8

Eisenstein Series Second Part

In Chap. 5, we already saw the Epstein zeta function, actually two zeta functions, one primitive and the other one completed by a Riemann zeta function. Indeed, let $Y \in \mathrm{Pos}_n$. We may form the two series

$$E^{\mathrm{pr}}(Y,s) = \sum_{a \text{ prim}} ([a]Y)^{-s} \quad \text{and} \quad E(Y,s) = \sum_{a \neq 0} ([a]Y)^{-s}$$

where the first sum is taken over $a \in {}^t\mathbf{Z}^n, a \neq 0$ and a primitive; while the second sum is taken over all $a \in {}^t\mathbf{Z}^n, a \neq 0$. Any $a \in {}^t\mathbf{Z}^n$ can be written uniquely in the form

$$a = da_1 \text{ with } d \in \mathbf{Z}^+ \text{ and } a_1 \text{ primitive} .$$

Therefore

$$E(Y,s) = \zeta_{\mathbf{Q}}(2s)E^{\mathrm{pr}}(Y,s) .$$

We have to extend this property to the more general Selberg Eisenstein series on Pos_n. This involves a more involved combinatorial formalism, about integral matrices in $\mathbf{Z}^{j,j+1}$ with $j = 1, \ldots, n-1$. Thus the first section is devoted to the linear algebra formalism of such integral matrices and their decompositions. After that, we define the general Eisenstein series and obtain various expressions for them which are used subsequently in deriving the analytic continuation and functional equations. For all this, we will follow Maass from [Maa 71] after [Maa 55], [Maa 56]. He did a great service to the mathematical community in providing us with a careful and detailed account. However, we have had to rethink through all the formulas because we use left characters instead of right characters as in Maass-Selberg, and also we introduce the Selberg variables $s = (s_1, \ldots, s_n)$ as late as possible. Indeed, we work with more general functions than characters, for application to more general types of Eisenstein series constructed with automorphic forms, or beyond with the heat kernel.

We note here one important feature about the structure of various fudge factors occurring in functional equations: they are eigenvalues of certain

Jay Jorgenson: *Pos$_n$(R) and Eisenstein Series*, Lect. Notes Math. **1868**, 133–162 (2005)
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operators, specifically three operators: a regularizing invariant differential operator, the gamma operator (convolution with the kernel of the gamma function on Pos_n), a Hecke-zeta operator. To bring out more clearly the structure of these operators and their role, we separate the explicit computation of their eigenvalues from the position these eigenvalues occupy as fudge factors. When the eigenfunctions are characters, these eigenvalues are respectively polynomials, products of ordinary gamma functions, and products of Riemann zeta functions, with the appropriate complex variables. Such eigenvalues are those occurring in the theory of the Selberg Eisenstein series, which are the most basic ones. However, Eisenstein series like other invariants from spectral theory (including analytic number theory) have an inductive "ladder" structure, and on higher rungs of their ladder, the eigenvalues are of course more complicated and require more elaborate explicit computations, which will be carried out in their proper place. On the other hand, the general formulas given in the present chapter will be applicable to these more general cases.

1 Integral Matrices and Their Chains

Throughout, we let:

$\Gamma_n = \mathrm{GL}_n(\mathbf{Z})$;
$\mathbf{M}_n^* =$ set of integral $n \times n$ matrices of rank n;
$\mathbf{M}^*(p,q) =$ set of integral $p \times q$ matrices of rank $\min(p,q)$;
$\Delta_n =$ set of upper triangular integral $n \times n$ matrices of rank n;
$\mathcal{T}_n = \Gamma_n \cap \Delta_n =$ group of upper triangular integral matrices of determinant ± 1.

We note that $\mathbf{M}_n^*$ and Δ_n are just sets of matrices, not groups. The diagonal components of an element in Δ_n are arbitrary integers $\neq 0$, so elements of Δ_n are not necessarily unipotent. On the other hand, the elements of $\mathcal{T}_n$ necessarily have ± 1 on the diagonal, so differ from unipotent elements precisely by such diagonal elements. Note that Δ_n is stable under the action of $\mathcal{T}_n$ on both sides, but we shall usually consider the left action. Thus we consider coset representatives in Γ_n for the coset space $\mathcal{T}_n\backslash\Gamma_n$ and also coset representatives $D \in \Delta_n$ of the coset $\mathcal{T}_n D$, which is a subset of Δ_n. Similarly, $\mathbf{M}_n^*$ is stable under the action of Γ_n on both sides, and we can consider the coset space $\Gamma_n\backslash\mathbf{M}_n^*$.

Lemma 1.1. *The natural inclusion $\Delta_n \hookrightarrow \mathbf{M}_n^*$ induces a bijection*

$$\mathcal{T}_n\backslash\Delta_n \to \Gamma_n\backslash\mathbf{M}_n^*$$

of the coset spaces.

Proof. By induction, and left to the reader. We shall work out formally a more complicated variation below.

The bijection of Lemma 1.1 is called **triangularization**.

Next we determine a natural set of coset representatives for $\mathcal{T}_n\backslash\Delta_n$.

Lemma 1.2. *A system of coset representatives of $\mathcal{T}_n\backslash\Delta_n$ consists of the matrices*

$$D = \begin{pmatrix} d_{11} & \dots & d_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_{nn} \end{pmatrix} = (d_{ij})$$

satisfying $d_{ij} = 0$ if $j < i$ (upper triangularity), $d_{jj} > 0$ all j, and

$$0 \leqq d_{ij} < d_{jj} \quad \text{for} \quad 1 \leqq i < j \leqq n\ .$$

In other words, in a vertical column, the components are $\geqq 0$ and strictly smaller than the diagonal component in this column.

Proof. In the first place, multiplying an arbitrary element $D \in \Delta_n$ by a diagonal matrix with ± 1 diagonal components, we can make the diagonal elements d_{jj}, $(j = 1, \dots, n)$ to be positive. Then we want to determine a nilpotent integral matrix X (upper triangular, zero on the diagonal) such that $(I + X)D$ is among the prescribed representatives, and furthermore, such X is uniquely determined. This amounts to the euclidean algorithm, and is done by induction, starting with the top left. Pictorially, given an upper triangular integral matrix with positive diagonal elements d_{ii} and strictly upper triangular elements y_{ij}, we want to find $X = (x_{ij})$ such that the product

$$\begin{pmatrix} 1 & x_{12} & \dots & & x_{1n} \\ 0 & 1 & \dots & & x_{2n} \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & 1 & x_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} d_{11} & y_{12} & \dots & & y_{1n} \\ 0 & d_{22} & \dots & & y_{2n} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & d_{n-1,n-1} & y_{n-1,n} \\ 0 & 0 & \dots & 0 & d_{nn} \end{pmatrix}$$

satisfies the inequalities in the lemma. We start at the top, so we first solve for x_{12} such that

$$0 \leqq y_{12} + x_{12}d_{22} < d_{22}\ .$$

This inequality has a unique integral solution x_{12}. We then solve inductively for $x_{13}, \dots, x_{1n}$; then we go down the rows to conclude the proof.

Lemma 1.3. *Given integers $d_{jj} > 0$ $(j = 1, \dots, n)$, the number of cosets $\mathcal{T}_nD$ with D having the given diagonal elements is*

$$\prod_{j=1}^{n} d_{jj}^{j-1}\ .$$

Proof. Immediate.

Remark. The previous lemmas have analogues for the right action of Γ_n on $\mathbf{M}_n^*$. First, Lemma 1.1 is valid without change for the right action of Γ_n on $\mathbf{M}_n^*$ and the right action of $\mathcal{T}_n$ on Δ_n. On the other hand, the inequalities defining coset representatives in Lemma 1.2 for the right have to read:

$$0 \leqq d_{ij} < d_{ii} \text{ for } 1 \leqq i < j \leqq n \ .$$

Then the number of cosets $D\mathcal{T}_n$ with D having given $d_{11}, \ldots, d_{nn} > 0$ is

$$\prod_{j=1}^{n} d_{jj}^{n-j} \ .$$

Next we deal with $\mathbf{M}^*(n-1,n)$ with n equal to a positive integer $\geqq 2$.

Lemma 1.4. *Let $C \in \mathbf{M}^*(n-1,n)$. There exist $\gamma_j \in \Gamma_j (j = n-1, n)$ such that*

$$\gamma_{n-1} C \gamma_n^{-1} = (0, D) \text{ with } D \in \Delta_{n-1},$$

that is D is upper triangular.

Proof. The proof is a routine induction. Let $n = 1$. Let $C \in \mathbf{M}^*(1,2)$, so $C = (b, c)$ is a pair of integers, one of which is $\neq 0$. Let us write $b = db_1$, $c = dc_1$ where (b_1, c_1) is primitive, i.e. b_1, c_1 are relatively prime, and d is a non-zero integer. We can complete a first column ${}^t(-c_1, b_1)$ to an element of $\mathrm{SL}_2(\mathbf{Z})$ to complete the proof. The rest is done by induction, using blocks. A more detailed argument will be given in a similar situation, namely the proof of Lemma 1.6.

We consider the coset space $\mathcal{T}_{n-1}\backslash\mathbf{M}^*(n-1,n)$. Given a coset $\mathcal{T}_{n-1}C$, by Lemma 1.4 we can find a coset representative of the form $(0, D)\gamma$ with $\gamma \in \Gamma_n$. We use such representatives to describe a fibration of $\mathcal{T}_{n-1}\backslash\mathbf{M}^*(n-1,n)$ over $\mathcal{T}_n\backslash\Gamma_n$ as follows.

Lemma 1.5. *Let $\pi : \mathcal{T}_{n-1}\backslash\mathbf{M}^*(n-1,n) \to \mathcal{T}_n\backslash\Gamma_n$ be the map which to each coset $\mathcal{T}_{n-1}C$ with representative $(0, D)\gamma$ associates the coset $\mathcal{T}_n\gamma$. This map π is a surjection on $\mathcal{T}_n\backslash\Gamma_n$, and the fibers are $\mathcal{T}_{n-1}\backslash\Delta_{n-1}$.*

Proof. Implicit in the statement of the lemma is that the association π as described is well defined, i.e. independent of the chosen representative. Suppose

$$(0, D)\gamma = (0, D')\gamma' \quad \text{with } D, D' \in \Delta_{n-1} \ .$$

Then $(0, D) = (0, D')\gamma'\gamma^{-1}$. Let $\tau = \gamma'\gamma^{-1} \in \Gamma_n$. Then the above equation shows that actually τ is triangular, and so lies in $\mathcal{T}_n$. This is done by an inductive argument, letting $\tau = (t_{ij})$ and starting with showing that $t_{21} = 0, \ldots t_{n+1,n+1} = 0$, and then proceeding inductively to the right with the second column, third column, etc. Thus γ, γ' are in the same coset of $\mathcal{T}_n\backslash\Gamma_n$, showing the map is well defined. We note that the surjectivity of π is immediate.

As to the fibers, if $\tau \in \mathcal{T}_n$ and $D \in \Delta_{n-1}$, then $(0, D)\tau$ again has the form $(0, D')$ with $D' \in \Delta_{n-1}$. Thus by definition, the fiber above a coset $\mathcal{T}_n\gamma$ consists precisely of cosets

$$\mathcal{T}_{n-1}(0, D) \quad \text{with} \quad D \in \Delta_{n-1} \ ,$$

which proves the lemma.

In Lemma 1.5, we note that for each $\gamma \in \Gamma_n$ we have a bijection

$$\mathcal{T}_{n-1}\backslash\Delta_{n-1} \to \text{ fiber above } \mathcal{T}_n\gamma,$$

induced by the representative map $D \mapsto (0, D)\gamma$.

The arguments of Lemma 1.4 and 1.5 will be pushed further inductively. The rest of this section follows the careful and elegant exposition in Maass [Maa 71].

Since we operate with the discrete group Γ on the left, we have to reverse the notation used in Selberg, Maass, and other authors, for example Langlands [Lgl 76], Appendix 1. Let $Y \in \text{Pos}_n$. If $Y_j = \text{Sub}_j(Y)$ is the lower right $j \times j$ square submatrix of Y, then we can express Y_j in the form

$$Y_j = [(0, I_j)]Y = (0, I_j)Y \begin{pmatrix} 0 \\ I_j \end{pmatrix} \ , \tag{1}$$

where I_j is the unit $j \times j$ matrix as usual. Note the operation on the left, and the fact that 0 denotes the $j \times (n-1)$ zero matrix, so that $(0, I_j)$ is a $j \times n$ matrix. If $Y = T^tT$ with an upper triangular matrix T, then $Y_j = T_j^tT_j$, where T_j is the lower right $j \times j$ submatrix of T.

From a given Y we obtain a sequence $(Y_n, Y_{n-1}, \ldots, Y_1)$ by the operation indicated in (1), starting with $Y_n = Y$. We call this sequence the **Selberg sequence of** Y. Given $\gamma \in \Gamma_n$, we shall also form the Selberg sequence with $Y_n = [\gamma]Y$. In some sense (to be formalized below) this procedure gives rise to "primitive" sequences. It will be necessary to deal with non-primitive sequences, and thus we are led to make more general definitions as follows.

By an **integral chain** (more precisely n**-chain**) we mean a finite sequence

$$\mathcal{C} = (\gamma, C_{n-1}, \ldots, C_1) \quad \text{with} \quad \gamma \in \Gamma_n \quad \text{and} \quad C_j \in \mathbf{M}^*(j, j+1)$$

for $j = 1, \ldots, n-1$. Let $\mathcal{C}$ be such a chain. Let $\mathcal{C}' = (\gamma, C'_{n-1}, \ldots, C'_1)$ be another chain. We define $\mathcal{C}$ **equivalent** to $\mathcal{C}'$ if either one of the following conditions are satisfied.

EQU 1. There exist $\gamma_j \in \Gamma_j$ $(j = 1, \ldots, n)$ such that

$$\gamma' = \gamma_n\gamma \quad \text{and} \quad C'_j = \gamma_j C_j \gamma_{j+1}^{-1} \quad \text{for} \quad j = 1, \ldots, n-1 \ . \tag{2}$$

EQU 2. There exist $\gamma_j \in \Gamma_j$ $(j = 1, \dots, n-1)$ such that

$$C'_j \dots C'_{n-1}\gamma' = \gamma_j C_j \dots C_{n-1}\gamma \quad \text{for} \quad j = 1, \dots, n-1\,. \tag{3}$$

It's obvious that (2) implies (3). Conversely, suppose **EQU 2** and (3). We then let $\gamma_n = \gamma'\gamma^{-1}$, and it follows inductively that (2) is satisfied.

A sequence $(\gamma, C_{n-1}, \dots, C_1)$ will be said to be **triangularized** if we have that $C_j = (0, D_j)$ with $D_j \in \Delta_j$ for $j = 1, \dots, n-1$. Thus the first column of C_j is zero.

The next lemmas give special representatives for equivalence classes.

Lemma 1.6. *Let $C_j \in \mathbf{Z}^{j,j+1}$ $(j = 1, \dots, n-1)$ be integral matrices. There exist elements $\gamma_j \in \Gamma_j$ $(j = 1, \dots, n)$ such that for $j = 1, \dots, n-1$ we have*

$$\gamma_j C_j \gamma_{j+1}^{-1} = (0, T_1) = \begin{pmatrix} 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & * \end{pmatrix},$$

that is, the first column on the right is 0*, and the rest is upper triangular, with $T_j \in \mathrm{Tri}_j^+$. Thus every chain is equivalent to a triangularized one.*

Proof. Induction. For $n = 2$, the assertion is obvious, but we note how it illustrates the proof in general. We just have $C_1 = (b, c)$ with numbers b, c. We have $\gamma_1 = 1$ and we write $b = db_1, c = dc_1$ with (b_1, c_1) relatively prime. Then we can complete a first column ${}^t(-c_1, b_1)$ to an element of $\mathrm{SL}_2(\mathbf{Z})$ to complete the proof. Now by induction, suppose $n \geqq 3$. There exist $\beta_2, \dots, \beta_n$ with $\beta_j \in \Gamma_j$ such that the first column of $C_j\beta_{j+1}^{-1}$ is 0 for $j = 1, \dots, n-1$. Then $\beta_j C_j \beta_{j+1}^{-1}$ also has first column equal to 0, and this also holds for $j = 1$. Hence without loss of generality, we may assume that C_j has first column equal to 0, that is

$$C_j = \begin{pmatrix} 0 & * * * \\ \vdots & \\ 0 & H_{j-1} \end{pmatrix} \quad \text{with} \quad H_{j-1} \in \mathbf{Z}^{j-1,j}\,.$$

By induction, there exists $\eta_{j-1} \in \Gamma_{j-1}$ $(j = 2, \dots, n)$ such that

$$\eta_{j-1} H_{j-1} \eta_j^{-1} = \begin{pmatrix} 0 & * \dots * \\ \vdots & \vdots \ddots \vdots \\ 0 & 0 \dots * \end{pmatrix}$$

where the matrix on the right has first column 0, and the rest upper triangular. We let

$$\gamma_j = \begin{pmatrix} 1 & 0 \\ & \\ 0 & \eta_{j-1} \end{pmatrix} \quad \text{for} \quad j = 1, \dots, n\,.$$

Then $\gamma_j \in \Gamma_j$ and matrix multiplication shows that

$$\gamma_j C_j \gamma_{j+1}^{-1} = \begin{pmatrix} 1 & 0 \\ & \\ 0 & \eta_{j-1} \end{pmatrix} \begin{pmatrix} 0 & * \\ & \\ 0 & H_{j-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ & \\ 0 & \eta_j^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & * \\ \vdots & \\ 0 & \eta_{j-1} H_{j-1} \eta_j^{-1} \end{pmatrix} .$$

This last matrix has the desired form $(0, T_j)$, thereby concluding the proof.

The next lemma will give a refinement by prescribing representatives even further.

Lemma 1.7. *For each coset of* $\mathcal{T}_n \backslash \Gamma_n, \mathcal{T}_{n-1} \backslash \Delta_{n-1}, \ldots, \mathcal{T}_1 \backslash \Delta_1$ *fix a coset representative. To each sequence*

$$(\gamma, D_{n-1}, \ldots, D_1)$$

whose components are among the fixed representatives, associate the chain

$$(\gamma, (0, D_{n-1}), \ldots, (0, D_1)) .$$

Then this association gives a bijection from the set of representative sequences to equivalence classes of chains, i.e. every chain is equivalent to exactly one formed as above, with the fixed representatives.

Proof. By Lemma 1.6, every equivalence class has a representative

$$(\gamma', (0, D'_{n-1}), \ldots, (0, D'_1))$$

with $\gamma' \in \Gamma_n$ and $D'_j \in \Delta_j$ for $j = n-1, \ldots, 1$. There is one element $\tau_n \in \mathcal{T}_n$ such that $\tau_n \gamma'$ is the fixed representative of the coset $\mathcal{T}_n \gamma'$. Then we select the unique τ_{n-1} such that if we put

$$(0, D_{n-1}) = \tau_{n-1}(0, D'_{n-1})\tau_n^{-1}$$

then D_{n-1} is the fixed representative of the coset $\mathcal{T}_{n-1} D_{n-1}$. We can then continue by induction. This shows that the stated association maps bijectively on the families of equivalence classes and proves the lemma.

A chain $(\gamma, C_{n-1}, \ldots, C_1)$ is called **primitive** if all the matrices C_j, with $j = 1, \ldots, n-1$, are primitive, that is, C_j can be completed to an element of Γ_{j+1} by an additional row. The property of being primitive depends only on the equivalence class of the chain, namely if this property holds for $\mathcal{C}$ then it holds for every chain equivalent to $\mathcal{C}$. Furthermore, if $(\gamma, (0, D_{n-1}), \ldots, (0, D_1))$ is a triangularized representative of an equivalence class, then it is primitive if and only if each $D_j \in \Gamma_j$. In the primitive case, we can choose the fixed coset representatives of $\mathcal{T}_j\backslash\Gamma_j$ $(j = 1, \ldots, n-1)$ to be the unit matrices I_j. The primitive chains of the form

$$(\gamma, (0, I_{n-1}), \ldots, (0, I_1)) \quad \text{with} \quad \gamma \in \mathcal{T}_n\backslash\Gamma_n$$

will be called **normalized primitive chains**. Alternatively, one can select a fixed set of representatives $\{\gamma\}$ for $\mathcal{T}_n\backslash\Gamma_n$, and the primitive chains formed with such γ are in bijection with the equivalence classes of all primitive chains. Formally, we state the result:

Lemma 1.8. *The map $\gamma \mapsto$ chains of $(\gamma, (0, I_{n-1}), \ldots, (0, I_1))$ induces a bijection*

$$\mathcal{T}_n\backslash\Gamma_n \to \text{ primitive equivalence classes of chains .}$$

2 The ζ_Q Fudge Factor

It will be convenient to put out of the way certain straightforward computations giving rise to the fudge factor involving the Riemann zeta function, so here goes. For a positive integer j we shall use the representatives of $\mathcal{T}_j\backslash\Delta_j$ from Lemma 1.2. We let $n \geqq 2$.

Let $\{z_1, z_2, \ldots\}$ be a sequence of complex variables. Let $m \geqq n$. On Pos_m we define the **Selberg power function** $q_z^{(n)}$ by the formula

$$q_z^{(n)}(S) = \prod_{j=1}^{n} |\text{Sub}_j(S)|^{z_j} \quad \text{with} \quad S \in \text{Pos}_m \,.$$

In particular, we may work with $q_z^{(n-1)}$ on Pos_n, or also with $q_z^{(n)}$ on Pos_n, depending on circumstances. In any case, we see that we may also write

$$q_z^{(n)} = \mathbf{d}_n^{z_n} \ldots \mathbf{d}_1^{z_1},$$

where $\mathbf{d}_j$ is the **partial determinant character**, namely

$$\mathbf{d}_j(S) = |\text{Sub}_j(S)| \,.$$

In the next lemma, we consider both interpretations of q_z. We shall look at values

$$q_z^{(n)}([(0, D)]S)$$

where $D \in \Delta_n$ is triangular, and $S \in \text{Pos}_m$. We note that this value is independent of the coset $\mathcal{T}_n D$ of D with respect to the triangular matrices with ± 1 on the diagonal. We shall sum over such cosets. More precisely, let φ be a $\mathcal{T}_n$-invariant function on Pos_n. Under conditions of absolute convergence, we define the **Hecke-zeta operator** on Pos_m by the formula

$$\mathbf{HZ}_n(\varphi) = \sum_{D \in \mathcal{T}_n \backslash \Delta_n} \varphi \circ [(0, D)],$$

that is for $S \in \text{Pos}_m$,

$$\mathbf{HZ}_n(\varphi)(S) = \sum_{D \in \mathcal{T}_n \backslash \Delta_n} \varphi([(0, D)]S) .$$

We consider what is essentially an eigenfunction condition:

EF HZ. There exists $\lambda_{\mathbf{HZ}}(\varphi)$ such that for all $S \in \text{Pos}_m$ we have

$$\mathbf{HZ}_n(\varphi)(S) = \lambda_{\mathbf{HZ}}(\varphi)\varphi(\text{Sub}_n S) .$$

Implicit in this definition is the assumption that the series involved converges absolutely. The next lemma gives a first example.

For any positive integer n, we make the general definition of the **Riemann zeta fudge factor at level** n,

$$\Phi_{\mathbf{Q},n}(z) = \prod_{i=1}^{n} \zeta_{\mathbf{Q}}(2(z_i + \ldots + z_n) - (n - i)) .$$

Lemma 2.1. *Let $S \in \text{Pos}_m$. Then*

$$\sum_{D \in \mathcal{T}_n \backslash \Delta_n} q_{-z}^{(n)}([(0, D)]S) = \Phi_{\mathbf{Q},n}(z) q_{-z}^{(n)}(S) .$$

In other words,

$$\lambda_{\mathbf{HZ}}(q_{-z}^{(n)}) = \Phi_{\mathbf{Q},n}(z) .$$

This relationship holds for $\text{Re}(z_i + \ldots + z_n) > (n - i + 1)/2, i = 1, \ldots, n$, *which is the domain of absolute convergence of the Hecke-zeta operator on* $q_{-z}^{(n)}$.

Proof. Directly from the definition of $q_{-z}^{(n)}$, we find

$$\begin{aligned} q_{-z}^{(n)}([(0, D)]S) &= \prod_{i=1}^{n} |[(0, I_i)(0, D)]S)|^{-z_i} \\ &= \prod_{i=1}^{n} |\text{Sub}_i(D)|^{-2z_i} |\text{Sub}_i(S)|^{-z_i} \\ &= \prod_{i=1}^{n} (d_{n-i+1} \cdots d_n)^{-2z_i} q_{-z}^{(n)}(S) , \end{aligned} \tag{1}$$

where $d_1, \ldots, d_n$ are the diagonal elements of D. Next we take the sum over all integral non-singular triangular D, from the set of representatives of Lemma 1.2, so

$$D = \begin{pmatrix} d_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_n \end{pmatrix} .$$

The sum over D can be replaced by a sum

$$\sum_{d_1,\dots,d_n=1}^{\infty} \prod_{k=1}^{n} d_k^{k-1}$$

by Lemma 1.3. With the substitution $k = n - i + 1$, the factor of $q_{-z}^{(n)}(S)$ in (1) can thus be expressed as

$$\begin{aligned} &\sum_D \prod_{i=1}^{n} (d_{n-i+1} \dots d_n)^{-2z_i} \\ (2) \qquad &= \sum_{d_1=1}^{\infty} \dots \sum_{d_n=1}^{\infty} \prod_{k=1}^{n} d_k^{-2(z_{n-k+1}+\dots+z_n)+k-1} \\ &= \Phi_{\mathbf{Q},n}(z) \end{aligned}$$

after reverting to indexing by i instead of $n - k + 1$. This proves the lemma.

Next we deal with a similar but more involved situation, for which we make a general definition of the **Riemann zeta fudge factors**, namely

$$\Phi_{\mathbf{Q},j}(z) = \Phi_{\mathbf{Q},j}(z_1, \dots, z_j) = \prod_{i=1}^{j} \zeta_{\mathbf{Q}}(2(z_i + \dots + z_j) + j - i)$$

and

$$\Phi_{\mathbf{Q}}^{(n)}(z_1, \dots, z_n) = \prod_{j=1}^{n} \Phi_{\mathbf{Q},j}(z) .$$

These products will occur as factors in relations among Eisenstein series later. In the next lemma, we let $\{D_j\}$ range over the representatives of $\mathcal{T}_j \backslash \Delta_j$ ($j = 1, \dots n$) as given in Lemma 1.2. We let $d_{\nu\nu}^{(j)}$ denote the diagonal elements of D_j, with the indexing $j - k + 1 \leqq \nu \leqq j$, which will fit the indexing in the literature. The indexing also fits our viewing D_j as a lower right square submatrix.

Lemma 2.3.

$$\begin{aligned} \sum_{D_n} \dots \sum_{D_1} \prod_{k=1}^{n} \prod_{j=k}^{n} \prod_{\nu=j-k+1}^{j} (d_{\nu\nu}^{(j)})^{-2z_k} &= \Phi_{\mathbf{Q}}^{(n)}(z) \\ &= \prod_{1 \leqq i \leqq j \leqq n} \zeta_{\mathbf{Q}}(2(z_i + \dots + z_j) + j - i) . \end{aligned}$$

Proof. For a fixed index j, we consider the sum on the left over the representatives $\{D_j\}$. The products inside the sum which are indexed by this value j then can be written

$$\sum_{D_j} \prod_{k=1}^{j} \prod_{\nu=j-k+1}^{j} (d_{\nu\nu}^{(j)})^{-2z_k} \, .$$

This is precisely the term evaluated in (2), and seen to be equal to $Z_{\mathbf{Q},j}(z)$. Taking the product over $j = 1, \ldots, n$ concludes the proof of the lemma.

3 Eisenstein Series

Next we shall apply chains as in Sect. 1 to elements of Pos_n. Let $Y \in \mathrm{Pos}_n$. Let $\mathcal{C}$ be a chain, $\mathcal{C} = (\gamma, C_{n-1}, \ldots, C_1)$. For each $j = 1, \ldots, n-1$ define

$$\mathcal{C}_j(Y) = [C_j \cdots C_{n-1}\gamma]Y, \quad \mathcal{C}_n(Y) = [\gamma](Y) \, .$$

Thus $\mathcal{C}_j(Y) = [C_j]\mathcal{C}_{j+1}(Y)$ for $j = 1, \ldots, n-1$.

Let $z_1, \ldots, z_{n-1}$ be $n-1$ complex variables. We define the **Selberg power function** $q_{\mathcal{C}}^{(n-1)} = q_{\mathcal{C}}$ (depending on the chain) by the formula

$$q_{\mathcal{C},z}^{(n-1)}(Y) = |\mathcal{C}_{n-1}(Y)|^{z_{n-1}} \ldots |\mathcal{C}_1(Y)|^{z_1} \, .$$

One may also define $q_{\mathcal{C}}^{(n)}$ with one more variable, namely

$$q_{\mathcal{C},z}^{(n)}(Y) = \prod_{j=1}^{n} |\mathcal{C}_j(Y)|^{z_j} \, .$$

Let $\mathcal{C}$ be equivalent to $\mathcal{C}'$. Then by (2) or (3) of Sect. 1 we have

$$\mathcal{C}'_j(Y) = [\gamma_j]\mathcal{C}_j(Y)$$

with γ_j having determinant ± 1, so $|\mathcal{C}'_j(Y)| = |\mathcal{C}_j(Y)|$. It follows that

$$q_{\mathcal{C}',z}^{(n-1)}(Y) = q_{\mathcal{C},z}^{(n-1)}(Y) \, ;$$

in other words, $q_{\mathcal{C},z}^{(n-1)}$ depends only on the equivalent class of $\mathcal{C}$. Hence the power function can be determined by using the representatives given by Lemma 1.7.

As in Sect. 1, we let $\mathcal{T}_n$ be the group of integral upper triangular $n \times n$ matrices with ± 1 on the diagonal. We define the **Selberg Eisenstein series**

$$E_{\mathcal{T},n}^{(n-1)}(Y, z) = \sum_{\mathcal{C}} q_{\mathcal{C},-z}^{(n-1)}(Y) \, ,$$

where the sum is taken over all equivalence classes of chains. We define the **primitive Selberg Eisenstein series** by the same sum taken only over the primitive equivalence classes, that is

$$E^{\mathrm{pr}(n-1)}_{\mathcal{T},n}(Y,z) = \sum_{\mathcal{C} \text{ primitive}} q^{(n-1)}_{\mathcal{C},-z}(Y) \ .$$

Furthermore, from Lemma 1.8, we know that a complete system of representatives for equivalence classes of primitive chains is given by

$$(\gamma, (0, I_{n-1}), \ldots, (0, I_1)) \quad \text{with} \quad \gamma \in \mathcal{T}_n \backslash \Gamma_n \ .$$

If $\mathcal{C}$ has the representative starting with γ, then we may write

$$q_{\mathcal{C},z}(Y) = q_z([\gamma]Y) \ .$$

We may thus write the primitive Eisenstein series in the form

$$(1) \qquad E^{\mathrm{pr}(n-1)}_{\mathcal{T},n}(Y,z) = \sum_{\gamma \in \mathcal{T}_n \backslash \Gamma_n} q^{(n-1)}_{-z}([\gamma]Y) \ .$$

This is essentially the Eisenstein series we have defined previously, except that we are summing $\bmod \mathcal{T}_n$ instead of $\bmod \Gamma_U$. However, we note that for any character ρ, and $\tau \in \mathcal{T}_n$ we have the invariance property

$$\rho([\tau]Y) = \rho(Y) \quad \text{for all} \quad Y \in \mathrm{Pos}_n \ .$$

Since $(\mathcal{T}_n : \Gamma_U) = 2^n$, denoting the old Eisenstein series by $E^{\mathrm{pr}}_U(Y, q_{-z})$, we get

$$(2) \qquad E^{\mathrm{pr}}_U(Y,z) = 2^n E^{\mathrm{pr}}_{\mathcal{T}}(Y,z) \ .$$

We recall explicitly that

$$E^{\mathrm{pr}}_U(Y,\rho) = \mathrm{Tr}_{\Gamma_U \backslash \Gamma}(\rho)(Y) = \sum_{\gamma \in \Gamma_U \backslash \Gamma} \rho([\gamma]Y) \ .$$

To make easier the formal manipulations with non-primitive series, we list some relations. For given $k = 1, \ldots, n-1$ we consider the product

$$(0, D_k) \ldots (0, D_{n-1}) = (0^{k,n-k}, T_k)$$

where $(\gamma, D_{n-1}, \ldots, D_1)$ is a chain equivalent to $\mathcal{C}$ and $D_j \in \Delta_j$. Thus T_k is a triangular $k \times k$ matrix. To determine more explicitly the Eisenstein series, we may assume without loss of generality that

$$\mathcal{C} = (\gamma, (0, D_{n-1}), \ldots, (0, D_1)) \ .$$

Then

(3) $$\mathcal{C}_k(Y) = [(0, T_k)\gamma]Y = [T_k][(0, I_k)\gamma]Y$$

and therefore

(4) $$|\mathcal{C}_k(Y)| = |T_k|^2 |\mathrm{Sub}_k([\gamma]Y)| \ .$$

Let $t_{\nu\nu}^{(k)}$ denote the diagonal elements of T_k. Then of course

(5) $$|T_k|^2 = \prod_{\nu=1}^{k} (t_{\nu\nu}^{(k)})^2 \ .$$

These products decomposition allow us to give a product expression for E in terms of E^{pr} and the Riemann zeta function via the formula

(6) $$\begin{aligned} q_{\mathcal{C},-z}^{(n-1)}(Y) &= \prod_{k=1}^{n-1} |\mathcal{C}_k(Y)|^{-z_k} \\ &= \prod_{k=1}^{n-1} |([\gamma]Y)_k|^{-z_k} \prod_{j=k}^{n-1} \prod_{\nu=j-k+1}^{j} (d_{\nu\nu}^{(j)})^{-2z_k} \ , \end{aligned}$$

here $d_{\nu\nu}^{(j)}$ are the diagonal elements of D_j.

Theorem 3.1. *The Eisenstein series $E_{U,n}^{(n-1)}(Y, z)$ converges absolutely for* $\mathrm{Re}(z_j) > 1$ $(j = 1, \ldots, n-1)$ *and satisfies the relation*

$$E_{U,n}^{(n-1)}(Y, z) = \Phi_{\mathbf{Q}}^{(n-1)}(z_1, \ldots, z_{n-1}) E_U^{\mathrm{pr}(n-1)}(Y, z) \ .$$

Proof. Both the relation and the convergence follow from (6) and Lemma 2.3 applied to $n-1$ instead of n, and Theorem 2.2 of Chap. 7.

Next, we have identities concerning the behavior of the Eisenstein series under the star involution. Recall that for any function φ on Pos_n, we define

$$\varphi^*(Y) = \varphi([\omega]Y^{-1}) = \varphi(\omega Y^{-1}\omega) \ .$$

Proposition 3.2. *Let φ be any U-invariant function such that its $\Gamma_U\backslash\Gamma$-trace converges absolutely. Then*

$$(\mathrm{Tr}_{\Gamma_U\backslash\Gamma}\varphi)(Y^{-1}) = (\mathrm{Tr}_{\Gamma_U\backslash\Gamma}(\varphi^*)(Y) \ .$$

In particular, if ρ is a left character, then

$$E_U^{\mathrm{pr}}(Y^{-1}, \rho) = E_U^{\mathrm{pr}}(Y, \rho^*) \ .$$

If $\{\gamma\}$ is a family of coset representatives of $\Gamma_U\backslash\Gamma$, then $\{\omega\, {}^t\gamma^{-1}\}$ is also such a family. Similarly for representatives of $\mathcal{T}\backslash\Gamma$.

Proof. As to the second statement, write $\Gamma = \bigcup \Gamma_U \gamma$. Let $\Gamma_{\bar{U}}$ be the lower triangular subgroup. Then

$$\begin{aligned}
\Gamma = \bigcup_{\gamma} {}^t\gamma \Gamma_{\bar{U}} &= \bigcup \Gamma_{\bar{U}} \; {}^t\gamma^{-1} && \text{(taking the inverse)} \\
&= \bigcup \omega \Gamma_{\bar{U}} \omega \omega \; {}^t\gamma^{-1} && \text{(because } \Gamma = \omega\Gamma \text{ and } \omega^2 = I) \\
&= \bigcup \Gamma_U \omega {}^t\gamma^{-1} && \text{(because } \omega\Gamma_{\bar{U}}\omega = \Gamma_U) \;.
\end{aligned}$$

This proves the second statement. Then the first formula comes out, namely:

$$\begin{aligned}
\mathrm{Tr}_{\Gamma_U \backslash \Gamma} \varphi(Y^{-1}) &= \sum_{\gamma \in \Gamma_U \backslash \Gamma} \varphi([\gamma] Y^{-1}) \\
&= \sum_{\gamma} \varphi(\gamma Y^{-1} \; {}^t\gamma) \\
&= \sum_{\gamma} \varphi^*(\omega({}^t\gamma^{-1} Y \gamma^{-1})\omega) \\
&= \mathrm{Tr}_{\Gamma_U \backslash \Gamma} \varphi^*(Y)
\end{aligned}$$

by the preceding result, thus proving the proposition.

The next two lemmas deal with similar identities with sums taken over cosets of matrices modulo the triangular group.

Lemma 3.3. *Let φ be a $\mathcal{T}_n$-invariant function such that the following sums are absolutely convergent, i.e. a left character on* Pos_n. *Let $S \in \mathrm{Pos}_{n+1}$. Then*

$$\sum_{A \in \mathbf{M}^*(n+1,n)/\mathcal{T}_n} \varphi^*((S[A])^{-1}) = \sum_{C \in \mathcal{T}_n \backslash \mathbf{M}^*(n,n+1)} \varphi([C]S) \;.$$

Proof. Inserting an ω inside the left side and using the definition of φ^*, together with $\varphi^{**} = \varphi$, we see that the left side is equal to

$$\sum_{A \in \mathbf{M}^*(n+1,n)/\mathcal{T}_n} \varphi(S[A][\omega]) = \sum_{A} \varphi(S[A\omega]) \;.$$

By definition, $\mathbf{M}^*(n+1,n) = \bigcup_A A\mathcal{T}_n$, with a family $\{A\}$ of coset representatives. Since $\mathbf{M}^*(n+1,n) = \mathbf{M}^*(n+1,n)\omega$, we also have

$$\bigcup_A A\mathcal{T}_n = \bigcup A\omega\omega\mathcal{T}_n\omega = \bigcup_{A\omega} A\omega\mathcal{T}_n^-$$

where $\mathcal{T}_n^-$ is the lower integral triangular group. Thus the family $\{A\omega\}$ is a family of coset representatives for $\mathbf{M}^*(n+1,n)/\mathcal{T}_n^-$. Writing

$$S[A\omega] = [\omega^t A]S,$$

we see that we can sum over the transposed matrices, and thus that the desired sum is equal to

$$\sum_{C \in \mathcal{T}_n \backslash \mathbf{M}^*(n,n+1)} \varphi([C]S) \ ,$$

which proves the lemma.

Instead of taking $\mathbf{M}^*(n+1,n)/\mathcal{T}_n$ we could also take $\mathbf{M}^*(n+1,n)/\Gamma_U$. Since $\mathcal{T}_n/\Gamma_U$ has order 2^n, we see that we have a relation similar to (2), namely

$$\text{(7)} \qquad \sum_{\Gamma_U \backslash \mathbf{M}^*(n,n+1)} \varphi^*([C]S) = 2^n \sum_{\mathcal{T}_n \backslash \mathbf{M}^*(n,n+1)} \varphi^*([C]S) \ .$$

Normalizing the series by taking sums mod $\mathcal{T}_n$ or mod Γ_U only introduces the simple factor 2^n each time.

We shall now develop further the series on the right in Lemma 3.3, by using the eigenvalue property **EF HZ** stated in Sect. 2.

Lemma 3.4. *Suppose that φ is $\mathcal{T}_n U$-invariant on* Pos_n*, and satisfies condition* **EF HZ** *(eigenfunction of Hecke-zeta operator). Then on* Pos_{n+1}*,*

$$\sum_{C \in \mathcal{T}_n \backslash \mathbf{M}^*(n,n+1)} \varphi \circ [C] = \lambda_{\mathbf{HZ}}(\varphi) \mathrm{Tr}_{\mathcal{T}_{n+1} \backslash \Gamma_{n+1}} (\varphi \circ \mathrm{Sub}_n) \ .$$

Proof. By the invariance assumption on φ, we can use the fibration of Lemma 1.5, and write the sum on the left evaluated at $S \in \mathrm{Pos}_{n+1}$ as

$$\sum_{\gamma \in \mathcal{T}_{n+1} \backslash \Gamma_{n+1}} \ \sum_{D \in \mathcal{T}_n \backslash \Delta_n} \varphi([(0,D)][\gamma]S) \ .$$

Then the inner sum is just the Hecke operator of φ, when evaluated at $\mathrm{Sub}_n[\gamma]S$. The result then falls out.

In particular, we may apply the lemma to the case when $\varphi = q_{-z}^{(n)}$, and we obtain:

Corollary 3.5. *Let $S \in \mathrm{Pos}_{n+1}$. Then*

$$\sum_{C \in \mathcal{T}_n \backslash \mathbf{M}^*(n,n+1)} q_{-z}^{(n)}([C]S) = \Phi_{\mathbf{Q},n}(z) E_{\mathcal{T},n+1}^{\mathrm{pr}(n)}(S, q_{-z}^{(n)}) \ .$$

Proof. Special case of Lemma 3.4, after applying Lemma 2.1 which determines the eigenvalue of the Hecke-zeta operator.

4 Adjointness and the $\Gamma_U \backslash \Gamma$-trace

We shall use differential operators introduced in Chap. 6. First, we observe that for $c > 0$, $Y \in \mathrm{Pos}_n$, $B \in \mathrm{Sym}_n$ we have by direct computation

$$(1) \qquad \left| \frac{\partial}{\partial Y} \right| e^{-c\mathrm{tr}(BY)} = (-c)^n \; |B| \; e^{-c\mathrm{tr}(BY)} \; .$$

In particular, the above expression vanishes if B is singular. In the applications, B will be semipositive, and the effect of applying $|\partial / \partial Y|$ will therefore be to eliminate such a term when B has rank $< n$.

As in Chap. 6 let the (first and second) **regularizing invariant differential operators** be

$$(2) \qquad Q = Q_n = |Y| \; \left| \frac{\partial}{\partial Y} \right| \quad \text{and} \quad \mathbf{D} = \mathbf{D}_n = |Y|^{-k} \tilde{Q}_n |Y|^{-k} Q_n \; .$$

Throughout we put $k = (n+1)/2$ and $\mathbf{D} = \mathbf{D}_n$ if we don't need to mention n. We recall that

$$(3) \qquad \tilde{\mathbf{D}}_n = |Y|^k \mathbf{D}_n |Y|^{-k} = \tilde{Q} |Y|^k Q |Y|^{-k} \; .$$

For $S \in \mathrm{Pos}_{n+1}$ we let

$$\theta(S, Y) = \sum_A e^{-\pi \mathrm{tr}(S[A]Y)}$$

where the sum is taken over $A \in \mathbf{Z}^{n+1,n}$. This is the standard theta series. We can differentiate term by term. By (1) and the subsequent remark, we note that

$$\mathbf{D}_Y \theta(S, Y) = \sum_{\mathrm{rk}(A)=n} \mathbf{D}_Y e^{-\pi (S[A]Y)} = \sum_{\mathrm{rk}(A)=n} \beta_{A,S}(Y) e^{-\pi \mathrm{tr}(S[A]Y)} \; ,$$

where $\beta_{A,S}$ (S being now fixed) is a function of Y with only polynomial growth, and so not affecting the convergence of the series. Although its coefficients are complicated, there is one simplifying effect to having applied the differential operator $\mathbf{D}$, namely we sum only over the matrices A of rank n. Thus we abbreviate as before, and for this section, we let:

$$\mathbf{M}^* = \mathbf{M}^{*(n+1,n)} = \text{ subset of elements in } \mathbf{Z}^{(n+1) \times n} \text{ of rank } n \; .$$

Then the sum expressing $\mathbf{D}_Y \theta(S, Y)$ is taken over $A \in \mathbf{M}^*$.

Note that both θ and $\mathbf{D}\theta$ are functions of two variables, and thus will be viewed as kernels, which induce integral operators by convolution, provided they are applied to functions for which the convolution integral is absolutely convergent.

We recall the **functional equation** for θ,

$$\theta(S^{-1}, Y^{-1}) = |S|^{n/2}|Y|^{(n+1)/2}\theta(S, Y) . \tag{4}$$

From (3), we then see that $\mathbf{D}\theta$ *satisfies the same functional equation*, that is

$$(\mathbf{D}\theta)(S^{-1}, Y^{-1}) = |S|^{n/2}|Y|^{(n+1)/2}(\mathbf{D}\theta)(S, Y) . \tag{5}$$

Here we have used the special value $k = (n+1)/2$.

We shall now derive an adjoint relation in the present context. For a U-invariant function φ on Pos_n, we recall the $\Gamma_U\backslash\Gamma$**-trace**, defined by

$$\mathrm{Tr}_{\Gamma_U\backslash\Gamma}(\varphi)(Y) = \sum_{\gamma\in\Gamma_U\backslash\Gamma} \varphi([\gamma]Y) .$$

For functions φ such that the $\Gamma_U\backslash\Gamma$-trace and the following integral are absolutely convergent, we can form the convolution on $\Gamma_n\backslash\mathrm{Pos}_n$:

$$(\mathbf{D}\theta * \mathrm{Tr}_{\Gamma_U\backslash\Gamma}\varphi)(S) = \int\limits_{\Gamma_n\backslash\mathrm{Pos}_n} (\mathbf{D}_Y\theta)(S, Y)\mathrm{Tr}_{\Gamma_U\backslash\Gamma}(\varphi)(Y)d\mu_n(Y) .$$

We abbreviate as before

$$\mathcal{P} = \mathrm{Pos}_n, \qquad \Gamma = \Gamma_n$$

to make certain computations formally clearer.

Lemma 4.1. *For an arbitrary U-invariant function φ on Pos_n insuring absolute convergence of the series and integral, we have with $k = (n+1)/2$:*

$$\begin{aligned}(\mathbf{D}\theta &* \mathrm{Tr}_{\Gamma_U\backslash\Gamma}\varphi)(S) \\ &= \sum_{A\in\mathbf{M}^*/\Gamma_U} 2(-1)^n|\pi S[A]| \int\limits_{\mathcal{P}} e^{-\pi\mathrm{tr}(S[A]Y)}|Y|^{k+1}Q(\varphi\mathbf{d}^{-k})(Y)d\mu(Y) .\end{aligned}$$

Thus the convolution on the left is a sum of gamma transforms.

Proof. The proof is similar to those encountered before. We have:

$$\begin{aligned}&\int\limits_{\Gamma\backslash\mathcal{P}} (\mathbf{D}_y\theta)(S, Y)\mathrm{Tr}_{\Gamma_U\backslash\Gamma}\varphi(Y)d\mu(Y) \\ &\quad = \int\limits_{\Gamma\backslash\mathcal{P}} \sum_{A\in\mathbf{M}^*} \mathbf{D}_Y e^{-\pi\mathrm{tr}(S[A]Y)}\mathrm{Tr}_{\Gamma_U\backslash\Gamma}\varphi(Y)d\mu(Y) \\ &\quad = \sum_{A\in\mathbf{M}^*/\Gamma} \int\limits_{\Gamma\backslash\mathcal{P}} \sum_{\gamma\in\Gamma} \mathbf{D}_Y e^{-\pi\mathrm{tr}(S[A\gamma]Y)}\mathrm{Tr}_{\Gamma_U\backslash\Gamma}\varphi(Y)d\mu(Y)\end{aligned}$$

$$= \sum_{A\in \mathbf{M}^*/\Gamma} 2 \int_{\mathcal{P}} \mathbf{D}_Y e^{-\pi \mathrm{tr}(S[A]Y)} \sum_{\gamma\in\Gamma_U\backslash\Gamma} \varphi([\gamma]Y) d\mu(Y)$$

$$= \sum_{A\in \mathbf{M}^*/\Gamma} \sum_{\gamma\in\Gamma_U\backslash\Gamma}$$

$$2 \int_{\mathcal{P}} |Y|^{-k} \tilde{Q}_Y (|Y|^{k+1} \left| \frac{\partial}{\partial Y} \right| e^{-\pi \mathrm{tr}(S[A]Y)}) \varphi([\gamma](Y)) d\mu(Y)$$

$$= \sum_{A\in \mathbf{M}^*/\Gamma} \sum_{\gamma\in\Gamma_U\backslash\Gamma}$$

$$2(-1)^n |\pi S[A]| \int_{\mathcal{P}} e^{-\pi \mathrm{tr}(S[A]Y)} |Y|^{k+1} Q_Y (|Y|^{-k} \varphi([\gamma]Y)) d\mu(Y) ,$$

using formula (2), and then transposing $\tilde{Q}_Y$ from the exponential term to the $\varphi \circ [\gamma](Y)$ term. Now we make the translation $Y \mapsto [\gamma^{-1}]Y$ in the integral over $\mathcal{P}$. Under this change, $\Gamma_U\backslash\Gamma \mapsto \Gamma/\Gamma_U$, and the expression is equal to

$$= \sum^{n}_{A\in \mathbf{M}^*/\Gamma} \sum_{\gamma^{-1}\in\Gamma/\Gamma_U} 2(-1)^n |\pi S[A]|$$

$$\int_{\mathcal{P}} e^{-\pi \mathrm{tr}(S[A\gamma^{-1}]Y)} |Y|^{k+1} Q_Y (|Y|^{-k} \varphi(Y)) d\mu(Y) .$$

The two sums over Γ/Γ_U and over $\mathbf{M}^*/\Gamma$ can be combined into a single sum with $A \in \mathbf{M}^*/\Gamma_U$, which yields the formula proving the lemma.

Looking at the integral expression on the right in the lemma, we see at once that it is a gamma transform. Furthermore, if $\varphi\mathbf{d}^{-k}$ is an eigenfunction of Q_n, then the integral can be further simplified, and this condition is satisfied in the case of immediate interest when φ is a character. However, it continues to be clearer to extract precisely what is being used of a more general function φ, which amounts to eigenfunction properties in addition to $\mathcal{T}_n U$-invariance and the absolute convergence of the series and integral involved. Thus we list these properties as follows.

EF Q. The function $\varphi\mathbf{d}^{-(n+1)/2}$ is an eigenfunction of Q_n.

EF Γ. The function $\varphi\mathbf{d}$ is an eigenfunction of the gamma transform, it being assumed that the integral defining this transform converges absolutely.

We use λ to denote eigenvalues. Specifically, let D be an invariant differential operator. Let φ be a D-eigenfunction. We let $\lambda_D(\varphi)$ be the eigenvalue so that

$$D\varphi = \lambda_D(\varphi)\varphi .$$

Similarly, we have the integral gamma operator, and for an eigenfunction φ, we let

$$\lambda_{\mathbf{\Gamma}}(\varphi) = \mathbf{\Gamma}_n(\varphi) \quad \text{so that} \quad \mathbf{\Gamma}\#\varphi = \lambda_{\mathbf{\Gamma}}(\varphi)\varphi\ .$$

In addition, we *define*

$$\Lambda_n(\varphi) = (-1)^n \lambda_Q(\varphi\mathbf{d}^{-(n+1)/2})\lambda_\Gamma(\varphi\mathbf{d})\ .$$

Theorem 4.2. *Assume that φ is $\mathcal{T}_n U$-invariant and satisfies the two properties* **EF Q** *and* **EF Γ**. *Then for $S \in \mathrm{Pos}_{n+1}$, under conditions of absolute convergence,*

$$(\mathbf{D}\theta * \mathrm{Tr}_{\Gamma_U\backslash\Gamma}(\varphi)(S) = 2\Lambda_n(\varphi) \sum_{A\in\mathbf{M}^*(n+1,n)/\Gamma_U} \varphi((\pi S[A])^{-1})\ .$$

Proof. By using the eigenfunction assumptions on the expression being summed on the right side of the equality in Lemma 4.1, and again if we set $k = (n+1)/2$, we obtain:

$$\begin{aligned}
&|\pi S[A]| \int_{\mathcal{P}} e^{-\pi \mathrm{tr}(S[A]Y)}|Y|^{k+1}\lambda_Q(\varphi\mathbf{d}^{-k})(\varphi\mathbf{d}^{-k})(Y)d\mu(Y)\\
&\qquad = \lambda_Q(\varphi\mathbf{d}^{-k})|\pi S[A]| \int_{\mathcal{P}} e^{-\pi\mathrm{tr}(S[A]Y)}(\varphi\mathbf{d})(Y)d\mu(Y)\\
&\qquad = \lambda_Q(\varphi\mathbf{d}^{-k})|\pi S[A]|\lambda_\Gamma(\varphi\mathbf{d})(\varphi\mathbf{d})((\pi S[A])^{-1})
\end{aligned}$$

by definition of the gamma transform and an eigenvalue, cf. Chap. 3, Proposition 2.2,

$$= \lambda_Q(\varphi\mathbf{d}^{-k})\lambda_\Gamma(\varphi\mathbf{d})\varphi((\pi S[A])^{-1})$$

because the determinant cancels. This proves the theorem.

Theorem 4.3. *Let φ be $\mathcal{T}_n U$-invariant, satisfying* **EF Q**, **EF Γ**, *and* **EF HZ**. *Then for $S \in \mathrm{Pos}_{n+1}$, when the series and integral are absolutely convergent,*

$$(\mathbf{D}\theta * \mathrm{Tr}_{\Gamma_U\backslash\Gamma}(\varphi^*)(S) = \Lambda_n(\varphi^*)\lambda_{\mathbf{HZ}}(\varphi)\mathrm{Tr}_{\Gamma_{U_{n+1}}\backslash\Gamma_{n+1}}(\varphi \circ Sub_n)(\pi S)\ .$$

Proof. We apply Theorem 4.2 to φ^* instead of φ. The sum in Theorem 4.2 can be further simplified as follows:

$$\begin{aligned}
&\sum_{A\in\mathbf{M}^*(n+1,n)/\Gamma_U} \varphi^*((\pi S[A])^{-1})\\
&\quad = 2^n \sum_{A\in\mathbf{M}^*(n+1,n)/\mathcal{T}_n} \varphi^*((\pi S[A])^{-1})\\
&\quad = 2^n\lambda_{\mathbf{HZ}}(\varphi)E^{\mathrm{pr}}_{\mathcal{T}}(\pi S, \varphi\circ\mathrm{Sub}_n) \quad \text{by Lemmas 3.3 and 3.4.}
\end{aligned}$$

The Eisenstein series here is on Pos_{n+1}, and going back to $\Gamma_{U_{n+1}}$ instead of $\mathcal{T}_{n+1}$ introduces the factor $1/2^{n+1}$, which multiplied by 2^n leaves $1/2$. This $1/2$ cancels the factor 2 occurring in Theorem 4.2. The relationship asserted in the theorem then falls out, thus concluding the proof.

Corollary 4.4. *Let* $\mathbf{D} = \mathbf{D}_n$ *be the invariant differential operator defined at the beginning of the section. Let* φ *be homogeneous of degree* w*, for instance* φ *is a character. Then for* $S \in \mathrm{Pos}_{n+1}$,

$$(\mathbf{D}\theta * \mathrm{Tr}_{\Gamma_U \backslash \Gamma_n} \varphi^*)(S) = \pi^w \Lambda_n(\varphi^*) \lambda_{\mathbf{HZ}}(\varphi) \mathrm{Tr}_{\Gamma_{U_{n+1}} \backslash \Gamma_{n+1}} (\varphi \circ Sub_n)(S) .$$

Proof. We just pull out the homogeneity factor from inside the expression in Theorem 4.3.

Remark. Remark Immediately from the definitions, one sees that for the Selberg power character, we have

$$\deg q_z^{(n)} = w_n(z) = \sum_{j=1}^{n} j z_j .$$

This character may be viewed as a character on Pos_m for any $m \geqq n$. The degree is the same in all cases. For application to the Eisenstein series, we use of course $q_{-z}^{(n)}$, which has degree $-w_n(z) = w_n(-z)$. Actually, in the next section we shall change variables, and get another expression for the degree in terms of the new variables.

The inductive formula of this section stems from the ideas presented by Maass [Maa 71], pp. 268–272, but we have seen how it is valid for much more general functions φ besides characters. Maass works only with the special characters coming from the Selberg power function, and normalizes these characters with s-variables. We carry out this normalization in the next section, as a preliminary to Maass' proof of the functional equation.

5 Changing to the $(s_1, \ldots, s_n)$-variables

We recall the Selberg power function of Chap. 3, Sect. 1, expressed in terms of two sets of complex variables

$$z = (z_1, \ldots, z_{n-1}) \text{ and } s = (s_1, \ldots, s_n) ,$$

namely

(1) $$|Y|^{s_n + (n-1)/4} q_{-z}^{(n-1)}(Y) = h_s(Y) = \prod_{i=1}^{n} (t_{n-i+1})^{2s_i + i - (n+1)/2} ,$$

where

$$z_j = s_{j+1} - s_j + \frac{1}{2} \quad \text{for } j = 1, \ldots, n-1,$$

or also

(2) $$q_{-z}^{(n-1)}(Y) = |Y|^{-s_n - (n-1)/4} h_s(Y) .$$

To determine the degree of homogeneity of h_s, we note that

$$Y \mapsto cY (c > 0) \text{ corresponds to } t \mapsto c^{1/2} t \, .$$

Then we find immediately:

$$\text{(3)} \qquad \deg h_s = \sum_{i=1}^{n} s_i \quad \text{and} \quad \deg h_s^* = -\deg h_s \, .$$

Throughout this section, we fix the notation. We let $\Gamma = \Gamma_n$, *and*

$$\begin{aligned} \zeta^{\mathrm{pr}}(Y, s) = E_U^{\mathrm{pr}}(Y, q_{-z}^{(n-1)}) &= \mathrm{Tr}_{\Gamma_U \backslash \Gamma} q_{-z}^{(n-1)}(Y) \\ &= |Y|^{-s_n - (n-1)/4} \, \mathrm{Tr}_{\Gamma_U \backslash \Gamma} h_s(Y) \, . \end{aligned}$$

Proposition 5.1. *We have in the appropriate domain (see the remark below):*

$$\zeta^{\mathrm{pr}}(Y^{-1}, s) = |Y|^{s_n - s_1 + (n-1)/2} \zeta^{\mathrm{pr}}(Y, s^*) \, ,$$

where $s^* = (-s_n, \ldots, -s_1)$, *so* $s_j^* = -s_{n-j+1}$.

Proof. We have

$$\begin{aligned} \zeta^{\mathrm{pr}}(Y^{-1}, s) &= |Y|^{s_n + (n-1)/4} \mathrm{Tr}_{\Gamma_U \backslash \Gamma} h_s(Y^{-1}) && \text{by (2)} \\ &= |Y|^{s_n + (n-1)/4} \mathrm{Tr}_{\Gamma_U \backslash \Gamma} h_s^*(Y) && \text{by Prop. 3.2} \\ &= |Y|^{s_n + (n-1)/4} \mathrm{Tr}_{\Gamma_U \backslash \Gamma} h_{s^*}(Y) && \text{by Chap. 3, Prop. 1.7} \\ &= |Y|^{s_n - s_1 + (n-1)/2} \zeta^{\mathrm{pr}}(Y, s^*) && \text{by (2)} \end{aligned}$$

because $\mathrm{Tr}_{\Gamma_U \backslash \Gamma} h_{s^*}(Y) = |Y|^{s_n^* + (n-1)/4} \zeta^{\mathrm{pr}}(Y, s^*)$ by (2). This concludes the proof.

Remark. The domain of absolute convergence of the Eisenstein series $E_U^{\mathrm{pr}}(Y, q_{-z}^{(n-1)})$ was proved to be $\mathrm{Re}(z_j) > 1$ for $j = 1, \ldots, n-1$, that is

$$\mathrm{Re}\left(s_{j+1} - s_j + \frac{1}{2}\right) > 1 \quad \text{for} \quad j = 1, \ldots, n-1 \, .$$

From the relation $s_k^* = -s_{n-k+1}$ we see that

$$s_{k+1}^* - s_k^* + \frac{1}{2} = s_j - s_{j-1} + \frac{1}{2} \quad \text{with} \quad j = n-k+1 \, .$$

Thus the domains of convergence in terms of the s^* and s variables are "the same" half planes.

We shall meet a systematic pattern as follows. Let $\psi = \psi(u)$ be a function of one variable. For $n \geqq 2$, we define

$$\psi_n(s) = \psi_n(s_1, \ldots, s_n) = \prod_{i=1}^{n-1} \psi(s_n - s_i + 1/2)$$

$$\psi^{(n)}(s) = \prod_{j=2}^{n} \psi_j(s_1, \ldots, s_j)\ .$$

We note the completely general fact:

Lemma 5.2. $\psi^{(n)}(s^*) = \psi^{(n)}(s)$.

This relation is independent of the function ψ, and is trivially verified from the definition of $\psi^{(n)}$. It will apply to three important special cases below. We start with the function $\psi(u) = \zeta_{\mathbf{Q}}(2u)$, where $\zeta_{\mathbf{Q}}$ is the Riemann zeta function. Then we use a special letter $Z_{\mathbf{Q}}$ and define

$$Z_{\mathbf{Q},n}(s) = \prod_{i=1}^{n-1} \zeta_{\mathbf{Q}}(2(s_n - s_i + 1/2))$$

$$Z_{\mathbf{Q}}^{(n)}(s) = \prod_{1 \leqq i < j \leqq n} \zeta_{\mathbf{Q}}(2(s_j - s_i + 1/2)).$$

Lemma 5.3. *With* $\Phi_{\mathbf{Q},n-1}$ *as in Lemma 2.1, we have*

$$\Phi_{\mathbf{Q},n-1}(z_1, \ldots, z_{n-1}) = Z_{\mathbf{Q},n}(s_1, \ldots, s_n)\ .$$

Proof. By definition,

$$\Phi_{\mathbf{Q},n-1}(z) = \prod_{i=1}^{n-1} \zeta_{\mathbf{Q}}(2(z_i + \ldots + z_{n-1}) - (n - i - 1))\ .$$

With the s-variables, we get a cancellation, namely

(4) $$z_i + \ldots + z_{n-1} = s_n - s_i + \frac{n-i}{2}\ .$$

This proves the lemma.

The non-primitive Eisenstein series $E_U^{(n-1)}(Y, z)$ is defined to be the product of the primitive Eisenstein series times $\Phi_{\mathbf{Q}}^{(n-1)}(z)$. Hence from the transfer to the s-variables in Lemma 5.3 we have

(5) $$\zeta(Y, s) = E_U^{(n-1)}(Y, z) = Z_{\mathbf{Q}}^{(n)}(s)\zeta^{\mathrm{pr}}(Y, s)\ .$$

Since $Z_{\mathbf{Q}}^{(n)}(s^*) = Z_{\mathbf{Q}}^{(n)}(s)$ by Lemma 5.2, it follows that Proposition 5.1 is valid if we replace the primitive Eisenstein series $\zeta^{\mathrm{pr}}(Y, s)$ by $\zeta(Y, s)$.

In connection with using Pos_{n+1} via Theorem 4.3 and Corollary 4.4, it is natural to consider $q_{-z}^{(n)}$ as well as $q_{-z}^{(n-1)}$.

Lemma 5.4. *Put* $z_n = s_{n+1} - s_n + 1/2$, *and let* $\varphi_{s,s_{n+1}}$ *be the character on* Pos_n *defined by*

$$\varphi_{s,s_{n+1}}(Y) = |Y|^{-s_{n+1}-(n+1)/4} h_s(Y) .$$

In other words, $\varphi_{s,s_{n+1}} = \mathbf{d}^{-s_{n+1}-(n+1)/4} h_s$. *Then on* Pos_n,

$$q_{-z}^{(n)} = \varphi_{s,s_{n+1}} .$$

Proof. By definition,

$$q_{-z}^{(n)}(Y) = |Y|^{-z_n} q_{-z}^{(n-1)}(Y) .$$

Substituting $z_n = s_{n+1} - s_n + \frac{1}{2}$ and using (2) yields the desired relation.

The Hecke-zeta eigenvalue is given by

$$\lambda_{\mathbf{HZ}}(\varphi_{s,s_{n+1}}) = Z_{\mathbf{Q},n+1}(s_1, \ldots, s_{n+1}) = Z_{\mathbf{Q},n+1}(s, s_{n+1}) . \tag{6}$$

This is just the formulation of Lemma 2.1 in the (s, s_{n+1}) variables. Furthermore,

$$w_n(z) = \deg \varphi_{s,s_{n+1}} = \sum_{i=1}^{n} \left(s_i - s_{n+1} - \frac{n+1}{4} \right) . \tag{7}$$

This is immediate from (3) and the homogeneity degree of the determinant.

We define various elementary functions from which we build others, and relate them to eigenvalues found in the preceding section. We let

$$g(u) = \pi^{-u}\Gamma(u) \quad \text{and} \quad F(u) = u(1-u)g(u) .$$

These are standard fudge factors in **one** variable u. Following the previous general pattern, we define

$$g_n(s) = g_n(s_1, \ldots, s_n) = \prod_{i=1}^{n-1} g(s_n - s_i + 1/2)$$

$$F_n(s) = F_n(s_1, \ldots, s_n) = \prod_{i=1}^{n-1} F(s_n - s_i + 1/2).$$

Finally, we define

$$g^{(n)}(s) = \prod_{j=1}^{n} g_j(s) \quad \text{and} \quad F^{(n)}(s) = \prod_{j=1}^{n} F_j(s) .$$

These definitions follow the same pattern that we used with the fudge factor involving the Riemann zeta function, i.e. $Z_{\mathbf{Q},n}(s)$ and $Z_{\mathbf{Q}}^{(n)}(s)$. In particular,

$$F_{n+1}(s_1, \ldots, s_{n+1}) = \frac{F^{(n+1)}(s_1, \ldots, s_{n+1})}{F^{(n)}(s_1, \ldots, s_n)}$$

and

$$F^{(n+1)}(s, s_{n+1}) = \prod_{j=1}^{n} F_{j+1}(s_1, \ldots, s_{j+1}) \, .$$

The next lemma is the analogue of Proposition 5.3 for the fudge factor that we are now dealing with.

Lemma 5.5. *We have the explicit determination*

$$F_{n+1}(s, s_{n+1}) = \pi^{w_n} \Lambda_n(\varphi^*_{s,s_{n+1}}) \, .$$

The exponent w_n *is the degree in (7), as a function of* $s_1, \ldots, s_{n+1}$.

Proof. This is a tedious verification.

We apply Corollary 4.4 to the character $q^{(n)}_{-z} = \varphi_{s,s_{n+1}}$. We note that

$$q^{(n)*}_{-z} = \varphi^*_{s,s_{n+1}} = \mathbf{d}^{s_{n+1}+(n+1)/4} \; h_{s^*} \, . \tag{8}$$

Lemma 5.6. *For* $Y \in \mathrm{Pos}_n$,

$$\mathrm{Tr}_{\Gamma_U \backslash \Gamma}(q^{(u)*}_{-z}(Y) = \mathrm{Tr}_{\Gamma_U \backslash \Gamma}(\varphi^*_{s,s_{n+1}})(Y) = |Y|^{s_{n+1}-s_1+n/2} \zeta^{\mathrm{pr}}(Y, s^*) \, .$$

Proof. By definition of $s^* = (-s_n, \ldots, s_1)$ we have

$$\zeta^{\mathrm{pr}}(Y, s^*) = |Y|^{s_1-(n-1)/4} \mathrm{Tr}_{\Gamma_U \backslash \Gamma} h^*_s(Y) = \mathrm{Tr}_{\Gamma_U \backslash \Gamma}(\mathbf{d}^{s_1-(n-1)/4} h_{s^*})(Y) \, .$$

Multiplying by $\mathbf{d}^{s_{n+1}-s_1+n/2}$ and using (8) concludes the proof.

For $S \in \mathrm{Pos}_{n+1}, \Gamma = \Gamma_n$, **define**

$$\xi(S; s, s_{n+1}) = (\mathbf{D}\theta * \mathrm{Tr}_{\Gamma_U \backslash \Gamma}(\varphi^*_{s,s_{n+1}}))(S) \, .$$

Thus by definition of the convolution and Lemma 5.6,

$$\xi(S; s, s_{n+1}) = \int_{\Gamma_n \backslash \mathcal{P}_n} \mathbf{D}\theta(S, Y) |Y|^{s_{n+1}-s_1+n/2} \zeta(Y, s^*) d\mu(Y) \, .$$

Let $\mathfrak{B}$ be the domain defined by the inequalities

$$\mathrm{Re}(s_{j+1} - s_j + 1/2) > 1 \;\text{ for }\; j = 1, \ldots, n-1 \;\text{ and }\; s_{n+1} \;\text{ arbitrary,}$$

while $\mathfrak{B}_1$ is defined by these inequalities together with

$$\mathrm{Re}(s_1 - s_{n+1} + 1/2) > 1 \, .$$

Estimates in Chap. 7 show that the integral for ξ converges absolutely in the domain $\mathfrak{B}_1$. In light of our definitions and Lemma 5.5, we may now reformulate Theorem 4.3 or rather Corollary 4.4 as follows.

Theorem 5.7. *In the domain defined by these inequalities, we have*

$$\begin{aligned} \xi(S; s, s_{n+1}) &= F_{n+1}(s, s_{n+1}) \zeta(S; s, s_{n+1}) \\ &= \frac{F^{n+1}(s_1, \ldots, s_{n+1})}{F^{(n)}(s_1, \ldots, s_n)} \zeta(S; s, s_{n+1}) \, . \end{aligned}$$

6 Functional Equation: Invariance under Cyclic Permutations

Here we follow Maass [Maa 71]. For the function $\xi(S; s, s_{n+1})$ defined at the end of the preceding section, we first have

Lemma 6.1. *For* $S \in \mathrm{Pos}_{n+1}$,

$$\xi(S^{-1}; s^*, -s_{n+1}) = |S|^{n/2}\xi(S; s, s_{n+1}) \ .$$

Proof. This result is proved by the Riemann method. The integral over $\Gamma_n\backslash\mathcal{P}_n$ is decomposed into a sum

$$\int\limits_{\Gamma_n\backslash\mathcal{P}_n} = \int\limits_{(\Gamma_n\backslash\mathcal{P}_n)(\geqq 1)} + \int\limits_{(\Gamma_n\backslash\mathcal{P}_n)(\leqq 1)} ,$$

where the parentheses $(\geqq 1)$ and $(\leqq 1)$ signify the subdomain where the determinant is $\geqq 1$ resp. $\leqq 1$. On the second integral, we make the change of variables $Y \mapsto Y^{-1}$. Then letting $\mathcal{F}_n = \Gamma_n\backslash\mathcal{P}_n$, we get:

(1)
$$\begin{aligned}&\xi(S; s, s_{n+1})\\ &\quad = \int\limits_{\mathcal{F}_n(\geqq 1)} \big\{\mathbf{D}\theta(S, Y)|Y|^{s_{n+1}-s_1+n/2}\zeta(Y, s^*)\\ &\qquad\qquad + \mathbf{D}\theta(S, Y^{-1})|Y|^{s_1-s_{n+1}-n/2}\zeta(Y^{-1}, s^*)\big\}d\mu(Y) \ .\end{aligned}$$

On the other hand,

(2)
$$\begin{aligned}&\xi(S^{-1}; s^*, -s_{n+1})\\ &\quad = \int\limits_{\mathcal{F}_n(\geqq 1)} \big\{\mathbf{D}\theta(S^{-1}, Y)|Y|^{-s_{n+1}+s_n+n/2}\zeta(Y, s)\\ &\qquad\qquad + \mathbf{D}\theta(S^{-1}, Y)|Y|^{s_{n+1}-s_n-n/2}\zeta(Y^{-1}, s)\big\}d\mu(Y) \ .\end{aligned}$$

We now use two previous functional equations. One is the functional equation for the regularized theta functions, namely Sect. 4, formulas (4) and (5), which read:

$$\mathbf{D}\theta(S^{-1}, Y^{-1}) = |S|^{n/2}|Y|^{(n+1)/2}\mathbf{D}\theta(S, Y)$$

$$\mathbf{D}\theta(S^{-1}, Y) = |S|^{n/2}|Y|^{-(n+1)/2}\mathbf{D}\theta(S, Y^{-1})$$

The other equation is stated in Proposition 5.1, which is valid with $\zeta(Y, s)$ instead of $\zeta^{\mathrm{pr}}(Y, s)$, because $Z_{\mathbf{Q}}^{(n)}(s^*) = Z_{\mathbf{Q}}^{(n)}(s)$ is the same factor needed to change the primitive Eisenstein series into the non-primitive one. Applying this proposition and the functional equation for the theta function shows directly and immediately that the two terms under the integral for $\xi(S^{-1}; s^*, s_{n+1})$ are changed precisely into the two terms which occur in the integral expression for $\xi(S; s, s_{n+1})$ multiplied by $|S|^{n/2}$. This concludes the proof.

Theorem 6.2. *Let $S \in \mathrm{Pos}_{n+1}$ and let*

$$\eta(S; s^{(n+1)}) = F^{(n+1)}(s_1, \dots, s_{n+1})|S|^{s_{n+1}}\zeta(S; s_1 \dots, s_{n+1}) .$$

Then $\eta(S; s_1, \dots, s_{n+1})$ is invariant under a cyclic permutation of the variables, that is

$$\eta(S; s_1, \dots, s_{n+1}) = \\ F^{(n+1)}(s_{n+1}, s_1, \dots, s_n)|S|^{s_n}\zeta(S; s_{n+1}, s_1, \dots, s_n) .$$

Furthermore, $\eta(S; s_1, \dots, s_{n+1})$ is holomorphic in the domain $\mathfrak{B}$.

Proof. By Theorem 5.7 and $F^{(n)}(s^*) = F^{(n)}(s)$, we have

$$\begin{aligned}\xi(S^{-1}; s^*, -s_{n+1}) &= \tfrac{F^{(n+1)}(s^*, -s_{n+1})}{F^{(n)}(s)}\zeta(S^{-1}; s^*, -s_{n+1}) \\ = \frac{F^{(n+1)}(s_{n+1}, s)}{F^{(n)}(s)} & \ |S|^{-s_{n+1}+s_n+n/2}\zeta(S; s_{n+1}, s_1, \dots, s_n)\end{aligned}$$

by Proposition 5.1, valid in the domain $\mathrm{Re}(S_{j+1} - s_j + \frac{1}{2}) > 1$ for each index $j = 1, \dots, n-1$, that is in the domain $\mathfrak{B}$. On the other hand,

$$|S|^{n/2}\xi(S; s, s_{n+1}) = \\ \frac{F^{(n+1)}(s_1, \dots, s_n, s_{n+1})}{F^{(n)}(s_1, \dots, s_n)}|S|^{n/2}\zeta(S, s_1, \dots, s_n, s_{n+1}) .$$

Using the definition of $\eta(S; s_1, \dots, s_{n+1})$ and cross multiplying, we apply Lemma 6.2 to conclude the proof.

Note. The three essential ingredients in the above proof are:

EIS 1. For each integer $n \geqq 3$ there is a fudge factor $F^{(n)}(s_1, \dots, s_n)$ such that for $S \in \mathrm{Pos}_{n+1}$ we have

$$\xi(S; s, s_{n+1}) = \frac{F^{(n+1)}(s_1, \dots, s_n, s_{n+1})}{F^{(n)}(s_1, \dots, s_n)}\zeta(S; s, s_{n+1}) .$$

Furthermore, $F^{(n)}(s^*) = F^{(n)}(s)$ (invariance under $s \mapsto s^*$).
See Lemma 5.2 and Theorem 5.7.

EIS 2. $\zeta(Y^{-1}, s) = |Y|^{s_n - s_1 + (n-1)/2}\zeta(Y, s^*)$ in the domain

$$\mathrm{Re}(s_{j+1} - s_j + 1/2) > 1 .$$

Ref: Proposition 5.1 and Lemma 5.2.

EIS 3. $\xi(S^{-1}; s^*, -s_{n+1}) = |S|^{n/2}\xi(S; s, s_{n+1})$

Ref: Theorem 6.1.

Finally, we prove the analytic continuation over all of $\mathbf{C}^{n+1}$ by means of a theorem in several complex variables. That is, we want:

Theorem 6.3. *The function $\eta(S, s_1, \ldots, s_{n+1})$ is holomorphic on all of $\mathbf{C}^{n+1}$.*

Proof. We reduce the result to a basic theorem in several complex variables. Let σ be the cyclic permutation

$$\sigma : (s_1, \ldots, s_{n+1}) \mapsto (s_{n+1}, s_1, \ldots, s_n) .$$

By Theorem 6.2, we know that η is holomorphic in the domain

$$\mathfrak{D} = \bigcup_{j=1}^{n} \sigma^j \mathfrak{B} \subset \mathbf{C}^{n+1} .$$

Let $\mathrm{pr}_{\mathbf{R}^{n+1}}(\mathfrak{D}) = \mathfrak{D}_{\mathbf{R}}$ be the projection on the real part. Since the inequalities defining $\mathfrak{D}$ involve only the real part, it follows that

$$\mathfrak{D} = \mathfrak{D}_{\mathbf{R}} + i\mathbf{R}^{n+1},$$

so $\mathfrak{D}$ is what is commonly called a tube domain. By Theorem 2.5.10 in Hörmander [Hör 66], it follows that η is holomorphic on the convex closure of the tube. But $\mathfrak{D}_{\mathbf{R}}$ contains a straight line parallel to the $(n+1)$-th axis of $\mathbf{R}^{n+1}$. This line can be mapped on a line parallel to the j-th axis of $\mathbf{R}^{n+1}$ for each j, by powers of σ. The convex closure of these lines in the real part $\mathbf{R}^{n+1}$ is all of $\mathbf{R}^{n+1}$, and by the theorem in Hörmander, it follows that the convex closure of $\mathfrak{D}$ is $\mathbf{C}^{n+1}$. This concludes the proof.

7 Invariance under All Permutations

In light of the theorems in Sect. 6, all that remains to be done is to prove the invariance of the function

$$\eta(Y; s_1, \ldots, s_n) = F^{(n)}(s_1, \ldots, s_n)|Y|^{s_n}\zeta(Y; s_1, \ldots, s_n)$$

under a transposition, and even under the transposition between the special variables s_1 and s_2. Then we shall obtain Selberg's theorem:

Theorem 7.1. *For $Y \in \mathrm{Pos}_n$, the function $\eta(Y; s_1, \ldots, s_n)$ is invariant under all permutations of the variables.*

Proof. The following proof follows Selberg's lines and is the one given in Maass [Maa 71]. We have $\zeta(Y;s) = E_U^{(n-1)}(Y,z)$ (the non-primitive Eisenstein series). The essential part of the proof will be to show that the function

$$\pi^{-s_1}\mathbf{\Gamma}(z_1)E_U^{(n-1)}(Y,z) = \pi^{-(s_2-s_1+1/2)}\mathbf{\Gamma}(s_2-s_1+1/2)\zeta(Y,s)$$

is invariant under the transpositon of s_1 and s_2. Before proving this, we show how it implies the theorem. As before, let

$$g(u) = \pi^{-u}\mathbf{\Gamma}(u)\ .$$

Then it follows that

$$\prod_{1\leqq i<j\leqq n} g(s_j - s_i + 1/2)|Y|^{s_n}\zeta(Y,s) = g^{(n)}(s)|Y|^{s_n}\zeta(Y,s)$$

is invariant under the transposition of s_1 and s_2. By Theorem 6.2 we conclude that this function is invariant under all permutations of $(s_1,\ldots,s_n)$. Theorem 7.1 follows by the factorization of $F^{(n)}(s_1,\ldots,s_n)$ given in Sect. 5.

To prove the invariance of $g(z_1)E_U^{(n-1)}(Y,z)$ under the transposition of s_1 and s_2, we go back to the definition of the Eisenstein series in terms of the z-variables, and we write this definition in the form

$$E_{\mathcal{T}}^{(n-1)}(Y,z) = \sum_{(C_n,\ldots,C_2)} \prod_{j=2}^{n-1} |\mathcal{C}_j(Y)|^{-z_j} E_{\mathcal{T}}^{(1)}(\mathcal{C}_2(Y),z_1) \quad \text{with} \quad C_n = \gamma\ .$$

The sum over $(C_n,\ldots,C_2)$ is over equivalence classes, whose definition for such truncated sequences is the same as for $(C_n,\ldots,C_1)$, except for disregarding the condition on C_1. The theorem was proved in Chap. 5, Theorem 4.1 in the case $n=2$, so we assume $n \geqq 3$. We write the Eisenstein series with one further splitting, that is

$$E_{\mathcal{T}}^{(n-1)}(Y,z) = \sum_{(C_n,\ldots,C_2)} \prod_{j=3}^{n-1} |\mathcal{C}_j(Y)|^{-z_j}|\mathcal{C}_2(Y)|^{-z_2} E_{\mathcal{T}}^{(1)}(\mathcal{C}_2(Y),z_1)\ .$$

Although the notation with the chains was the clearest previously, it now becomes a little cumbersome, so we abbreviate

$$\mathcal{C}_j(Y) = Y_j \quad \text{for} \quad j = 1,\ldots,n\ .$$

Then we rewrite the above expressions for the Eisenstein series in the form

$$(1) \qquad E_{\mathcal{T}}^{(n-1)}(Y,z) = \sum_{(Y_n,\ldots,Y_2)} \prod_{j=2}^{n-1} |Y_j|^{-z_j}\ E_{\mathcal{T}}^{(1)}(Y_2,z_1)$$

$$(2) \qquad = \sum_{(Y_n,\ldots,Y_2)} \prod_{j=3}^{n-1} |Y_j|^{-z_j} |Y_2|^{-z_2} E_{\mathcal{T}}^{(1)}(Y_2, z_1) \ .$$

With the change of variables $z_j = s_{j+1} - s_j + 1/2$, we can also write

$$|Y_2|^{-z_2} E_{\mathcal{T}}^{(1)}(Y_2, z_1) = 2^2 |Y_2|^{s_2 - s_3 + 1/2} \zeta(Y_2; s_1, s_2) \ .$$

By Theorem 4.1 of Chap. 5, the function

$$\eta_2(Y; s_1, s_2) = \pi^{-z_1} \mathbf{\Gamma}(z_1) |Y|^{s_2} \zeta(Y_2; s_1, s_2)$$

is invariant under permutation of s_1 and s_2, because in the notation of this reference,

$$E(Y_2, z) = E^{(1)}(Y_2, z) = \zeta(Y_2; s_1, s_2) \ .$$

Thus formally, we conclude that

$$\pi^{-z_1} \mathbf{\Gamma}(z_1) \zeta(Y; s_1, \ldots, s_2) = \pi^{-z_1} \mathbf{\Gamma}(z_1) E^{(n-1)}(Y, z)$$

is invariant under the permutation of s_1 and s_2. The only thing to watch for is that this permutation can be done while preserving the convergence of the series expression (2) for $E^{(n-1)}(Y, z)$. Thus one has to select an appropriate domain of absolute convergence, so that all the above expressions make sense. Maass does this as follows. We start with the inductive lowest dimensional piece,

$$\mathbf{\Lambda}_2(Y, z_1) = \pi^{-z_1} \mathbf{\Gamma}(z_1) E^{(1)}(Y, z_1),$$

which is the first case studied in Chap. 5, Sect. 3. We gave an estimate for this function in the strip $\mathrm{Str}(-2, 3)$, that is

$$-2 < \mathrm{Re}(z_1) < 3 \ ,$$

away from 0 and 1, specifically outside the discs of radius 1 centered ar 0 and 1, as in Corollary 3.8 of Chap. 5.

Next, we consider the series

$$(3) \qquad \pi^{-z_1} \mathbf{\Gamma}(z_1) E^{(n-1)}(y, Z) = \sum_{(Y_n,\ldots,Y_2)} \prod_{j=2}^{n-1} |Y_j|^{-z_j} \mathbf{\Lambda}_2(Y_2, z_1) \ .$$

By Theorem 3.1, mostly Theorem 2.2 of Chap. 7, the series in (3) converges absolutely for $\mathrm{Re}(z_j) > 1, \ j = 1, \ldots, n-1$. Similarly, by Chap. 7, Theorem 3.1, we also know that the series

$$(4) \qquad \sum_{(Y_n,\ldots,Y_2)} \prod_{j=2}^{n-1} |Y_j|^{-z_j}$$

converges absolutely for $\mathrm{Re}(z_j) > 1$ $(3 \leqq j \leqq n-1)$ and $\mathrm{Re}(z_2) > 3/2$. By Chap. 5, Corollary 3.8 (put there for the present purpose), adding up the power of $|Y_2|$, in the sbove strip outside the unit discs around $0, 1$, it follows that the Eisenstein series from (1) converges absolutely in the domain

$\mathfrak{D}_1$ = points in $\mathbf{C}^n$ with z_1 in the strip $\mathrm{Str}(-2, 3)$ outside the discs of radius 1 around $0, 1$; and

$$\mathrm{Re}(z_2) > 7/2; \qquad \mathrm{Re}(z_j) > 1 \quad \text{for} \quad j = 3, \ldots, n-1 \, .$$

Let

$\mathfrak{D}_2$ = subdomain of $\mathfrak{D}_1$ satisfying the further inequality $\mathrm{Re}(z_2) > 6$.

In terms of the variables z, we want to prove the functional equation

$$\sum_{(Y_n,\ldots,Y_2)} \prod_{j=3}^{n-1} |Y_j|^{-z_j} |Y_2|^{-z_2} \mathbf{\Lambda}_2(Y_2, z_1)$$
$$= \sum_{(Y_n,\ldots,Y_2)} \prod_{j=3}^{n-1} |Y_j|^{-z_j} |Y_2|^{-z_1-z_2+1/2} \mathbf{\Lambda}_2(Y_2, 1-z_1) \, .$$

The series on both sides are convergent in $\mathfrak{D}_2$, so the formal argument is now justified, and we have proved that

$$\pi^{-z_1} \mathbf{\Gamma}(z_1) E^{(n-1)}(Y, z)$$

is invariant under the equivalent transformations:

$$z_1 \mapsto 1 - z_1, \;\; z_2 \mapsto z_1 + z_2 - 1/2, \;\; z_j \mapsto z_j \;\; (j = 3, \ldots, n-1), s_n \mapsto s_n \; ,$$

or

$$\text{transposition of } s_1 \text{ and } s_2 \; .$$

This concludes the proof of Theorem 7.1.

Remark. Just as Maass gave convergence criteria for Eisenstein series with more general parabolic groups [Maa 71], Sect. 7, he also gives the analytic continuation and functional equation for these more general groups at the end of Sect. 17, pp. 279–299.