7

Geometric and Analytic Estimates

In Chap. 1 and 2, we dealt at length with estimates concerning various coordinates on Pos_n and the volume on Pos_n. Here we come to deal with the metric itself, and the application of coordinate estimates to the convergence of certain Dirichlet series called Eisenstein series. Further properties of such series will then be treated in the next chapter. On the whole we follow the exposition in Maass [Maa 71], Sect. 3, Sect. 7 and especially Sect. 10, although we make somewhat more efficient use of the invariant measure in Iwasawa coordinates, thereby introducing some technical simplifications.

1 The Metric and Iwasawa Coordinates

The basic differential geometry of the space Pos_n is given in Chap. XI of [La 99] and will not be reproduced here. We merely recall the basic definition. We view Sym_n (vector space of real symmetric $n \times n$ matrices) as the tangent space at every point Y of Pos_n. The Riemannian metric is defined at the point Y by the formula

$$ds^2 = \mathrm{tr}((Y^{-1}dY)^2) \text{ also written } \mathrm{tr}(Y^{-1}dY)^2 \, .$$

This means that if $t \mapsto Y(t)$ is a C^1 curve in Pos_n, then

$$\langle Y'(t), Y'(t)\rangle_{Y(t)} = \mathrm{tr}(Y(t)^{-1}Y'(t))^2 \, ,$$

where $Y'(t)$ is the naive derivative of the map of a real interval into Pos_n, viewed as an open subset of Sym_n.

The two basic properties of this Riemannian metric are:

Theorem 1.1. *Let* Sym_n *have the positive definite scalar product given by* $\langle M, M_1\rangle = \mathrm{tr}(MM_1)$. *Then the exponential map* $\exp : \mathrm{Sym}_n \to \mathrm{Pos}_n$ *is metric semi-increasing, and is metric preserving on lines from the origin.*

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Theorem 1.2. *The Riemannian distance between any two points $Y, Z \in \mathrm{Pos}_n$ is given by the formula*

$$\mathrm{dist}(Y,Z) = \sum (\log\ a_i)^2 ,$$

where $a_1, \dots, a_n$ are the roots of the polynomial $\det(tY - Z)$.

See [La 99], Chap. XI, Theorems 1.2, 1.3 and 1.4. In the present section, we shall consider the distance formula in the context of Iwasawa-Jacobi coordinates.

As in Chap. 2, Sect. 2 the partial Iwasawa-Jacobi coordinates of an element $Y \in \mathrm{Pos}_n$ are given by the expression

$$Y = [u(X)] \begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix}, \quad u(X) = \begin{pmatrix} I_p & X \\ 0 & I_q \end{pmatrix} ,$$

with $X \in \mathbf{R}^{p,q}, W \in \mathrm{Pos}_p$ and $V \in \mathrm{Pos}_q$. Matrix multiplication shows that

$$Y = \begin{pmatrix} W + [X]V & XV \\ V^t X & V \end{pmatrix} . \tag{1}$$

In particular, by definition

$$V = \mathrm{Sub}_q(Y)$$

is the lower right square submatrix which we have used in connection with the Selberg power function, cf. Chap. 3, Sect. 1. We shall need the matrix for Y^{-1}, given by

$$\begin{aligned} Y^{-1} &= \begin{pmatrix} W^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix} [u(-X)] \\ &= \begin{pmatrix} W^{-1} & -W^{-1}X \\ -{}^tXW^{-1} & V^{-1} + [{}^tX]W^{-1} \end{pmatrix} . \end{aligned} \tag{2}$$

Theorem 1.3. *The metric on Pos_n admits the decomposition*

$$\mathrm{tr}(Y^{-1}dY)^2 = \mathrm{tr}(W^{-1}dW)^2 + \mathrm{tr}(V^{-1}dV)^2 + 2\mathrm{tr}(W^{-1}[dX]V) .$$

All three terms on the right are $\geqq 0$. In particular,

$$\mathrm{tr}(V^{-1}dV)^2 \leqq \mathrm{tr}(Y^{-1}dY)^2 ,$$

and the map

$$\mathrm{Pos}_n \to \mathrm{Pos}_q \quad \textit{given by} \quad Y \mapsto \mathrm{Sub}_q(Y) = V$$

is metric decreasing.

Proof. We copy Maass [Maa 71], Sect. 3. We start with

$$(3) \qquad dY = \begin{pmatrix} dW + [X]dV + dX \cdot V\,{}^tX + XV \cdot d\,{}^tX & dX \cdot V + XdV \\ dV \cdot {}^tX + V \cdot d\,{}^tX & dV \end{pmatrix} .$$

With the abbreviation

$$dY \cdot Y^{-1} = \begin{pmatrix} L_0 & L_1 \\ L_2 & L_3 \end{pmatrix}$$

we have

$$(4) \qquad \begin{aligned} \operatorname{tr}(Y^{-1}dY)^2 &= \operatorname{tr}(dY \cdot Y^{-1} \cdot dY \cdot Y^{-1}) \\ &= \operatorname{tr}(L_0^2 + L_1L_2) + \operatorname{tr}(L_2L_1 + L_3^2) . \end{aligned}$$

A straightforward calculation yields

$$(5) \qquad \begin{aligned} L_0 &= dW \cdot W^{-1} + XV \cdot d\,{}^tX \cdot W^{-1} \\ L_1 &= -dW \cdot w^{-1}X - XV \cdot d\,{}^tX \cdot W^{-1}X + dX + X \cdot dV \cdot V^{-1} \\ L_2 &= V \cdot dX \cdot W^{-1} \\ L_3 &= dV \cdot V^{-1} - V \cdot d\,{}^tX \cdot W^{-1}X . \end{aligned}$$

The formula giving the decomposition of $\operatorname{tr}(Y^{-1}dY)^2$ as a sum of three terms then follows immediately from (4) and the values for the components in (5). As to the positivity, the only possible question is about the third term on the right of the formula. For this, we write $W = A^2$ and $V = B^2$ with positive A, B. Let $Z = B \cdot d\,{}^tX \cdot A^{-1}$. Then

$$\operatorname{tr}(W^{-1}[dX]V) = \operatorname{tr}(Z\,{}^tZ) ,$$

which shows that the third quadratic form is positive definite and concludes the proof.

Let $G = \mathrm{GL}_n(\mathbf{R})$ as usual. It is easily verified that the action of G on Pos_n is metric preserving, so G has a representation as a group of Riemannian automorphisms of Pos_n. Again cf. [La 99] Chap. XI, Theorem 1.1. Here we are interested in the behavior of the determinant $|Y|$ as a function of distance. Consider first a special case, taking distances from the origin $I = I_n$. By Theorem 1.2, we know that if $Y \in \mathbf{B}_r(I)$ (Riemannian ball of radius r centered at I), then

$$\operatorname{dist}(Y, I)^2 = \sum (\log\ a_i)^2 < r^2 .$$

It then follows that there exists a number $c_n(r)$, such that for $Y \in \mathbf{B}_r(I)$, we have

$$(6) \qquad \frac{1}{c_n(r)} < |Y| < c_n(r) .$$

Indeed, the determinant is equal to the product of the characteristic roots,

$$|Y| = a_1 \dots a_n \ .$$

With the Schwarz inequality, we take $c_n(r) = e^{\sqrt{n}r}$. Note that from an upper bound for $|Y|$, we get a lower bound automatically because $Y \mapsto Y^{-1}$ is an isometry. From another point of view, we also have $(\log\ a_i)^2 = (\log\ a_i^{-1})^2$.

In the above estimate, we took a ball around I. But the transitive action of G on Pos_n gives us more uniformity. Indeed:

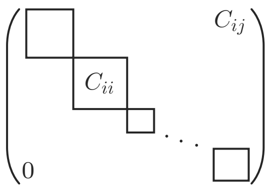
Lemma 1.4. *For any pair* $Y, Z \in \mathrm{Pos}_n$ *with* $\mathrm{dist}(Y, Z) < r$, *we have*

$$c_n(r)^{-1} < \frac{|Z|}{|Y|} < c_n(r).$$

Proof. We have

$$|tZ - Y| = |Y|\ |tY^{-1}Z - I|\ .$$

The roots of this polynomial are the same as the roots of the polynomial $|t[Y^{-\frac{1}{2}}]Z \in \mathbf{B}_r(I)$, so the lemma follows from the corresponding statement translated to the origin I.



We shall also be interested in the subdeterminants $\mathrm{Sub}_j(Y)$ of Y. By Theorem 1.3, we know that the association $Y \mapsto \mathrm{Sub}_j(Y)$ is metric decreasing. Hence we may extend the uniformity of Lemma 1.4 as follows.

Lemma 1.5. *For* $g \in \mathrm{GL}_n(\mathbf{R})$ *and all pairs* $Y, Z \in \mathrm{Pos}_n$ *with* $\mathrm{dist}\ (Y, Z) \leqq r$, *and all* $j = 1, \dots, n$ *we have*

$$c_n(r)^{-1}|\mathrm{Sub}_j[g]Y| < |\mathrm{Sub}_j[g]Z| < c_n(r)|\mathrm{Sub}_j[g]Y|\ .$$

Briefly: $|\mathrm{Sub}_j[g]Z| \gg\ll_r |\mathrm{Sub}_j[g]Y|$.

Next, let

$$\mathcal{D}_r = \Bigg\{ Y \in \mathrm{Pos}_n \text{ such that}$$

$$|Y| < c_n(r) \text{ and } |\mathrm{Sub}_j Y| > \frac{1}{c_n(r)} \text{ for } j = 1, \dots, n \Bigg\}.$$

Lemma 1.6. *For all* $\gamma \in \Gamma = \mathrm{GL}_n(\mathbf{Z})$ *we have*

$$\mathbf{B}_r([\gamma]I) \subset \mathcal{D}_r \ .$$

Proof. Let $Y \in B_r([\gamma]I)$. Then $[\gamma^{-1}]Y \in \mathbf{B}_r(I)$, and we can apply (6), as well as $|[\gamma^{-1}]I| = 1$ to prove the inequality $|Y| < c_n(r)$. For the other inequality, by the distance decreasing property, we have

$$\mathrm{dist}(\mathrm{Sub}_j[\gamma]I, \mathrm{Sub}_j Y) \leqq \mathrm{dist}([\gamma]I, Y) < r \ .$$

Hence by Lemma 1.4,

$$|\mathrm{Sub}_j Y| > \frac{1}{c_n(r)} |\mathrm{Sub}_j([\gamma]I)| \geqq \frac{1}{c_n(r)}$$

because $[\gamma]I$ is an integral matrix, with determinant $\geqq 1$. This concludes the proof.

The set of elements $[\gamma]I$ with $\gamma \in \Gamma$ is discrete in Pos_n. We call $r > 0$ a **radius of discreteness** for Γ if $\mathrm{dist}([\gamma]I, I) < 2r$ implies $\gamma = \pm I$, that is $[\gamma]$ acts trivially on Pos_n. We shall need:

Lemma 1.7. *Let* $\gamma, \gamma' \in \Gamma$, *and let* r *be a radius of discreteness for* Γ. *If there is an element* $Y \in \mathrm{Pos}_n$ *in the intersection of the balls* $\mathbf{B}_r([\gamma]I)$ *and* $\mathbf{B}_r([\gamma']I)$, *then* $[\gamma] = [\gamma']$, *that is* $\gamma = \pm\gamma$.

Proof. By hypothesis, $\mathrm{dist}([\gamma]I, [\gamma']I) < 2r$, so

$$\mathrm{dist}([\gamma^{-1}\gamma']I, I) < 2r \ ,$$

and the lemma follows.

2 Convergence Estimates for Eisenstein Series

We shall need a little geometry concerning the action of the unipotent group on Pos_n, so we start with an independent discussion of this geometry.

An element $Y \in \mathrm{Pos}_n$ can be written uniquely in the form

$$Y = [u(X)]A \quad \text{with } u(X) = I_n + X \ ,$$

and

$$A = \begin{pmatrix} a_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{nn} \end{pmatrix}, a_{ii} > 0 \ ,$$

and $X = (x_{ij})$ is strictly upper triangular. We call (X, A) the **full Iwasawa coordinates** for Y on Pos_n.

Let $\Gamma = \mathrm{GL}_n(\mathbf{Z})$ as usual, Γ_U = subgroup of unipotent elements in Γ, so the upper triangular integral matrices with every diagonal element equal to 1. Thus $\gamma \in \Gamma_U$ can be written $\gamma = I_n + X$ with an integral matrix X.

It is easy to construct a fundamental domain for $\Gamma_U \backslash \mathrm{Pos}_n$. First we note that a fundamental domain for the real unipotent group $\mathrm{Uni}^+(\mathbf{R})$ modulo the integral subgroup Γ_U consists of all elements $u(X)$ such that $0 \leqq x_{ij} < 1$. We leave the proof to the reader. In an analogous discrete situation when all matrices are integral, we shall carry out the inductive argument in Lemma 1.2 of Chap. 8, using the euclidean algorithm. In the present real situation, one uses a "continuous" euclidean algorithm, as it were. Then we define:

$$\mathcal{F}_U = \text{set of elements } [u(X)]A \in \mathrm{Pos}_n \text{ with } 0 \leqq x_{ij} < 1 \, .$$

From the uniqueness of the Iwasawa coordinates, we conclude that $\mathcal{F}_U$ is a strict fundamental domain for $\Gamma_U \backslash \mathrm{Pos}_n$.

The main purpose of this section is to prove the convergence of a certain series called an Eisenstein series. We shall prove it by an integral test, depending on the finiteness of a certain integral, which we now describe in a fairly general context.

Let $c > 0$. We define the subset $\mathcal{D}(c)$ of Pos_n to be:

$$\mathcal{D}(c) = \{Y \in \mathrm{Pos}_n, |Y| < c \text{ and } |\mathrm{Sub}_j Y| > 1/c \text{ for all } j = 1, \ldots, n\} \, .$$

We recall the Selberg power function

$$q_{-z}^{(n)}(Y) = \prod_{j=1}^{n} |\mathrm{Sub}_j Y|^{-z_j} \, .$$

We are interested in the integral of this power function over a set

$$\mathcal{D}(c) \cap \mathcal{F}_U \, .$$

To test absolute convergence, it suffices to do so when all z_j are real. The next lemma will prove absolute convergence when $Re(z_j) > 1$.

Lemma 2.1. *Let $b > 1$. Then*

$$\int\limits_{\mathcal{D}(c)\cap\mathcal{F}_U} \prod_{j=1}^{n} |\mathrm{Sub}_j Y|^{-b} d\mu_n(Y) < \infty \, .$$

Proof. In Chap. 2, Proposition 2.4, we computed the invariant measure $d\mu_n(Y)$ in terms of the Iwasawa coordinates, and found

$$d\mu_n(Y) = \prod_{i=1}^{n} a_{ii}^{i-(n+1)/2} \prod_{i=1}^{n} \frac{da_{ii}}{a_{ii}} \prod_{i<j} dx_{ij} \, . \tag{1}$$

We note that $|\mathrm{Sub}_j Y| = a_{n-j+1} \dots a_n$, writing $a_i = a_{ii}$. Hence, if we take $\varepsilon > 0$ and set $b = 1 + \varepsilon$, we have

$$\prod_{j=1}^{n} |\mathrm{Sub}_j Y|^{-b} = \prod_{i=1}^{n} a_i^{-i-\varepsilon i} \ . \tag{2}$$

The effect of intersecting $\mathcal{D}(c)$ with $\mathcal{F}_u$ is to bound the x_{ij}-coordinates. Thus the convergence of the integral depends only on the a_i-coordinates. To concentrate on them, we let

$$d\mu_{n,A} = \prod_{i=1}^{n} a_i^{i-(n+1)/2} \prod_{i=1}^{n} \frac{da_i}{a_i} \ .$$

We let $\mathcal{D}_A(c)$ be the region in the A-space defined by the inequalities

$$\frac{1}{c} < a_1 \cdots a_n < c \quad \text{and} \quad a_n > \frac{1}{c} \ ,$$
$$a_n a_{n-1} > \frac{1}{c}, \quad \dots, \quad a_n a_{n-1} \cdots a_1 > \frac{1}{c} \ .$$

Thus $\mathcal{D}_A(c)$ is a region in the n-fold product of the positive multiplicative group, and the convergence of the integral in our lemma is reduced to the convergence of an integral in a euclidean region, so to calculus. Taking the product of the expressions in (1) and (2), and integrating over $\mathcal{D}_A(c)$, we see that the finiteness of the integral in our lemma is reduced to proving the finiteness

$$\int_{\mathcal{D}_A(c)} \prod a_{ii}^{-(\varepsilon i + 1 + (n+1)/2)} \prod da_{ii} < \infty \ . \tag{3}$$

Just to see what's going on, suppose $n = 2$ and the variables are

$$a_1 = u \ \text{ and } \ a_2 = v \ .$$

The region is defined by the inequalities

$$\frac{1}{c} < uv < c \quad \text{and} \quad v > \frac{1}{c} \ .$$

The integral can be rewritten as the repeated integral

$$\int_{1/c}^{\infty} \left(\int_{1/cv}^{c/v} u^{-(1+\varepsilon+(n+1)/2)} du \right) v^{-(2\varepsilon+1+(n+1)/2)} dv \ .$$

The inner integral with respect to u can be evaluated, and up to a constant factor, it produces a term

$$v^{\varepsilon+(n+1)/2}$$

which cancels the similar expression in the outer v-integral. Thus finally the convergence is reduced to

$$\int_{1/c}^{\infty} \frac{1}{v^{1+\varepsilon}} dv < \infty$$

which is true. Having n variables only complicates the notation but not the idea, which is to integrate successively with respect to da_n, then da_{n-1}, and so forth until da_1, which we leave to the reader to conclude the proof of Lemma 2.1.

Next we combine the metric estimates from the last section with the measure estimates which we have just considered. Let r be a radius of discreteness for Γ, defined at the end of the last section. Then

$$\mathcal{D}_r = \mathcal{D}(c_n(r)) ,$$

where $\mathcal{D}(c)$ is the set we considered in Lemma 2.1.

Let $\{\gamma_m\}$ (with $m = 1, 2, \ldots$) be a family of coset representatives for $\pm\Gamma_U\backslash\Gamma$. For each m we let $\tau_{mk}(k = 1, \ldots, d_m)$ be a minimal number of elements of $\pm\Gamma_U$ such that

$$\mathbf{B}_r([\gamma_m]I) \subset \bigcup_{k=1}^{d_m} [\tau_{mk}]\mathcal{F}_U .$$

In particular, the intersection

$$\mathcal{S}_{mk} = \mathbf{B}_r([\gamma_m]I) \cap [\tau_{mk}]\mathcal{F}_U$$

is not empty for each m, k. The set $\mathcal{D}$ defined above is stable under the action of Γ_U. Hence translating the sets $\mathcal{S}_{mk}$ back into $\mathcal{F}_U$ we conclude that

$$[\tau_{mk}^{-1}]\mathcal{S}_{mk} \subset \mathcal{D}_r \cap \mathcal{F}_U \text{ for all } m, k . \tag{4}$$

By Lemma 1.7, the sets $[\tau_{mk}^{-1}]\mathcal{S}_{mk}$ are disjoint, for pairs (m, k) defined as above.

We are now ready to apply the geometry to estimate certain series.

Let ρ be a character. The primitive **Eisenstein series** is defined by

$$E_U^{\mathrm{pr}}(Y, \rho) = \sum_{\gamma \in \Gamma_U\backslash\Gamma} \rho([\gamma]Y) .$$

We shall be concerned with the character equal to the Selberg power function, that is $q_{-z}^{(n-1)}$, so that by definition,

$$E_U^{\mathrm{pr}(n-1)}(Y, z) = \sum_{\gamma \in \Gamma_U\backslash\Gamma} \prod_{j=1}^{n-1} |\mathrm{Sub}_j[\gamma]Y|^{-z_j} .$$

First, note that any $Y \in \mathrm{Pos}_n$ lies in some ball $\mathbf{B}_r(I)$, and by Lemma 1.5, we see that the convergence of the series for any given Y is equivalent to the convergence with $Y = I$. We also have uniformity of convergence in a ball of fixed radius. In addition, we note that

$$|\mathrm{Sub}_n[\gamma]Y| = |[\gamma]Y| = |Y| \text{ for all } \gamma \in \Gamma \,.$$

Thus the convergence of the above Eisenstein series is equivalent with the convergence of

$$E_U^{\mathrm{pr}(n)}(Y, z) = \sum_{\gamma \in \Gamma_U \backslash \Gamma} \prod_{j=1}^{n} |\mathrm{Sub}_j[\gamma]Y|^{-z_j} \,.$$

Furthermore, z_n has no effect on the convergence. The main theorem is:

Theorem 2.2. *The Eisenstein series converges absolutely for all* z_j *with* $\mathrm{Re}(z_j) > 1$ *for* $j = 1, \ldots, n-1$.

Proof. First we replace z_j by a fixed number $b > 1$. We prove the convergence for $Y = I$, but we shall immediately take an average, namely we use the inequalities for $Y \in \mathbf{B}_r(I)$, with r a radius of discreteness for Γ:

$$\begin{aligned}
(5) \qquad E(I, b) &= \sum_{\gamma \in \Gamma_U \backslash \Gamma} \prod_{j=1}^{n} |\mathrm{Sub}_j([\gamma]I)|^{-b} \\
&\ll \sum_{\gamma \in \Gamma_U \backslash \Gamma} \int\limits_{\mathbf{B}_r(I)} \prod_{j=1}^{n} |\mathrm{Sub}_j[\gamma]Y|^{-b} d\mu(Y) \\
&\ll \sum_{\gamma \in \Gamma_U \backslash \Gamma} \int\limits_{\mathbf{B}_r([\gamma]I)} \prod_{j=1}^{n} |\mathrm{Sub}_j Y|^{-b} d\mu(Y) \,.
\end{aligned}$$

We combine the inclusion (4) with the estimate in (5). We use the fact that

$$|\mathrm{Sub}_j[\tau]Y| = |\mathrm{Sub}_j Y| \quad \text{for} \quad \tau \in \Gamma_U \,,$$

and we translate each integral back into $\mathcal{F}_U$. We then obtain from (5)

$$\begin{aligned}
E(I, b) &\ll_n \sum_{m=1}^{\infty} \sum_{k=1}^{d_m} \int\limits_{[\tau_{mk}^{-1}]\mathcal{S}_{mk}} \prod_{j=1}^{n} |\mathrm{Sub}_j Y|^{-b} d\mu(Y) \\
&\ll_n \int\limits_{\mathcal{D}_r \cap \mathcal{F}_{\mathcal{U}}} \prod_{j=1}^{n} |\mathrm{Sub}_j Y|^{-b} d\mu_n(Y) \,.
\end{aligned}$$

The sign $\ll_n$ means that the left side is less than the right side times a constant depending only on n. We have used here the fact already determined that the sets $[\tau_{mk}^{-1}]\mathcal{S}_{mk}$ are disjoint and contained in $\mathcal{D}_r \cap \mathcal{F}_U$. The finiteness of the integral was proved in Lemma 2.1, which thereby concludes the proof of Theorem 2.2.

3 A Variation and Extension

In the application of Chap. 8, one needs convergence of a modified Eisenstein series, specifically the following case.

Theorem 3.1. *The series*

$$\sum_{\gamma\in\Gamma_U\backslash\Gamma}\prod_{j=2}^{n}|\mathrm{Sub}_j Y|^{-z_j}$$

converges absolutely for $\mathrm{Re}(z_2) > 3/2$ *and* $\mathrm{Re}(z_j) > 1$ *with* $j \geqq 3$.

The proof is the same as the proof of Theorem 2.2. One uses the same set $\mathcal{D}(c)$. Lemma 2.1 has its analogue for the product with one term omitted. The calculus computation comes out as stated. For instance, for $n = 3$, the region $\mathcal{D}(c)$ is defined by the inequalities

$$\frac{1}{c} < uvw < c, \quad vw > \frac{1}{c}, \quad w > \frac{1}{c}\ .$$

The series is dominated by the repeated integral

$$\int\limits_{1/c}^{\infty}\int\limits_{1/wc}^{\infty}\int\limits_{1/vwc}^{c/vw}(vw)^{-3/2-\varepsilon}u^{(n+1)/2}v^{1-(n+1)/2}w^{2-(n+1)/2}du dv dw\ ,$$

which comes out up to a constant factor to be

$$\int\limits_{1/c}^{\infty}w^{-1-\varepsilon}dw\ .$$

For various reasons, including the above specific application, Maass extends the convergence theorem still further as follows [Maa 71].

Let

$$0 = k_0 < k_1 < \ldots < k_m < k_{m+1} = n$$

be a sequence of integers which we call an **integral partition** P of n. Let

$$n_i = k_i - k_{i-1},\ i = 1, \ldots, m+1\ .$$

Then $n = n_1 + \ldots + n_{m+1}$ is a partition of n in the number theoretic sense. Matrices consisting of blocks of size n_i (with $i = 1, \ldots, m+1$) on the diagonal generalize diagonal matrices. We let:

Γ_P = Subgroup of Γ consisting of elements which are upper diagonal over such block matrices, in other words, elements $\gamma = (C_{ij})$ $C_{ii} \in \Gamma_{n_i}$ for $1 \leqq i \leqq m+1$ and $C_{ij} = 0$ for $1 \leqq j < i \leqq m+1$.

In the previous cases, we have $k_j = j$, $n_j = 1$ for all $j = 1, \dots, n$, and $m+1 = n$. The description of the groups associated with a partition as above is slightly more convenient than to impose further restriction, but we note that in this case the diagonal elements may be ± 1, so we are dealing with the group T rather than the unipotent group U.

A group such that Γ_P above is also called a **parabolic subgroup**.

We define the **Eisenstein series** as a function of variables $z_1, \dots, z_m$ by

$$E_P(Y, z) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} \prod_{i=1}^{m} |\mathrm{Sub}_{k_i}[\gamma]Y|^{-z_i} \, .$$

Theorem 3.2. ([Maa 71], Sect. 7) *This Eisenstein series is absolutely convergent for*

$$\mathrm{Re}(z_i) > \frac{1}{2}(n_{i+1} + n_i) = \frac{1}{2}(k_{i+1} - k_{i-1}), i = 1, \dots, m \, .$$

Proof. One has to go through the same steps as in the preceding section, with the added complications of the more elaborate partition. One needs the Iwasawa-Jacobi coordinates with blocks,

$$Y = [u(X)] \begin{pmatrix} W_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & W_{m+1} \end{pmatrix} \text{ and } u(X) = \begin{pmatrix} I_{n_1} & \dots & X_{ij} \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_{n_{m+1}} \end{pmatrix} .$$

The measure is given by

$$d\mu_n(Y) = \prod_{i=1}^{m+1} |W_i|^{(k_i - k_{i-1} - n)} \, d\mu(W_i) \prod_{1 \leqq i < j \leqq m+1} d\mu_{\mathrm{euc}}(X_{ij}) \, .$$

The fundamental domain for Γ_P consists of those Y whose coordinates satisfy:

$W_i \in$ Fundamental domain for Γ_{n_i} $(i = 1, \dots, m+1)$ in Pos_{n_i}.
X_{ij} has coordinates $0 \leqq x_{\nu\mu} < 1$.

The domain $\mathcal{D}(c)$ is now

$$\mathcal{D}_P(c) = \Big\{ Y \text{ such that } W_i > 0 \text{ for all } i = 1, \dots, m+1;$$
$$\prod_{i=1}^{m+1} |W_i| < c, |W_m| > \frac{1}{c}, |W_m||W_{m-1}| > \frac{1}{c}, \dots, |W_m| \dots |W_1| > \frac{1}{c}. \Big\}$$

thus we merely replace a_i by $|W_i|$ throughout the previous definition. Maass gives his proof right away with the more complicated notation, and readers can refer to it.

Note that Theorem 3.1 is a special case of Theorem 3.2. However, the notation of Theorem 3.1 is simpler, and we thought it worth while to state it and indicate its proof separately, using the easier notation for the Eisenstein series.

The subgroup Γ_P is usually called a **parabolic subgroup**. Such subgroups play an essential role in the compactification of $\Gamma_n \backslash \mathrm{Pos}_n$, and in the subsequent spectral eigenfunction decomposition.