\left(1 - \frac{1}{2\mu - 1}\right)^2 \le \left(\frac{2(\mu - 1)}{2\mu - 1}\right)^2 \le \left(\frac{2(\mu - 1)}{\mu}\right)^2 .$$

By Lemma 7.3.1, $||g_1||_C \ge e^{-1}(1-1/\mu)$, so $||\Delta||_C \le (2(1-1/\mu))^2 \le 4e^2||g_1||_C^2$.

2. $\tau > 2\mu - 1$. Since $\tau \le \mu^2$ we have

$$||t^{2\mu-1} - t^\tau|| \le 1 - \frac{2\mu - 1}{\tau} \le 1 - \frac{2\mu - 1}{\mu^2} = \left(1 - \frac{1}{\mu}\right)^2 \le e^2||g_1||_C^2 .$$

Now, using the previous case and triangle inequality we finish the proof of 7.4.1 □

As a consequence of Lemma 7.4.1 we obtain estimates for the inclination of the elements of $M(\Lambda)$ and their differences with respect to the sup-norm.

Theorem 7.4.2 *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ and put $e_k = t^{\lambda_k}$, $g_k = e_{k+1} - e_k$, $k = 1, 2, \ldots$. Then we have, with respect to the sup-norm $||\cdot||_C$,*

(a) $(\widehat{e_k, e_{k+1}}) \sim 1 - \lambda_k/\lambda_{k+1}$ *as $k \to \infty$. In particular, if Λ is a standard sequence, then $M(\Lambda)$ is closing.*

(b) $(\widehat{g_k, g_{k+1}}) \le 10e^2(\widehat{e_k, e_{k+1}})$ *provided that*

$$2\lambda_{k+1} \le \lambda_k + \lambda_{k+2} \quad \text{and} \quad \lambda_k\lambda_{k+2} \le \lambda_{k+1}^2 .$$

If the latter conditions hold for all k and Λ is standard then $\{g_k\}_{k=1}^\infty$ is closing, too.

Proof. (a) is a direct consequence of Lemma 7.3.1. To prove (b) fix k. Then we have

$$(g_k, \widehat{g_{k+1}}) \leq \frac{||g_k - g_{k+1}||_C}{||g_k||_C} = \frac{||\Delta||_C}{||g_k||_C} \leq 5e^2||g_k||_C$$

and, with the notion of angle (see 1.2),

$$||g_k||_C = ||e_{k+1} - e_k||_C = \varphi(e_k, e_{k+1}) \leq 2(e_k, \widehat{e_{k+1}}) \ .$$

This finishes part (b) of the theorem. □

For example $\Lambda = \{k^2\}_{k=1}^{\infty}$ satisfies the assumptions of Theorem 7.4.2 (b).

Now we turn to the L_p-case for $1 \leq p < \infty$. Recall that

$$||t^\lambda||_{L_p} = (\lambda p + 1)^{-1/p} \ .$$

Lemma 7.4.3 *For $0 < \lambda < \mu$ we obtain*

$$\frac{1}{2}\left(\frac{\lambda}{2^p\mu}\right)^{\frac{p\lambda/\mu+1/\mu}{p^2(1-\lambda/\mu)}} \leq ||(\lambda p+1)^{1/p}t^\lambda - (\mu p+1)^{1/p}t^\mu||_{L_p}$$
$$\leq 1 - \left(\frac{\lambda}{\mu}\right)^{1/p} + p^{1/p}\left(1 - \frac{\lambda}{\mu}\right)^{1/p} .$$

Proof. We have, with $a = (\lambda p+1)^{1/p}$ and $b = (\mu p+1)^{1/p}$,

$$\begin{aligned}
||bt^\mu - at^\lambda||_{L_p} &\leq (b-a)||t^\mu||_{L_p} + a||t^\lambda - t^\mu||_{L_p} \\
&\leq \left(1 - \frac{a}{b}\right) + a\left(\int_0^1 (t^\lambda - t^\mu)dt\right)^{1/p} \\
&\leq 1 - \frac{a}{b} + a\left(\frac{1}{\lambda+1} - \frac{1}{\mu+1}\right)^{1/p} \\
&\leq 1 - \left(\frac{\lambda}{\mu}\right)^{1/p} + \left(\frac{\lambda p+1}{\lambda+1}\right)^{1/p}\left(1 - \frac{\lambda}{\mu}\right)^{1/p} \\
&\leq 1 - \left(\frac{\lambda}{\mu}\right)^{1/p} + p^{1/p}\left(1 - \frac{\lambda}{\mu}\right)^{1/p} .
\end{aligned}$$

Here we used $|t^\lambda - t^\mu|^p \leq (t^\lambda - t^\mu)$ and $(\lambda c+1)/(\mu c+1) \geq \lambda/\mu$ for any $c > 0$.

Put $t_0 = (a/(2b))^{1/(\mu-\lambda)}$. Then $at^\lambda - bt^\mu \geq at^\lambda/2$ if $0 \leq t \leq t_0$. This implies

$$\begin{aligned}
||bt^\mu - at^\lambda||_{L_p} &\geq \frac{a}{2}\left(\int_0^{t_0} t^{\lambda p}dt\right)^{1/p} \\
&= \frac{1}{2}t_0^{\lambda+1/p} \\
&= \frac{1}{2}\left(\frac{\lambda p+1}{2^p(\mu p+1)}\right)^{\frac{1+\lambda p}{p^2(\mu-\lambda)}} \\
&\geq \frac{1}{2}\left(\frac{\lambda}{2^p\mu}\right)^{\frac{p\lambda/\mu+1/\mu}{p^2(1-\lambda/\mu)}} .
\end{aligned}$$

□

We obtain, using

$$\widehat{(x,y)} \le ||x-y|| \le 2\widehat{(x,y)} \quad \text{if } ||x||=1 \text{ and } ||y||=1\ ,$$

Theorem 7.4.4 *The following are equivalent*
(i) $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ *is standard*
(ii) Λ *is not block-lacunary*
(iii) $M(\Lambda)$ *is closing in* $C[0,1]$
(iv) $M(\Lambda)$ *is closing in* L_p *if* $1 \le p < \infty$

Proof. $(i) \Leftrightarrow (ii)$ follows from 7.1.2, $(i) \Leftrightarrow (iii)$ follows from 7.4.2 (a) and 7.3.1. $(i) \Leftrightarrow (iv)$ follows from 7.4.3. Note, if

$$\frac{1}{2}\left(\frac{\lambda_k}{2^p\lambda_{k+1}}\right)^{\frac{\lambda_k/\lambda_{k+1}+1/(p\lambda_{k+1})}{p(1-\lambda_k/\lambda_{k+1})}}$$

tends to 0, then $\lim_{k\to\infty}\lambda_k/\lambda_{k+1}=1$. □

7.5 The Inclination of Positive Octants of Müntz Sequences

The results of the preceding section allow us to get estimates of the inclinations of the positive octants for any Müntz sequence.

For any sequence $\bar{e} = \{e_k\}_{k=1}^{\infty}$ in a Banach space and integers n, m with $m \le n$ define the *positive* (m,n)*-octant* as

$$\Omega_{m,n}(\bar{e}) = \left\{ \sum_{k=m}^{n} \alpha_k e_k \ :\ \alpha_k \ge 0,\ k=m,\dots,n \right\}\ ,$$

and *positive normed* (m,n)*-octant* as

$$\tilde{\Omega}_{m,n}(\bar{e}) = \left\{ x \ :\ x = \sum_{k=m}^{n} \alpha_k e_k,\ \alpha_k \ge 0,\ k=m,\dots,n,\ ||x|| \le 1 \right\}\ .$$

Now we consider again an increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ of positive real numbers and put $e_k = t^{\lambda_k}$. The error of approximation of Müntz polynomials with positive coefficients can be estimated by the following theorem.

Theorem 7.5.1 *For all integers* n *and* m *with* $0 \le m < n$ *we have, with respect to the sup-norm on* $[0,1]$*,*

$$(a)\quad \widehat{(\tilde{\Omega}_{1,m}, \tilde{\Omega}_{m+1,n})} = ||t^{\lambda_m} - t^{\lambda_{m+1}}||_C$$

$$(b)\quad \frac{1}{2}||t^{\lambda_m} - t^{\lambda_{m+1}}||_C \le \widehat{(\Omega_{1,m}, \Omega_{m+1,n})} \le ||t^{\lambda_m} - t^{\lambda_{m+1}}||_C$$

Hence

$$(\Omega_{1,m},\widehat{\Omega_{m+1,n}}) \sim 1-\frac{\lambda_m}{\lambda_{m+1}} \text{ and } (\tilde{\Omega}_{1,m},\widehat{\tilde{\Omega}_{m+1,n}}) \sim 1-\frac{\lambda_m}{\lambda_{m+1}} \text{ as } m\to\infty$$

Proof. (a): Put $a = (\tilde{\Omega}_{1,m},\widehat{\tilde{\Omega}_{m+1,n}})$. Take polynomials $p(t) = \sum_{k=1}^{m}\alpha_k t^{\lambda_k}$ and $q(t) = \sum_{k=m+1}^{n}\beta_k t^{\lambda_k}$ with

$$\alpha_k \geq 0,\ \beta_k \geq 0 \text{ for all } k,\ ||p||_C = \sum_{k=1}^{m}\alpha_k = 1, (\tilde{\Omega}_{1,m},\widehat{\tilde{\Omega}_{m+1,n}}) = ||p-q||_C$$

and $||q||_C \leq 1$. Hence $\sum_{k=m+1}^{n}\beta_k \leq 1$. We obtain

$$||t^{\lambda_m}-t^{\lambda_{m+1}}||_C \geq a \geq p(t)-q(t) \geq t^{\lambda_m} - \sum_{k=m+1}^{n}\beta_k t^{\lambda_{m+1}} \geq t^{\lambda_m}-t^{\lambda_{m+1}} \geq 0$$

for all $t\in[0,1]$. This implies $a = ||t^{\lambda_m}-t^{\lambda_{m+1}}||_C$.
(b): Let a be as in (a) and put $b = (\Omega_{1,m},\widehat{\Omega_{m+1,n}})$. Take polynomials $p\in\Omega_{1,m}$ and $q\in\Omega_{m+1,n}$ such that $||p||_C = 1$ and $b = ||p-q||_C$. Then we have $||q||_C \leq b+1$. Hence

$$b \geq \left\|p-\frac{q}{||q||_C}\right\|_C - ||q||_C + 1 \geq a-b$$

and we obtain $a/2 \leq b$. The right-hand inequality of (b) follows directly from the definitions.

The last assertion of Theorem 7.5.1 is a consequence of 7.4.2. □

We also obtain the straightforward.

Proposition 7.5.2 *Let $E = C$ or $E = L_p$, $1\leq p<\infty$. Then for any Müntz sequence $M(\Lambda)$ and any Müntz polynomial $p(t)=\sum_k \alpha_k t^{\lambda_k}$ we have*

$$||p||_E \leq \left\|\sum_k |\alpha_k| t^{\lambda_k}\right\|_E .$$