Approximation of Weak Limits and Related Problems

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Preface

This lecture course is concerned with some mathematical problems originated from the theory of compressible Navier-Stokes equations (cf.[9],[15],[16]).

The lecture notes consist of three sections. We discuss the problem of strong approximation of weak limits in section I and prove, firstly, that weak limit of some sequence of functions in Orlicz space can be approximated in strong sense (in norm) by the subsequence of averaged functions if the radius of averaging tends to zero slowly enough. This result allows us to control the order of approximation by weakly converging sequence. In particular, it justifies the smoothing approach near singularities in computation of non-smooth solutions to partial differential equations. Secondly, we consider the weak converging sequence of approximate solutions to averaged Navier-Stokes equations for incompressible fluids and obtain strong convergence to the solution of the limiting equations.

The section II contents the recent results [23],[25] in theory of transport equations in Orlicz spaces. We introduce special class of convex functions (Young functions) and corresponding Orlicz spaces. It allows us to obtain exact well-posedness (existence and uniqueness) results for linear transport equations in Orlicz spaces and describe the optimal conditions for coefficients. It is worthy to be mentioned that this class of Young functions (of fast growth at infinity, greater than any polynomial) is connected with Gronwall-type inequality and Osgood's uniqueness theorem (cf.[26]) for Cauchy problem in

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ordinary differential equations theory. Namely, we obtain optimal conditions for coefficients in Gronwall inequality. At the same time it gives the relations between Orlicz-Sobolev spaces and Osgood's condition in uniqueness problem.

The section III concerns with some problems related to compactness arguments. Besides the classical compactness (cf.[1],[2],[3]) so called compensated compactness ([4]-[7]) is also under consideration. We expose new version of classic compactness theorem which, in fact, is a particular case of compensated compactness. On the other side, we give a new and very simple proof of "div-curl" lemma (the mostly important tool in applications of compensated compactness theory to nonlinear P.D.E's.) which reduces the compensated compactness to the current one. Finally, a new viewpoint on the general compensated compactness theorem (Theorem of L.Tartar) is suggested: vanishing of quadratic form on the kernel of operator instead of algebraic conditions. Such approach, perhaps, can be more convenient in some cases when it is possible to describe the kernel of differential operator.

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1 Strong Approximation of Weak Limits by Averagings

1.1 Notations and Basic Notions from Orlicz Function Spaces Theory

Let $\Omega \subset \mathbb{R}^n$ be bounded domain with smooth boundary Γ, and $\mathbf{x} = (x_1, \ldots, x_n)$ are the points of Ω. By $L^1(\Omega)$ we denote the space of integrable functions on Ω. $L^\infty(\Omega)$ is the space of essentially bounded functions, $L^p(\Omega)$, $1 < p < \infty$ – the Lebesgue space of functions which are integrable in power p. We shall use also the Orlicz function spaces, and remind the basic notions (cf. [18]).

Let m(r) be defined on $[0, \infty)$ function, continuous from the right, non-negative, non-decreasing and such that

$$m(0) = 0, \quad m(r) \to \infty \text{ as } r \to \infty. \tag{1.1}$$

The convex function (Young function)

$$M(t) = \int_0^t m(r)dr \tag{1.2}$$

produces Orlicz class $K_M(\Omega)$ containing the functions $f(\mathbf{x}) \in L^1(\Omega)$ such that $M(|f(\mathbf{x})|)$ belong to $L^1(\Omega)$, too.The liner span of $K_M(\Omega)$ endowed with the norm

$$\|f\|_{L_M(\Omega)} = \inf \left\{ \lambda > 0 \,\middle|\, \int\limits_{\Omega} M\left(\frac{|f(\mathbf{x})|}{\lambda}\right) d\mathbf{x} \leq 1 \right\} \tag{1.3}$$

is called as Orlicz space $L_M(\Omega)$associated with Young function $M(t)$. The closure of $L^\infty(\Omega)$ in the norm (1.3) yields, in general, another Orlicz space $E_M(\Omega)$, and the inclusions take place

$$E_M(\Omega) \subseteq K_M(\Omega) \subseteq L_M(\Omega). \tag{1.4}$$

One says function $M(t)$ satisfies Δ_2-*condition* (cf.[18]) if there exist constants $C > 0$ and $t_0 > 0$ such that

$$M(2t) \leq C\ M(t) \quad \text{for } \forall t \geq t_0. \tag{1.5}$$

Three sets E_M, K_M and L_M coincide if and only if $M(t)$ satisfies Δ_2-condition. As rule, it's possible to compare two Young functions $M_1(t)$ and $M_2(t)$, namely, $M_2(t)$ dominates $M_1(t)$ if there exist constants $C > 0$ and $\alpha > 0$ such that

$$M_1(\alpha t) \leq M_2(t) \quad \text{for } \forall\ t \geq t_0 = t_0(\alpha, C). \tag{1.6}$$

In this case

$$E_{M_2} \subseteq E_{M_1} \text{ and } L_{M_2} \subseteq L_{M_1}.$$

If each function M_k, $k = 1, 2$, dominates the other one, then M_1 and M_2 are equivalent, $M_1 \cong M_2$, and corresponding Orlicz spaces are the same. Further, one says function M_2 dominates M_1 essentially if

$$\lim_{t \to \infty} \frac{M_1(\beta t)}{M_2(t)} = 0, \quad \forall \beta = \text{const} > 0. \tag{1.7}$$

In this case the strong embeddings

$$E_{M_2} \subset E_{M_1}, \quad L_{M_2} \subset L_{M_1} \tag{1.8}$$

are valid.
Denote by

$$n(r) = m^{-1}(r) \tag{1.9}$$

the inverse function to $m(r)$ and introduce Young function

$$N(t) = \int\limits_0^t n(r) dr. \tag{1.10}$$

This function is called as complementary convex function to $M(t)$, and it's equivalent to the next one:

$$N(t) \cong \sup_{r>0} \{tr - M(r)\} . \tag{1.11}$$

Two Orlicz spaces L_M and L_N are supplementary, and for any $f(\mathbf{x}) \in L_M(\Omega)$, $g(\mathbf{x}) \in L_N(\Omega)$ there exists the integral

$$< f, g > \equiv \int_\Omega f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} \tag{1.12}$$

which defines the linear continuous functional on $E_N(\Omega)$ with fixed $f \in L_M(\Omega)$ and any $g \in E_M(\Omega)$. It gives the notion of weak convergence (actually, weak* convergence) in $L_M(\Omega)$: the sequence $\{f_n(\mathbf{x})\}$converges weakly to $f(\mathbf{x}) \in L_M(\Omega)$, $f_n \rightharpoonup f$ if

$$< f_n, g > \to < f, g > \quad \text{for each } g \in E_N(\Omega). \tag{1.13}$$

At the same time it's possible to define mean convergence in $L_M(\Omega)$:

$$f_n \to f \quad \text{in mean value, if}$$

$$\int_\Omega M(|f_n - f|) d\mathbf{x} \to 0 \quad \text{as } n \to \infty. \tag{1.14}$$

If $M(t)$ satisfies Δ_2-condition, mean 2convergence is equivalent to the strong-convergence, i.e. in norm of $L_M(\Omega)$. Otherwise, mean convergence is stronger than weak, but weaker than strong one. As an important and interesting examples we shall use three Young functions: $M_1(t) = t^p$,$1 < p < \infty$, $M_2(t) = e^t - t - 1$ and $M_3(t) = (1+t)\ln(1+t) - t$. The first function $M_1(t)$ yields the Lebesgue space $L^p(\Omega)$, the second one $M_2(t)$ produces two Orlicz spaces L_{M_2} and E_{M_2} because $M_2(t)$ doesn't satisfy Δ_2-condition, and $M_3(t)$ is slowly increasing function which is essentially dominated by$M_1(t)$. The Orlicz space $L_{M_3}(\Omega)$ is located between $L^1(\Omega)$ and any $L^p(\Omega)$, $p > 1$.

Finally, for any $f(\mathbf{x}) \in L^1(\Omega)$ and $h > 0$, let us denote

$$f_h(\mathbf{x}) = \frac{1}{2} h^N \int_\Omega f(\mathbf{y}) \omega\left(\frac{\mathbf{x} - \mathbf{y}}{h}\right) d\mathbf{y} \tag{1.15}$$

an averaging of $f(\mathbf{x})$ where $\omega(\mathbf{z})$ is the kernel of averaging:

$$\omega(\mathbf{z}) \in C_0^\infty(\mathbb{R}^N), \quad \omega(\mathbf{z}) \geq 0, \quad \int_{R^N} \omega(\mathbf{z}) d\mathbf{z} = 1.$$

Formula (1.15) often is called as the mollification of function f and, in fact, it is the convolution of f with the mollifier $\frac{1}{h^N}\omega(\mathbf{z}/h)$.
It's the well-known fact that

$$\|f_h - f\|_{L_M(\Omega)} \to 0 \quad \text{as } h \to 0 \tag{1.16}$$

if $M(t)$ satisfies Δ_2-condition, and

$$\int_\Omega M(|f_h - f|)d\mathbf{x} \to \quad \text{as } h \to 0 \tag{1.17}$$

for any $M(t)$, i.e.the sequence of $\{f_h\}$ approximates f in the sense of mean convergence.

1.2 Strong Approximation of Weak Limits

Let us consider some sequence of functions $\{f_n(\mathbf{x})\}$, $n = 1, 2, \ldots$, from Orlicz space $L_M(\Omega)$ such that $f_n \rightharpoonup f$ weakly in $L_M(\Omega)$, as $n \to \infty$. For each $f_n(\mathbf{x})$ we construct the family of averaged functions $(f_n)_h(\mathbf{x})$.

Theorem 1.1 If $M(t)$ satisfies Δ_2-condition then there exists subsequence $(f_m)_{h_m}$ such that

$$(f_m)_{h_m} \to f \quad \text{strongly in } L_M(\Omega)$$

$$\text{as } m \to \infty,\ h \to 0.$$

For any $M(t)$ there exists subsequence $(f_m)_{h_m}$ converging to f in mean value, i.e. in the sense (1.14).

Proof. Step 1. Simple example.

In order to understand the problem we consider firstly, one very simple example of the sequence $f_n(x) = \sin(nx)$, $n = 1, 2, \ldots$, $x \in \mathbb{R}^1$, $\Omega = (0, 2\pi)$, of periodic functions. In this case we have

$$f_n(x) \rightharpoonup 0 \text{ weakly in } L^2(0, 2\pi).$$

Let us take the Steklov averaging

$$(f_n)_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f_n(\xi)d\xi. \tag{1.18}$$

It's easy to calculate

$$(f_n)_h(x) = \frac{\sin nh}{nh} \sin(nx). \tag{1.19}$$

and thereby to obtain:

a) If $h = h_n \to 0$ as $n \to \infty$, but $nh_n \to \infty$ (for example, $h_n = n^{-\alpha}$, $0 < \alpha < 1$) then

$$(f_n)_{h_n} \to 0 \text{ as } n \to \infty,$$

i.e. $(f_n)_{h_n}(x)$ tends to 0 strongly when $h_n \to 0$ slowly enough.

b) If $nh_n \to const \neq 0$ (or are bounded), then $(f_n)_{h_n} \rightharpoonup 0$ weakly only.

Step 2. One-dimensional case, Steklov averaging.

Now consider the case of $N = 1$, $x \in \mathbb{R}^1$, and $f_n(x) \rightharpoonup f(x)$ weakly in $L_M(\Omega)$, where$M(t)$ satisfies Δ_2-condition.

For the sake of simplicity we assume $f_n(x)$, $f(x)$ to be T-periodic, $T = \text{const} > 0$, and, moreover, without loss of generality, one can admit $f(x) \equiv 0$, since it is possible to consider the sequence of differences $f_n - f$ instead of f_n, i.e.

$$f_n(x) \rightharpoonup 0 \text{ weakly in } L_M(0,T). \tag{1.20}$$

Let us construct the family of functions

$$(f_n)_h(x) = \frac{1}{2h} \int\limits_{x-h}^{x+h} f_n(\xi)\xi \tag{1.21}$$

and the sequence

$$F_n(x) = \int\limits_0^x f_n(\xi)d\xi \tag{1.22}$$

which doesn't depend on h.

It means

$$(f_n)_h(x) = \frac{1}{2h}(F_n(x+h) - F_n(x-h)). \tag{1.23}$$

The sequence $\{F_n(x)\}$ possesses the estimates

$$\sup_x |F_n(x)| \le C, \quad \|F_n'()x\|_{L_M(0,T)} \le C \tag{1.24}$$

with constant C independent on n.

Compactness theorem yields $F_n(x) \to 0$ strongly in $L_M(0,T)$, i.e.

$$\|F_n(x)\|_{L_M(0,T)} \le C_n, \quad C_n \to 0 \text{ as } n \to \infty. \tag{1.25}$$

If we take $F_n(x+h)$ or $F_n(x-h)$ (displacements of $F_n(x)$) then

$$\|F_n(x+h)\|_{L_M} \le C_n \cdot C, \ \ \|F_n(x-h)\|_{L_M} \le C_n \cdot C \tag{1.26}$$

with C independent on $h, h \in (01]$.

It gives

$$\|(f_n)_h\|_{L_M(0,T)} \le C \cdot \frac{C_n}{h}. \tag{1.27}$$

So, if we take $h = h_n$ such that $C_n \cdot h_n^{-1} \to 0$ as $n \to \infty$, for example, $h_n = C_n^\beta, 0 < \beta = \mathit{const} < 1$, then

$$(f_n)_{h_n} \to 0 \ \textit{strongly in } L_M(0,T) \tag{1.28}$$

It proves the theorem 1 in 1-dimensional case.

Step 3. The general case.

Let the sequence $\{f_n(\mathbf{x})\}$, $\mathbf{x} \in \Omega \subset \mathbb{R}^N$, be weakly converging in $L_M(\Omega)$ to $f(\mathbf{x}) \equiv 0$,where $M(t)$ satisfies Δ_2-condition.

We extend $f_n(\mathbf{x})$ by 0 outside of Ω and consider the family $(f_n)_h(\mathbf{x})$ given by the formula (1.15)

$$(f_n)_h(\mathbf{x}) = \frac{1}{h^n} \int\limits_{\mathbb{R}^N} f_n(\mathbf{y})\omega\left(\frac{\mathbf{x}-\mathbf{y}}{h}\right) d\mathbf{y} \tag{1.29}$$

with arbitrary kernel of averaging $\omega(\mathbf{z})$.

At the beginning we fix some $h = h_0$, for example $h_0 = 1$, and consider the sequence

$$F_n(\mathbf{x}) = \int\limits_{\mathbb{R}^N} f_n(\mathbf{y})\omega(\mathbf{x}-\mathbf{y})d\mathbf{y} \equiv \int\limits_{\mathbb{R}^N} f_n(\mathbf{x}-\mathbf{z})\omega(\mathbf{z})d\mathbf{z}$$

which doesn't depend on h.

As in one-dimensional case we conclude by the compactness theorem

$$F_m(\mathbf{x}) \to 0 \text{ strongly in } L_M(\Omega),$$

i.e. $\|F_m\|_{L_M(\Omega)} \leq C_m$, $C_m \to 0$ as $m \to \infty$. If we take the family

$$F_{m\ h}(\mathbf{x}) \equiv \int\limits_{\mathbb{R}^N} f_m(\mathbf{y})\omega\left(\frac{\mathbf{x}-\mathbf{y}}{h}\right) d\mathbf{y}$$

then for $h \leq h_0 = 1$

$$\|F_{mh}\|_{L_M(\Omega)} \leq C\|F_m\|_{L_M(\Omega)}$$

with constant C independent on h, i.e.

$$\|F_{mh}\|_{L_M(\Omega)} \leq C \cdot C_m.$$

In view of $(f_n)_h = h^{-N}F_{nh}$ it means that if we choose $h = h_m$ such that

$$C_m h_m^{-N} \to 0 \text{ as } m \to \infty,\ h_m \to 0, \tag{1.30}$$

for instance, $h_m^N = C_m^\beta$, $0 < \beta < 1$, then$(f_m)_{h_m} \to 0$ strongly in $L_M(\Omega)$.

The theorem 1.1 is proved for the case of Young function $M(t)$ satisfying Δ_2-condition. For any $M(t)$ the same proof is acceptable if one considers the mean convergence (1.14) instead of strong or strong convergence in any Banach function space where the set of smooth functions is dense, for example, in the space $E_M(\Omega)$.

Remark 1.1

The main significance of theorem 1.1 seems to be useful for justification of the smoothing approach in computation of non-smooth solutions to P.D.E's. Big oscillations occur near singularities. The appearance of oscillations can be connected with weak convergence of approximate solutions to the exact one. The procedure of "smoothing" means, in fact, an averaging, and the theorem 1.1 indicates that the radius of averaging must be big enough according to assumption (1.30).

1.3 Applications to Navier-Stokes Equations

It this section we illustrate the theorem 1.1 by one simple example of Navier-Stokes equations for viscous incompressible fluid. We consider the sequence $\{\mathbf{u}_n\}$ of solutions to Navier-Stokes equations [20]:

$$\begin{gathered}\frac{\partial \mathbf{u}_n}{\partial t} + (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n + \nabla p_n = \nu \Delta \mathbf{u}_n + \mathbf{f}_n, \\ \mathrm{div}\mathbf{u}_n = 0, \quad (\mathbf{x}, t) \in Q = \Omega \times (0,T), \quad \Omega \subset \mathbb{R}^3, \quad \Gamma = \partial\Omega.\end{gathered} \tag{1.31}$$

which are complemented with the initial and boundary data

$$\mathbf{u}_n \Big|_{t=0} = \mathbf{u}_n^0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \ \mathbf{u}_n \Big|_{\Gamma} = 0, \quad t \in (0,T) \tag{1.32}$$

Let us suppose

$$\mathbf{u}_n^0(\mathbf{x}) \rightharpoonup \mathbf{u}^0(\mathbf{x}) \text{ weakly in } L^2(\Omega),$$

$$\mathbf{f}_n \rightharpoonup \mathbf{f} \text{ weakly in } \quad L^2(0,T;L^2(\Omega)) \quad as \quad n \to \infty.$$

In view of well-known a priori estimate

$$\sup_{0<t<T} \|\mathbf{u}_n(t)\|_{L^2(\Omega)} + \int_0^T \|\nabla \mathbf{u}_n(t)\|^2_{L^2(\Omega)} dt \le C \tag{1.33}$$

with constant C independent on n, we may admit

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)) \cap L^p(0,T;L^2(\Omega)) \tag{1.34}$$

with any p, $1 \le p < \infty$.

According to the theorem 1.1 it's possible to extract subsequence $\{(\mathbf{u}_m)_{h_m}\}$ of averaged functions (with respect all independent variables or to spatial variables only) such that

$$(\mathbf{u}_m)_{h_m} \to \mathbf{u} \quad \text{strongly in } L^2(0,T;W^{1,2}(\Omega)) \cap L^p(0,T;L^2(\Omega)),$$

$$(\mathbf{f}_m)_{h_m} \to \mathbf{f} \text{ strongly in} L^2(0,T;L^2(\Omega)).$$

And the question arises here: are $\{(\mathbf{u}_m)_{h_m}\}$ approximate solutionsto the limiting equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f}, \quad \mathrm{div}\mathbf{u} = 0 \tag{1.35}$$

in some strong sense?

To give answer to this question we apply the operator of averaging to equations (1.31) and obtain the system (1.35) for $(\mathbf{u})_{h_m}$ with new right part

$$\mathbf{F}_m = (\mathbf{f}_m)_{h_m} + [((\mathbf{u}_m)_{h_m} \cdot \nabla)(\mathbf{u}_m)_{h_m} - ((\mathbf{u}_m \cdot \nabla)\mathbf{u}_m)_{h_m}] \equiv$$

$$(\mathbf{f}_m)_{h_m} + \varphi_m.$$

Theorem 1.2 There exists subsequence $(\mathbf{u}_m)_{h_m}$ such that the difference φ_m tends to zero in the norm of space $L^p(0,T;L^q(\Omega))$ with exponents (p,q), $p \in [1,2]$, $q \in [1,3/2]$, $1/p + 3/2q > 2$.

The proof follows from theorem 1.1 because the sequence $\{(\mathbf{u}_n \cdot \nabla)\mathbf{u}_n\}$ is bounded in $L^p(0,T;L^q(\Omega))$ with such exponents (cf.[20]).

2 Transport Equations in Orlicz Spaces

2.1 Statement of Problem

Here we are concerned with the spatially periodic Cauchy problem for the linear first-order equation (such as the well-known continuity equation in fluid mechanics)

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = h, \qquad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad t \in (0,T) \tag{2.1}$$

complemented with the initial data

$$\rho|_{t=0} = \rho_0(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n \tag{2.2}$$

Besides, we consider the adjoint problem

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta = g, \qquad \mathbf{x} \in \mathbb{R}^n, \quad t \in (0,T) \tag{2.3}$$

$$\zeta|_{t=T} = \zeta_T(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n \tag{2.4}$$

Here $\rho(\mathbf{x},t)$ and $\zeta(\mathbf{x},t)$ are unknown functions, while $\mathbf{u} = (u_1, \dots, u_n)$, h, g, ρ_0, and ζ_T are given ones, and we assume, for the sake of simplicity, all functions being periodic with respect to all spatial variables x_k, $k = 1, 2, \dots, n$.

We denote by Ω the period set, $\Omega = \prod_{k=1}^{n}(0,T_k)$, $0 < T_k < \infty$, and by $Q = \Omega\times(0,T)$ the domain of functions in the space of independent variables $(\mathbf{x}, t)$.

Our main goal is to search for the minimal, as it is possible, conditions for the smoothness of the coefficients $\mathbf{u} = (u_1, \dots, u_n)$ (forming the velocity vector in mechanics) to provide the existence and uniqueness of generalized solutions to the problems (2.1),(2.2) and (2.3),(2.4). There is a particular interest in compressible Navier-Stokes equations concerned with conditions namely for the divergency of the velocity vector ($\mathrm{div}\mathbf{u} = \sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i}$) as far as this quantity and the density play the important role in the estimating of solutions (cf.[9],[15],[16]).

For the case of solutions ρ, ζ from Lebesgue spaces $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, such conditions for $\mathrm{div}\mathbf{u}$ were obtained by R.J.DiPerna and P.L.Lions [23], namely

$$\mathrm{div}\mathbf{u} \in L^1(0,T,L^\infty(\Omega)) \tag{2.5}$$

and these conditions seem to be optimal except the case of $p = 1$ for the problem (2.1), (2.2) and $p = \infty$ for the problem (2.3), (2.4), respectively.

We use Orlicz function spaces associated with Young functions of low and fast growth at the infinity instead of $L^p(\Omega)$ for construction of generalized solutions to the problems (2.1),(2.2) and (2.3),(2.4), and replace the condition (2.5) with the assumption of integrability in certain Orlicz spaces. We remind the Hölder inequality

$$\int_D f(\mathbf{x})g(\mathbf{x})d\mathbf{x} \leqslant C\|f\|_{L_M(D)} \cdot \|g\|_{L_N(D)} \tag{2.6}$$

for conjugate Orlicz spaces. We use Orlicz spaces associated with Young function of rather fast increase. Namely, let us introduce the class $\mathcal{K}$ of convex functions M such that (cf.[16],[25])

$$\mathcal{K} = \left\{ M(r) \;\middle|\; \int^{\infty}\frac{\ln M(r)}{r^2}dr = \infty \right\} \tag{2.7}$$

where the last condition means the integral to be diverging at the infinity. The class $\mathcal{K}$ contains functions increasing at the infinity faster than polinomials and non slower than the exponent. Roughly speaking class $\mathcal{K}$ consists of Young functions like that

$$M(t) = exp\{\frac{t}{ln^{\alpha_1}t\cdot ln^{\alpha_2}lnt\cdots \underbrace{ln^{\alpha_m}ln\cdots lnt}_{m\ times}}\}$$

If all $\alpha_i \leq 1$ then $M(t) \in \mathcal{K}$, but if at least one $\alpha_j > 1$ then M(t) does not belong to $\mathcal{K}$. Two another (but equivalent) definitions of class $\mathcal{K}$ in terms of inverse function $M^{-1}(t)$ and supplementary function $N(t)$ are the following ones (cf.[25])

$$\mathcal{K} = \left\{ M(r) \;\middle|\; \int^{\infty} \frac{1}{rM^{-1}(r)} dr = \infty \right\} = \left\{ M(r) \;\middle|\; \int^{\infty} \frac{1}{N(r)} dr = \infty \right\}$$

Remark that, given function H, the composition $H \circ M \in \mathcal{K}$ if and only if the inverse function H^{-1} increases at the infinity non faster than polinomials.

2.2 Existence and Uniqueness Theorems

Let us explain the main idea to obtain a priori estimates for the solution $\rho(\mathbf{x}, t)$ to the problem (2.1),(2.2). If ρ is a classic solution to the equation (2.1) and $\Phi(\rho)$ is an arbitrary smooth function of ρ then the equality

$$\frac{\partial \Phi(\rho)}{\partial t} + \mathrm{div}(\Phi(\rho)\mathbf{u}) + [\rho\Phi'(\rho) - \Phi(\rho)]\mathrm{div}\mathbf{u} = \Phi'(\rho) \cdot h \tag{2.8}$$

holds besides of (2.1); here $\Phi'(\rho) = \dfrac{d\Phi}{d\rho}$. So, if the function $\Phi(\rho)$ is such that

$$r\Phi'(r) - \Phi(r) \prec\prec \Phi(r) \tag{2.9}$$

then there exists Young function $M(s)$ to provide the relation

$$M(r\Phi'(r) - \Phi(r)) \simeq \Phi(r) \tag{2.10}$$

and one can apply the Hölder inequality (2.6) to the third term in equation (2.8). Then the complementary function $N(r)$ gives the corresponding Orlicz space for div$\mathbf{u}$ to be from. The remarkable fact is that (2.9) and (2.10) are possible if and only if $N(r)$ belongs to the class $\mathcal{K}$ and $\Phi(r)$ is a function of slow growth (less than any power $r^{1+\varepsilon}$ with positive ε).

Proposition 2.1. *Let $N(r) \in \mathcal{K}$, and Φ satisfies the condition (2.9). Then a priori estimates*

$$\|\rho\|_{L^\infty(0,T,L_\Phi(\Omega))} \leqslant C\left(1+\int_Q N(|\mathrm{div}\mathbf{u}|)d\mathbf{x}dt\right) \cdot \left[\|\rho_0\|_{L_\Phi(\Omega)} + \|h\|_{L^1(0,T,L_\Phi(\Omega))}\right] \tag{2.11}$$

$$\|\zeta\|_{L^\infty(0,T,L_\Psi(\Omega))} \leqslant C\left(1+\int_Q N(|\mathrm{div}\mathbf{u}|)d\mathbf{x}dt\right) \cdot \left[\|\zeta_T\|_{L_\Psi(\Omega)} + \|g\|_{L^1(0,T,L_\Psi(\Omega))}\right] \tag{2.12}$$

hold for the solutions of the problems (2.1),(2.2) and (2.3),(2.4) respectively, where Ψ is the complementary function to Φ.

A priori estimates (2.11) and (2.12) allow us to prove the existence of solutions under the assumption on div$\mathbf{u}$ to be from the Orlicz class $K_N(Q)$ (or from Orlicz space $E_N(Q)$) associated with some $N \in \mathcal{K}$ and $\mathbf{u}$ being from $L^1(0, T, L_\Phi(\Omega))$ or $L^1(0, T, L_\Psi(\Omega))$ for the problems (2.1),(2.2) and (2.3),(2.4) respectively.

Theorem 2.1. *If* $\rho_0 \in L_\Phi(\Omega), \quad \zeta_T \in L_\Psi(\Omega),$

$$h \in L^1(0, T, L_\Phi(\Omega)), \qquad g \in L^1(0, T, L_\Psi(\Omega)),$$

$$\mathbf{u} \in L^1(0, T, L_\Phi(\Omega)) \quad \textit{or} \quad \mathbf{u} \in L^1(0, T, L_\Psi(\Omega))$$

and div$\mathbf{u} \in K_N(Q)$ *with* $N \in \mathcal{K}$*, then there exist solutions to the problems (2.1),(2.2) or (2.3),(2.4) respectively.*

Scetch of proof. We approximate all prescribed functions with the sequences of smooth functions and construct the sequence of classic solutions which contents weakly converging subsequence in view of (2.11),(2.12). Passing to the limit yields the existence of weak solutions. (More details one can find in [25].)

To provide the uniqueness of solution it is nesessary to complete the above conditions with additional smoothness of $\mathbf{u}$.

Theorem 2.2 *If the conditions of Theorem 2.1 are fulfilled and*

$$\mathbf{u} \in L^1(0, T, W^{1,1}(\Omega))$$

then the generalized solutions of the problems (2.1),(2.2) and (2.3),(2.4) are unique.

The proof relies upon the following results given below.

2.3 Gronwall-type Inequality and Osgood Uniqueness Theorem

To prove the uniqueness in Theorem 2.2 we reduce the problem to the inequality of Gronwall type

$$\int_\Omega |\psi| d\mathbf{x} \leqslant C \int_0^t \int_\Omega |\psi| \cdot |\mathrm{div}\mathbf{u}| d\mathbf{x} ds \tag{2.13}$$

for the difference ψ of the solutions.

This inequality yields $\psi \equiv 0$ if and only if div$\mathbf{u} \in K_N(Q)$ with some $N \in \mathcal{K}$.
Indeed, denote $\alpha(t) = \int_\Omega \psi(x,t)dx$, $\quad A(t) = \int_0^t \alpha(s)ds$ and

$$\beta(t) = \begin{cases} 0, & \text{if } \alpha(t) = 0, \\ \frac{1}{\alpha(t)} \int_\Omega f(x,t)\psi(x,t)dx, & \text{if } \alpha(t) \neq 0 \end{cases}$$

Then (2.13) takes the form

$$\alpha(t) \leq \int_0^t \alpha(s)\beta(s)ds \tag{2.14}$$

It gives after changing of variables:

$$N(\frac{\alpha(t)}{A(t)}) \leq N(\frac{1}{A(t)} \int_0^{A(t)} \beta(A^{-1}(\tau))d\tau)$$

$$\leq \frac{1}{A(t)} \int_0^{A(t)} N(\beta(A^{-1}(\tau)))d\tau = \frac{1}{A(t)} \int_0^t N(\beta(s))\alpha(s)ds \tag{2.15}$$

Here we have used Jensen's inequality for convex functions

$$M(\frac{1}{mes\Omega} \int_\Omega f(\mathbf{x})d\mathbf{x}) \leq \frac{1}{mes\Omega} \int_\Omega M(f(\mathbf{x}))d\mathbf{x}$$

$$\forall f \in L_M(\Omega), \forall M(t) - convex.$$

Inequalities (2.14),(2.15) lead to differential inequality

$$\frac{dA}{dt} \leq A(t)N^{-1}(\frac{1}{A(t)}), \quad A(0) = 0 \tag{2.16}$$

By Osgood uniqueness theorem (2.16) implies $A(t) \equiv 0$ and then $\psi = 0$ a.e.
In this connection let us consider Cauchy problem for the system of ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t), \qquad \mathbf{x}|_{t=0} = \mathbf{x}_0 \in \mathbb{R}^n \tag{2.17}$$

We suppose $\mathbf{u}(\mathbf{x}, t) \in L^1(0, T, W^1 L_N(\Omega))$ with some $N \in \mathcal{K}$ (that means: gradient of $\mathbf{u}$ with respect to $\mathbf{x}$ belongs to $L^1(0, T, L_N(\Omega))$).

By the embedding theorem (cf.[24]) the field $\mathbf{u}$ possesses the continuity modulus (in the generalized Hölder sense)

$$\sigma(s) = \int_{s^{-n}}^{\infty} \frac{N^{-1}(\xi)}{\xi^{1+1/n}} d\xi \tag{2.18}$$

By Osgood uniqueness theorem the solution of (2.17) is unique if and only if (cf.[26], Ch.3, Corollary 6.2)

$$\int_0 \frac{ds}{\sigma(s)} = \infty \tag{2.19}$$

In view of (2.18) this condition is equivalent to $N(r)$ being from $\mathcal{K}$.

Proposition 2.2 *If* $\dfrac{\partial u_i}{\partial x_j} \in L^1(0, T, L_N(\Omega))$, $i, j = 1, 2, \ldots, n$, *then the uniqueness of solution to Cauchy problem (2.17) is equivalent to the assumption* $N(r) \in \mathcal{K}$.

2.4 Conclusive Remarks

The nesessity of conditions for the coefficients in well-posedness theorems we illustrate by the following example.

Example 2.1 Let $n = 2$, and $\gamma(s)$ is continuous odd function such that $\gamma(0) = 0$, and $\gamma(s) \to \infty$ as $s \to \infty$, and $\displaystyle\int_0 \frac{ds}{\gamma(s)} < \infty$. We construct a vector field on the plane $\mathbb{R}^2$ by the formula

$$\mathbf{u} = \mathbf{u}(x, y) = (\gamma(x), -y\gamma'(x)) \tag{2.20}$$

Then the system of equations for the trajectories

$$\frac{dx}{dt} = \gamma(x), \qquad \frac{dy}{dt} = -y\gamma'(x) \tag{2.21}$$

yields the solutions

$$x(t) = \pm\Gamma^{-1}(t), \qquad y(t) = C\exp\left(-\int_1^t \gamma'(\Gamma^{-1}(s))ds \right) \tag{2.22}$$

(here $\Gamma(z) = \displaystyle\int_0^z \frac{ds}{\gamma(s)}$), forming a double surface containing the axis $\{ (x, y, t) \,|\, x = t = 0 \}$. In particular, taking $\gamma(x) = 2\sqrt{x}$, we obtain the family of solutions

$$x = t^2, \qquad y = \frac{C}{t} \tag{2.23}$$

Thus, the function

$$\zeta(x, y, t) = \begin{cases} 1, & -\Gamma^{-1}(t) < x < \Gamma^{-1}(t) \\ 0, & \text{otherwise} \end{cases} \tag{2.24}$$

is a nontrivial solution to the transport equation (2.3) vith zero initial data.

Such a non-uniqueness example appeared due to the non-embeddings

$$\mathbf{u} \notin L^1(0,T,W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)) \quad \text{and} \quad \mathrm{div}\mathbf{u} \in K_M(Q) \ \text{with}\ M \notin \mathcal{K}$$

Example 2.2 Set

$$\zeta = \chi_{\mathcal{C}}, \quad \rho = \beta^{-n}(t)\chi_{\mathcal{C}}, \quad \mathbf{u} = \frac{\beta'(t)}{\beta(t)}\mathbf{x} \tag{2.25}$$

with $\chi_{\mathcal{C}}$ standing for the charasteristic function of $\mathcal{C}$, and $\mathcal{C}$, in its turn, is the cone $\mathcal{C} = \{\, (\mathbf{x},t) \,|\, |\mathbf{x}| \leqslant \beta(t)\,\}$ with a positive function $\beta(t)$ vanishing as $t = T$.
Since

$$\text{meas supp}\,\rho(t,\cdot) = \text{meas supp}\,\zeta(t,\cdot) \to 0 \quad \text{as}\ \ t \to T$$

then the formula (2.25) defines the example of nontrivial solution to the homogeneous problem (2.3),(2.4) and decaying solution to the problem (2.1),(2.2). Such an example is possible due to the non-embedding

$$\mathrm{div}\mathbf{u} \in L^1(0,T;L_{M,\mathrm{loc}}(\mathbb{R}^n)) \qquad \text{with} \quad M \in \mathcal{K}$$

Remark on "Div-condition". Here we consider the connection appearing between the functions $\Phi(r)$ satisfying (2.9) and corresponding Young functions $M(r)$ (or their complementaries $N(s)$) from (2.10) which describe the function space for div**u**. Given function $M(r)$ produces the class of functions $\Phi(r)$ with the property (2.9) which have the same growth with respect to the linear function, with the power-type difference only. Namely, two functions $\Phi_1(r)$ and $\Phi_2(r)$ satisfy (2.10) with the same $M(r)$ if and only if there exists $q = \text{const} > 0$ such that

$$\frac{\Phi_1(r)}{r} = \left(\frac{\Phi_2(r)}{r}\right)^q, \qquad r > 0 \tag{2.26}$$

This explains, in particular, the condition (2.5) for the case of Lebesgue spaces L^p.

3 Some Remarks on Compensated Compactness Theory

3.1 Introduction

Compactness method (as it's named by J.-L.Lions in his famous book [1]) concerns with the solving of some boundary value problem

$$L(u) \ = \ 0 \tag{3. 1}$$

by the construction of sequence of "approximate" solutions $\{u_n\}$

$$L_n(u_n) = 0, \quad n = 1, 2, \ldots \tag{3. 2}$$

and passing to the limit as $n \to \infty$.

As rule, the sequence $\{u_n\}$ converges in certain sense to some element u, but it's not so evident for u to be the solution of (3.1), in particular, in the case of nonlinear problem (3.1).

The simple example

$$u_n(x) = sin(nx) \quad on \quad [0, 2\pi]$$

illustrates the typical situation:

$$u_n(x) \rightharpoonup 0 \ \ weakly\ in\ L^2(0, 2\pi) \quad as \quad n \to \infty$$

but $u_n^2(x) \rightharpoonup \frac{1}{2}$.

Very often one has quadratic nonlinearity

$$u_n \rightharpoonup u, \quad v_n \rightharpoonup v \quad weakly \quad in\ L^2(\Omega)$$

Then

$$u_n \cdot v_n \rightharpoonup \chi \quad in\ sense\ of\ distributions \quad D'(\Omega)$$

And the question is wheather equality

$$\chi(\mathbf{x}) = u(\mathbf{x}) \cdot v(\mathbf{x}) \quad in \quad D'(\Omega)$$

holds or not?

This question possesses the positive answer if one of two sequences $\{u_n\}$ or $\{v_n\}$ converges in strong sense, i.e. if

$$u_n \to u \quad or \quad v_n \to v \quad in\ norm\ of\ L^2(\Omega)$$

So, the main goal of compactness method is to obtain a strong convergence at least for one sequence, and the principal tool here is based on the Aubin- Simon compactness theorem (cf.[1],Ch.2, theorem 5.1, [2],[3]), which is elucidated in the next section for the convenience of reader.

As a one of very useful tool in solving of non-linear P.D.E.'s is so called theory of compensated compactness (cf.[4]-[7]). Roughly speaking, this theory allows us to pass to the limit in weakly converging sequences under minimal conditions which don't provide in general case strong convergence of any sequence either $\{u_n\}$ or$\{v_n\}$.

The mostly important for the applications to nonlinear P.D.E.'s theory is one particular version of compensated compactness, namely, so called "div-curl" lemma [4],[5],[7]. Numerous interesting results were obtained on the base of this lemma (cf. [8]-[17]).

3.2 Classical Compactness (Aubin-Simon Theorem)

Now we recall here one well-known and widely-used compactness argument (cf. [1], Ch. 2, Theorem 5.1). Let $B_0,\ B\ and\ B_1$ be three Banach spaces such that

$$B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1$$

Here $\hookrightarrow$ means continuous embedding while $\hookrightarrow\hookrightarrow$ is continuous and compact one.

If the sequence$\{u_n(t)\}$ is bounded in $L^{p_0}(0,T;\ B_0),\quad 1 \le p_0 \le \infty$, and the sequence of derivatives $\{du_n/dt\}$ is bounded in $L^{p_1}(0,T;\ B_1),\quad 1 \le p_1 \le \infty$, then $\{u_n(t)\}$ is compact in $L^p(0,T;\ B)$ with $1 \le p \le p_0$ in the case of $p_0 < \infty$ and $1 \le p < \infty$ in the case of $p_0 = \infty,\ p_1 = 1$.

This theorem was proved firstly by J.P.Aubin [2] in the case $1 < p_0, p_1 < \infty$ and for the limiting cases by J.Simon [3].

The same result takes the place if the sequence of time-derivatives of the other sequence $\{v_n(t)\}$ instead of $\{u_n(t)\}$ is bounded (cf.[16], lemma 6).

Let$V$ and W be two Banach function spaces defined on the bounded domain $\Omega \in \mathbb{R}^N$ and such that $D(\Omega) \hookrightarrow V \hookrightarrow\hookrightarrow W \hookrightarrow D'(\Omega)$ We suppose, of course, for any Banach space$B$ located between $D(\Omega)$and $D'(\Omega)$ the product with infinitely differentiable functions is defined:

$$\forall u \in B, \forall \phi \in D(\Omega) \quad \exists (u \cdot \phi) \in B.$$

In particular, it's valid for the spaces V and W as well as for their conjugate spaces V' and W' which satisfy the embedding relations

$$D(\Omega) \hookrightarrow W' \hookrightarrow\hookrightarrow V' \hookrightarrow D'(\Omega)$$

Finally, let V_1' be some Banach space, arbitrary wide

$$V' \hookrightarrow V_1' \hookrightarrow D'(\Omega).$$

Now we suppose that there exist two sequences $\{u_n(t)\}$ and $\{v_n(t)\}$ such that

$$u_n(t) \to u(t) \quad weakly\ in\ L^p(0,T;W), 1 < 0 \le \infty$$

$$v_n(t) \to v(t) \quad weakly\ in\ L^q(0,T;W'), q \ge \frac{p}{p-1}$$

Then one can define the sequence of products $\{u_n(t) \cdot v_n(t)\}$ as a sequence in the space $L^1(0,T;D'(\Omega)$ by the rule

$$< u_n \cdot v_n, \phi >=< v_n, u_n \cdot \phi >, \quad \forall \phi \in D(\Omega),$$

where $< g, \phi >$ designates the value of distribution g on the test function ϕ. We assume also the sequence $\{u_n \cdot v_n\}$ to be converging in $D'(0,T;D'(\Omega)$

$$u_n \cdot v_n \to \chi \quad in\ D'(0,T;D'(\Omega)$$

If we suppose additionally $\{u_n(t)\}$ is bounded in $L^p(0,T;V)$ and $\{v_n'(t)\}$ is bounded in $L^{p_1}(0,T;V_1')$, $1 \leq p_1 \leq \infty$ then the equality $\chi = u \cdot v$ holds in sense of $D'(0,T;D'(\Omega))$. This proposition is a particular case of compensated compactness theory (see subsection 4 below) but it's proof can be reduced to Aubin - Simon theorem if we take $B_0 = W', B = V' \quad and \quad B_1 = V_1'$.

To apply this version of compactness theorem to P.D.E. theory we define, firstly, the sequence $\{v_n\}$ which admits a priori estimates for time-derivatives. The other factor in non-linear term gives the sequence $\{u_n\}$. It indicates the spaces V and W'. Then we have to check the compactness of embedding V into W or, if it's easier, W' into V'. (See, for example, application to compressible Navier-Stokes equations in [16], subsection 5.2, and also recent paper [31])

3.3 Compensated Compactness – "div-curl" Lemma

Let $\Omega \subset \mathbb{R}^N$ be bounded domain with boundary Γ, and let $\{\mathbf{w}_n^1\}$ and $\{\mathbf{w}_n^2\}$, $n = 1, 2, \ldots$, be two sequences of vector fields on Ω such that

$$\mathbf{w}_n^1 \rightharpoonup \mathbf{w}^1, \quad \mathbf{w}_n^2 \rightharpoonup \mathbf{w}^2 \ \textit{weakly in } L^2(\Omega) \quad \textit{as } n \to \infty. \tag{3. 3}$$

Then the sequence of scalar products $Q(\mathbf{w}_n^1, \ \mathbf{w}_n^2) = \sum_{i=1}^N (w_n^1)_i (w_n^2)_i$ is bounded in $L^1(\Omega)$, so

$$Q(\mathbf{w}_n^1(\mathbf{x}), \mathbf{w}_n^2(\mathbf{x})) \rightharpoonup \chi(\mathbf{x}) \quad \textit{in } D'(\Omega) \tag{3. 4}$$

And the question is: wheather equality

$$\chi(\mathbf{x}) \ = \ Q(\mathbf{w}^1(\mathbf{x}), \mathbf{w}^2(\mathbf{x})) \quad \textit{in } \ D'\Omega \tag{3. 5}$$

holds or not?

Let us introduce two operators of vector analysis:

$$div\mathbf{a} \ = \ \sum_{i=1}^N \frac{\partial a_i}{\partial x_i}, \quad (curl\mathbf{a})_{ij} \ = \ \frac{\partial a_i}{\partial x_j} \ - \ \frac{\partial a_j}{\partial x_i}, \quad i,j \ = \ 1,2,\ldots,N. \tag{3. 6}$$

Proposition 3.1 (F.Murat[4],[5], see also [7])

If the sequence $\{div\mathbf{w}_n^1\}$ is compact in $H_{loc}^{-1}(\Omega)$ and $\{curl\mathbf{w}_n^2\}$ is compact in $(H_{loc}^{-1}(\Omega))^m$, $m \ = \ N(N-1)/2$, then the equality (3.5) is valid.

Remark 3.1 If div and curl of one sequence $\{\mathbf{w}_n^1\}$ or $\{\mathbf{w}_n^2\}$ are compact, then this sequence is strongly compact in $L^2(\Omega)$ as a solution to elliptic problem for Laplace operator (see decomposition (3.7) below).

Proof. Introduce two function spaces:

$$J(\Omega) \ = \ \{\mathbf{u} \in L^2(\Omega) \mid \quad div\mathbf{u} \ = \ 0\}$$

$$G(\Omega) \;=\; \{\mathbf{v} \in L^2(\Omega) \mid \quad \mathbf{v} \;=\; \nabla\psi, \quad \psi \in \overset{\circ}{H}{}^1 (\Omega)\}$$

which make Helmholtz-Weyl decomposition (cf.[19]-[21])

$$L^2(\Omega) \;=\; J(\Omega) \;\oplus\; G(\Omega) \tag{3. 7}$$

(To find ψ for given $\mathbf{w}$ one has to solve Dirichlet problem for Poisson equation with right hand part $div\mathbf{w}$. Such operator is continious map from $H^{-1}(\Omega)$ into $\overset{\circ}{H}{}^1 (\Omega)(cf.[1],[13])$. In particular, if the sequence $\{div\mathbf{w}_n\}$ is compact in $H^{-1}(\Omega)$ then the corresponding sequence $\{\mathbf{v}_n\}$ in decomposition (3.7) is compact in $L^2(\Omega)$.)

Now one can represent

$$\mathbf{w}^1_n \;=\; \mathbf{u}^1_n \;+\; \mathbf{v}^1_n, \quad \mathbf{w}^2_n \;=\; \mathbf{u}^2_n \;+\; \mathbf{v}^2_n$$

where $\mathbf{u}^k_n \in J(\Omega), \quad \mathbf{v}^k_n \in G(\Omega), \quad k = 1, 2.$. Then one has

$$\{div\mathbf{v}^1_n \;=\; div\mathbf{w}^1_n\}$$

-compact set in $H^{-1}_{loc}(\Omega)$, according to assumption, and $curl\mathbf{v}^1_n \;\equiv\; 0$. It means the set $\{\mathbf{v}^1_n\}$ is compact in $L^2(\Omega)$.

By the same way

$$\{curl\mathbf{u}^2_n \;=\; curl\mathbf{w}^2_n\}$$

is compact set in $(H^{-1}_{loc}(\Omega))^m, \quad m = 1/2N(N-1)$ and $div\mathbf{u}^2_n \;\equiv\; 0$.

It implies compactness of $\{\mathbf{u}^2_n\}$ in $L^2(\Omega)$. So, the scalar product can be rewritten as follows

$$Q(\mathbf{w}^1_n, \mathbf{w}^2_n) \;=\; Q(\mathbf{u}^1_n, \mathbf{u}^2_n) \;+\; Q(\mathbf{v}^1_n, \mathbf{v}^2_n) \;+\; (\mathbf{u}^1_n \cdot \mathbf{v}^2_n) \;+\; (\mathbf{u}^2_n \cdot \mathbf{v}^1_n)$$

After integrating over Ω two last terms vanish since $J(\Omega)$ and $G(\Omega)$ are orthogonal subspaces. And two first terms content strongly converging sequences $\{\mathbf{u}^2_n\}$ and $\{\mathbf{v}^1_n\}$ what allows to pass to the limit in (3.4),(3.5) and to prove proposition 3.1. In addition we remark that test function $\phi \in D(\Omega)$ can be included in any sequence $\{\mathbf{w}^1_n\}$ or $\{\mathbf{w}^2_n\}$. It is also the reason why the smoothness of boundary Γ is not so important.

As it has been noted, our proof relies upon decomposition (3.7) which itself is based on the optimal estimates for the solutions of Dirichlet problem. We are able to prove "div-curl" lemma for the wide class of Orlicz spaces instead of $L^2(\Omega)$ (cf.[27],[28].) Indeed, let $\{\mathbf{w}^1_n\}$ and $\{\mathbf{w}^2_n\}$be two sequences such that

$$\mathbf{w}^1_n \rightharpoonup \mathbf{w}^1 \quad weakly\ in\ L_M(\Omega), \quad \mathbf{w}^2_n \rightharpoonup \mathbf{w}^2 \quad weakly\ in\ L_N(\Omega)$$

where $M(t)$ and $N(t)$ are mutually complementary Young functions.

Then we have the convergence (3.4), and to prove (3.5) it is necessary to provide decomposition (3.7) and strong compactness of $\{\mathbf{v}^1_n\}$ and $\{\mathbf{u}^2_n\}$ in $L_M(\Omega)$ and $L_N(\Omega)$, respectively. We assume additionally the set $\{div\ \mathbf{w}^1_n\}$

to be compact in Orlicz-Sobolev space $W^{-1}L_M(\Omega)$ which is conjugate for $\overset{\circ}{W}{}^1 E_N(\Omega)$, and the set $\{curl\ \mathbf{w}_n^2\}$ to be compact in $W^{-1}L_N(\Omega)$ (conjugate for $\overset{\circ}{W}{}^1 E_M(\Omega)$), respectively.

One can use decomposition (3.7) and corresponding optimal estimates if Young function M(t) or N(t) satisfies additional condition (cf.[29],[30])

$$1 < p_1 = const \leq \frac{tM'(t)}{M(t)} \leq p_2 = const < \infty,$$

or

$$1 < q_1 = const \leq \frac{tN'(t)}{N(t)} \leq q_2 = const < \infty$$

This means that the spaces $L_M(\Omega)$ and $L_N(\Omega)$ are located between Lebesgue spaces $(L^{p_1}(\Omega),\ L^{p_2}(\Omega))$ and $(L^{q_1}(\Omega),\ L^{q_2}(\Omega))$, respectively. Optimal estimates are obtained in [29],[30] by interpolation of estimates for Lebesgue spaces. For the limiting cases $p_1 = 1$ or $q_1 = 1$ and $p_2 = \infty$ or $q_2 = \infty$ the optimal results are not proved still and it remains as an interesting open problem.

3.4 Compensated Compactness-theorem of L. Tartar

Let

$$Q(\mathbf{u}) \ = \ \sum_{i,j=1}^{p} b_{ij}\, u_i\, u_j \tag{3. 8}$$

be arbitrary quadratic form with constant coefficients, ($b_{ij} = const$) on $\mathbb{R}^p$.

Let $\{\mathbf{u}_n(\mathbf{x})\}$, $\mathbf{x} \in \Omega \subset \mathbb{R}^N$ be some sequence such that

$$\mathbf{u}_n(\mathbf{x}) \rightharpoonup \mathbf{u}(\mathbf{x}) \quad weakly\ in\ \ (L^2(\Omega))^p \quad as\ n \to \infty \tag{3. 9}$$

Then

$$Q(\mathbf{u}_n(\mathbf{x})) \ \rightharpoonup \chi(\mathbf{x}) \quad in\ D'(\Omega) \tag{3. 10}$$

and the question is

$$\chi(\mathbf{x}) \ = \ Q(\mathbf{u}(\mathbf{x})) \quad in\ \ D'(\Omega)\ ? \tag{3. 11}$$

Suppose, some additional information is known, namely, let

$$A:\ (L^2(\Omega))^p \ \to \ (H^{-1}(\Omega))^q$$

be linear bounded operator of the form

$$A_k(\mathbf{u}) \ = \ \sum_{i=1}^{p}\sum_{j=1}^{N} a_{kij}\frac{\partial u_i}{\partial x_j}, \quad k \ = \ 1, 2, \ldots, q \tag{3. 12}$$

where, for simplicity, a_{kij} are real constants.

Finally, let us introduce the set

$$\Lambda = \{\lambda \in \mathbb{R}^p |\ \exists \xi \in \mathbb{R}^N, \xi \neq 0,\ \sum_{i=1}^{p}\sum_{j=1}^{N} a_{kij}\lambda_i\xi_j \ = \ 0,\ \forall k = \ 1,2,\ldots,q\} \tag{3. 13}$$

Proposition 3.2 (L.Tartar[6])

Assume (3.9), (3.10) and additionally

$$\{A(\mathbf{u}_n)\}\ \textit{is compact in}\ \ (H^{-1}_{loc}(\Omega))^q \tag{3. 14}$$

and

$$Q(\lambda) \ = \ 0, \quad \forall\lambda \ \in \ \Lambda \tag{3. 15}$$

Then the equality (3.11) holds.

Remark 3.2. "div-curl" lemma is a particular case of this proposition.

Proof. To make a proof looking like the proof of "div-curl" lemma we assume at the beginning operator A to be closed. It means the set ImA^* is closed, where $A^* :\ (\overset{\circ}{H}{}^1\ (\Omega))^q \ \rightarrow\ (L^2(\Omega))^p$ is an adjoint operator, or, by other words, second order operator $A \circ A^* :\ (\overset{\circ}{H}{}^1\ (\Omega))^q \ \rightarrow (H^{-1}(\Omega))^q$ is strongly elliptic.

We shall use decomposition (cf.[22])

$$L^2(\Omega) \ = \ kerA \ \oplus \ ImA^* \tag{3. 16}$$

instead of (3.7). Then for $\forall\, n \ = \ 1,2,\ldots$ one has

$$\mathbf{u}_n \ = \ \mathbf{v}_n \ + \ \mathbf{w}_n, \quad \mathbf{v}_n \ \in \ kerA, \quad \mathbf{w}_n \ \in \ ImA^*$$

The set $\{\mathbf{w}_n\}$ is compact in $(L^2(\Omega))^p$ since operator A is invertible on ImA^*, and $\{A(\mathbf{w}_n)\} \ = \ A(\mathbf{u}_n)\}$ is compact in $(H^{-1}(\Omega))^q$ according to assumption.

Rewrite quadratic form Q:

$$Q(\mathbf{u}_n) \ = \ Q(\mathbf{v}_n) \ + \ Q(\mathbf{w}_n) \ + \ \tilde{Q}(\mathbf{v}_n,\mathbf{w}_n) \ + \ \tilde{Q}(\mathbf{w}_n,\mathbf{v}_n) \tag{3. 17}$$

where

$$\tilde{Q}(\mu,\nu) \ = \ \sum_{i,j=1}^{p} b_{ij}\mu_i\nu_j$$

is bilinear form on $\mathbb{R}^p \times \mathbb{R}^p$ such that $Q(\mathbf{u}) \ = \ \tilde{Q}(\mathbf{u},\mathbf{u})$.

Formula (3.17) means that only one question is in limiting passing, namely, does $Q(\mathbf{v}_n)$ converge to $Q(\mathbf{v})$ if $\mathbf{v}_n \in kerA$?

Fourier transformation applied to equalities

$$A_k(\mathbf{v}_n) \ = \ 0 \quad k \ = \ 1,2,\ldots,q$$

yields

$$\sum_{i=1}^{p}\sum_{j=1}^{N} a_{kij}\ (\hat{v}_n)_i\xi_j \ = \ 0$$

where $\hat{\mathbf{v}_n}$ is Fourier image of $\mathbf{v}_n$ extended by 0 on the whole $\mathbb{R}^N$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ - parameters of Fourier transformation

It means both $Re(\hat{\mathbf{v}_n})$ and $Im(\hat{\mathbf{v}_n})$ belong to the set Λ , and according to assumption (15)

$$\tilde{Q}(\hat{\mathbf{v}_n}, \bar{\hat{\mathbf{v}_n}}) \ = \ 0.$$

It gives by Plancherel-Parseval identity

$$\int_{\mathbb{R}^N} Q(\mathbf{v}_n)d\mathbf{x} \ = \ C_N \int_{\mathbb{R}^N} \tilde{Q}(\hat{\mathbf{v}_n}, \bar{\hat{\mathbf{v}_n}})d\xi \ = \ 0$$

Passing to the limit in (3.17) we obtain the result of proposition 3.2 in the case of closed set ImA^* in $(L^2(\Omega))^p$. Note that construction of sequence $\mathbf{w}_n$ does not depend explicitly on the norm of inverse operatorA^{-1}. In general case we can approximate each $\mathbf{w}_n$ from closure of ImA^* by sequence $\{\mathbf{w}_{nm}\}$ from ImA^* and pass to the limit by standard diagonal procedure. In conclusion of this section we underline once again that proposition 3.2 is not a new mathematical result, but the sense of condition (3.15) as vanishing of form Q on subspace $kerA$ is a new viewpoint which can be usefull in some cases when $kerA$ admits the simple description.

3.5 Generalizations and Examples

New approach to compensated compactness theory based on decomposition (3.16) allows us to give clear and short proofs of main known theorems formulated in propositions 3.1 and 3.2. At the same time we can give some natural generalizations of theory on the other cases of compensated compactness arguments.

a). We shall start from generalization of “div-curl” lemma as a most important for the theory of nonlinear partial differential equations.

Consider two operators of vector analysis (3.6):

$$A_1(\mathbf{u}) \ = \ div\mathbf{u}, \qquad A_2(\mathbf{u}) \ = \ curl\mathbf{u}$$

as linear bounded operators from $L^2(\Omega)$ onto $H^{-1}(\Omega)$.The kernel of operator A_1 is the subspace $J(\Omega)$ in Helmholtz-Weyl decomposition (3.7). It's easy to calculate the adjoint operator for A_1, namely, operator-gradient: $A_1^* = -\nabla$, which acts as bounded linear one from $\overset{\circ}{H}{}^1(\Omega)$ onto $L^2(\Omega)$ and $ImA_1^* = G(\Omega)$ - the second subspace in decomposition (7). Moreover, $G(\Omega) = kerA_2$, so (3.7) is a particular case of decomposition (3.16):

$$L^2(\Omega) \ = \ J(\Omega) \oplus G(\Omega) \ = \ kerA_1 \oplus ImA_1^* \ = \ ImA_2^* \oplus kerA_2$$

Now let us consider two arbitrary linear bounded operators instead of (3.6):

$$A_1 : (L^2(\Omega))^N \to (H^{-1}(\Omega))^p, \quad A_2 : (L^2(\Omega))^N \to (H^{-1}(\Omega))^q \tag{3. 18}$$

Then the following assertion takes place.

Proposition 3.3 Assume (3.3) and (3.4), and additionally let the sets

$$\{A_1(\mathbf{w}_n^1)\} \quad and \quad \{A_2(\mathbf{w}_n^2)\}$$

are compact in $(H_{loc}^{-1}(\Omega))^p \quad and \quad (H_{loc}^{-1}(\Omega)^q$, respectively.

Then equality (3.5) holds, if operators A_1 and A_2 satisfy the condition

$$kerA_i \subseteq ImA_j^*, \quad i \neq j \tag{3. 19}$$

Proof. At the first step we use decomposition (3.16) with operator A_1 and represent $\mathbf{w}_n^1$ as follows

$$\mathbf{w}_n^1 = \mathbf{u}_n^1 + \mathbf{v}_n^1, \quad \mathbf{u}_n^1 \in kerA_1, \quad \mathbf{v}_n^1 \in ImA_1^* \tag{3. 20}$$

Then $\mathbf{v}_n^1 \to \mathbf{v}^1$ strongly in $L^2(\Omega)$, and in the scalar product

$$(\mathbf{w}_n^1 \cdot \mathbf{w}_n^2) = (\mathbf{u}_n^1 \cdot \mathbf{w}_n^2) + (\mathbf{v}_n^1 \cdot \mathbf{w}_n^2)$$

one can pass to the limit in second term. In the first term we use decomposition (3.16) with operator A_2 and represent $\mathbf{w}_n^2$:

$$\mathbf{w}_n^2 = \mathbf{u}_n^2 + \mathbf{v}_n^2, \quad \mathbf{u}_n^2 \in kerA_2, \quad \mathbf{v}_n^2 \in ImA_2^*$$

where the sequence $\{\mathbf{v}_n^2\}$ is compact (strongly) in $L^2(\Omega)$. It means that

$$(\mathbf{u}_n^1 \cdot \mathbf{w}_n^2) = (\mathbf{u}_n^1 \cdot \mathbf{u}_n^2) + (\mathbf{u}_n^1 \cdot \mathbf{v}_n^2)$$

and we can pass to the limit in the last term, while the first one vanishes after integrating over Ω since $\mathbf{u}_n^1$ and $\mathbf{u}_n^2$ are orthogonal in view of condition (3.19). It proves the proposition 3.3

If assumption (3.19) doesn't take place and subspaces $kerA_1$ and $kerA_2$ have nontrivial intersection, i.e.

$$S = kerA_1 \bigcap kerA_2 \neq \{0\} \tag{3. 21}$$

then (3.5) doesn't take place, in general. In this case some additional conditions are needed, but on the subspace S only. For example, another operator

$$A_3 : (L^2(\Omega))^N \to (H^{-1}(\Omega))^r$$

is compact on S. If $kerA_3 \bigcap S \neq \{0\}$ then one has to continue the procedure adding new operators.

Example 3.1
Consider the simple case of N=2 and $A_1 = A_2 = div$, i.e.

$$A_1(\mathbf{w}) = A_2(\mathbf{w}) = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2}$$

Then $S = kerA_1 \bigcap kerA_2 = J(\Omega) \neq \{0\}$. Taking operator A_3 of the form

$$A_3(\mathbf{w}) = \frac{\partial w_2}{\partial x_1} + a\,\frac{\partial w_1}{\partial x_2} + b\,\frac{\partial w_2}{\partial x_2}$$

with arbitrary real parameters a and b , we obtain equality (5), in assumption of compactness of the set $\{A_3(\mathbf{w}_n^k)\}$ in $H^{-1}_{loc}(\Omega)$, if $b^2 + 4a < 0$. If $b^2 + 4a \geq 0$, then another conditions are required on $kerA_3 \bigcap J(\Omega)$.

b). Most part of applications to nonlinear P.D.E. problems concerns with the case when operator A in (3.12) is defined by a priori estimates connected with conservation laws, and quadratic form (3.8) is related to non-linearity of the system of equations, so these two given objects are independent, in some sense, each on other, and don't satisfy the crucial assumption (3.15). Such situation is needed for some additional information on $kerA$, e.g. another operator $B : L^2(\Omega) \rightarrow (H^{-1}(\Omega))^r$ has locally compact image.

Example 3.2 Let $N = 2,\ p = 2,\ q = 1, \quad A = div$ and

$$Q(\mathbf{u}) = u_1^2 + u_1 u_2.$$

Then

$$\Lambda = \{\lambda \in \mathbb{R}^2 \mid \exists \xi \in \mathbb{R}^2, \quad \xi \neq 0, \quad \lambda_1\xi_1 + \lambda_2\xi_2 = 0\}$$

which means $\Lambda = \mathbb{R}^2$, while

$$Q(\lambda) = \lambda_1(\lambda_1 + \lambda_2), \quad and \quad Q(\lambda) \neq 0 \quad \forall\ \lambda \in \Lambda = \mathbb{R}^2$$

Since $kerA = J(\Omega)$ one can add any operator

$$B(\mathbf{u}) = \sum_{i=1}^{2} a_i \frac{\partial u_1}{\partial x_i} + \sum_{i=1}^{2} b_i \frac{\partial u_2}{\partial x_i}$$

to be compact in $H^{-1}_{loc}(\Omega)$ on the sequence $\{\mathbf{u}_n\}$ from subspace $J(\Omega)$ under condition

$$(a_1 - b_2)^2 + 4a_2b_1 < 0$$

which provides operator B to be elliptic on $J(\Omega)$.

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