Classical and Free Infinite Divisibility and Lévy Processes

Ole E. Barndorff-Nielsen[1] and Steen Thorbjørnsen[2]

[1] Dept. of Mathematical Sciences
University of Aarhus
Ny Munkegade
DK-8000 Århus, Denmark
oebn@imf.au.dk

[2] Dept. of Mathematics & Computer Science
University of Southern Denmark
Campusvej 55
DK-5230 Odense, Denmark
steenth@imada.sdu.dk

O.E. Barndorff-Nielsen and S. Thorbjørnsen: *Classical and Free Infinite Divisibility and Lévy Processes*, Lect. Notes Math. **1866**, 33–159 (2006)
www.springerlink.com

1 Introduction

The present lecture notes have grown out of a wish to understand whether certain important concepts of classical infinite divisibility and Lévy processes, such as selfdecomposability and the Lévy-Itô decomposition, have natural and interesting analogues in free probability. The study of this question has led to new links between classical and free Lévy theory, and to some new results in the classical setting, that seem of independent interest. The new concept of Upsilon mappings have a key role in both respects. These are regularizing mappings from the set of Lévy measures into itself or, otherwise interpreted, mappings of the class of infinitely divisible laws into itself. One of these mappings, Υ, provides a direct connection to the Lévy-Khintchine formula of free probability.

The next Section recalls a number of concepts and results from the classical framework, and in Section 3 the basic Upsilon mappings Υ_0 and Υ are introduced and studied. They are shown to be smooth, injective and regularizing, and their relation to important subclasses of infinitely divisible laws is discussed. Subsequently Υ_0 and Υ are generalized to one-parameter families of mappings $(\Upsilon_0^\alpha)_{\alpha\in[0,1]}$ and $(\Upsilon^\alpha)_{\alpha\in[0,1]}$ with similar properties, and which interpolate between Υ_0 (resp. Υ) and the identity mapping on the set of Lévy measures (resp. the class of infinitely divisible laws). Other types of Upsilon mappings are also considered, including some generalizations to higher dimensions. Section 4 gives an introduction to non-commutative probability, particularly free infinite divisibility, and then takes up some of the above-mentioned questions concerning links between classical and free Lévy theory. The discussion of such links is continued in Section 5, centered around the Upsilon mapping Υ and the closely associated Bercovici-Pata mapping Λ. The final Section 6 discusses free stochastic integration and establishes a free analogue of the Lévy-Ito representation.

The material presented in these lecture notes is based on the authors' papers [BaTh02a], [BaTh02b], [BaTh02c], [BaTh04a], [BaTh04b] and [BaTh05].

2 Classical Infinite Divisibility and Lévy Processes

The classical theory of infinite divisibility and Lévy processes was founded by Kolmogorov, Lévy and Khintchine in the Nineteen Thirties. The monographs [Sa99] and [Be96],[Be97] are main sources for information on this theory. For some more recent results, including various types of applications, see [BaMiRe01].

Here we recall some of the most basic facts of the theory, and we discuss a hierarchy of important subclasses of the space of infinitely divisible distributions.

2.1 Basics of Infinite Divisibility

The class of infinitely divisible probability measures on the real line will here be denoted by $\mathcal{ID}(*)$. A probability measure μ on $\mathbb{R}$ belongs to $\mathcal{ID}(*)$ if there exists, for each positive integer n, a probability measure μ_n, such that

$$\mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ terms}},$$

where $*$ denotes the usual convolution of probability measures.

We recall that a probability measure μ on $\mathbb{R}$ is infinitely divisible if and only if its characteristic function (or Fourier transform) f_μ has the Lévy-Khintchine representation:

$$\log f_\mu(u) = \mathrm{i}\gamma u + \int_{\mathbb{R}} \Big(\mathrm{e}^{\mathrm{i}ut} - 1 - \frac{\mathrm{i}ut}{1+t^2}\Big)\frac{1+t^2}{t^2}\,\sigma(\mathrm{d}t), \quad (u \in \mathbb{R}), \tag{2.1}$$

where γ is a real constant and σ is a finite measure on $\mathbb{R}$. In that case, the pair (γ, σ) is uniquely determined, and is termed the generating pair for μ.

The function $\log f_\mu$ is called the *cumulant transform* for μ and is also denoted by C_μ, as we shall do often in the sequel.

In the literature, there are several alternative ways of writing the above representation. In recent literature, the following version seems to be preferred (see e.g. [Sa99]):

$$\log f_\mu(u) = \mathrm{i}\eta u - \tfrac{1}{2}au^2 + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\big)\ \rho(\mathrm{d}t), \quad (u \in \mathbb{R}), \tag{2.2}$$

where η is a real constant, a is a non-negative constant and ρ is a Lévy measure on $\mathbb{R}$ according to Definition 2.1 below. Again, a, ρ and η are uniquely determined by μ and the triplet (a, ρ, η) is called the *characteristic triplet* for μ.

Definition 2.1. *A Borel measure ρ on $\mathbb{R}$ is called a Lévy measure, if it satisfies the following conditions:*

$$\rho(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}} \min\{1, t^2\}\ \rho(\mathrm{d}t) < \infty.$$

The relationship between the two representations (2.1) and (2.2) is as follows:

$$\begin{aligned} a &= \sigma(\{0\}), \\ \rho(\mathrm{d}t) &= \frac{1+t^2}{t^2} \cdot 1_{\mathbb{R}\setminus\{0\}}(t)\ \sigma(\mathrm{d}t), \\ \eta &= \gamma + \int_{\mathbb{R}} t\Big(1_{[-1,1]}(t) - \frac{1}{1+t^2}\Big)\ \rho(\mathrm{d}t). \end{aligned} \tag{2.3}$$

2.2 Classical Lévy Processes

For a (real-valued) random variable X defined on a probability space $(\Omega, \mathcal{F}, P)$, we denote by $L\{X\}$ the distribution[1] of X.

Definition 2.2. *A real valued stochastic process $(X_t)_{t\geq 0}$, defined on a probability space $(\Omega, \mathcal{F}, P)$, is called a Lévy process, if it satisfies the following conditions:*

(i) *whenever $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the increments*

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}},$$

are independent random variables.

(ii) *$X_0 = 0$, almost surely.*

(iii) *for any s, t in $[0, \infty[$, the distribution of $X_{s+t} - X_s$ does not depend on s.*

(iv) *(X_t) is stochastically continuous, i.e. for any s in $[0, \infty[$ and any positive ϵ, we have: $\lim_{t\to 0} P(|X_{s+t} - X_s| > \epsilon) = 0$.*

(v) *for almost all ω in Ω, the sample path $t \mapsto X_t(\omega)$ is right continuous (in $t \geq 0$) and has left limits (in $t > 0$).*

[1] L stands for "the law of".

If a stochastic process $(X_t)_{t\geq 0}$ satisfies conditions (i)-(iv) in the definition above, we say that (X_t) is a *Lévy process in law*. If (X_t) satisfies conditions (i), (ii), (iv) and (v) (respectively (i), (ii) and (iv)) it is called an *additive process* (respectively an *additive process in law*). Any Lévy process in law (X_t) has a modification which is a Lévy process, i.e. there exists a Lévy process (Y_t), defined on the same probability space as (X_t), and such that $X_t = Y_t$ with probability one, for all t. Similarly any additive process in law has a modification which is a genuine additive process. These assertions can be found in [Sa99, Theorem 11.5].

Note that condition (iv) is equivalent to the condition that $X_{s+t} - X_s \to 0$ in distribution, as $t \to 0$. Note also that under the assumption of (ii) and (iii), this condition is equivalent to saying that $X_t \to 0$ in distribution, as $t \searrow 0$.

The concepts of infinitely divisible probability measures and of Lévy processes are closely connected, since there is a one-to-one correspondance between them. Indeed, if (X_t) is a Lévy process, then $L\{X_t\}$ is infinitely divisible for all t in $[0,\infty[$, since for any positive integer n

$$X_t = \sum_{j=1}^{n} (X_{jt/n} - X_{(j-1)t/n}),$$

and hence, by (i) and (iii) of Definition 2.2,

$$L\{X_t\} = \underbrace{L\{X_{t/n}\} * L\{X_{t/n}\} * \cdots * L\{X_{t/n}\}}_{n \text{ terms}}.$$

Moreover, for each t, $L\{X_t\}$ is uniquely determined by $L\{X_1\}$ via the relation $L\{X_t\} = L\{X_1\}^t$ (see [Sa99, Theorem 7.10]). Conversely, for any infinitely divisible distribution μ on $\mathbb{R}$, there exists a Lévy process (X_t) (on some probability space $(\Omega, \mathcal{F}, P)$), such that $L\{X_1\} = \mu$ (cf. [Sa99, Theorem 7.10 and Corollary 11.6]).

2.3 Integration with Respect to Lévy Processes

We start with a general discussion of the existence of stochastic integrals w.r.t. (classical) Lévy processes and their associated cumulant functions. Some related results are given in [ChSh02] and [Sa00], but they do not fully cover the situation considered below.

Throughout, we shall use the notation $C\{u \ddagger X\}$ to denote the cumulant function of (the distribution of) a random variable X, evaluated at the real number u.

Recall that a sequence (σ_n) of *finite* measures on $\mathbb{R}$ is said to converge weakly to a finite measure σ on $\mathbb{R}$, if

$$\int_{\mathbb{R}} f(t)\, \sigma_n(\mathrm{d}t) \to \int_{\mathbb{R}} f(t)\, \sigma(\mathrm{d}t), \quad \text{as } n \to \infty, \tag{2.4}$$

for any bounded continuous function $f\colon \mathbb{R} \to \mathbb{C}$. In that case, we write $\sigma_n \xrightarrow{\text{w}} \sigma$, as $n \to \infty$.

Remark 2.3. Recall that a sequence (x_n) of points in a metric space (M, d) converges to a point x in M, if and only if every subsequence $(x_{n'})$ has a subsequence $(x_{n''})$ converging to x. Taking $M = \mathbb{R}$ it is an immediate consequence of (2.4) that $\sigma_n \xrightarrow{\text{w}} \sigma$ if and only if any subsequence $(\sigma_{n'})$ has a subsequence $(\sigma_{n''})$ which converges weakly to σ. This observation, which we shall make use of in the folowing, follows also from the fact, that weak convergence can be viewed as convergence w.r.t. a certain metric on the set of bounded measures on $\mathbb{R}$ (the Lévy metric).

Lemma 2.4. *Let $(X_{n,m})_{n,m\in\mathbb{N}}$ be a family of random variables indexed by $\mathbb{N}\times\mathbb{N}$ and all defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that*

$$\forall u \in \mathbb{R}\colon \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tu}\, L\{X_{n,m}\}(\mathrm{d}t) \to 1, \quad \textit{as } n, m \to \infty. \tag{2.5}$$

Then $X_{n,m} \xrightarrow{\text{P}} 0$, as $n, m \to \infty$, in the sense that

$$\forall \epsilon > 0\colon P(|X_{n,m}| > \epsilon) \to 0, \quad \textit{as } n, m \to \infty. \tag{2.6}$$

Proof. This is, of course, a variant of the usual continuity theorem for characteristic functions. For completeness, we include a proof.

To prove (2.6), it suffices, by a standard argument, to prove that $L\{X_{n,m}\} \xrightarrow{\text{w}} \delta_0$, as $n, m \to \infty$, i.e. that

$$\forall f \in C_b(\mathbb{R})\colon \int_{\mathbb{R}} f(t)\, L\{X_{n,m}\}(\mathrm{d}t) \longrightarrow \int_{\mathbb{R}} f(t)\, \delta_0(\mathrm{d}t) = f(0), \quad \text{as } n, m \to \infty, \tag{2.7}$$

where $C_b(\mathbb{R})$ denotes the space of continuous bounded functions $f\colon \mathbb{R} \to \mathbb{R}$.

So assume that (2.7) is not satisfied. Then we may choose f in $C_b(\mathbb{R})$ and ϵ in $]0, \infty[$ such that

$$\forall N \in \mathbb{N}\ \exists n, m \geq N\colon \Big| \int_{\mathbb{R}} f(t)\, L\{X_{n,m}\}(\mathrm{d}t) - f(0) \Big| \geq \epsilon.$$

By an inductive argument, we may choose a sequence $n_1 \leq n_2 < n_3 \leq n_4 < \cdots$, of positive integers, such that

$$\forall k \in \mathbb{N}\colon \Big| \int_{\mathbb{R}} f(t)\, L\{X_{n_{2k}, n_{2k-1}}\}(\mathrm{d}t) - f(0) \Big| \geq \epsilon. \tag{2.8}$$

On the other hand, it follows from (2.5) that

$$\forall u \in \mathbb{R}\colon \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tu}\, L\{X_{n_{2k}, n_{2k-1}}\}(\mathrm{d}t) \to 1, \quad \text{as } k \to \infty,$$

so by the usual continuity theorem for characteristic functions, we find that $L\{X_{n_{2k}, n_{2k-1}}\} \xrightarrow{\text{w}} \delta_0$. But this contradicts (2.8). ◻

Lemma 2.5. *Assume that $0 \le a < b < \infty$, and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Let, further, $(X_t)_{t\ge 0}$ be a (classical) Lévy process, and put $\mu = L\{X_1\}$. Then the stochastic integral $\int_a^b f(t)\,\mathrm{d}X_t$ exists as the limit, in probability, of approximating Riemann sums. Furthermore, $L\{\int_a^b f(t)\,\mathrm{d}X_t\} \in \mathcal{ID}(*)$, and*

$$C\big\{u \ddagger \textstyle\int_a^b f(t)\,\mathrm{d}X_t\big\} = \displaystyle\int_a^b C_\mu(uf(t))\,\mathrm{d}t,$$

for all u in $\mathbb{R}$.

Proof. This is well-known, but, for completeness, we sketch the proof: By definition (cf. [Lu75]), $\int_a^b f(t)\,\mathrm{d}X_t$ is the limit in probability of the Riemann sums:

$$R_n := \sum_{j=1}^n f(t_j^{(n)})\big(X_{t_j^{(n)}} - X_{t_{j-1}^{(n)}}\big),$$

where, for each n, $a = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = b$ is a subdivision of $[a,b]$, such that $\max_{j=1,2,\dots,n}(t_j^{(n)} - t_{j-1}^{(n)}) \to 0$ as $n \to \infty$. Since (X_t) has stationary, independent increments, it follows that for any u in $\mathbb{R}$,

$$\begin{aligned}
C\{u \ddagger R_n\} &= \sum_{j=1}^n C\big\{f(t_j^{(n)})u \ddagger \big(X_{t_j^{(n)}} - X_{t_{j-1}^{(n)}}\big)\big\} \\
&= \sum_{j=1}^n C\big\{f(t_j^{(n)})u \ddagger X_{t_j^{(n)} - t_{j-1}^{(n)}}\big\} \\
&= \sum_{j=1}^n C_\mu\big(f(t_j^{(n)})u\big) \cdot (t_j^{(n)} - t_{j-1}^{(n)}),
\end{aligned}$$

where, in the last equality, we used [Sa99, Theorem 7.10]. Since C_μ and f are both continuous, it follows that

$$C\big\{u \ddagger \textstyle\int_a^b f(t)\,\mathrm{d}X_t\big\} = \displaystyle\lim_{n\to\infty} \sum_{j=1}^n C_\mu\big(f(t_j^{(n)})u\big) \cdot (t_j^{(n)} - t_{j-1}^{(n)}) = \int_a^b C_\mu(f(t)u)\,\mathrm{d}t,$$

for any u in $\mathbb{R}$. □

Proposition 2.6. *Assume that $0 \le a < b \le \infty$, and let $f\colon]a,b[\to \mathbb{R}$ be a continuous function. Let, further, $(X_t)_{t\ge 0}$ be a classical Lévy process, and put $\mu = L\{X_1\}$. Assume that*

$$\forall u \in \mathbb{R}\colon \int_a^b \big|C_\mu(uf(t))\big|\,\mathrm{d}t < \infty.$$

Then the stochastic integral $\int_a^b f(t)\,\mathrm{d}X_t$ *exists as the limit, in probability, of the sequence* $(\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t)_{n\in\mathbb{N}}$*, where* (a_n) *and* (b_n) *are arbitrary sequences in* $]a,b[$ *such that* $a_n \le b_n$ *for all* n *and* $a_n \searrow a$ *and* $b_n \nearrow b$ *as* $n \to \infty$*.*

Furthermore, $L\{\int_a^b f(t)\,\mathrm{d}X_t\} \in \mathcal{ID}(*)$ *and*

$$C\{u \ddagger \textstyle\int_a^b f(t)\,\mathrm{d}X_t\} = \int_a^b C_\mu(uf(t))\,\mathrm{d}t, \tag{2.9}$$

for all u *in* $\mathbb{R}$*.*

Proof. Let (a_n) and (b_n) be arbitrary sequences in $]a,b[$, such that $a_n \le b_n$ for all n and $a_n \searrow a$ and $b_n \nearrow b$ as $n \to \infty$. Then, for each n, consider the stochastic integral $\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t$. Since the topology corresponding to convergence in probability is complete, the convergence of the sequence $(\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t)_{n\in\mathbb{N}}$ will follow, once we have verified that it is a Cauchy sequence. Towards this end, note that whenever $n > m$ we have that

$$\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t - \int_{a_m}^{b_m} f(t)\,\mathrm{d}X_t = \int_{a_n}^{a_m} f(t)\,\mathrm{d}X_t + \int_{b_m}^{b_n} f(t)\,\mathrm{d}X_t,$$

so it suffices to show that

$$\int_{a_n}^{a_m} f(t)\,\mathrm{d}X_t \xrightarrow{\mathrm{P}} 0 \quad \text{and} \quad \int_{b_m}^{b_n} f(t)\,\mathrm{d}X_t \xrightarrow{\mathrm{P}} 0, \quad \text{as } n,m \to \infty.$$

By Lemma 2.4, this, in turn, will follow if we prove that

$$\forall u \in \mathbb{R}\colon C\{u \ddagger \textstyle\int_{a_n}^{a_m} f(t)\,\mathrm{d}X_t\} \longrightarrow 0, \qquad \text{as } n,m \to \infty,$$

and

$$\forall u \in \mathbb{R}\colon C\{u \ddagger \textstyle\int_{b_m}^{b_n} f(t)\,\mathrm{d}X_t\} \longrightarrow 0, \qquad \text{as } n,m \to \infty. \tag{2.10}$$

But for n,m in $\mathbb{N}$, $m < n$, it follows from Lemma 2.5 that

$$\big|C\{u \ddagger \textstyle\int_{a_n}^{a_m} f(t)\,\mathrm{d}X_t\}\big| \le \displaystyle\int_{a_n}^{a_m} \big|C_\mu(uf(t))\big|\,\mathrm{d}t, \tag{2.11}$$

and since $\int_a^b |C_\mu(uf(t))|\,\mathrm{d}t < \infty$, the right hand side of (2.11) tends to 0 as $n,m \to \infty$. Statement (2.10) follows similarly.

To prove that $\lim_{n\to\infty} \int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t$ does not depend on the choice of sequences (a_n) and (b_n), let (a_n') and (b_n') be sequences in $]a,b[$, also satisfying that $a_n' \le b_n'$ for all n, and that $a_n' \searrow a$ and $b_n' \nearrow b$ as $n \to \infty$. We may then, by an inductive argument, choose sequences $n_1 < n_2 < n_3 < \cdots$ and $m_1 < m_2 < m_3 \cdots$ of positive integers, such that

$$a_{n_1} > a_{m_1}' > a_{n_2} > a_{m_2}' > \cdots, \quad \text{and} \quad b_{n_1} < b_{m_1}' < b_{n_2} < b_{m_2}' < \cdots.$$

Consider then the sequences (a''_k) and (b''_k) given by:

$$a''_{2k-1} = a_{n_k},\ a''_{2k} = a'_{m_k}, \quad \text{and} \quad b''_{2k-1} = b_{n_k},\ b''_{2k} = b'_{m_k}, \qquad (k \in \mathbb{N}).$$

Then $a''_k \le b''_k$ for all k, and $a''_k \searrow a$ and $b''_k \nearrow b$ as $k \to \infty$. Thus, by the argument given above, all of the following limits exist (in probability), and, by "sub-sequence considerations", they have to be equal:

$$\begin{aligned}\lim_{n\to\infty} \int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t &= \lim_{k\to\infty} \int_{a_{n_k}}^{b_{n_k}} f(t)\,\mathrm{d}X_t = \lim_{k\to\infty} \int_{a''_{2k-1}}^{b''_{2k-1}} f(t)\,\mathrm{d}X_t \\ &= \lim_{k\to\infty} \int_{a''_k}^{b''_k} f(t)\,\mathrm{d}X_t = \lim_{k\to\infty} \int_{a''_{2k}}^{b''_{2k}} f(t)\,\mathrm{d}X_t \\ &= \lim_{k\to\infty} \int_{a'_{m_k}}^{b'_{m_k}} f(t)\,\mathrm{d}X_t = \lim_{n\to\infty} \int_{a'_n}^{b'_n} f(t)\,\mathrm{d}X_t,\end{aligned}$$

as desired.

To verify, finally, the last statements of the proposition, let (a_n) and (b_n) be sequences as above, so that, by definition, $\int_a^b f(t)\,\mathrm{d}X_t = \lim_{n\to\infty} \int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t$ in probability. Since $\mathcal{ID}(*)$ is closed under weak convergence, this implies that $L\{\int_a^b f(t)\,\mathrm{d}X_t\} \in \mathcal{ID}(*)$. To prove (2.9), we find next, using Gnedenko's theorem (cf. [GnKo68, §19, Theorem 1] and Lemma 2.5, that

$$\begin{aligned}C\{u \ddagger \textstyle\int_a^b f(t)\,\mathrm{d}X_t\} &= \lim_{n\to\infty} C\{u \ddagger \textstyle\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t\} \\ &= \lim_{n\to\infty} \int_{a_n}^{b_n} C_\mu(uf(t))\,\mathrm{d}t = \int_a^b C_\mu(uf(t))\,\mathrm{d}t,\end{aligned}$$

for any u in $\mathbb{R}$, and where the last equality follows from the assumption that $\int_a^b |C_\mu(uf(t))|\,\mathrm{d}t < \infty$. This concludes the proof. □

2.4 The Classical Lévy-Itô Decomposition

The Lévy-Itô decomposition represents a (classical) Lévy process (X_t) as the sum of two independent Lévy processes, the first of which is continuous (and hence a Brownian motion) and the second of which is, loosely speaking, the sum of the jumps of (X_t). In order to rigorously describe the sum of jumps part, one needs to introduce the notion of Poisson random measures. Before doing so, we introduce some notation: For any λ in $[0,\infty]$ we denote by $\text{Poiss}^*(\lambda)$ the (classical) Poisson distribution with mean λ. In particular, $\text{Poiss}^*(0) = \delta_0$ and $\text{Poiss}^*(\infty) = \delta_\infty$.

Definition 2.7. *Let $(\Theta, \mathcal{E}, \nu)$ be a σ-finite measure space and let $(\Omega, \mathcal{F}, P)$ be a probability space. A Poisson random measure on $(\Theta, \mathcal{E}, \nu)$ and defined on $(\Omega, \mathcal{F}, P)$ is a mapping $N\colon \mathcal{E} \times \Omega \to [0,\infty]$, satisfying the following conditions:*

(i) *For each E in $\mathcal{E}$, $N(E) = N(E, \cdot)$ is a random variable on $(\Omega, \mathcal{F}, P)$.*
(ii) *For each E in $\mathcal{E}$, $L\{N(E)\} = \mathrm{Poiss}^*(\nu(E))$.*
(iii) *If $E_1, \dots, E_n$ are disjoint sets from $\mathcal{E}$, then $N(E_1), \dots, N(E_n)$ are independent random variables.*
(iv) *For each fixed ω in Ω, the mapping $E \mapsto N(E, \omega)$ is a (positive) measure on $\mathcal{E}$.*

In the setting of Definition 2.7, the measure ν is called the *intensity measure* for the Poisson random measure N. Let $(\Theta, \mathcal{E}, \nu)$ be a σ-finite measure space, and let N be a Poisson random measure on it (defined on some probability space $(\Omega, \mathcal{F}, P)$). Then for any $\mathcal{E}$-measurable function $f\colon \Theta \to [0, \infty]$, we may, for all ω in Ω, consider the integral $\int_\Theta f(\theta)\, N(\mathrm{d}\theta, \omega)$. We obtain, thus, an everywhere defined mapping on Ω, given by: $\omega \mapsto \int_\Theta f(\theta)\, N(\mathrm{d}\theta, \omega)$. This observation is the starting point for the theory of integration with respect to Poisson random measures, from which we shall need the following basic properties:

Proposition 2.8. *Let N be a Poisson random measure on the σ-finite measure space $(\Theta, \mathcal{E}, \nu)$, defined on the probability space $(\Omega, \mathcal{F}, P)$.*

(i) *For any positive $\mathcal{E}$-measurable function $f\colon \Theta \to [0, \infty]$, $\int_\Theta f(\theta)\, N(\mathrm{d}\theta)$ is an $\mathcal{F}$-measurable positive function, and*

$$\mathbb{E}\Big\{ \int_\Theta f(\theta)\, N(\mathrm{d}\theta) \Big\} = \int_\Theta f \,\mathrm{d}\nu.$$

(ii) *If f is a real-valued function in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$, then $f \in \mathcal{L}^1(\Theta, \mathcal{E}, N(\cdot, \omega))$ for almost all ω in Ω, $\int_\Theta f(\theta)\, N(\mathrm{d}\theta) \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ and*

$$\mathbb{E}\Big\{ \int_\Theta f(\theta)\, N(\mathrm{d}\theta) \Big\} = \int_\Theta f \,\mathrm{d}\nu.$$

The proof of the above proposition follows the usual pattern, proving it first for simple (positive) $\mathcal{E}$-measurable functions and then, via an approximation argument, obtaining the results in general. We shall adapt the same method in developing integration theory with respect to free Poisson random measures in Section 6.4 below.

We are now in a position to state the Lévy-Itô decomposition for classical Lévy processes. We denote the Lebesgue measure on $\mathbb{R}$ by Leb.

Theorem 2.9 (Lévy-Itô Decomposition). *Let (X_t) be a classical (genuine) Lévy process, defined on a probability space $(\Omega, \mathcal{F}, P)$, and let ρ be the Lévy measure appearing in the generating triplet for $L\{X_1\}$.*

(i) *Assume that $\int_{-1}^{1} |x|\, \rho(\mathrm{d}x) < \infty$. Then (X_t) has a representation in the form:*

$$X_t \overset{\text{a.s.}}{=} \gamma t + \sqrt{a} B_t + \int_{]0,t] \times \mathbb{R}} x\, N(\mathrm{d}s, \mathrm{d}x), \tag{2.12}$$

where $\gamma \in \mathbb{R}$*,* $a \geq 0$*,* (B_t) *is a Brownian motion and* N *is a Poisson random measure on* $(]0,\infty[\times\mathbb{R}, \mathrm{Leb}\otimes\rho)$*. Furthermore, the last two terms on the right hand side of* (2.12) *are independent Lévy processes on* $(\Omega, \mathcal{F}, P)$*.*

(ii) *If* $\int_{-1}^{1} |x|\,\rho(\mathrm{d}x) = \infty$*, then we still have a decomposition like* (2.12)*, but the integral* $\int_{]0,t]\times\mathbb{R}} x\,N(\mathrm{d}s,\mathrm{d}x)$ *no longer makes sense and has to be replaced by the limit:*

$$Y_t = \lim_{\epsilon\searrow 0}\Big[\int_{]0,t]\times(\mathbb{R}\setminus[-\epsilon,\epsilon])} xN(\mathrm{d}u,\mathrm{d}x) - \int_{]0,t]\times([-1,1]\setminus[-\epsilon,\epsilon])} x\mathrm{Leb}\otimes\rho(\mathrm{d}u,\mathrm{d}x)\Big].$$

The process (Y_t) *is, again, a Lévy process, which is independent of* (B_t)*.*

The symbol $\stackrel{\text{a.s.}}{=}$ in (2.12) means that the two random variables are equal with probability 1 (a.s. stands for “almost surely”). The Poisson random measure N appearing in the right hand side of (2.12) is, specifically, given by

$$N(E,\omega) = \#\big\{s \in\,]0,\infty[\,\big|\, (s, \Delta X_s(\omega)) \in E\big\},$$

for any Borel subset E of $]0,\infty[\times(\mathbb{R}\setminus\{0\})$, and where $\Delta X_s = X_s - \lim_{u\nearrow s} X_u$. Consequently, the integral in the right hand side of (2.12) is, indeed, the sum of the jumps of X_t until time t: $\int_{]0,t]\times\mathbb{R}} x\,N(\mathrm{d}s,\mathrm{d}x) = \sum_{s\leq t}\Delta X_s$. The condition $\int_{-1}^{1}|x|\,\rho(\mathrm{d}x) < \infty$ ensures that this sum converges. Without that condition, one has to consider the “compensated sums of jumps” given by the process (Y_t). For a proof of Theorem 2.9 we refer to [Sa99].

2.5 Classes of Infinitely Divisible Probability Measures

In the following, we study, in various connections, dilations of Borel measures by constants. If ρ is a Borel measure on $\mathbb{R}$ and c is a non-zero real constant, then the dilation of ρ by c is the measure $D_c\rho$ given by

$$D_c\rho(B) = \rho(c^{-1}B),$$

for any Borel set B. Furthermore, we put $D_0\rho = \delta_0$ (the Dirac measure at 0). We shall also make use of terminology like

$$D_c\rho(\mathrm{d}x) = \rho(c^{-1}\mathrm{d}x),$$

whenever $c \neq 0$. With this notation at hand, we now introduce several important classes of infinitely divisible probability measures on $\mathbb{R}$.

In classical probability theory, we have the following fundamental hierarchy:

$$\mathcal{G}(*) \subset \mathcal{S}(*) \subset \mathcal{R}(*) \subset \mathcal{T}(*) \subset \left\{\begin{matrix}\mathcal{L}(*)\\ \mathcal{B}(*)\end{matrix}\right\} \subset \mathcal{ID}(*) \subset \mathcal{P}, \tag{2.13}$$

where

(i) $\mathcal{P}$ is the class of all probability measures on $\mathbb{R}$.
(ii) $\mathcal{ID}(*)$ is the class of infinitely divisible probability measures on $\mathbb{R}$ (as defined above).
(iii) $\mathcal{L}(*)$ is the class of selfdecomposable probability measures on $\mathbb{R}$, i.e.

$$\mu \in \mathcal{L}(*) \iff \forall c \in]0,1[\ \exists \mu_c \in \mathcal{P}\colon \mu = D_c\mu * \mu_c.$$

(iv) $\mathcal{B}(*)$ is the *Goldie-Steutel-Bondesson class*, i.e. the smallest subclass of $\mathcal{ID}(*)$, which contains all mixtures of positive and negative exponential distributions[2] and is closed under convolution and weak limits.
(v) $\mathcal{T}(*)$ is the *Thorin Class*, i.e. the smallest subclass of $\mathcal{ID}(*)$, which contains all positive and negative Gamma distributions[2] and is closed under convolution and weak limits.
(vi) $\mathcal{R}(*)$ is the class of tempered stable distributions, which will defined below in terms of the Lévy-Khintchine representation.
(vii) $\mathcal{S}(*)$ is the class of stable probability measures on $\mathbb{R}$, i.e.

$$\begin{aligned}\mu \in \mathcal{S}(*) \iff &\{\psi(\mu) \mid \psi\colon \mathbb{R} \to \mathbb{R},\ \text{increasing affine transformation}\}\\ &\text{is closed under convolution } *.\end{aligned}$$

(viii) $\mathcal{G}(*)$ is the class of Gaussian (or normal) distributions on $\mathbb{R}$.

The classes of probability measures, defined above, are all of considerable importance in classical probability and are of major applied interest. In particular the classes $\mathcal{S}(*)$ and $\mathcal{L}(*)$ have received a lot of attention. This is, partly, explained by their characterizations as limit distributions of certain types of sums of independent random variables. Briefly, the stable laws are those that occur as limiting distributions for $n \to \infty$ of affine transformations of sums $X_1 + \cdots + X_n$ of independent identically distributed random variables (subject to the assumption of uniform asymptotic neglibility). Dropping the assumption of identical distribution one arrives at the class $\mathcal{L}(*)$. Finally, the class $\mathcal{ID}(*)$ of all infinitely divisible distributions consists of the limiting laws for sums of independent random variables of the form $X_{n1} + \cdots + X_{nk_n}$ (again subject to the assumption of uniform asymptotic neglibility).

An alternative characterization of selfdecomposability says that (the distribution of) a random variable Y is selfdecomposable if and only if for all c in $]0,1[$ the characteristic function f of Y can be factorised as

$$f(\zeta) = f(c\zeta)f_c(\zeta), \tag{2.14}$$

for some characteristic function f_c (which then, as can be proved, necessarily corresponds to an infinitely divisible random variable Y_c). In other words, considering Y_c as independent of Y we have a representation in law

[2] A negative exponential (resp. Gamma) distribution is of the form $D_{-1}\mu$, where μ is a positive exponential (resp. Gamma) distribution.

$$Y \stackrel{\mathrm{d}}{=} cY + Y_c$$

(where the symbol $\stackrel{\mathrm{d}}{=}$ means that the random variables on the left and right hand side have the same distribution). This latter formulation makes the idea of selfdecomposability of immediate appeal from the viewpoint of mathematical modeling. Yet another key characterization is given by the following result which was first proved by Wolfe in [Wo82] and later generalized and strengthened by Jurek and Verwaat ([JuVe83], cf. also Jurek and Mason, [JuMa93, Theorem 3.6.6]): A random variable Y has law in $\mathcal{L}(*)$ if and only if Y has a representation of the form

$$Y \stackrel{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-t} \, \mathrm{d}X_t, \tag{2.15}$$

where X_t is a Lévy process satisfying $\mathbb{E}\{\log(1 + |X_1|)\} < \infty$. The process $X = (X_t)_{t \geq 0}$ is termed the *background driving Lévy process* or the BDLP corresponding to Y.

There is a very extensive literature on the theory and applications of stable laws. A standard reference for the theoretical properties is [SaTa94], but see also [Fe71] and [BaMiRe01]. In comparison, work on selfdecomposability has up till recently been somewhat limited. However, a comprehensive account of the theoretical aspects of selfdecomposability, and indeed of infinite divisibility in general, is now available in [Sa99]. Applications of selfdecomposability are discussed, inter alia, in [BrReTw82], [Ba98], [BaSh01a] and [BaSh01b].

The class $\mathcal{R}(*)$, its d-dimensional version $\mathcal{R}^d(*)$, and the associated Lévy processes and Ornstein-Uhlenbeck type processes were introduced and studied extensively by Rosinski (see [Ros04]), following earlier works by other authors on special instances of this kind of stochastic objects (see references in [Ros04]). These processes are of considerable interest as they exhibit stable like behaviour over short time spans and - in the Lévy process case - Gaussian behaviour for long lags. That paper also develops powerful series representations of shot noise type for the processes.

By $\mathcal{ID}^+(*)$ we denote the class of infinitely divisible probability measures, which are concentrated on $[0, \infty[$. The classes $\mathcal{S}^+(*), \mathcal{R}^+(*), \mathcal{T}^+(*), \mathcal{B}^+(*)$ and $\mathcal{L}^+(*)$ are defined similarly. The class $\mathcal{T}^+(*)$, in particular, is the class of measures which was originally studied by O. Thorin in [Th77]. He introduced it as the smallest subclass of $\mathcal{ID}(*)$, which contains the Gamma distributions and is closed under convolution and weak limits. This group of distributions is also referred to as generalized gamma convolutions and have been extensively studied by Bondesson in [Bo92]. (It is noteworthy, in the present context, that Bondesson uses Pick functions, which are essentially Cauchy transforms, as a main tool in his investigations. The Cauchy transform also occur as a key tool in the study of free infinite divisibility; see Section 4.4).

Example 2.10. An important class of generalized Gamma convolutions are the generalized inverse Gaussian distributions: Assume that λ in $\mathbb{R}$ and γ, δ in

$[0,\infty[$ satisfy the conditions: $\lambda < 0 \Rightarrow \delta > 0$, $\lambda = 0 \Rightarrow \gamma, \delta > 0$ and $\lambda > 0 \Rightarrow \gamma > 0$. Then the *generalized inverse Gaussian distribution* $\mathrm{GIG}(\lambda, \delta, \gamma)$ is the distribution on $\mathbb{R}_+$ with density (w.r.t. Lebesgue measure) given by

$$g(t; \lambda, \delta, \gamma) = \frac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)} t^{\lambda-1} \exp\big\{ -\tfrac{1}{2}(\delta^2 t^{-1} + \gamma^2 t)\big\}, \quad t \geq 0,$$

where K_λ is the modified Bessel function of the third kind and with index λ. For all λ, δ, γ (subject to the above restrictions) $\mathrm{GIG}(\lambda, \delta, \gamma)$ belongs to $\mathcal{T}^+(*)$, and it is not stable unless $\lambda = -\frac{1}{2}$ and $\gamma = 0$. For special choices of the parameters, one obtains the gamma distributions (and hence the exponential and χ^2 distributions), the inverse Gaussian distributions, the reciprocal inverse Gaussian distributions[3] and the reciprocal gamma distributions.

Example 2.11. A particularly important group of examples of selfdecomposable laws, supported on the whole real line, are the marginal laws of subordinated Brownian motion with drift, when the subordinator process is generated by one of the generalized gamma convolutions. The induced selfdecomposability of the marginals follows from a result due to Sato (cf. [Sa00]).

We introduce next some notation that will be convenient in Section 3.3 below. There, we shall also consider translations of the measures in the classes $\mathcal{T}^+(*)$, $\mathcal{L}^+(*)$ and $\mathcal{ID}^+(*)$. For a real constant c, we consider the mapping $\tau_c \colon \mathbb{R} \to \mathbb{R}$ given by

$$\tau_c(x) = x + c, \qquad (x \in \mathbb{R}),$$

i.e. τ_c is translation by c. For a Borel measure μ on $\mathbb{R}$, we may then consider the translated measure $\tau_c(\mu)$ given by

$$\tau_c(\mu)(B) = \mu(B - c),$$

for any Borel set B in $\mathbb{R}$. Note, in particular, that if μ is infinitely divisible with characteristic triplet (a, ρ, η), then $\tau_c(\mu)$ is infinitely divisible with characteristic triplet $(a, \rho, \eta + c)$.

Definition 2.12. *We introduce the following notation:*

$$\mathcal{ID}^+_\tau(*) = \{\mu \in \mathcal{ID}(*) \mid \exists c \in \mathbb{R} \colon \tau_c(\mu) \in \mathcal{ID}^+(*)\}$$

$$\mathcal{L}^+_\tau(*) = \{\mu \in \mathcal{ID}(*) \mid \exists c \in \mathbb{R} \colon \tau_c(\mu) \in \mathcal{L}^+(*)\} = \mathcal{ID}^+_\tau \cap \mathcal{L}(*)$$

$$\mathcal{T}^+_\tau(*) = \{\mu \in \mathcal{ID}(*) \mid \exists c \in \mathbb{R} \colon \tau_c(\mu) \in \mathcal{T}^+(*)\} = \mathcal{ID}^+_\tau \cap \mathcal{T}(*).$$

[3] The inverse Gaussian distributions and the reciprocal inverse Gaussian distributions are, respectively, the first and the last passage time distributions to a constant level by a Brownian motion with drift.

Remark 2.13. The probability measures in $\mathcal{ID}^+(*)$ are characterized among the measures in $\mathcal{ID}(*)$ as those with characteristic triplets in the form $(0,\rho,\eta)$, where ρ is concentrated on $[0,\infty[$, $\int_{[0,1]} t\,\rho(\mathrm{d}t)<\infty$ and $\eta\geq\int_{[0,1]} t\,\rho(\mathrm{d}t)$ (cf. [Sa99, Theorem 24.11]). Consequently, the class $\mathcal{ID}^+_\tau(*)$ can be characterized as that of measures in $\mathcal{ID}(*)$ with generating triplets in the form $(0,\eta,\rho)$, where ρ is concentrated on $[0,\infty[$ and $\int_{[0,1]} t\,\rho(\mathrm{d}t)<\infty$.

Characterization in Terms of Lévy Measures

We shall say that a nonnegative function k with domain $\mathbb{R}\backslash\{0\}$ is *monotone on* $\mathbb{R}\backslash\{0\}$ if k is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$. And we say that k is *completely monotone on* $\mathbb{R}\backslash\{0\}$ if k is of the form

$$k(t)=\begin{cases}\int_0^\infty \mathrm{e}^{-ts}\nu(\mathrm{d}s), & \text{for } t>0\\ \int_{-\infty}^0 \mathrm{e}^{-ts}\nu(\mathrm{d}s), & \text{for } t<0\end{cases} \tag{2.16}$$

for some Borel measure ν on $\mathbb{R}\backslash\{0\}$. Note in this case that ν is necessarily a Radon measure on $\mathbb{R}\setminus\{0\}$. Indeed, for any compact subset K of $]0,\infty[$, we may consider the strictly positive number $m:=\inf_{s\in K}\mathrm{e}^{-s}$. Then,

$$\nu(K)\leq m^{-1}\int_K \mathrm{e}^{-s}\,\nu(\mathrm{d}s)\leq m^{-1}\int_0^\infty \mathrm{e}^{-s}\,\nu(\mathrm{d}s)=m^{-1}k(1)<\infty.$$

Similarly, $\nu(K)<\infty$ for any compact subset of K of $]-\infty,0[$.

With the notation just introduced, we can now state simple characterizations of the Lévy measures of each of the classes $\mathcal{S}(*),\mathcal{T}(*),\mathcal{R}(*),\mathcal{L}(*),\mathcal{B}(*)$ as follows. In all cases the Lévy measure has a density r of the form

$$r(t)=\begin{cases}c_+t^{-a_+}k(t), & \text{for } t>0,\\ c_-\,|t|^{-a_-}\,k(t), & \text{for } t<0,\end{cases} \tag{2.17}$$

where a_+,a_-,c_+,c_- are non-negative constants and where $k\geq 0$ is monotone on $\mathbb{R}\backslash\{0\}$.

- The Lévy measures of $\mathcal{S}(*)$ are characterized by having densities r of the form (2.17) with $a_\pm=1+\alpha$, $\alpha\in\,]0,2[$, and k constant on $\mathbb{R}_{<0}$ and on $\mathbb{R}_{>0}$.
- The Lévy measures of $\mathcal{R}(*)$ are characterized by having densities r of the form (2.17) with $a_\pm=1+\alpha$, $\alpha\in\,]0,2[$, and k completely monotone on $\mathbb{R}\backslash\{0\}$ with $k(0+)=k(0-)=1$.
- The Lévy measures of $\mathcal{T}(*)$ are characterized by having densities r of the form (2.17) with $a_\pm=1$ and k completely monotone on $\mathbb{R}\backslash\{0\}$.
- The Lévy measures of $\mathcal{L}(*)$ are characterized by having densities r of the form (2.17) with $a_\pm=1$ and k monotone on $\mathbb{R}\backslash\{0\}$.
- The Lévy measures of $\mathcal{B}(*)$ are characterized by having densities r of the form (2.17) with $a_\pm=0$ and k completely monotone on $\mathbb{R}\backslash\{0\}$.

In the case of $\mathcal{S}(*)$ and $\mathcal{L}(*)$ these characterizations are well known, see for instance [Sa99]. For $\mathcal{T}(*)$, $\mathcal{R}(*)$ and $\mathcal{B}(*)$ we indicate the proofs in Section 3.

3 Upsilon Mappings

The term Upsilon mappings is used to indicate a class of one-to-one regularizing mappings from the set of Lévy measures into itself or, equivalently, from the set of infinitely divisible distributions into itself. They are defined as deterministic integrals but have a third interpretation in terms of stochastic integrals with respect to Lévy processes. In addition to the regularizing effect, the mappings have simple relations to the classes of infinitely divisible laws discussed in the foregoing section. Some extensions to multivariate settings are briefly discussed at the end of the section.

3.1 The Mapping Υ_0

Let ρ be a Borel measure on $\mathbb{R}$, and consider the family $(D_x\rho)_{x>0}$ of Borel measures on $\mathbb{R}$. Assume that ρ has density r w.r.t. some σ-finite Borel measure σ on $\mathbb{R}$: $\rho(\mathrm{d}t) = r(t)\,\sigma(\mathrm{d}t)$. Then $(D_x\rho)_{x>0}$ is a Markov kernel, i.e. for any Borel subset B of $\mathbb{R}$, the mapping $x \mapsto D_x\rho(B)$ is Borel measurable. Indeed, for any x in $]0,\infty[$ we have

$$D_x\rho(B) = \rho(x^{-1}B) = \int_{\mathbb{R}} 1_{x^{-1}B}(t)r(t)\,\sigma(\mathrm{d}t) = \int_{\mathbb{R}} 1_B(xt)r(t)\,\sigma(\mathrm{d}t).$$

Since the function $(t,x) \mapsto 1_B(tx)r(t)$ is a Borel function of two variables, and since σ is σ-finite, it follows from Tonelli's theorem that the function $x \mapsto \int_{\mathbb{R}} 1_B(xt)r(t)\,\sigma(\mathrm{d}t)$ is a Borel function, as claimed.

Assume now that ρ is Borel measure on $\mathbb{R}$, which has a density r w.r.t. some σ-finite Borel measure on $\mathbb{R}$. Then the above considerations allow us to define a new Borel measure $\tilde{\rho}$ on $\mathbb{R}$ by:

$$\tilde{\rho} = \int_0^\infty (D_x\rho)\mathrm{e}^{-x}\,\mathrm{d}x, \tag{3.1}$$

or more precisely:

$$\tilde{\rho}(B) = \int_0^\infty D_x\rho(B)\mathrm{e}^{-x}\,\mathrm{d}x,$$

for any Borel subset B of $\mathbb{R}$. In the following we usually assume that ρ is a σ-finite, although many of the results are actually valid in the slightly more general situation, where ρ is only assumed to have a (possibly infinite) density w.r.t. a σ-finite measure. In fact, we are mainly interested in the case where ρ is a Lévy measure (recall that Lévy measures are automatically σ-finite).

Definition 3.1. *Let $\mathfrak{M}(\mathbb{R})$ denote the class of all positive Borel measure on $\mathbb{R}$ and let $\mathfrak{M}_L(\mathbb{R})$ denote the subclass of all Lévy measure on $\mathbb{R}$. We then define a mapping $\Upsilon_0 \colon \mathfrak{M}_L(\mathbb{R}) \to \mathfrak{M}(\mathbb{R})$ by*

$$\Upsilon_0(\rho) = \int_0^\infty (D_x\rho)\mathrm{e}^{-x}\,\mathrm{d}x, \qquad (\rho \in \mathfrak{M}_L(\mathbb{R})).$$

As we shall see at the end of this section, the range of Υ_0 is actually a genuine subset of $\mathfrak{M}_L(\mathbb{R})$ (cf. Corollary 3.10 below).

In the following we consider further, for a measure ρ on $\mathbb{R}$, the transformation of $\rho_{|\mathbb{R}\setminus\{0\}}$ by the mapping $x \mapsto x^{-1}\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$ (here $\rho_{|\mathbb{R}\setminus\{0\}}$ denotes the restriction of ρ to $\mathbb{R}\setminus\{0\}$). The transformed measure will be denoted by ω and occasionally also by $\underleftarrow{\rho}$. Note that ω is σ-finite if ρ is, and that ρ is a Lévy measure if and only if $\rho(\{0\}) = 0$ and ω satisfies the property:

$$\int_{\mathbb{R}} \min\{1, s^{-2}\}\,\omega(\mathrm{d}s) < \infty. \tag{3.2}$$

Theorem 3.2. *Let ρ be a σ-finite Borel measure on $\mathbb{R}$, and consider the Borel function $\tilde{r}\colon \mathbb{R}\setminus\{0\} \to [0,\infty]$, given by*

$$\tilde{r}(t) = \begin{cases} \int_{]0,\infty[} s\mathrm{e}^{-ts}\,\omega(\mathrm{d}s), & \text{if } t > 0,\\ \int_{]-\infty,0[} |s|\mathrm{e}^{-ts}\,\omega(\mathrm{d}s), & \text{if } t < 0, \end{cases} \tag{3.3}$$

where ω is the transformation of $\rho_{|\mathbb{R}\setminus\{0\}}$ by the mapping $x \mapsto x^{-1}\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$.

Then the measure $\tilde{\rho}$, defined in (3.1), is given by:

$$\tilde{\rho}(\mathrm{d}t) = \rho(\{0\})\delta_0(\mathrm{d}t) + \tilde{r}(t)\,\mathrm{d}t.$$

Proof. We have to show that

$$\tilde{\rho}(B) = \rho(\{0\})\delta_0(B) + \int_{B\setminus\{0\}} \tilde{r}(t)\,\mathrm{d}t, \tag{3.4}$$

for any Borel set B of $\mathbb{R}$. Clearly, it suffices to verify (3.4) in the two cases $B \subseteq [0,\infty[$ and $B \subseteq\,]-\infty,0]$. If $B \subseteq [0,\infty[$, we find that

$$\begin{aligned}\tilde{\rho}(B) &= \int_0^\infty \Big(\int_{[0,\infty[} 1_B(s)\,D_x\rho(\mathrm{d}s)\Big)\mathrm{e}^{-x}\,\mathrm{d}x\\ &= \int_0^\infty \Big(\int_{[0,\infty[} 1_B(sx)\,\rho(\mathrm{d}s)\Big)\mathrm{e}^{-x}\,\mathrm{d}x\\ &= \int_{[0,\infty[} \Big(\int_0^\infty 1_B(sx)\mathrm{e}^{-x}\,\mathrm{d}x\Big)\,\rho(\mathrm{d}s).\end{aligned}$$

Using, for $s > 0$, the change of variable $u = sx$, we find that

$$\begin{aligned}\tilde{\rho}(B) &= \Big(1_B(0)\int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x\Big)\rho(\{0\}) + \int_{]0,\infty[} \Big(\int_0^\infty 1_B(u)\mathrm{e}^{-u/s}s^{-1}\,\mathrm{d}u\Big)\,\rho(\mathrm{d}s)\\ &= \rho(\{0\})\delta_0(B) + \int_0^\infty 1_B(u)\Big(\int_{]0,\infty[} s^{-1}\mathrm{e}^{-u/s}\,\rho(\mathrm{d}s)\Big)\,\mathrm{d}u\\ &= \rho(\{0\})\delta_0(B) + \int_0^\infty 1_B(u)\Big(\int_{]0,\infty[} s\mathrm{e}^{-us}\,\omega(\mathrm{d}s)\Big)\,\mathrm{d}u,\end{aligned}$$

as desired. The case $B \subseteq \,]-\infty, 0]$ is proved similarly or by applying, what we have just established, to the set $-B$ and the measure $D_{-1}\rho$. $\square$

Corollary 3.3. *Let ρ be a σ-finite Borel measure on $\mathbb{R}$ and consider the measure $\tilde{\rho}$ given by (3.1). Then*

$$\tilde{\rho}(\{t\}) = \begin{cases} 0, & \text{if } t \in \mathbb{R} \setminus \{0\}, \\ \rho(\{0\}), & \text{if } t = 0. \end{cases}$$

Corollary 3.4. *Let $r \colon \mathbb{R} \to [0, \infty[$ be a non-negative Borel function and let ρ be the measure on $\mathbb{R}$ with density r w.r.t. Lebesgue measure: $\rho(\mathrm{d}t) = r(t)\,\mathrm{d}t$. Consider further the measure $\tilde{\rho}$ given by (3.1). Then $\tilde{\rho}$ is absolutely continuous w.r.t. Lebesgue measure and the density, $\tilde{r}$, is given by*

$$\tilde{r}(t) = \begin{cases} \int_0^\infty y^{-1} r(y^{-1}) \mathrm{e}^{-ty}\,\mathrm{d}y, & \text{if } t > 0, \\ \int_{-\infty}^0 -y^{-1} r(y^{-1}) \mathrm{e}^{-ty}\,\mathrm{d}y, & \text{if } t < 0. \end{cases}$$

Proof. This follows immediately from Theorem 3.2 together with the fact that the measure ω has density

$$s \mapsto s^{-2} r(s^{-1}), \qquad (s \in \mathbb{R} \setminus \{0\}),$$

w.r.t. Lebesgue measure. $\square$

Corollary 3.5. *Let ρ be a Lévy measure on $\mathbb{R}$. Then the measure $\Upsilon_0(\rho)$ is absolutely continuous w.r.t. Lebesgue measure. The density, $\tilde{r}$, is given by (3.3) and is a C^∞-function on $\mathbb{R} \setminus \{0\}$.*

Proof. We only have to verify that $\tilde{r}$ is a C^∞-function on $\mathbb{R} \setminus \{0\}$. But this follows from the usual theorem on differentiation under the integral sign, since, by (3.2),

$$\int_{]0,\infty[} s^p \mathrm{e}^{-ts}\,\omega(\mathrm{d}s) < \infty \quad \text{and} \quad \int_{]-\infty,0[} |s|^p \mathrm{e}^{ts}\,\omega(\mathrm{d}s) < \infty,$$

for any t in $]0, \infty[$ and any p in $\mathbb{N}$. $\square$

Proposition 3.6. *Let ρ be a σ-finite measure on $\mathbb{R}$, let $\tilde{\rho}$ be the measure given by (3.1) and let ω be the transformation of $\rho_{|\mathbb{R}\setminus\{0\}}$ under the mapping $t \mapsto t^{-1}$. We then have*

$$\tilde{\rho}([t, \infty[) = \int_0^\infty \mathrm{e}^{-ts}\,\omega(\mathrm{d}s), \qquad (t \in \,]0, \infty[), \tag{3.5}$$

and

$$\tilde{\rho}(]-\infty, t]) = \int_{-\infty}^0 \mathrm{e}^{-ts}\,\omega(\mathrm{d}s), \qquad (t \in \,]-\infty, 0[). \tag{3.6}$$

Proof. Using Theorem 3.2 we find, for $t > 0$, that

$$\tilde{\rho}([t,\infty[) = \int_t^{\infty} \Big(\int_{]0,\infty[} s\mathrm{e}^{-us}\, \omega(\mathrm{d}s) \Big)\, \mathrm{d}u = \int_{]0,\infty[} \Big(\int_t^{\infty} \mathrm{e}^{-us} s\, \mathrm{d}u \Big)\, \omega(\mathrm{d}s)$$

$$= \int_{]0,\infty[} \Big(\int_{ts}^{\infty} \mathrm{e}^{-x}\, \mathrm{d}x \Big)\, \omega(\mathrm{d}s) = \int_{]0,\infty[} \mathrm{e}^{-ts}\, \omega(\mathrm{d}s),$$

where we have used the change of variable $x = us$. Formula (3.6) is proved similarly. □

Corollary 3.7. *The mapping $\Upsilon_0 \colon \mathfrak{M}_L(\mathbb{R}) \to \mathfrak{M}(\mathbb{R})$ is injective.*

Proof. Suppose $\rho \in \mathfrak{M}_L(\mathbb{R})$ and let ω be the transformation of $\rho_{|\mathbb{R}\setminus\{0\}}$ be the mapping $t \mapsto t^{-1}$. Let, further, ω_+ and ω_- denote the restrictions of ω to $]0,\infty[$ and $]-\infty,0[$, respectively. By (3.2) it follows then that the Laplace transform for ω_+ is well-defined on all of $]0,\infty[$. Furthermore, (3.5) shows that this Laplace transform is uniquely determined by $\tilde{\rho}$. Hence, by uniqueness of Laplace transforms (cf. [Fe71, Theorem 1a, Chapter XIII.1]), ω_+ is uniquely determined by $\tilde{\rho}$. Arguing similarly for the measure $D_{-1}\omega_-$, it follows that $D_{-1}\omega_-$ (and hence ω_-) is uniquely determined by $\tilde{\rho}$. Altogether, ω (and hence ρ) is uniquely determined by $\tilde{\rho}$. □

Proposition 3.8. *Let ρ be a σ-finite measure on $\mathbb{R}$ and let $\tilde{\rho}$ be the measure given by (3.1). Then for any p in $[0,\infty[$, we have that*

$$\int_{\mathbb{R}} |t|^p\, \tilde{\rho}(\mathrm{d}t) = \Gamma(p+1) \int_{\mathbb{R}} |t|^p\, \rho(\mathrm{d}t).$$

In particular, the p'th moment of $\tilde{\rho}$ and ρ exist simultaneously, in which case

$$\int_{\mathbb{R}} t^p\, \tilde{\rho}(\mathrm{d}t) = \Gamma(p+1) \int_{\mathbb{R}} t^p\, \rho(\mathrm{d}t). \tag{3.7}$$

Proof. Let p from $[0,\infty[$ be given. Then

$$\int_{\mathbb{R}} |t|^p\, \tilde{\rho}(\mathrm{d}t) = \int_0^{\infty} \Big(\int_{\mathbb{R}} |t|^p\, D_x\rho(\mathrm{d}t) \Big) \mathrm{e}^{-x}\, \mathrm{d}x \int_0^{\infty} \Big(\int_{\mathbb{R}} |tx|^p\, \rho(\mathrm{d}t) \Big) \mathrm{e}^{-x}\, \mathrm{d}x$$

$$= \int_{\mathbb{R}} |t|^p \Big(\int_0^{\infty} x^p \mathrm{e}^{-x}\, \mathrm{d}x \Big)\, \rho(\mathrm{d}t) = \Gamma(p+1) \int_{\mathbb{R}} |t|^p\, \rho(\mathrm{d}t).$$

If the integrals above are finite, we can perform the same calculation without taking absolute values, and this establishes (3.7). □

Proposition 3.9. *Let ρ be a σ-finite Borel measure on $\mathbb{R}$ and let $\tilde{\rho}$ be the measure given by (3.1). We then have*

$$\int_{\mathbb{R}\setminus[-1,1]} 1\,\tilde{\rho}(\mathrm{d}t) = \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{-1/|t|}\,\rho(\mathrm{d}t) \tag{3.8}$$

$$\int_{[-1,1]} t^2\,\tilde{\rho}(\mathrm{d}t) = \int_{\mathbb{R}\setminus\{0\}} 2t^2 - \mathrm{e}^{-1/|t|}(1+2|t|+2t^2)\,\rho(\mathrm{d}t). \tag{3.9}$$

In particular

$$\int_{\mathbb{R}} \min\{1,t^2\}\,\tilde{\rho}(\mathrm{d}t) = \int_{\mathbb{R}\setminus\{0\}} 2t^2\big(1-\mathrm{e}^{-1/|t|}(|t|^{-1}+1)\big)\,\rho(\mathrm{d}t), \tag{3.10}$$

and consequently

$$\int_{\mathbb{R}} \min\{1,t^2\}\,\tilde{\rho}(\mathrm{d}t) < \infty \iff \int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t) < \infty. \tag{3.11}$$

Proof. We note first that

$$\begin{aligned}
\int_{\mathbb{R}\setminus[-1,1]} 1\,\tilde{\rho}(\mathrm{d}t) &= \int_0^\infty \Big(\int_{\mathbb{R}} 1_{]1,\infty[}(|t|)\,D_x\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_0^\infty \Big(\int_{\mathbb{R}} 1_{]1,\infty[}(|tx|)\,\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_{\mathbb{R}\setminus\{0\}} \Big(\int_{1/|t|}^\infty \mathrm{e}^{-x}\,\mathrm{d}x\Big)\rho(\mathrm{d}t) \\
&= \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{-1/|t|}\,\rho(\mathrm{d}t),
\end{aligned}$$

which proves (3.8). Regarding (3.9) we find that

$$\begin{aligned}
\int_{[-1,1]} t^2\,\tilde{\rho}(\mathrm{d}t) &= \int_0^\infty \Big(\int_{\mathbb{R}} 1_{[0,1]}(|t|)t^2\,D_x\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_0^\infty \Big(\int_{\mathbb{R}} 1_{[0,1]}(|tx|)t^2x^2\,\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_{\mathbb{R}\setminus\{0\}} \Big(\int_0^{1/|t|} x^2\mathrm{e}^{-x}\,\mathrm{d}x\Big)t^2\,\rho(\mathrm{d}t) \\
&= \int_{\mathbb{R}\setminus\{0\}} \big(2-\mathrm{e}^{-1/|t|}(t^{-2}+2|t|^{-1}+2)\big)t^2\,\rho(\mathrm{d}t) \\
&= \int_{\mathbb{R}\setminus\{0\}} 2t^2 - \mathrm{e}^{-1/|t|}(1+2|t|+2t^2)\,\rho(\mathrm{d}t),
\end{aligned}$$

as claimed. Combining (3.8) and (3.9), we immediately get (3.10). To deduce finally (3.11), note first that for any positive u, we have by second order Taylor expansion

$$\frac{2}{u^2}\big(1-\mathrm{e}^{-u}(u+1)\big)=\frac{2\mathrm{e}^{-u}}{u^2}\big(\mathrm{e}^{u}-u+1\big)=\mathrm{e}^{\xi-u}, \tag{3.12}$$

for some number ξ in $]0,u[$. It follows thus that

$$\forall t\in\mathbb{R}\setminus\{0\}\colon 0<2t^2\big(1-\mathrm{e}^{-1/|t|}(|t|^{-1}+1)\big)\le 1, \tag{3.13}$$

and from the upper bound together with (3.10), the implication "$\Leftarrow$" in (3.11) follows readily. Regarding the converse implication, note that (3.12) also shows that

$$\lim_{|t|\to\infty} 2t^2\big(1-\mathrm{e}^{-1/|t|}(|t|^{-1}+1)\big)=1,$$

and together with the lower bound in (3.13), this implies that

$$\inf_{t\in\mathbb{R}\setminus[-1,1]} 2t^2\big(1-\mathrm{e}^{-1/|t|}(|t|^{-1}+1)\big)>0. \tag{3.14}$$

Note also that

$$\lim_{t\to 0} 2\big(1-\mathrm{e}^{-1/|t|}(|t|^{-1}+1)\big)=2\lim_{u\to\infty}\big(1-\mathrm{e}^{-u}(u+1)\big)=2,$$

so that

$$\inf_{t\in[-1,1]\setminus\{0\}} 2\big(1-\mathrm{e}^{-1/|t|}(|t|^{-1}+1)\big)>0. \tag{3.15}$$

Combining (3.14),(3.15) and (3.10), the implication "$\Rightarrow$" in (3.11) follows. This completes the proof. □

Corollary 3.10. *For any Lévy measure ρ on $\mathbb{R}$, $\Upsilon_0(\rho)$ is again a Lévy measure on $\mathbb{R}$. Moreover, a Lévy measure υ on $\mathbb{R}$ is in the range of Υ_0 if and only if the function $F_\upsilon\colon\mathbb{R}\setminus\{0\}\to[0,\infty[$ given by*

$$F_\upsilon(t)=\begin{cases}\upsilon(]-\infty,t]), & \text{if } t<0,\\ \upsilon([t,\infty[), & \text{if } t>0,\end{cases}$$

is completely monotone (cf. (2.16)).

Proof. It follows immediately from (3.11) that $\Upsilon(\rho)$ is a Lévy measure if ρ is.

Regarding the second statement of the corollary, we already saw in Proposition 3.6 that $F_{\Upsilon(\rho)}$ is completely monotone for any Lévy measure ρ on $\mathbb{R}$. Assume conversely that υ is a Lévy measure on $\mathbb{R}$, such that F_υ is completely monotone, i.e.

$$\upsilon([t,\infty[)=\int_0^\infty \mathrm{e}^{-ts}\,\omega(\mathrm{d}s), \qquad (t\in\,]0,\infty[),$$

and

$$\upsilon(]-\infty,t]) = \int_{-\infty}^{0} \mathrm{e}^{-ts}\,\omega(\mathrm{d}s), \qquad (t \in]-\infty,0[).$$

for some Radon measure ω on $\mathbb{R}\setminus\{0\}$. Now let ρ be the transformation of ω by the mapping $t \mapsto t^{-1}\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$. Then ρ is clearly a Radon measure on $\mathbb{R}\setminus\{0\}$, too. Setting $\rho(\{0\}) = 0$, we may thus consider ρ as a σ-finite measure on $\mathbb{R}$. Applying then Proposition 3.6 to ρ, it follows that $\tilde{\rho}$ and υ coincide on all intervals in the form $]-\infty,-t]$ or $[t,\infty[$ for $t>0$. Since also $\tilde{\rho}(\{0\} = 0 = \upsilon(\{0\})$ by Corollary 2.3, we conclude that $\tilde{\rho} = \upsilon$. Combining this with formula (3.11), it follows finally that ρ is a Lévy measure and that $\upsilon = \tilde{\rho} = \Upsilon_0(\rho)$. □

Proposition 3.11. *Let ρ be a σ-finite measure concentrated on $[0,\infty[$ and let $\tilde{\rho}$ be the measure given by (3.1). We then have*

$$\int_{]1,\infty[} 1\,\tilde{\rho}(\mathrm{d}t) = \int_{]0,\infty[} \mathrm{e}^{-1/t}\,\rho(\mathrm{d}t), \tag{3.16}$$

$$\int_{[0,1]} t\,\tilde{\rho}(\mathrm{d}t) = \int_{]0,\infty[} t(1-\mathrm{e}^{-1/t}) - \mathrm{e}^{-1/t}\,\rho(\mathrm{d}t). \tag{3.17}$$

In particular

$$\int_{[0,\infty[} \min\{1,t\}\,\tilde{\rho}(\mathrm{d}t) = \int_{]0,\infty[} t(1-\mathrm{e}^{-1/t})\,\rho(\mathrm{d}t), \tag{3.18}$$

and therefore

$$\int_{[0,\infty[} \min\{1,t\}\,\tilde{\rho}(\mathrm{d}t) < \infty \iff \int_{[0,\infty[} \min\{1,t\}\,\rho(\mathrm{d}t) < \infty. \tag{3.19}$$

Proof. Note first that (3.18) follows immediately from (3.16) and (3.17). To prove (3.16), note that by definition of $\tilde{\rho}$, we have

$$\begin{aligned}
\int_{]1,\infty[} 1\,\tilde{\rho}(\mathrm{d}t) &= \int_0^\infty \Big(\int_{[0,\infty[} 1_{]1,\infty[}(t)\,D_x\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_0^\infty \Big(\int_{[0,\infty[} 1_{]1,\infty[}(tx)\,\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_{]0,\infty[} \Big(\int_{1/t}^\infty \mathrm{e}^{-x}\,\mathrm{d}x\Big)\,\rho(\mathrm{d}t) \\
&= \int_{]0,\infty[} \mathrm{e}^{-1/t}\,\rho(\mathrm{d}t).
\end{aligned}$$

Regarding (3.17), we find similarly that

$$\begin{aligned}
\int_{[0,1]} t\,\tilde{\rho}(\mathrm{d}t) &= \int_0^\infty \Big(\int_{[0,1]} t\,D_x\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_0^\infty \Big(\int_{[0,\infty[} tx1_{[0,1]}(tx)\,\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_{]0,\infty[} t\Big(\int_0^{1/t} x\mathrm{e}^{-x}\,\mathrm{d}x\Big)\,\rho(\mathrm{d}t) \\
&= \int_{]0,\infty[} t\big(1-\mathrm{e}^{-1/t}(\tfrac{1}{t}+1)\big)\,\rho(\mathrm{d}t) \\
&= \int_{]0,\infty[} t(1-\mathrm{e}^{-1/t})-\mathrm{e}^{-1/t}\,\rho(\mathrm{d}t).
\end{aligned}$$

Finally, (3.19) follows from (3.18) by noting that

$$0 \le t(1-\mathrm{e}^{-1/t}) = -\frac{\mathrm{e}^{-1/t}-1}{1/t} \le 1, \qquad \text{whenever } t>0,$$

and that

$$\lim_{t\searrow 0}(1-\mathrm{e}^{-1/t}) = 1 = \lim_{t\to\infty} t(1-\mathrm{e}^{-1/t}).$$

This concludes the proof. □

3.2 The Mapping Υ

We now extend the mapping Υ_0 to a mapping Υ from $\mathcal{ID}(*)$ into $\mathcal{ID}(*)$.

Definition 3.12. *For any μ in $\mathcal{ID}(*)$, with characteristic triplet (a,ρ,η), we take $\Upsilon(\mu)$ to be the element of $\mathcal{ID}(*)$ whose characteristic triplet is $(2a,\tilde{\rho},\tilde{\eta})$ where*

$$\tilde{\eta} = \eta + \int_0^\infty \Big(\int_{\mathbb{R}} t\,\big(1_{[-1,1]}(t) - 1_{[-x,x]}(t)\big)\;D_x\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \tag{3.20}$$

and

$$\tilde{\rho} = \Upsilon_0(\rho) = \int_0^\infty (D_x\rho)\mathrm{e}^{-x}\mathrm{d}x. \tag{3.21}$$

Note that it is an immediate consequence of Proposition 3.9 that the measure $\tilde{\rho}$ in Definition 3.12 is indeed a Lévy measure. We verify next that the integral in (3.20) is well-defined.

Lemma 3.13. *Let ρ be a Lévy measure on $\mathbb{R}$. Then for any x in $]0,\infty[$, we have that*

$$\int_{\mathbb{R}} \big|ux\cdot\big(1_{[-1,1]}(ux) - 1_{[-x,x]}(ux)\big)\big|\,\rho(\mathrm{d}u) < \infty.$$

Furthermore,

$$\int_0^\infty \Big(\int_{\mathbb{R}} \big| ux \cdot \big(1_{[-1,1]}(ux) - 1_{[-x,x]}(ux) \big) \big| \, \rho(\mathrm{d}u) \Big) \mathrm{e}^{-x} \, \mathrm{d}x < \infty.$$

Proof. Note first that for any x in $]0, \infty[$ we have that

$$\begin{aligned}
&\int_{\mathbb{R}} \big| ux \cdot \big(1_{[-1,1]}(ux) - 1_{[-x,x]}(ux) \big) \big| \, \rho(\mathrm{d}u) \\
&\qquad = \int_{\mathbb{R}} \big| ux \cdot \big(1_{[-x^{-1},x^{-1}]}(u) - 1_{[-1,1]}(u) \big) \big| \, \rho(\mathrm{d}u) \\
&\qquad = \begin{cases} x \int_{\mathbb{R}} |u| \cdot 1_{[-x^{-1},x^{-1}] \setminus [-1,1]}(u) \, \rho(\mathrm{d}u), & \text{if } x \le 1, \\ x \int_{\mathbb{R}} |u| \cdot 1_{[-1,1] \setminus [-x^{-1},x^{-1}]}(u) \, \rho(\mathrm{d}u), & \text{if } x > 1. \end{cases}
\end{aligned}$$

Note then that whenever $0 < \epsilon < K$, we have that

$$|u| \cdot 1_{[-K,K] \setminus [-\epsilon,\epsilon]}(u) \le \min\{K, \tfrac{u^2}{\epsilon}\} \le \max\{K, \epsilon^{-1}\} \min\{u^2, 1\},$$

for any u in $\mathbb{R}$. Hence, if $0 < x \le 1$, we find that

$$\begin{aligned}
&x \int_{\mathbb{R}} \big| u \cdot \big(1_{[-x^{-1},x^{-1}]}(u) - 1_{[-1,1]}(u) \big) \big| \, \rho(\mathrm{d}u) \\
&\qquad \le x \max\{x^{-1}, 1\} \int_{\mathbb{R}} \min\{u^2, 1\} \, \rho(\mathrm{d}u) = \int_{\mathbb{R}} \min\{u^2, 1\} \, \rho(\mathrm{d}u) < \infty,
\end{aligned}$$

since ρ is a Lévy measure. Similarly, if $x \ge 1$,

$$\begin{aligned}
&x \int_{\mathbb{R}} \big| u \cdot \big(1_{[-1,1]}(u) - 1_{[-x^{-1},x^{-1}]}(u) \big) \big| \, \rho(\mathrm{d}u) \\
&\qquad \le x \max\{1, x\} \int_{\mathbb{R}} \min\{u^2, 1\} \, \rho(\mathrm{d}u) = x^2 \int_{\mathbb{R}} \min\{u^2, 1\} \, \rho(\mathrm{d}u) < \infty.
\end{aligned}$$

Altogether, we find that

$$\begin{aligned}
&\int_0^\infty \Big(\int_{\mathbb{R}} \big| ux \cdot \big(1_{[-1,1]}(ux) - 1_{[-x,x]}(ux) \big) \big| \, \rho(\mathrm{d}u) \Big) \mathrm{e}^{-x} \, \mathrm{d}x \\
&\qquad \le \int_{\mathbb{R}} \min\{u^2, 1\} \, \rho(\mathrm{d}u) \cdot \Big(\int_0^1 \mathrm{e}^{-x} \, \mathrm{d}x + \int_1^\infty x^2 \mathrm{e}^{-x} \, \mathrm{d}x \Big) < \infty,
\end{aligned}$$

as asserted. □

Remark 3.14. In connection with (3.20), note that it follows from Lemma 3.13 above that the integral

$$\int_0^\infty \Big(\int_{\mathbb{R}} u\big(1_{[-1,1]}(u) - 1_{[-x,x]}(u)\big)\, D_x\rho(\mathrm{d}u)\Big)\mathrm{e}^{-x}\,\mathrm{d}x,$$

is well-defined. Indeed,

$$\begin{aligned}\int_0^\infty \Big(\int_{\mathbb{R}} \big|u\big(1_{[-1,1]}(u) - 1_{[-x,x]}(u)\big)\big|\, D_x\rho(\mathrm{d}u)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\ = \int_0^\infty \Big(\int_{\mathbb{R}} \big|ux\big(1_{[-1,1]}(ux) - 1_{[-x,x]}(ux)\big)\big|\, \rho(\mathrm{d}u)\Big)\mathrm{e}^{-x}\,\mathrm{d}x.\end{aligned}$$

Having established that the definition of Υ is meaningful, we prove next a key formula for the cumulant transform of $\Upsilon(\mu)$ (Theorem 3.17 below). From that formula we derive subsequently a number of important properties of Υ. We start with the following technical result.

Lemma 3.15. *Let ρ be a Lévy measure on $\mathbb{R}$. Then for any number ζ in $]-\infty,0[$, we have that*

$$\int_0^\infty \Big(\int_{\mathbb{R}} \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx 1_{[-1,1]}(t)\big|\, \rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x < \infty.$$

Proof. Let ζ from $]-\infty,0[$ and x in $[0,\infty[$ be given. Note first that

$$\begin{aligned}\int_{\mathbb{R}\setminus[-1,1]} \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx 1_{[-1,1]}(t)\big|\, \rho(\mathrm{d}t) &= \int_{\mathbb{R}\setminus[-1,1]} \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1\big|\, \rho(\mathrm{d}t) \\ &\le 2\int_{\mathbb{R}\setminus[-1,1]} \min\{1,t^2\}\rho(\mathrm{d}t) \\ &\le 2\int_{\mathbb{R}} \min\{1,t^2\}\rho(\mathrm{d}t).\end{aligned}$$

To estimate $\int_{-1}^1 |\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx|\,\rho(\mathrm{d}t)$, we note that for any real number t, it follows by standard second order Taylor expansion that

$$\big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx\big| \le \frac{1}{\sqrt{2}}(\zeta tx)^2,$$

and hence

$$\begin{aligned}\int_{-1}^1 \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx\big|\,\rho(\mathrm{d}t) &\le \frac{1}{\sqrt{2}}(\zeta x)^2 \int_{-1}^1 t^2\,\rho(\mathrm{d}t) \\ &\le \frac{1}{\sqrt{2}}(\zeta x)^2 \int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t).\end{aligned}$$

Altogether, we find that for any number x in $[0,\infty[$,

$$\int_{\mathbb{R}} \left|e^{i\zeta tx} - 1 - i\zeta tx 1_{[-1,1]}(t)\right| \rho(dt) \le \Big(2 + \frac{1}{\sqrt{2}}(\zeta x)^2\Big) \int_{\mathbb{R}} \min\{1, t^2\} \rho(dt),$$

and therefore

$$\begin{aligned}\int_0^\infty \Big(\int_{\mathbb{R}} \left|e^{i\zeta tx} - 1 - i\zeta tx 1_{[-1,1]}(t)\right| \rho(dt)\Big) e^{-x} \, dx \\ \le \int_{\mathbb{R}} \min\{1, t^2\} \rho(dt) \int_0^\infty \Big(2 + \frac{1}{\sqrt{2}}(\zeta x)^2\Big) e^{-x} \, dx < \infty,\end{aligned}$$

as desired. □

Theorem 3.16. *Let μ be a measure in $\mathcal{ID}(*)$ with characteristic triplet (a, ρ, η). Then the cumulant function of $\Upsilon(\mu)$ is representable as*

$$C_{\Upsilon(\mu)}(\zeta) = i\eta\zeta - a\zeta^2 + \int_{\mathbb{R}} \Big(\frac{1}{1 - i\zeta t} - 1 - i\zeta t 1_{[-1,1]}(t)\Big) \rho(dt), \tag{3.22}$$

for any ζ in $\mathbb{R}$.

Proof. Recall first that for any $z \in \mathbb{C}$ with $\mathrm{Re} z < 1$ we have

$$\frac{1}{1 - z} = \int_0^\infty e^{zx} e^{-x} dx,$$

implying that for ζ real with $\zeta \le 0$

$$\frac{1}{1 - i\zeta t} - 1 - i\zeta t 1_{[-1,1]}(t) = \int_0^\infty \left(e^{i\zeta tx} - 1 - i\zeta tx 1_{[-1,1]}(t)\right) e^{-x} dx. \tag{3.23}$$

Now, let μ from $\mathcal{ID}(*)$ be given and let (a, ρ, η) be the characteristic triplet for μ. Then by the above calculation

$$\begin{aligned}&\int_{\mathbb{R}} \Big(\frac{1}{1 - i\zeta t} - 1 - i\zeta t 1_{[-1,1]}(t)\Big) \rho(dt) \\
&= \int_{\mathbb{R}} \Big(\int_0^\infty \left(e^{i\zeta tx} - 1 - i\zeta tx 1_{[-1,1]}(t)\right) e^{-x} \, dx\Big) \rho(dt) \\
&= \int_0^\infty \Big(\int_{\mathbb{R}} \left(e^{i\zeta u} - 1 - i\zeta u 1_{[-x,x]}(u)\right) \rho(x^{-1} du)\Big) e^{-x} \, dx \\
&= \int_0^\infty \Big(\int_{\mathbb{R}} \left(e^{i\zeta u} - 1 - i\zeta u 1_{[-1,1]}(u)\right) \rho(x^{-1} du)\Big) e^{-x} \, dx \\
&\qquad + i\zeta \int_0^\infty \Big(\int_{\mathbb{R}} u\left(1_{[-1,1]}(u) - 1_{[-x,x]}(u)\right) \rho(x^{-1} du)\Big) e^{-x} \, dx \\
&= \int_{\mathbb{R}} \left(e^{i\zeta u} - 1 - i\zeta u 1_{[-1,1]}(u)\right) \tilde{\rho}(du) \\
&\qquad + i\zeta \int_0^\infty \Big(\int_{\mathbb{R}} u\left(1_{[-1,1]}(u) - 1_{[-x,x]}(u)\right) \rho(x^{-1} du)\Big) e^{-x} \, dx,\end{aligned}$$

where we have changed the order of integration in accordance with Lemma 3.15. Comparing the above calculation with Definition 3.12, the theorem follows readily. □

Theorem 3.17. *For any μ in $\mathcal{ID}(*)$ we have*

$$C_{\Upsilon(\mu)}(z) = \int_0^\infty C_\mu(zx)\mathrm{e}^{-x}\,\mathrm{d}x, \qquad (z \in \mathbb{R}).$$

Proof. Let (a, ρ, η) be the characteristic triplet for μ. For arbitrary z in $\mathbb{R}$, we then have

$$\begin{aligned}
&\int_0^\infty C_\mu(zx)\mathrm{e}^{-x}\,\mathrm{d}x \\
&\quad= \int_0^\infty \Big(\mathrm{i}\eta zx - \frac{1}{2}az^2x^2 + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}tzx} - 1 - \mathrm{i}tzx1_{[-1,1]}(t)\big)\,\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&\quad= \mathrm{i}\eta z\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x - \frac{1}{2}az^2\int_0^\infty x^2\mathrm{e}^{-x}\,\mathrm{d}x \\
&\qquad\qquad + \int_{\mathbb{R}}\Big(\int_0^\infty \big(\mathrm{e}^{\mathrm{i}tzx} - 1 - \mathrm{i}tzx1_{[-1,1]}(t)\big)\mathrm{e}^{-x}\,\mathrm{d}x\Big)\,\rho(\mathrm{d}t) \\
&\quad= \mathrm{i}\eta z - az^2 + \int_{\mathbb{R}}\Big(\frac{1}{1-\mathrm{i}zt} - 1 - \mathrm{i}zt1_{[-1,1]}(t)\Big)\,\rho(\mathrm{d}t),
\end{aligned} \tag{3.24}$$

where the last equality uses (3.23). According to Theorem 3.16, the resulting expression in (3.24) equals $C_{\Upsilon(\mu)}(z)$, and the theorem follows. □

Based on Theorem 3.17 we establish next a number of interesting properties for Υ.

Proposition 3.18. *The mapping $\Upsilon\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$ has the following properties:*

(i) *Υ is injective.*
(ii) *For any measures μ, ν in $\mathcal{ID}(*)$, $\Upsilon(\mu * \nu) = \Upsilon(\mu) * \Upsilon(\nu)$.*
(iii) *For any measure μ in $\mathcal{ID}(*)$ and any constant c in $\mathbb{R}$, $\Upsilon(D_c\mu) = D_c\Upsilon(\mu)$.*
(iv) *For any constant c in $\mathbb{R}$, $\Upsilon(\delta_c) = \delta_c$.*
(v) *Υ is continuous w.r.t. weak convergence*[4]*.*

Proof. (i) This is an immediate consequence of the definition of Υ together with the injectivity of Υ_0 (cf. Corollary 3.7).

(ii) Suppose $\mu_1, \mu_2 \in \mathcal{ID}(*)$. Then for any z in $\mathbb{R}$ we have by Proposition 3.17

[4]In fact, it can be proved that Υ is a homeomorphism onto its range with respect to weak convergence; see [BaTh04c].

$$C_{\Upsilon(\mu_1*\mu_2)}(z) = \int_0^\infty C_{\mu_1*\mu_2}(zx)\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^\infty \big(C_{\mu_1}(zx) + C_{\mu_2}(zx)\big)\mathrm{e}^{-x}\,\mathrm{d}x$$
$$= C_{\Upsilon(\mu_1)}(z) + C_{\Upsilon(\mu_2)}(z) = C_{\Upsilon(\mu_1)*\Upsilon(\mu_2)}(z),$$

which verifies statement (ii)

(iii) Suppose $\mu \in \mathcal{ID}(*)$ and $c \in \mathbb{R}$. Then for any z in $\mathbb{R}$,

$$C_{\Upsilon(D_c\mu)}(z) = \int_0^\infty C_{D_c\mu}(zx)\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^\infty C_\mu(czx)\mathrm{e}^{-x}\,\mathrm{d}x$$
$$= C_{\Upsilon(\mu)}(cz) = C_{D_c\Upsilon(\mu)}(z),$$

which verifies (iii).

(iv) Let c from $\mathbb{R}$ be given. For z in $\mathbb{R}$ we then have

$$C_{\Upsilon(\delta_c)}(z) = \int_0^\infty C_{\delta_c}(zx)\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^\infty \mathrm{i}czx\mathrm{e}^{-x}\,\mathrm{d}x = \mathrm{i}cz = C_{\delta_c}(z),$$

which verifies (iv).

(v) Although we might give a direct proof of (v) at the present stage (see the proof of Theorem 3.40), we postpone the proof to Section 5.3, where we can give an easy argument based on the continuity of the Bercovici-Pata bijection Λ (introduced in Section 5.1) and the connection between Υ and Λ (see Section 5.2).

Corollary 3.19. *The mapping $\Upsilon\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$ preserves stability and selfdecomposability. More precisely, we have*

$$\Upsilon(\mathcal{S}(*)) = \mathcal{S}(*) \quad \textit{and} \quad \Upsilon(\mathcal{L}(*)) \subseteq \mathcal{L}(*).$$

Proof. Suppose $\mu \in \mathcal{S}(*)$ and that $c, c' > 0$ and $d, d' \in \mathbb{R}$. Then

$$(D_c\mu * \delta_d) * (D_{c'}\mu * \delta_{d'}) = D_{c''}\mu * \delta_{d''},$$

for suitable c'' in $]0,\infty[$ and d'' in $\mathbb{R}$. Using now (ii)-(iv) of Proposition 3.18, we find that

$$\begin{aligned}\big(D_c\Upsilon(\mu) * \delta_d\big) * \big(D_{c'}\Upsilon(\mu) * \delta_{d'}\big) &= \big(\Upsilon(D_c\mu) * \Upsilon(\delta_d)\big) * \big(\Upsilon(D_{c'}\mu) * \Upsilon(\delta_{d'})\big)\\ &= \Upsilon(D_c\mu * \delta_d) * \Upsilon(D_{c'}\mu * \delta_{d'})\\ &= \Upsilon\big((D_c\mu * \delta_d) * (D_{c'}\mu * \delta_{d'})\big)\\ &= \Upsilon\big(D_{c''}\mu * \delta_{d''})\big)\\ &= D_{c''}\Upsilon(\mu) * \delta_{d''},\end{aligned}$$

which shows that $\Upsilon(\mu) \in \mathcal{S}(*)$. This verifies the inclusion $\Upsilon(\mathcal{S}(*)) \subseteq \mathcal{S}(*)$. To prove the converse inclusion, we use Corollary 3.4 (the following argument, in fact, also shows the inclusion just verified above). As described in Section 2.5, the stable laws are characterized by having Lévy measures in the form $r(t)\,\mathrm{d}t$, where

$$r(t) = \begin{cases} c_+ t^{-1-\alpha}, & \text{for } t > 0,\\ c_-|t|^{-1-\alpha}, & \text{for } t < 0,\end{cases}$$

with $\alpha \in\,]0,2[$ and $c_+, c_- \geq 0$. Using Corollary 3.4, it follows then that for μ in $\mathcal{S}(*)$, the Lévy measure for $\Upsilon(\mu)$ takes the form $\tilde{r}(t)\,\mathrm{d}t$, with $\tilde{r}(t)$ given by

$$\begin{aligned}\tilde{r}(t) &= \begin{cases} \int_0^\infty y^{-1} r(y^{-1}) \mathrm{e}^{-ty}\,\mathrm{d}y, & \text{if } t > 0,\\ \int_{-\infty}^0 -y^{-1} r(y^{-1}) \mathrm{e}^{-ty}\,\mathrm{d}y, & \text{if } t < 0,\end{cases}\\ &= \begin{cases} c_+ \Gamma(1+\alpha) t^{-1-\alpha}, & \text{if } t > 0,\\ c_- \Gamma(1+\alpha)|t|^{-1-\alpha}, & \text{if } t < 0,\end{cases}\end{aligned} \tag{3.25}$$

where the second equality follows by a standard calculation. Formula (3.25) shows, in particular, that any measure in $\mathcal{S}(*)$ is the image by Υ of another measure in $\mathcal{S}(*)$.

Assume next that $\mu \in \mathcal{L}(*)$. Then for any c in $]0,1[$, there exists a measure μ_c in $\mathcal{ID}(*)$, such that $\mu = D_c\mu * \mu_c$. Using now (ii)-(iii) of Proposition 3.18, we find that

$$\Upsilon(\mu) = \Upsilon(D_c\mu * \mu_c) = \Upsilon(D_c\mu) * \Upsilon(\mu_c) = D_c\Upsilon(\mu) * \Upsilon(\mu_c),$$

which shows that $\Upsilon(\mu) \in \mathcal{L}(*)$. □

Remark 3.20. By the definition of Υ and Corollary 3.5 it follows that the Lévy measure for any probability measure in the range $\Upsilon(\mathcal{ID}(*))$ of Υ has a C^∞ density w.r.t. Lebesgue measure. This implies that the mapping $\Upsilon\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$ *is not surjective.* In particular it is apparent that the (classical) Poisson distributions are not in the image of Υ, since the characteristic triplet for the Poisson distribution with mean $c > 0$ is $(0, c\delta_1, c)$. In [BaMaSa04], it was proved that the full range of Υ is the Goldie-Steutel-Bondesson class $\mathcal{B}(*)$. In Theorem 3.27 below, we show that $\Upsilon(\mathcal{L}(*)) = \mathcal{T}(*)$.

We end this section with some results on properties of distributions that are preserved by the mapping Υ. The first of these results is an immediate consequence of Proposition 3.11.

Corollary 3.21. *Let μ be a measure in $\mathcal{ID}(*)$. Then $\mu \in \mathcal{ID}^+_\tau(*)$ if and only if $\Upsilon(\mu) \in \mathcal{ID}^+_\tau(*)$.*

Proof. For a measure μ in $\mathcal{ID}(*)$ with Lévy measure ρ, $\Upsilon(\mu)$ has Lévy measure $\Upsilon_0(\rho) = \tilde{\rho}$. Hence, the corollary follows immediately from formula (3.19) and the characterization of $\mathcal{ID}^+_\tau(*)$ given in Remark 2.13. □

The next result shows that the mapping Υ has the same property as that of Υ_0 exhibited in Proposition 3.8.

Proposition 3.22. *For any measure μ in $\mathcal{ID}(*)$ and any positive number p, we have*

$$\mu \text{ has } p\text{'th moment} \iff \Upsilon(\mu) \text{ has } p\text{'th moment.}$$

Proof. Let μ in $\mathcal{ID}(*)$ be given and put $\nu = \Upsilon(\mu)$. Let (a, ρ, η) be the characteristic triplet for μ and $(2a, \tilde{\rho}, \tilde{\eta})$ the characteristic triplet for ν (in particular $\tilde{\rho} = \Upsilon_0(\rho)$. Now by [Sa99, Corollary 25.8] we have

$$\int_{\mathbb{R}} |x|^p \, \mu(\mathrm{d}x) < \infty \iff \int_{[-1,1]^c} |x|^p \, \rho(\mathrm{d}x) < \infty, \tag{3.26}$$

and

$$\int_{\mathbb{R}} |x|^p \, \nu(\mathrm{d}x) < \infty \iff \int_{[-1,1]^c} |x|^p \, \tilde{\rho}(\mathrm{d}x) < \infty. \tag{3.27}$$

Note next that

$$\begin{aligned} \int_{[-1,1]^c} |x|^p \tilde{\rho}(\mathrm{d}x) &= \int_0^\infty \Big(\int_{[-1,1]^c} |x|^p \, D_y\rho(\mathrm{d}x) \Big) \mathrm{e}^{-y} \, \mathrm{d}y \\ &= \int_0^\infty \Big(\int_{\mathbb{R}} |xy|^p 1_{[-1,1]^c}(xy) \, \rho(\mathrm{d}x) \Big) \mathrm{e}^{-y} \, \mathrm{d}y \\ &= \int_{\mathbb{R}} |x|^p \Big(\int_{1/|x|}^\infty y^p \mathrm{e}^{-y} \, \mathrm{d}y \Big) \, \rho(\mathrm{d}x), \end{aligned} \tag{3.28}$$

where we interpret $\int_{1/|x|}^\infty y^p \mathrm{e}^{-y} \, \mathrm{d}y$ as 0, when $x = 0$.

Assume now that μ has p'th moment. Then by (3.26), $\int_{[-1,1]^c} |x|^p \, \rho(\mathrm{d}x) < \infty$, and by (3.28)

$$\begin{aligned} &\int_{[-1,1]^c} |x|^p \tilde{\rho}(\mathrm{d}x) \\ &\qquad \le \int_{[-1,1]} |x|^p \Big(\int_{1/|x|}^\infty y^p \mathrm{e}^{-y} \, \mathrm{d}y \Big) \, \rho(\mathrm{d}x) + \Gamma(p+1) \int_{[-1,1]^c} |x|^p \, \rho(\mathrm{d}x). \end{aligned}$$

By (3.27), it remains thus to show that

$$\int_{[-1,1]} |x|^p \Big(\int_{1/|x|}^\infty y^p \mathrm{e}^{-y} \, \mathrm{d}y \Big) \, \rho(\mathrm{d}x) < \infty. \tag{3.29}$$

If $p \ge 2$, then this is obvious:

$$\int_{[-1,1]} |x|^p \Big(\int_{1/|x|}^\infty y^p \mathrm{e}^{-y} \, \mathrm{d}y \Big) \, \rho(\mathrm{d}x) \le \Gamma(p+1) \int_{[-1,1]} |x|^p \, \rho(\mathrm{d}x) < \infty,$$

since ρ is a Lévy measure. For p in $]0, 2[$ we note first that for any numbers t, q in $]0, \infty[$ we have

$$\int_t y^p \mathrm{e}^{-y}\,\mathrm{d}y = \int_t^\infty \frac{y^{p+q}}{y^q}\mathrm{e}^{-y}\,\mathrm{d}y \le t^{-q}\int_t^\infty y^{p+q}\mathrm{e}^{-y}\,\mathrm{d}y \le t^{-q}\Gamma(p+q+1).$$

Using this with $t = 1/|x|$, we find for any positive q that

$$\int_{[-1,1]} |x|^p \Big(\int_{1/|x|}^\infty y^p \mathrm{e}^{-y}\,\mathrm{d}y\Big)\,\rho(\mathrm{d}x) \le \Gamma(p+q+1)\int_{[-1,1]} |x|^{p+q}\,\rho(\mathrm{d}x).$$

Choosing $q = 2-p$ we find as desired that

$$\int_{[-1,1]} |x|^p \Big(\int_{1/|x|}^\infty y^p \mathrm{e}^{-y}\,\mathrm{d}y\Big)\,\rho(\mathrm{d}x) \le \Gamma(3)\int_{[-1,1]} |x|^2\,\rho(\mathrm{d}x) < \infty,$$

since ρ is a Lévy measure.

Assume conversely that $\nu = \Upsilon(\mu)$ has p'th moment. Then by (3.27), we have $\int_{[-1,1]^c} |x|^p\,\tilde{\rho}(\mathrm{d}x) < \infty$, and by (3.26) we have to show that $\int_{[-1,1]^c} |x|^p\,\rho(\mathrm{d}x) < \infty$. For this, note that whenever $|x| > 1$ we have

$$\int_{1/|x|}^\infty y^p \mathrm{e}^{-y}\,\mathrm{d}y \ge \int_1^\infty y^p \mathrm{e}^{-y}\,\mathrm{d}y \in \,]0,\infty[.$$

Setting $c(p) = \int_1^\infty y^p \mathrm{e}^{-y}\,\mathrm{d}y$ and using (3.28) we find thus that

$$\begin{aligned}\int_{[-1,1]^c} |x|^p\,\rho(\mathrm{d}x) &\le \frac{1}{c(p)}\int_{[-1,1]^c} |x|^p \Big(\int_{1/|x|}^\infty y^p \mathrm{e}^{-y}\,\mathrm{d}y\Big)\,\rho(\mathrm{d}x)\\ &\le \frac{1}{c(p)}\int_{[-1,1]^c} |x|^p\,\tilde{\rho}(\mathrm{d}x) < \infty,\end{aligned}$$

as desired. □

3.3 Relations between Υ_0, Υ and the Classes $\mathcal{L}(*)$, $\mathcal{T}(*)$

In this section we establish a close connection between the mapping Υ and the relationship between the classes $\mathcal{T}(*)$ and $\mathcal{L}(*)$. More precisely, we prove that $\Upsilon(\mathcal{L}(*)) = \mathcal{T}(*)$ and also that $\Upsilon(\mathcal{L}_\tau^+(*)) = \mathcal{T}_\tau^+(*)$. We consider the latter equality first.

The Positive Thorin Class

We start by establishing the following technical result on the connection between complete monotonicity and Lévy densities for measures in $\mathcal{ID}^+(*)$.

Lemma 3.23. *Let ν be a Borel measure on $[0,\infty[$ such that*

$$\forall t > 0\colon \int_{[0,\infty[} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s) < \infty,$$

and note that ν is necessarily a Radon measure. Let $q\colon]0,\infty[\, \to [0,\infty[$ be the function given by:

$$q(t) = \frac{1}{t}\int_{[0,\infty[} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s), \qquad (t>0).$$

Then q satisfies the condition

$$\int_0^\infty \min\{1,t\}q(t)\,\mathrm{d}t < \infty, \tag{3.30}$$

if and only if ν satisfies the following three conditions:

(a) $\nu(\{0\}) = 0$,
(b) $\int_{]0,1]} |\log(t)|\,\nu(\mathrm{d}t) < \infty$,
(c) $\int_{[1,\infty[} \frac{1}{t}\,\nu(\mathrm{d}t) < \infty$.

Proof. We note first that

$$\begin{aligned}\int_0^1 tq(t)\,\mathrm{d}t &= \int_0^1\int_{[0,\infty[} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s)\,\mathrm{d}t = \int_{[0,\infty[}\Big(\int_0^1 \mathrm{e}^{-ts}\,\mathrm{d}t\Big)\,\nu(\mathrm{d}s)\\ &= \nu(\{0\}) + \int_{]0,\infty[} \tfrac{1}{s}(1-\mathrm{e}^{-s})\,\nu(\mathrm{d}s).\end{aligned} \tag{3.31}$$

Note next that

$$\begin{aligned}\int_1^\infty q(t)\,\mathrm{d}t &= \int_1^\infty \tfrac{1}{t}\int_{[0,\infty[} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s)\,\mathrm{d}t = \int_{[0,\infty[}\Big(\int_1^\infty \tfrac{1}{t}\mathrm{e}^{-ts}\,\mathrm{d}t\Big)\,\nu(\mathrm{d}s)\\ &= \int_{[0,\infty[}\Big(\int_s^\infty \tfrac{1}{t}\mathrm{e}^{-t}\,\mathrm{d}t\Big)\,\nu(\mathrm{d}s) = \int_0^\infty \tfrac{1}{t}\mathrm{e}^{-t}\Big(\int_{[0,t]} 1\,\nu(\mathrm{d}s)\Big)\,\mathrm{d}t\\ &= \int_0^\infty \tfrac{1}{t}\mathrm{e}^{-t}\nu([0,t])\,\mathrm{d}t.\end{aligned} \tag{3.32}$$

Assume now that (3.30) is satisfied. It follows then from (3.32) that

$$\infty > \int_0^1 \tfrac{1}{t}\mathrm{e}^{-t}\nu([0,t])\,\mathrm{d}t \geq \mathrm{e}^{-1}\int_0^1 \tfrac{1}{t}\nu([0,t])\,\mathrm{d}t.$$

Here, by partial (Stieltjes) integration,

$$\begin{aligned}\int_0^1 \tfrac{1}{t}\nu([0,t])\,\mathrm{d}t &= \Big[\log(t)\nu([0,t])\Big]_0^1 - \int_{]0,1]} \log(t)\,\nu(\mathrm{d}t)\\ &= \lim_{t\searrow 0} |\log(t)|\nu([0,t]) + \int_{]0,1]} |\log(t)|\,\nu(\mathrm{d}t),\end{aligned}$$

so we may conclude that

$$\lim_{t\searrow 0} |\log(t)|\nu([0,t]) < \infty \quad \text{and} \quad \int_{]0,1]} |\log(t)|\,\nu(\mathrm{d}t) < \infty,$$

and this implies that (a) and (b) are satisfied. Regarding (c), note that it follows from (3.30) and (3.31) that

$$\infty > \int_0^1 tq(t)\,\mathrm{d}t \geq \int_{[1,\infty[} \tfrac{1}{s}(1-\mathrm{e}^{-s})\,\nu(\mathrm{d}s) \geq (1-\mathrm{e}^{-1})\int_{[1,\infty[} \tfrac{1}{s}\,\nu(\mathrm{d}s),$$

and hence (c) follows.

Assume conversely that ν satisfies conditions (a), (b) and (c). Then by (3.31) we have

$$\int_0^1 tq(t)\,\mathrm{d}t = \int_{]0,\infty[} \tfrac{1}{s}(1-\mathrm{e}^{-s})\,\nu(\mathrm{d}s) \leq \int_{]0,1[} 1\,\nu(\mathrm{d}s) + \int_{[1,\infty[} \tfrac{1}{s}\,\nu(\mathrm{d}s),$$

where we have used that $\frac{1}{s}(1-\mathrm{e}^{-s}) \leq 1$ for all positive s. Thus, by (b) and (c), $\int_0^1 tq(t)\,\mathrm{d}t < \infty$. Regarding $\int_1^\infty q(t)\,\mathrm{d}t$, note that for any s in $]0,1]$ we have (using (a))

$$\begin{aligned} 0 \leq |\log(s)|\nu([0,s]) &= \int_{]0,s]} \log(s^{-1})\,\nu(\mathrm{d}u) \leq \int_{]0,s]} \log(u^{-1})\,\nu(\mathrm{d}u) \\ &= \int_{]0,s]} |\log(u)|\,\nu(\mathrm{d}u), \end{aligned}$$

and hence it follows from (b) that $|\log(s)|\nu([0,s]) \to 0$ as $s \searrow 0$. By partial integration we obtain thus that

$$\begin{aligned} \infty > \int_{]0,1]} |\log(s)|\,\nu(\mathrm{d}s) &= \Big[|\log(s)|\nu([0,s])\Big]_0^1 + \int_0^1 \tfrac{1}{s}\nu([0,s])\,\mathrm{d}s \\ &= \int_0^1 \tfrac{1}{s}\nu([0,s])\,\mathrm{d}s \\ &\geq \int_0^1 \tfrac{1}{s}\mathrm{e}^{-s}\nu([0,s])\,\mathrm{d}s. \end{aligned}$$

By (3.32) and (b) it remains, thus, to show that $\int_1^\infty \frac{1}{s}\mathrm{e}^{-s}\nu([0,s])\,\mathrm{d}s < \infty$. For that, it obviously suffices to prove that $\frac{1}{s}\nu([0,s]) \to 0$ as $s \to \infty$. Note, towards this end, that whenever $s \geq t \geq 1$, we have

$$\tfrac{1}{s}\nu([0,s]) = \tfrac{1}{s}\nu([0,t]) + \int_{]t,s]} \tfrac{1}{s}\,\nu(\mathrm{d}u) \leq \tfrac{1}{s}\nu([0,t]) + \int_{]t,s]} \tfrac{1}{u}\,\nu(\mathrm{d}u),$$

and hence, for any t in $[1,\infty[$,

$$\limsup_{s\to\infty} \tfrac{1}{s}\nu([0,s]) \le \int_{]t,\infty[} \tfrac{1}{u}\,\nu(\mathrm{d}u).$$

Letting finally $t\to\infty$, it follows from (c) that

$$\limsup_{s\to\infty} \tfrac{1}{s}\nu([0,s]) = 0,$$

as desired. □

Theorem 3.24. *The mapping Υ maps the class $\mathcal{L}_\tau^+(*)$ onto the class $\mathcal{T}_\tau^+(*)$, i.e.*

$$\Upsilon(\mathcal{L}_\tau^+(*)) = \mathcal{T}_\tau^+(*).$$

Proof. Assume that $\mu \in \mathcal{L}_\tau^+(*)$ with generating triplet (a,ρ,η). Then, by Remark 2.13, $a=0$, ρ is concentrated on $[0,\infty[$, and $\int_0^\infty \min\{1,t\}\,\rho(\mathrm{d}t) < \infty$. Furthermore, since μ is selfdecomposable, $\rho(\mathrm{d}t) = r(t)\,\mathrm{d}t$ for some density function $r\colon [0,\infty[\to[0,\infty[$, satisfying that the function $q(t) = tr(t)$ $(t\ge 0)$ is decreasing (cf. the last paragraph in Section 2.5).

Now the measure $\Upsilon(\mu)$ has generating triplet $(0,\tilde\rho,\tilde\eta)$, where $\tilde\rho$ has density $\tilde r$ given by

$$\tilde r(t) = \int_0^\infty q(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s, \qquad (t\ge 0),$$

(cf. Corollary 3.4). We already know from Corollary 3.21 that $\Upsilon(\mu)\in\mathcal{ID}_\tau^+(*)$, so it remains to show that the function $t\mapsto t\tilde r(t)$ is completely monotone, i.e. that

$$t\tilde r(t) = \int_{[0,\infty[} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s), \qquad (t>0),$$

for some (Radon) measure ν on $[0,\infty[$. Note for this, that the function $s\mapsto q(s^{-1})$ is increasing on $]0,\infty[$. This implies, in particular, that $s\mapsto q(s^{-1})$ has only countably many points of discontinuity, and hence, by changing r on a Lebesgue null-set, we may assume that $s\mapsto q(s^{-1})$ is increasing and right continuous. Note finally that $q(s^{-1})\to 0$ as $s\searrow 0$. Indeed, since $s\mapsto q(s^{-1})$ is increasing, the limit $\beta = \lim_{s\searrow 0} q(s^{-1})$ exists and equals $\inf_{s>0} q(s^{-1})$. Since $sr(s) = q(s)\to\beta$ as $s\to\infty$ and $\int_1^\infty r(s)\,\mathrm{d}s < \infty$, we must have $\beta = 0$. We may now let ν be the Stieltjes measure corresponding to the function $s\mapsto q(s^{-1})$, i.e.

$$\nu(]-\infty,s]) = \begin{cases} q(s^{-1}), & \text{if } s>0,\\ 0, & \text{if } s\le 0.\end{cases}$$

Then, whenever $t\in\,]0,\infty[$ and $0<a<b<\infty$, we have by partial integration

$$\int_a^b q(s^{-1})t\mathrm{e}^{-ts}\,\mathrm{d}s = \Big[-q(s^{-1})\mathrm{e}^{-ts}\Big]_a^b + \int_{]a,b]} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s). \tag{3.33}$$

Here $q(a^{-1})\mathrm{e}^{-ta} \to 0$ as $a \searrow 0$. Furthermore, since $\int_0^\infty q(s^{-1})t\mathrm{e}^{-ts}\,\mathrm{d}s = t\tilde{r}(t) < \infty$, it follows from (3.33) that $\gamma = \lim_{b\to\infty} q(b^{-1})\mathrm{e}^{-bt}$ exists in $[0,\infty]$. Now $sr(s)\mathrm{e}^{-t/s} = q(s)\mathrm{e}^{-t/s} \to \gamma$ as $s \searrow 0$, and since $\int_0^1 sr(s)\,\mathrm{d}s < \infty$, this implies that $\gamma = 0$. Letting, finally, $a \to 0$ and $b \to \infty$ in (3.33), we may now conclude that

$$t\tilde{r}(t) = \int_0^\infty q(s^{-1})t\mathrm{e}^{-ts}\,\mathrm{d}s = \int_{]0,\infty[} \mathrm{e}^{-ts}\nu(\mathrm{d}s), \qquad (t > 0),$$

as desired.

Assume conversely that $\tilde{\mu} \in \mathcal{T}_\tau^+(*)$ with generating triplet $(a, \tilde{\rho}, \tilde{\eta})$. Then $a = 0$, $\tilde{\rho}$ is concentrated on $[0,\infty[$ and $\int_0^\infty \min\{1,t\}\,\tilde{\rho}(\mathrm{d}t) < \infty$. Furthermore, $\tilde{\rho}$ has a density $\tilde{r}$ in the form

$$\tilde{r}(t) = \frac{1}{t}\int_{[0,\infty[} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s), \qquad (t > 0),$$

for some (Radon) measure ν on $[0,\infty[$, satisfying conditions (a),(b) and (c) of Lemma 3.23.

We define next a function $r\colon]0,\infty[\to [0,\infty[$ by

$$r(s) = \tfrac{1}{s}\nu([0,\tfrac{1}{s}]), \qquad (s > 0). \tag{3.34}$$

Furthermore, we put

$$q(s) = sr(s) = \nu([0,\tfrac{1}{s}]), \qquad (s > 0),$$

and we note that q is decreasing on $]0,\infty[$ and that $q(s^{-1}) = \nu([0,s])$. Note also that, since $\nu(\{0\}) = 0$ (cf. Lemma 3.23),

$$0 \le \nu([0,s])\mathrm{e}^{-ts} \le \nu([0,s]) \to 0, \qquad \text{as } s \searrow 0,$$

for any $t > 0$. Furthermore, since $\int_{[1,\infty[} \frac{1}{s}\,\nu(\mathrm{d}s) < \infty$ (cf. Lemma 3.23), it follows as in the last part of the proof of Lemma 3.23 that $\frac{1}{s}\nu([0,s]) \to 0$ as $s \to \infty$. This implies, in particular, that $q(s^{-1})\mathrm{e}^{-ts} = \nu([0,s])\mathrm{e}^{-ts} = \frac{1}{s}\nu([0,s])s\mathrm{e}^{-ts} \to 0$ as $s \to \infty$ for any positive t. By partial integration, we now conclude that

$$\int_0^\infty q(s^{-1})t\mathrm{e}^{-ts}\,\mathrm{d}s = \Big[-q(s^{-1})\mathrm{e}^{-ts}\Big]_0^\infty + \int_{]0,\infty[} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s) = t\tilde{r}(t),$$

for any positive t. Hence,

$$\tilde{r}(t) = \int_0^\infty q(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s = \int_0^\infty s^{-1}r(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s, \qquad (t > 0),$$

and by Corollary 3.4, this means that

$$\tilde{\rho} = \int_0^\infty (D_x\rho)\mathrm{e}^{-x}\,\mathrm{d}x,$$

where $\rho(\mathrm{d}t) = r(t)\,\mathrm{d}t$. Note that since ν is a Radon measure, r is bounded on compact subsets of $]0,\infty[$, and hence ρ is σ-finite. We may thus apply Proposition 3.11 to conclude that $\int_0^\infty \min\{1,t\}\,\rho(\mathrm{d}t) < \infty$, so in particular ρ is a Lévy measure. Now, let μ be the measure in $\mathcal{ID}(*)$ with generating triplet $(0,\rho,\eta)$, where

$$\eta = \tilde{\eta} - \int_0^\infty \Big(\int_{\mathbb{R}} t\big(1_{[-1,1]}(t) - 1_{[-x,x]}(t)\big)\,D_x\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x.$$

Then $\Upsilon(\mu) = \tilde{\mu}$ and $\mu \in \mathcal{ID}^+_\tau(*)$ (cf. Corollary 3.21). Moreover, since $tr(t) = q(t)$ is a decreasing function of t, it follows that μ is selfdecomposable (cf. the last paragraph of Section 2.5). This concludes the proof. $\square$

The General Thorin Class

We start again with some technical results on complete monotonicity.

Lemma 3.25. *Let ν be a Borel measure on $[0,\infty[$ satisfying that*

$$\forall t > 0\colon \int_{[0,\infty[} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s) < \infty,$$

and note that ν is a Radon measure on $[0,\infty[$. Let further $q\colon]0,\infty[\to [0,\infty[$ be the function given by

$$q(t) = \frac{1}{t}\int_{[0,\infty[} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s), \qquad (t > 0). \tag{3.35}$$

Then q is a Lévy density (i.e. $\int_0^\infty \min\{1,t^2\}q(t)\,\mathrm{d}t < \infty$) if and only if ν satisfies the following three conditions:

(a) $\nu(\{0\}) = 0$.
(b) $\int_{]0,1[} |\log(t)|\,\nu(\mathrm{d}t) < \infty$.
(c) $\int_{[1,\infty[} \frac{1}{t^2}\,\nu(\mathrm{d}t) < \infty$.

Proof. We note first that

$$\int_0^1 t^2 q(t)\,\mathrm{d}t = \int_0^1 t\Big(\int_{[0,\infty[} \mathrm{e}^{-ts}\,\nu(\mathrm{d}s)\Big)\,\mathrm{d}t = \int_{[0,\infty[}\Big(\int_0^1 t\mathrm{e}^{-ts}\,\mathrm{d}t\Big)\,\nu(\mathrm{d}s)$$
$$= \frac{1}{2}\nu(\{0\}) + \int_{]0,\infty[} \frac{1}{s^2}(1 - \mathrm{e}^{-s} - s\mathrm{e}^{-s})\,\nu(\mathrm{d}s). \tag{3.36}$$

Exactly as in the proof of Lemma 3.23 we have also that

$$\int_1^\infty q(t)\,\mathrm{d}t = \int_0^\infty \frac{1}{t}\mathrm{e}^{-t}\nu([0,t])\,\mathrm{d}t. \tag{3.37}$$

Assume now that q is a Lévy density. Exactly as in the proof of Lemma 3.23, formula (3.37) then implies that ν satisfies conditions (a) and (b). Regarding (c), note that by (3.36),

$$\infty > \int_0^1 t^2 q(t)\,\mathrm{d}t \geq \int_{[1,\infty[} \frac{1}{s^2}(1-\mathrm{e}^{-s}-s\mathrm{e}^{-s})\,\nu(\mathrm{d}s) \geq (1-2\mathrm{e}^{-1})\int_{[1,\infty[} \frac{1}{s^2}\,\nu(\mathrm{d}s),$$

where we used that $s \mapsto 1-\mathrm{e}^{-s}-s\mathrm{e}^{-s}$ is an increasing function on $[0,\infty[$. It follows thus that (c) is satisfied too.

Assume conversely that ν satisfies (a),(b) and (c). Then by (3.36) we have

$$\int_0^1 t^2 q(t)\,\mathrm{d}t = \int_{]0,\infty[} \frac{1}{s^2}(1-\mathrm{e}^{-s}-s\mathrm{e}^{-s})\,\nu(\mathrm{d}s) \leq \int_{]0,1[} 1\,\nu(\mathrm{d}s) + \int_{[1,\infty[} \frac{1}{s^2}\,\nu(\mathrm{d}s),$$

where we used that $s^{-2}(1-\mathrm{e}^{-s}-s\mathrm{e}^{-s}) = \int_0^1 t\mathrm{e}^{-ts}\,\mathrm{d}t \leq 1$ for all positive s. Hence, using (c) (and the fact that ν is a Radon measure on $[0,\infty[$), we see that $\int_0^1 t^2 q(t)\,\mathrm{d}t < \infty$.

Regarding $\int_1^\infty q(t)\,\mathrm{d}t$, we find by application of (a) and (b), exactly as in the proof of Lemma 3.23, that

$$\infty > \int_{[0,1]} |\log(s)|\,\nu(\mathrm{d}s) \geq \int_0^1 \frac{1}{s}\mathrm{e}^{-s}\nu([0,s])\,\mathrm{d}s.$$

By (3.37), it remains thus to show that $\int_1^\infty \frac{1}{s}\mathrm{e}^{-s}\nu([0,s])\,\mathrm{d}s < \infty$, and this clearly follows, if we prove that $s^{-2}\nu([0,s]) \to 0$ as $s \to \infty$ (since ν is a Radon measure). The latter assertion is established similarly to the last part of the proof of Lemma 3.23: Whenever $s \geq t \geq 1$, we have

$$\frac{1}{s^2}\nu([0,s]) \leq \frac{1}{s^2}\nu([0,t]) + \int_{]t,s]} \frac{1}{u^2}\,\nu(\mathrm{d}u),$$

and hence for any t in $[1,\infty[$,

$$\limsup_{s\to\infty} \frac{1}{s^2}\nu([0,s]) \leq \int_{]t,\infty[} \frac{1}{u^2}\nu(\mathrm{d}u). \tag{3.38}$$

Letting finally $t \to \infty$ in (3.38), it follows from (c) that

$$\limsup_{s\to\infty} s^{-2}\nu([0,s]) = 0.$$

This completes the proof. ∎

Corollary 3.26. *Let ν be a Borel measure on $\mathbb{R}$ satisfying that*

$$\forall t \in \mathbb{R} \setminus \{0\}\colon \int_{\mathbb{R}} \mathrm{e}^{-|ts|}\, \nu(\mathrm{d}s) < \infty,$$

and note that ν is necessarily a Radon measure on $\mathbb{R}$. Let $q\colon \mathbb{R} \setminus \{0\} \to [0, \infty[$ be the function defined by:

$$q(t) = \begin{cases} \frac{1}{t} \int_{[0,\infty[} \mathrm{e}^{-ts}\, \nu(\mathrm{d}s), & \text{if } t > 0, \\ \frac{1}{|t|} \int_{]-\infty,0]} \mathrm{e}^{-ts}\, \nu(\mathrm{d}s), & \text{if } t < 0. \end{cases}$$

Then q is a Lévy density (i.e. $\int_{\mathbb{R}} \min\{1, t^2\} q(t)\,\mathrm{d}t < \infty$), if and only if ν satisfies the following three conditions:

(d) $\nu(\{0\}) = 0$.
(e) $\int_{[-1,1]\setminus\{0\}} \big|\log|t|\big|\, \nu(\mathrm{d}t) < \infty$.
(f) $\int_{\mathbb{R}\setminus]-1,1[} \frac{1}{t^2}\, \nu(\mathrm{d}t) < \infty$.

Proof. Let ν_+ and ν_- be the restrictions of ν to $[0, \infty[$ and $]-\infty, 0]$, respectively. Let, further, $\check{\nu}_-$ be the transformation of ν_- by the mapping $s \mapsto -s$, and put $\check{q}(t) = q(-t)$. Note then that

$$\check{q}(t) = \frac{1}{t} \int_{[0,\infty[} \mathrm{e}^{-ts}\, \check{\nu}_-(\mathrm{d}s), \qquad (t > 0).$$

By application of Lemma 3.25, we now have

$$\begin{aligned} q \text{ is a Lévy density on } \mathbb{R} &\iff q \text{ and } \check{q} \text{ are Lévy densities on } [0, \infty[\\ &\iff \nu_+ \text{ and } \check{\nu}_- \text{ satisfy (a),(b) and (c) of Lemma 3.25} \\ &\iff \nu \text{ satisfies (d),(e) and (f).} \end{aligned}$$

This proves the corollary. □

Theorem 3.27. *The mapping Υ maps the class of selfdecomposable distributions on $\mathbb{R}$ onto the generalized Thorin class, i.e.*

$$\Upsilon(\mathcal{L}(*)) = \mathcal{T}(*).$$

Proof. We prove first that $\Upsilon(\mathcal{L}(*)) \subseteq \mathcal{T}(*)$. So let μ be a measure in $\mathcal{L}(*)$ and consider its generating triplet (a, ρ, η). Then $a \geq 0$, $\eta \in \mathbb{R}$ and $\rho(\mathrm{d}t) = r(t)\,\mathrm{d}t$ for some density function, $r(t)$, satisfying that the function

$$q(t) := |t| r(t), \qquad (t \in \mathbb{R}),$$

is increasing on $]-\infty,0[$ and decreasing on $]0,\infty[$. Next, let $(2a,\tilde{\rho},\tilde{\eta})$ be the generating triplet for $\Upsilon(\mu)$. From Lemma 3.4 we know that $\tilde{\rho}$ has the following density w.r.t. Lebesgue measure:

$$\tilde{r}(t)=\begin{cases}\int_0^{\infty}q(y^{-1})\mathrm{e}^{-ty}\,\mathrm{d}y, & \text{if } t>0,\\ \int_{-\infty}^{0}q(y^{-1})\mathrm{e}^{-ty}\,\mathrm{d}y, & \text{if } t<0.\end{cases}$$

Note that the function $y\mapsto q(y^{-1})$ is increasing on $]0,\infty[$. Thus, as in the proof of Theorem 3.24, we may, by changing $r(t)$ on a null-set, assume that $y\mapsto q(y^{-1})$ is increasing and right-continuous on $]0,\infty[$. Furthermore, since $\int_1^{\infty}\frac{1}{s}q(s)\,\mathrm{d}s=\int_1^{\infty}r(s)\,\mathrm{d}s<\infty$, it follows as in the proof of Theorem 3.24 that $q(y^{-1})\to 0$ as $y\searrow 0$. Thus, we may let ν_+ be the Stieltjes measure corresponding to the function $y\mapsto q(y^{-1})$ on $]0,\infty[$, i.e.

$$\nu_+(]-\infty,y])=\begin{cases}0, & \text{if } y\le 0,\\ q(y^{-1}), & \text{if } y>0.\end{cases}$$

Now, whenever $t>0$ and $0<b<c<\infty$, we have by partial Stieltjes integration that

$$t\int_b^c q(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s=\Big[-\mathrm{e}^{-ts}q(s^{-1})\Big]_b^c+\int_b^c\mathrm{e}^{-ts}\,\nu_+(\mathrm{d}s). \tag{3.39}$$

Here, $\mathrm{e}^{-tb}q(b^{-1})\le q(b^{-1})\to 0$ as $b\searrow 0$. Since $\int_0^{\infty}q(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s=\tilde{r}(t)<\infty$, (3.39) shows, furthermore, that the limit

$$\gamma:=\lim_{c\to\infty}\mathrm{e}^{-tc}q(c^{-1})=\lim_{s\searrow 0}\mathrm{e}^{-t/s}sr(s)$$

exists in $[0,\infty]$. Since $\int_0^{\infty}s^2r(s)\,\mathrm{d}s<\infty$, it follows that we must have $\gamma=0$. From (3.39), it follows thus that

$$t\tilde{r}(t)=t\int_0^{\infty}q(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s=\int_0^{\infty}\mathrm{e}^{-ts}\,\nu_+(\mathrm{d}s). \tag{3.40}$$

Replacing now $r(s)$ by $r(-s)$ for s in $]0,\infty[$, the argument just given yields the existence of a measure $\check{\nu}_-$ on $[0,\infty[$, such that (after changing r on a null-set)

$$\check{\nu}_-(]-\infty,y])=\begin{cases}0, & \text{if } y\le 0,\\ q(-y^{-1}), & \text{if } y>0.\end{cases}$$

Furthermore, the measure $\check{\nu}_-$ satisfies the identity

$$t\int_0^{\infty}q(-s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s=\int_0^{\infty}\mathrm{e}^{-ts}\,\check{\nu}_-(\mathrm{d}s),\qquad (t>0).$$

Next, let ν_- be the transformation of $\check{\nu}_-$ by the mapping $s\mapsto -s$. For t in $]-\infty,0[$ we then have

$$\begin{aligned}|t|\tilde{r}(t) &= |t|\int_{-\infty}^{0} q(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s = |t|\int_{0}^{\infty} q(-s^{-1})\mathrm{e}^{-|t|s}\,\mathrm{d}s \\ &= \int_{0}^{\infty} \mathrm{e}^{-|t|s}\,\check{\nu}_{-}(\mathrm{d}s) = \int_{-\infty}^{0} \mathrm{e}^{-ts}\,\nu_{-}(\mathrm{d}s).\end{aligned} \tag{3.41}$$

Putting finally $\nu = \nu_{+} + \nu_{-}$, it follows from (3.40) and (3.41) that

$$|t|\tilde{r}(t) = \begin{cases} \int_0^\infty \mathrm{e}^{-ts}\,\nu(\mathrm{d}s), & \text{if } t > 0, \\ \int_{-\infty}^0 \mathrm{e}^{-ts}\,\nu(\mathrm{d}s), & \text{if } t < 0, \end{cases}$$

and this shows that $\Upsilon(\mu) \in \mathcal{T}(*)$, as desired (cf. the last paragraph in Section 2.5).

Consider, conversely, a measure $\tilde{\mu}$ in $\mathcal{T}(*)$ with generating triplet $(a, \tilde{\rho}, \tilde{\eta})$. Then $a \geq 0$, $\tilde{\eta} \in \mathbb{R}$ and $\tilde{\rho}$ has a density, $\tilde{r}$, w.r.t. Lebesgue measure such that

$$|t|\tilde{r}(t) = \begin{cases} \int_0^\infty \mathrm{e}^{-ts}\,\nu(\mathrm{d}s), & \text{if } t > 0, \\ \int_{-\infty}^0 \mathrm{e}^{-ts}\,\nu(\mathrm{d}s), & \text{if } t < 0, \end{cases}$$

for some (Radon) measure ν on $\mathbb{R}$ satisfying conditions (d),(e) and (f) of Corollary 3.26. Define then the function $r\colon \mathbb{R}\setminus\{0\} \to [0,\infty[$ by

$$r(s) = \begin{cases} \frac{1}{s}\nu([0,\frac{1}{s}]), & \text{if } s > 0, \\ \frac{1}{|s|}\nu([\frac{1}{s},0]), & \text{if } s < 0, \end{cases}$$

and put furthermore

$$q(t) = |s|r(s) = \begin{cases} \nu([0,\frac{1}{s}]), & \text{if } s > 0, \\ \nu([\frac{1}{s},0]), & \text{if } s < 0. \end{cases} \tag{3.42}$$

Note that since $\nu(\{0\}) = 0$ (cf. Corollary 3.26), we have

$$\forall t > 0\colon \nu([0,s])\mathrm{e}^{-ts} \leq \nu([0,s]) \to 0, \quad \text{as } s \searrow 0,$$

and

$$\forall t < 0\colon \nu([s,0])\mathrm{e}^{-ts} \leq \nu([s,0]) \to 0, \quad \text{as } s \nearrow 0.$$

Furthermore, since $\int_{\mathbb{R}\setminus[-1,1]} \frac{1}{s^2}\,\nu(\mathrm{d}s) < \infty$, it follows as in the last part of the proof of Lemma 3.25 that

$$\lim_{s\to\infty} s^{-2}\nu([0,s]) = 0 = \lim_{s\to-\infty} s^{-2}\nu([s,0]).$$

In particular it follows that

$$\forall t > 0\colon \lim_{s\to\infty} \nu([0,s])\mathrm{e}^{-ts} = 0, \quad \text{and that} \quad \forall t < 0\colon \lim_{s\to-\infty} \nu([s,0])\mathrm{e}^{-ts} = 0.$$

By partial Stieltjes integration, we find now for $t > 0$ that

$$\begin{aligned} t\int_0^\infty q(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s &= \Big[-q(s^{-1})\mathrm{e}^{-ts}\Big]_0^\infty + \int_0^\infty \mathrm{e}^{-ts}\,\nu(\mathrm{d}s) \\ &= \int_0^\infty \mathrm{e}^{-ts}\,\nu(\mathrm{d}s) = t\tilde{r}(t). \end{aligned} \tag{3.43}$$

Denoting by $\check{\nu}$ the transformation of ν by the mapping $s \mapsto -s$, we find similarly for $t < 0$ that

$$\begin{aligned} |t|\tilde{r}(t) &= \int_{-\infty}^0 \mathrm{e}^{-ts}\,\nu(\mathrm{d}s) = \int_0^\infty \mathrm{e}^{-|t|s}\,\check{\nu}(\mathrm{d}s) \\ &= \Big[\mathrm{e}^{-|t|s}q(-s^{-1})\Big]_0^\infty + |t|\int_0^\infty \mathrm{e}^{-|t|s}q(-s^{-1})\,\mathrm{d}s = |t|\int_{-\infty}^0 \mathrm{e}^{-ts}q(s^{-1})\,\mathrm{d}s. \end{aligned} \tag{3.44}$$

Combining now (3.43) and (3.44) it follows that

$$\tilde{r}(t) = \begin{cases} \int_0^\infty q(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s, & \text{if } t > 0, \\ \int_{-\infty}^0 q(s^{-1})\mathrm{e}^{-sy}\,\mathrm{d}s, & \text{if } t < 0. \end{cases}$$

By Corollary 3.4 we may thus conclude that $\tilde{\rho}(\mathrm{d}t) = \int_0^\infty (D_x\rho)\mathrm{e}^{-x}\,\mathrm{d}x$, where $\rho(\mathrm{d}t) = r(t)\,\mathrm{d}t$. Since ν is a Radon measure, r is bounded on compact subsets of $\mathbb{R}\setminus\{0\}$, so that ρ is, in particular, σ-finite. By Proposition 3.9, it follows then that $\int_{\mathbb{R}} \min\{1, t^2\}\,\rho(\mathrm{d}t) < \infty$, so that ρ is actually a Lévy measure and $\Upsilon_0(\rho) = \tilde{\rho}$.

Let, finally, μ be the measure in $\mathcal{ID}(*)$ with generating triplet $(\frac{1}{2}a, \rho, \eta)$, where

$$\eta = \tilde{\eta} - \int_0^\infty \Big(\int_{\mathbb{R}} t\big(1_{[-1,1]}(t) - 1_{[-x,x]}(t)\big)\,D_x\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x.$$

Then $\Upsilon(\mu) = \tilde{\mu}$, and since q is increasing on $]-\infty, 0[$ and decreasing on $]0, \infty[$ (cf. (3.42)), we have that $\mu \in \mathcal{L}(*)$. This concludes the proof. □

3.4 The Mappings Υ_0^α and Υ^α, $\alpha \in [0, 1]$

As announced in Section 1, we now introduce two families of mappings $\{\Upsilon_0^\alpha\}_{0\le\alpha\le1}$ and $\{\Upsilon^\alpha\}_{0\le\alpha\le1}$ that, respectively, generalize Υ_0 and Υ, with $\Upsilon_0^0 = \Upsilon_0$, $\Upsilon^0 = \Upsilon$ and with Υ_0^1 and Υ^1 the identity mappings on $\mathfrak{M}_L$ and $\mathcal{ID}(*)$, respectively. The Mittag-Leffler function takes a natural role in this.

A review of relevant properties of the Mittag-Leffler function is given. The transformation Υ_0^α is defined in terms of the associated stable law and is shown to be injective, with absolutely continuous images. Then Υ_0^α is extended to a mapping $\Upsilon^\alpha : \mathcal{ID}(*) \to \mathcal{ID}(*)$, in analogy with the extension of Υ_0 to Υ, and properties of Υ^α are discussed. Finally, stochastic representations of Υ and Υ^α are given.

The Mittag-Leffler Function

The Mittag-Leffler function of negative real argument and index $\alpha > 0$ is given by

$$E_\alpha(-t) = \sum_{k=0}^{\infty} \frac{(-t)^k}{\Gamma(\alpha k + 1)}, \qquad (t > 0). \tag{3.45}$$

In particular we have $E_1(-t) = \mathrm{e}^{-t}$, and if we define E_0 by setting $\alpha = 0$ on the right hand side of (3.45) then $E_0(-t) = (1+t)^{-1}$ (whenever $|t| < 1$).

The Mittag-Leffler function is infinitely differentiable and completely monotone if and only if $0 < \alpha \leq 1$. Hence for $0 < \alpha \leq 1$ it is representable as a Laplace transform and, in fact, for α in $]0,1[$ we have (see [Fe71, p. 453])

$$E_\alpha(-t) = \int_0^\infty \mathrm{e}^{-tx} \zeta_\alpha(x)\,\mathrm{d}x, \tag{3.46}$$

where

$$\zeta_\alpha(x) = \alpha^{-1} x^{-1-1/\alpha} \sigma_\alpha(x^{-1/\alpha}), \qquad (x > 0), \tag{3.47}$$

and σ_α denotes the density function of the positive stable law with index α and Laplace transform $\exp(-\theta^\alpha)$. Note that, for $0 < \alpha < 1$, the function $\zeta_\alpha(x)$ is simply the probability density obtained from $\sigma_\alpha(y)$ by the transformation $x = y^{-\alpha}$. In other words, if we denote the distribution functions determined by ζ_α and σ_α by Z_α and S_α, respectively, then

$$Z_\alpha(x) = 1 - S_\alpha(x^{-1/\alpha}). \tag{3.48}$$

As kindly pointed out to us by Marc Yor, ζ_α has a direct interpretation as the probability density of $l_1^{(\alpha)}$ where $l_t^{(\alpha)}$ denotes the local time of a Bessel process with dimension $2(1-\alpha)$. The law of $l_1^{(\alpha)}$ is called the *Mittag-Leffler distribution*. See [MoOs69] and [ChYo03, p. 114]; cf. also [GrRoVaYo99]. Defining $\zeta_\alpha(x)$ as e^{-x} for $\alpha = 0$ and as the Dirac density at 1 when $\alpha = 1$, formula (3.46) remains valid for all α in $[0,1]$.

For later use, we note that the probability measure $\zeta_\alpha(x)\,\mathrm{d}x$ has moments of all orders. Indeed, for α in $]0,1[$ and any p in $\mathbb{N}$ we have

$$\int_0^\infty x^p \zeta_\alpha(x)\,\mathrm{d}x = \int_0^\infty x^{-p\alpha} \sigma_\alpha(x)\,\mathrm{d}x,$$

where clearly $\int_1^\infty x^{-p\alpha}\sigma_\alpha(x)\,\mathrm{d}x < \infty$. Furthermore, by partial integration,

$$\begin{aligned}\int_0^1 x^{-p\alpha}\sigma_\alpha(x)\,\mathrm{d}x &= \left[x^{-p\alpha} S_\alpha(x)\right]_0^1 + p\alpha \int_0^1 x^{-p\alpha-1} S_\alpha(x)\,\mathrm{d}x \\ &= S_\alpha(1) + p\alpha \int_0^1 x^{-p\alpha-1} S_\alpha(x)\,\mathrm{d}x < \infty,\end{aligned}$$

where we make use (twice) of the relation

$$\mathrm{e}^{x^{-\alpha}} S_\alpha(x) \to 0, \quad \text{as } x \searrow 0,$$

(cf. [Fe71, Theorem 1, p.448]). Combining the observation just made with (3.45) and (3.46), we obtain the formula

$$\int_0^\infty x^k \zeta_\alpha(x)\,\mathrm{d}x = \frac{k!}{\Gamma(\alpha k+1)}, \qquad (k \in \mathbb{N}_0), \tag{3.49}$$

which holds for all α in $[0,1]$.

The Mapping Υ_0^α

As before, we denote by $\mathfrak{M}$ the class of all Borel measures on $\mathbb{R}$, and $\mathfrak{M}_L$ is the subclass of all Lévy measures on $\mathbb{R}$.

Definition 3.28. *For any α in $]0,1[$, we define the mapping $\Upsilon_0^\alpha\colon \mathfrak{M}_L \to \mathfrak{M}$ by the expression:*

$$\Upsilon_0^\alpha(\rho) = \int_0^\infty (D_x\rho)\zeta_\alpha(x)\,\mathrm{d}x, \qquad (\rho \in \mathfrak{M}_L). \tag{3.50}$$

We shall see, shortly, that Υ_0^α actually maps $\mathfrak{M}_L$ into itself. In the sequel, we shall often use $\tilde{\rho}_\alpha$ as shorthand notation for $\Upsilon_0^\alpha(\rho)$. Note that with the interpretation of $\zeta_\alpha(x)\mathrm{d}x$ for $\alpha = 0$ and 1, given above, the formula (3.50) specializes to $\Upsilon_0^1(\rho) = \rho$ and $\Upsilon_0^0(\rho) = \Upsilon_0(\rho)$.

Using (3.47), the formula (3.50) may be reexpressed as

$$\tilde{\rho}_\alpha(\mathrm{d}t) = \int_0^\infty \rho(x^\alpha \mathrm{d}t)\sigma_\alpha(x)\,\mathrm{d}x. \tag{3.51}$$

Note also that $\tilde{\rho}_\alpha(\mathrm{d}t)$ can be written as

$$\tilde{\rho}_\alpha(\mathrm{d}t) = \int_0^\infty \rho\big(\tfrac{1}{R_\alpha(y)}\mathrm{d}t\big)\,\mathrm{d}y,$$

where R_α denotes the inverse function of the distribution function Z_α of $\zeta_\alpha(x)\,\mathrm{d}x$.

Theorem 3.29. *The mapping Υ_0^α sends Lévy measures to Lévy measures.*

For the proof of this theorem we use the following technical result:

Lemma 3.30. *For any Lévy measure ρ on $\mathbb{R}$ and any positive x, we have*

$$\int_{\mathbb{R}\setminus[-1,1]} 1\, D_x\rho(\mathrm{d}t) \le \max\{1, x^2\} \int_{\mathbb{R}} \min\{1, t^2\}\,\rho(\mathrm{d}t), \tag{3.52}$$

and also

$$\int_{[-1,1]} t^2\, D_x\rho(\mathrm{d}t) \le \max\{1, x^2\} \int_{\mathbb{R}} \min\{1, t^2\}\,\rho(\mathrm{d}t). \tag{3.53}$$

Proof. Note first that

$$\int_{\mathbb{R}\setminus[-1,1]} 1\, D_x\rho(\mathrm{d}t) = D_x\rho(\mathbb{R}\setminus[-1,1]) = \rho(\mathbb{R}\setminus[-x^{-1},x^{-1}]).$$

If $0 < x \leq 1$, then

$$\rho(\mathbb{R}\setminus[-x^{-1},x^{-1}]) \leq \rho(\mathbb{R}\setminus[-1,1]) \leq \int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t),$$

and if $x > 1$,

$$\begin{aligned}\rho(\mathbb{R}\setminus[-x^{-1},x^{-1}]) &\leq \int_{[-1,1]\setminus[-x^{-1},x^{-1}]} x^2t^2\,\rho(\mathrm{d}t) + \int_{\mathbb{R}\setminus[-1,1]} 1\,\rho(\mathrm{d}t)\\ &\leq x^2\int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t).\end{aligned}$$

This verifies (3.52). Note next that

$$\int_{[-1,1]} t^2\, D_x\rho(\mathrm{d}t) = \int_{\mathbb{R}} x^2t^2 1_{[-x^{-1},x^{-1}]}(t)\rho(\mathrm{d}t).$$

If $x \geq 1$, we find that

$$\int_{\mathbb{R}} x^2t^2 1_{[-x^{-1},x^{-1}]}(t)\,\rho(\mathrm{d}t) \leq x^2\int_{\mathbb{R}} t^2 1_{[-1,1]}(t)\,\rho(\mathrm{d}t) \leq x^2\int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t),$$

and, if $0 < x < 1$,

$$\begin{aligned}&\int_{\mathbb{R}} x^2t^2 1_{[-x^{-1},x^{-1}]}(t)\,\rho(\mathrm{d}t)\\ &\qquad= x^2\int_{-1}^{1} t^2\,\rho(\mathrm{d}t) + x^2\int_{\mathbb{R}} t^2 1_{[-x^{-1},x^{-1}]\setminus[-1,1]}(t)\,\rho(\mathrm{d}t)\\ &\qquad\leq x^2\int_{-1}^{1} t^2\,\rho(\mathrm{d}t) + x^2\int_{\mathbb{R}} x^{-2} 1_{[-x^{-1},x^{-1}]\setminus[-1,1]}(t)\,\rho(\mathrm{d}t)\\ &\qquad\leq \int_{-1}^{1} t^2\,\rho(\mathrm{d}t) + \int_{\mathbb{R}} 1_{\mathbb{R}\setminus[-1,1]}(t)\,\rho(\mathrm{d}t)\\ &\qquad= \int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t).\end{aligned}$$

This verifies (3.53). □

Proof of Theorem 3.29. Let ρ be a Lévy measure on $\mathbb{R}$ and consider the measure $\tilde{\rho}_\alpha = \Upsilon^\alpha(\rho)$. Using Lemma 3.30 and (3.49) we then have

$$\begin{aligned}\int_{\mathbb{R}} \min\{1,t^2\}\,\tilde{\rho}_\alpha(\mathrm{d}t) &= \int_0^\infty \Big(\int_{\mathbb{R}} \min\{1,t^2\}\, D_x\rho(\mathrm{d}t)\Big)\zeta_\alpha(x)\,\mathrm{d}x\\ &= \int_0^\infty 2\max\{1,x^2\}\Big(\int_{\mathbb{R}} \min\{1,t^2\}\rho(dt)\Big)\zeta_\alpha(x)\,\mathrm{d}x\\ &= 2\int_{\mathbb{R}} \min\{1,t^2\}\rho(dt)\int_0^\infty 2\max\{1,x^2\}\zeta_\alpha(x)\,\mathrm{d}x < \infty,\end{aligned}$$

as desired. □

Absolute Continuity

As in Section 3.1, we let ω denote the transformation of the Lévy measure ρ by the mapping $x \mapsto x^{-1}$.

Theorem 3.31. *For any Lévy measure ρ the Lévy measure $\tilde{\rho}_\alpha$ given by (3.50) is absolutely continuous with respect to Lebesgue measure. The density $\tilde{r}_\alpha$ is the function on $\mathbb{R}\setminus\{0\}$ given by*

$$\tilde{r}_\alpha(t) = \begin{cases} \int_0^\infty s\zeta_\alpha(st)\,\omega(\mathrm{d}s), & \text{if } t > 0,\\ \int_{-\infty}^0 |s|\zeta_\alpha(st)\,\omega(\mathrm{d}s), & \text{if } t < 0.\end{cases}$$

Proof. It suffices to prove that the restrictions of $\tilde{\rho}_\alpha$ to $]-\infty,0[$ and $]0,\infty[$ equal those of $\tilde{r}_\alpha(t)\,\mathrm{d}t$. For a Borel subset B of $]0,\infty[$, we find that

$$\begin{aligned}\int_B \tilde{r}_\alpha(t)\,\mathrm{d}t &= \int_B \Big(\int_0^\infty s\zeta_\alpha(st)\,\omega(\mathrm{d}s)\Big)\,\mathrm{d}t = \int_0^\infty \Big(\int_0^\infty s1_B(t)\zeta_\alpha(st)\,\mathrm{d}t\Big)\,\omega(\mathrm{d}s)\\ &= \int_0^\infty \Big(\int_0^\infty 1_B(s^{-1}u)\zeta_\alpha(u)\,\mathrm{d}u\Big)\,\omega(\mathrm{d}s),\end{aligned}$$

where we have used the change of variable $u = st$. Changing again the order of integration, we have

$$\begin{aligned}\int_B \tilde{r}_\alpha(t)\,\mathrm{d}t &= \int_0^\infty \Big(\int_0^\infty 1_B(s^{-1}u)\,\omega(\mathrm{d}s)\Big)\zeta_\alpha(u)\,\mathrm{d}u\\ &= \int_0^\infty \Big(\int_0^\infty 1_B(su)\,\rho(\mathrm{d}s)\Big)\zeta_\alpha(u)\,\mathrm{d}u\\ &= \int_0^\infty \rho(u^{-1}B)\zeta_\alpha(u)\,\mathrm{d}u = \tilde{\rho}_\alpha(B).\end{aligned}$$

One proves similarly that the restriction to $]-\infty,0[$ of $\tilde{\rho}_\alpha$ equals that of $\tilde{r}_\alpha(t)\,\mathrm{d}t$. □

Corollary 3.32. *Letting, as above, Z_α denote the distribution function for the probability measure $\zeta_\alpha(t)\,\mathrm{d}t$, we have*

$$\tilde{\rho}_\alpha([t,\infty[) = \int_0^\infty (1 - Z_\alpha(st))\,\omega(\mathrm{d}s) = \int_0^\infty S_\alpha((ts)^{-1/\alpha})\,\omega(\mathrm{d}s) \tag{3.54}$$

for t in $]0,\infty[$, and

$$\tilde{\rho}_\alpha(]-\infty,t]) = \int_{-\infty}^0 (1 - Z_\alpha(st))\,\omega(\mathrm{d}s) = \int_{-\infty}^0 S_\alpha((ts)^{-1/\alpha})\,\omega(\mathrm{d}s) \tag{3.55}$$

for t in $]-\infty,0[$.

Proof. For t in $[0,\infty[$ we find that

$$\begin{aligned}
\tilde{\rho}_\alpha([t,\infty[) &= \int_t^\infty \Big(\int_0^\infty s\zeta_\alpha(su)\,\omega(\mathrm{d}s)\Big)\,\mathrm{d}u \\
&= \int_0^\infty \Big(\int_0^\infty s\zeta_\alpha(su)1_{[t,\infty[}(u)\,\mathrm{d}u\Big)\,\omega(\mathrm{d}s) \\
&= \int_0^\infty \Big(\int_0^\infty \zeta_\alpha(w)1_{[t,\infty[}(s^{-1}w)\,\mathrm{d}w\Big)\,\omega(\mathrm{d}s) \\
&= \int_0^\infty \Big(\int_0^\infty \zeta_\alpha(w)1_{[st,\infty[}(w)\,\mathrm{d}w\Big)\,\omega(\mathrm{d}s) \\
&= \int_0^\infty (1 - Z_\alpha(st))\,\omega(\mathrm{d}s) \\
&= \int_0^\infty S_\alpha((st)^{-1/\alpha})\,\omega(\mathrm{d}s),
\end{aligned}$$

where the last equality follows from (3.48). Formula (3.55) is proved similarly. □

Injectivity of Υ_0^α

In order to show that the mappings $\Upsilon_\alpha\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$ are injective, we first introduce a Laplace like transform: Let ρ be a Lévy measure on $\mathbb{R}$, and as above let ω be the transformation of ρ by the mapping $t \mapsto t^{-1}\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$. Then ω satisfies

$$\omega(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min\{1, t^{-2}\}\,\omega(\mathrm{d}t) < \infty. \tag{3.56}$$

For any $\theta, \beta > 0$ we then define

$$\mathcal{L}_\beta(\theta \ddagger \omega) = \int_{\mathbb{R}} \mathrm{e}^{-\theta|t|^\beta}\,\omega(\mathrm{d}t).$$

It follows immediately from (3.56) that $\mathcal{L}_\beta(\theta \ddagger \omega)$ is a finite, positive number for all $\theta, \beta > 0$. For $\beta = 1$, we recover the usual Laplace transform.

Proposition 3.33. *Let α be a fixed number in $]0,1[$, let ρ be a Lévy measure on $\mathbb{R}$, and put $\tilde{\rho}_\alpha = \Upsilon_0^\alpha(\rho)$. Let further ω and $\tilde{\omega}_\alpha$ denote, respectively, the transformations of ρ and $\tilde{\rho}_\alpha$ by the mapping $t \mapsto t^{-1}\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$. We then have*

$$\mathcal{L}_{1/\alpha}(\theta^{1/\alpha} \ddagger \tilde{\omega}_\alpha) = \mathcal{L}_1(\theta \ddagger \omega), \qquad (\theta \in\,]0,\infty[).$$

Proof. Recall first from Theorem 3.31 that $\tilde{\rho}_\alpha(\mathrm{d}t) = \tilde{r}_\alpha(t)\,\mathrm{d}t$, where

$$\tilde{r}_\alpha(t) = \begin{cases} \int_0^\infty s\zeta_\alpha(st)\,\omega(\mathrm{d}s), & \text{if } t > 0,\\ \int_{-\infty}^0 |s|\zeta_\alpha(st)\,\omega(\mathrm{d}s), & \text{if } t < 0.\end{cases}$$

Consequently, $\tilde{\omega}_\alpha$ has the following density w.r.t. Lebesgue measure:

$$\tilde{r}_\alpha(t^{-1})t^{-2} = \begin{cases} \int_0^\infty st^{-2}\zeta_\alpha(st^{-1})\,\omega(\mathrm{d}s), & \text{if } t > 0,\\ \int_{-\infty}^0 |s|t^{-2}\zeta_\alpha(st^{-1})\,\omega(\mathrm{d}s), & \text{if } t < 0.\end{cases}$$

For any positive θ, we then find

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-\theta t^{1/\alpha}}\,\tilde{\omega}_\alpha(\mathrm{d}t)\\
&\quad= \int_0^\infty \mathrm{e}^{-\theta t^{1/\alpha}}\Big(\int_0^\infty st^{-2}\zeta_\alpha(st^{-1})\,\omega(\mathrm{d}s)\Big)\,\mathrm{d}t\\
&\quad= \int_0^\infty \Big(\int_0^\infty \mathrm{e}^{-\theta t^{1/\alpha}}t^{-2}\zeta_\alpha(st^{-1})\,\mathrm{d}t\Big)\,s\omega(\mathrm{d}s)\\
&\quad= \int_0^\infty \Big(\int_0^\infty \mathrm{e}^{-\theta t^{1/\alpha}}t^{-2}\big[\alpha^{-1}(st^{-1})^{-1-1/\alpha}\sigma_\alpha((st^{-1})^{-1/\alpha})\big]\,\mathrm{d}t\Big)\,s\omega(\mathrm{d}s)\\
&\quad= \frac{1}{\alpha}\int_0^\infty \Big(\int_0^\infty \mathrm{e}^{-\theta t^{1/\alpha}}t^{-1+1/\alpha}\sigma_\alpha(s^{-1/\alpha}t^{1/\alpha})\,\mathrm{d}t\Big)\,s^{-1/\alpha}\omega(\mathrm{d}s),
\end{aligned}$$

where we have used (3.47). Applying now the change of variable: $u = s^{-1/\alpha}t^{1/\alpha}$, we find that

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\theta t^{1/\alpha}}\,\tilde{\omega}_\alpha(\mathrm{d}t) &= \int_0^\infty \Big(\int_0^\infty \mathrm{e}^{-\theta s^{1/\alpha}u}\sigma_\alpha(u)\,\mathrm{d}u\Big)\,\omega(\mathrm{d}s)\\
&= \int_0^\infty \mathrm{e}^{-(\theta s^{1/\alpha})^\alpha}\,\omega(\mathrm{d}s) \qquad (3.57)\\
&= \int_0^\infty \mathrm{e}^{-\theta^\alpha s}\,\omega(\mathrm{d}s),
\end{aligned}$$

where we used that the Laplace transform of $\sigma_\alpha(t)\,\mathrm{d}t$ is given by

$$\int_0^\infty \mathrm{e}^{-\eta t}\sigma_\alpha(t)\,\mathrm{d}t = \mathrm{e}^{-\eta^\alpha}, \qquad (\eta > 0),$$

(cf. [Fe71, Theorem 1, p. 448]). Applying next the above calculation to the measure $\check{\omega} := D_{-1}\omega$, we find for any positive θ that

$$\begin{aligned}\int_{-\infty}^0 \mathrm{e}^{-\theta|t|^{1/\alpha}}\,\tilde{\omega}_\alpha(\mathrm{d}t) &= \int_{-\infty}^0 \mathrm{e}^{-\theta|t|^{1/\alpha}}\Big(\int_{-\infty}^0 |s|t^{-2}\zeta_\alpha(st^{-1})\,\omega(\mathrm{d}s)\Big)\,\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-\theta t^{1/\alpha}}\Big(\int_0^\infty st^{-2}\zeta_\alpha(st^{-1})\,\check{\omega}(\mathrm{d}s)\Big)\,\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-\theta^\alpha s}\,\check{\omega}(\mathrm{d}s) \\ &= \int_{-\infty}^0 \mathrm{e}^{-\theta^\alpha|s|}\,\omega(\mathrm{d}s).\end{aligned} \tag{3.58}$$

Combining formulae (3.57) and (3.58), it follows immediately that $\mathcal{L}_{1/\alpha}(\theta \ddagger \tilde{\omega}_\alpha) = \mathcal{L}_1(\theta^\alpha \ddagger \omega)$, for any positive θ. □

Corollary 3.34. *For each α in $]0,1[$, the mapping $\Upsilon_0^\alpha\colon \mathfrak{M}_L \to \mathfrak{M}_L$ is injective.*

Proof. With notation as in Proposition 3.33, it follows immediately from that same proposition that the (usual) Laplace transform of ω is uniquely determined by $\tilde{\rho}_\alpha = \Upsilon_0^\alpha(\rho)$. As in the proof of Corollary 3.7, this implies that ω, and hence ρ, is uniquely determined by $\Upsilon_0^\alpha(\rho)$. □

The Mapping Υ^α

Our next objective is to "extend" Υ_0^α to a mapping $\Upsilon^\alpha\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$.

Definition 3.35. *For a probability measure μ in $\mathcal{ID}(*)$ with generating triplet (a,ρ,η), we let $\Upsilon^\alpha(\mu)$ denote the measure in $\mathcal{ID}(*)$ with generating triplet $(c_\alpha a, \tilde{\rho}_\alpha, \eta_\alpha)$, where $\tilde{\rho}_\alpha = \Upsilon_0^\alpha(\rho)$ is defined by (3.50) while*

$$c_\alpha = \frac{2}{\Gamma(2\alpha+1)} \qquad \textit{for} \quad 0 \le \alpha \le 1$$

and

$$\eta_\alpha = \frac{\eta}{\Gamma(\alpha+1)} + \int_0^\infty \Big(\int_{\mathbb{R}} t\big(1_{[-1,1]}(t) - 1_{[-x^{-1},x^{-1}]}(t)\big)\rho(x^{-1}\mathrm{d}t)\Big)\zeta_\alpha(x)\,\mathrm{d}x. \tag{3.59}$$

To see that the integral in (3.59) is well-defined, we note that it was shown, although not explicitly stated, in the proof of Lemma 3.13 that

$$\int_{\mathbb{R}} |ux|\big|1_{[-1,1]}(ux) - 1_{[-x,x]}(ux)\big|\,\rho(\mathrm{d}x) \le \max\{1,x^2\}\int_0^\infty \min\{1,u^2\}\,\rho(\mathrm{d}u).$$

Together with (3.49), this verifies that η_α is well-defined. Note also that since Υ_0^α is injective (cf. Corollary 3.34), it follows immediately from the definition above that so is Υ^α. The choice of the constants c_α and η_α is motivated by the following two results, which should be seen as analogues of Theorems 3.16 and 3.17. In addition, the choice of c_α and η_α is essential to the stochastic interpretation of Υ^α given in Theorem 3.44 below. Note that for $\alpha = 0$, we recover the mapping Υ, whereas putting $\alpha = 1$ produces the identity mapping on $\mathcal{ID}(*)$.

Theorem 3.36. *Let μ be a measure in $\mathcal{ID}(*)$ with characteristic triplet (a,ρ,η). Then the cumulant function of $\Upsilon^\alpha(\mu)$ is representable as*

$$C_{\Upsilon^\alpha(\mu)}(\zeta) = \frac{\mathrm{i}\eta\zeta}{\Gamma(\alpha+1)} - \tfrac{1}{2}c_\alpha a\zeta^2 + \int_{\mathbb{R}} \big(E_\alpha(\mathrm{i}\zeta t) - 1 - \mathrm{i}\zeta\tfrac{t}{\Gamma(\alpha+1)}1_{[-1,1]}(t)\big)\,\rho(\mathrm{d}t), \tag{3.60}$$

for any ζ in $\mathbb{R}$, and where E_α is the Mittag-Leffler function.

Proof. For every $0 \le \alpha \le 1$ we note first that for any ζ in $\mathbb{R}$,

$$E_\alpha(\mathrm{i}\zeta t) - 1 - \mathrm{i}\zeta\frac{t}{\Gamma(\alpha+1)}1_{[-1,1]}(t) = \int_0^\infty \big(\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx1_{[-1,1]}(t)\big)\zeta_\alpha(x)\,\mathrm{d}x, \tag{3.61}$$

which follows immediately from the above-mentioned properties of E_α and the probability density ζ_α (including the interpretation of $\zeta_\alpha(x)\mathrm{d}x$ for $\alpha = 0$ or 1). Note in particular that $\int_0^\infty x\zeta_\alpha(x)\mathrm{d}x = \frac{1}{\Gamma(\alpha+1)}$ (cf. (3.49)).

We note next that it was established in the proof of Lemma 3.15 that

$$\int_0^\infty \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx1_{[-1,1]}(t)\big|\,\rho(\mathrm{d}t) \le \Big(2 + \frac{1}{\sqrt{2}}(\zeta x)^2\Big)\int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t).$$

Together with Tonelli's theorem, (3.61) and (3.49), this verifies that the integral in (3.60) is well-defined, and that it is permissible to change the order of integration in the following calculation:

$$\begin{aligned}
&\int_{\mathbb{R}} \big(E_\alpha(\mathrm{i}\zeta t) - 1 - \mathrm{i}\zeta \tfrac{t}{\Gamma(\alpha+1)} 1_{[-1,1]}(t)\big)\,\rho(\mathrm{d}t) \\
&\quad = \int_{\mathbb{R}} \Big(\int_0^\infty \big(\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx 1_{[-1,1]}(t)\big)\zeta_\alpha(x)\,\mathrm{d}x\Big)\rho(\mathrm{d}t) \\
&\quad = \int_0^\infty \Big(\int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\zeta u} - 1 - \mathrm{i}\zeta u 1_{[-x^{-1},x^{-1}]}(u)\big)\,\rho(x^{-1}\mathrm{d}u)\Big)\zeta_\alpha(x)\,\mathrm{d}x \\
&\quad = \int_0^\infty \Big(\int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\zeta u} - 1 - \mathrm{i}\zeta u 1_{[-1,1]}(u)\big)\,\rho(x^{-1}\mathrm{d}u)\Big)\zeta_\alpha(x)\,\mathrm{d}x \\
&\qquad\qquad + \mathrm{i}\zeta \int_0^\infty \Big(\int_{\mathbb{R}} u\big(1_{[-1,1]}(u) - 1_{[-x^{-1},x^{-1}]}(u)\big)\,\rho(x^{-1}\mathrm{d}u)\Big)\zeta_\alpha(x)\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\zeta u} - 1 - \mathrm{i}\zeta u 1_{[-1,1]}(u)\big)\,\tilde{\rho}_\alpha(\mathrm{d}u) \\
&\qquad\qquad + \mathrm{i}\zeta \int_0^\infty \Big(\int_{\mathbb{R}} u\big(1_{[-1,1]}(u) - 1_{[-x^{-1},x^{-1}]}(u)\big)\,\rho(x^{-1}\mathrm{d}u)\Big)\zeta_\alpha(x)\,\mathrm{d}x.
\end{aligned}$$

Comparing the above calculation with Definition 3.35, the theorem follows readily. □

Proposition 3.37. *For any α in $]0,1[$ and any measure μ in $\mathcal{ID}(*)$ we have*

$$C_{\Upsilon^\alpha(\mu)}(z) = \int_0^\infty C_\mu(zx)\zeta_\alpha(x)\,\mathrm{d}x, \qquad (z \in \mathbb{R}).$$

Proof. Let (a,ρ,η) be the characteristic triplet for μ. For arbitrary z in $\mathbb{R}$, we then have

$$\begin{aligned}
&\int_0^\infty C_\mu(zx)\zeta_\alpha(x)\,\mathrm{d}x \\
&= \int_0^\infty \Big(\mathrm{i}\eta zx - \frac{1}{2}az^2x^2 + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}tzx} - 1 - \mathrm{i}tzx 1_{[-1,1]}(t)\big)\,\rho(\mathrm{d}t)\Big)\zeta_\alpha(x)\,\mathrm{d}x \\
&= \mathrm{i}\eta z \int_0^\infty x\zeta_\alpha(x)\,\mathrm{d}x - \frac{1}{2}az^2 \int_0^\infty x^2\zeta_\alpha(x)\,\mathrm{d}x \\
&\qquad\qquad + \int_{\mathbb{R}} \Big(\int_0^\infty \big(\mathrm{e}^{\mathrm{i}tzx} - 1 - \mathrm{i}tzx 1_{[-1,1]}(t)\big)\zeta_\alpha(x)\,\mathrm{d}x\Big)\,\rho(\mathrm{d}t) \\
&= \frac{\mathrm{i}\eta z}{\Gamma(\alpha+1)} - \frac{az^2}{\Gamma(2\alpha+1)} + \int_{\mathbb{R}} \big(E_\alpha(\mathrm{i}zt) - 1 - \mathrm{i}z \tfrac{t}{\Gamma(\alpha+1)} 1_{[-1,1]}(t)\big)\,\rho(\mathrm{d}t),
\end{aligned} \tag{3.62}$$

where the last equality uses (3.49) as well as (3.61). According to Theorem 3.36, the resulting expression in (3.62) equals $C_{\Upsilon^\alpha(\mu)}(z)$, and the proposition follows. □

Properties of Υ^α

We prove next that the mappings Υ^α posses properties similar to those of Υ established in Proposition 3.18.

Proposition 3.38. *For each α in $]0,1[$, the mapping $\Upsilon^\alpha\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$ has the following algebraic properties:*

(i) *For any μ_1, μ_2 in $\mathcal{ID}(*)$, $\Upsilon^\alpha(\mu_1 * \mu_2) = \Upsilon^\alpha(\mu_1) * \Upsilon^\alpha(\mu_2)$.*
(ii) *For any μ in $\mathcal{ID}(*)$ and any c in $\mathbb{R}$, $\Upsilon^\alpha(D_c\mu) = D_c\Upsilon^\alpha(\mu)$.*
(iii) *For any c in $\mathbb{R}$, $\Upsilon^\alpha(\delta_c) = \delta_c$.*

Proof. Suppose $\mu_1, \mu_2 \in \mathcal{ID}(*)$. Then for any z in $\mathbb{R}$ we have by Proposition 3.37

$$\begin{aligned} C_{\Upsilon^\alpha(\mu_1 * \mu_2)}(z) &= \int_0^\infty C_{\mu_1 * \mu_2}(zx)\zeta_\alpha(x)\,\mathrm{d}x \\ &= \int_0^\infty \big(C_{\mu_1}(zx) + C_{\mu_2}(zx)\big)\zeta_\alpha(x)\,\mathrm{d}x \\ &= C_{\Upsilon^\alpha(\mu_1)}(z) + C_{\Upsilon^\alpha(\mu_2)}(z) = C_{\Upsilon^\alpha(\mu_1)*\Upsilon^\alpha(\mu_2)}(z), \end{aligned}$$

which verifies statement (i). Statements (ii) and (iii) follow similarly by applications of Proposition 3.37. ☐

Corollary 3.39. *For each α in $[0,1]$, the mapping $\Upsilon^\alpha\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$ preserves the notions of stability and selfdecomposability, i.e.*

$$\Upsilon^\alpha(\mathcal{S}(*)) \subseteq \mathcal{S}(*) \quad \text{and} \quad \Upsilon^\alpha(\mathcal{L}(*)) \subseteq \mathcal{L}(*).$$

Proof. This follows as in the proof of Corollary 3.19. ☐

Theorem 3.40. *For each α in $]0,1[$, the mapping $\Upsilon^\alpha\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$ is continuous with respect to weak convergence*[5].

For the proof of this theorem we use the following

Lemma 3.41. *For any real numbers ζ and t we have*

$$\left|\mathrm{e}^{\mathrm{i}\zeta t} - 1 - \frac{\mathrm{i}\zeta t}{1+t^2}\right| \frac{1+t^2}{t^2} \le 5\max\{1, |\zeta|^2\}. \tag{3.63}$$

Proof. For $t = 0$ the left hand side of (3.63) is interpreted as $\frac{1}{2}\zeta^2$, and the inequality holds trivially. Thus, we assume that $t \neq 0$, and clearly we may assume that $\zeta \neq 0$ too.

For t in $\mathbb{R} \setminus [-1,1]$, note that $\frac{1+t^2}{t^2} \le 2$, and hence

[5] In fact, it can be proved that Υ^α is a homeomorphism onto its range with respect to weak convergence; see [BaTh04c].

$$\Big|\mathrm{e}^{\mathrm{i}\zeta t}-1-\frac{\mathrm{i}\zeta t}{1+t^2}\Big|\frac{1+t^2}{t^2}\le(1+1)\frac{1+t^2}{t^2}+\Big|\frac{\mathrm{i}\zeta}{t}\Big|\le 4+|\zeta|\le 5\max\{1,|\zeta|^2\}.$$

For t in $[-1,1]\setminus\{0\}$, note first that

$$\begin{aligned}\Big(\mathrm{e}^{\mathrm{i}\zeta t}-1-\frac{\mathrm{i}\zeta t}{1+t^2}\Big)\frac{1+t^2}{t^2}&=\Big(\mathrm{e}^{\mathrm{i}\zeta t}-1-\mathrm{i}\zeta t+\mathrm{i}\zeta t\frac{t^2}{1+t^2}\Big)\frac{1+t^2}{t^2}\\&=\Big(\big(\cos(\zeta t)-1\big)+\mathrm{i}\big(\sin(\zeta t)-\zeta t\big)\Big)\frac{1+t^2}{t^2}+\mathrm{i}\zeta t.\end{aligned}\tag{3.64}$$

Using the mean value theorem, there is a real number ξ_1 strictly between 0 and t, such that

$$\frac{\cos(\zeta t)-1}{t^2}=\frac{1}{t}\Big(\frac{\cos(\zeta t)-1}{t}\Big)=-\frac{1}{t}\sin(\zeta\xi_1)\zeta,$$

and hence

$$\Big|\frac{\cos(\zeta t)-1}{t^2}\Big|=\Big|\zeta^2\cdot\frac{\xi_1}{t}\cdot\frac{\sin(\zeta\xi_1)}{\zeta\xi_1}\Big|\le|\zeta|^2.\tag{3.65}$$

Appealing once more to the mean value theorem, there are, for any non-zero real number x, real numbers ξ_2 between 0 and x and ξ_3 between 0 and ξ_2, such that

$$\frac{\sin(x)}{x}-1=\cos(\xi_2)-1=-\xi_2\sin(\xi_3),\qquad\text{and hence}\qquad\Big|\frac{\sin(x)}{x}-1\Big|\le|x|.$$

As a consequence

$$\frac{1}{t^2}\cdot\big|\sin(\zeta t)-\zeta t\big|=\frac{1}{t^2}\cdot|\zeta t|\cdot\Big|\frac{\sin(\zeta t)}{\zeta t}-1\Big|\le\frac{1}{t^2}\cdot|\zeta t|^2=|\zeta|^2.\tag{3.66}$$

Combining (3.64)-(3.66), it follows for t in $[-1,1]\setminus\{0\}$ that

$$\Big|\mathrm{e}^{\mathrm{i}\zeta t}-1-\frac{\mathrm{i}\zeta t}{1+t^2}\Big|\frac{1+t^2}{t^2}\le\big(|\zeta|^2+|\zeta|^2\big)\cdot 2+|\zeta|\le 5\max\{1,|\zeta|^2\}.$$

This completes the proof. □

Corollary 3.42. *Let μ be an infinitely divisible probability measure on $\mathbb{R}$ with generating pair (γ,σ) (see Section 2.1). Then for any real number ζ we have*

$$\big|C_\mu(\zeta)\big|\le(|\gamma|+5\sigma(\mathbb{R}))\max\{1,|\zeta|^2\}.$$

Proof. This follows immediately from Lemma 3.41 and the representation:

$$C_\mu(\zeta)=\mathrm{i}\gamma\zeta+\int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}\zeta t}-1-\frac{\mathrm{i}\zeta t}{1+t^2}\Big)\frac{1+t^2}{t^2}\,\sigma(\mathrm{d}t).\qquad\blacksquare$$

Proof of Theorem 3.40. Let (μ_n) be a sequence of measures from $\mathcal{ID}(*)$, and suppose that $\mu_n \xrightarrow{\text{w}} \mu$ for some measure μ in $\mathcal{ID}(*)$. We need to show that $\Upsilon^\alpha(\mu_n) \xrightarrow{\text{w}} \Upsilon_\alpha(\mu)$. For this, it suffices to show that

$$C_{\Upsilon^\alpha(\mu_n)}(z) \longrightarrow C_{\Upsilon^\alpha(\mu)}(z), \qquad (z \in \mathbb{R}). \tag{3.67}$$

By Proposition 3.37,

$$C_{\Upsilon^\alpha(\mu_n)}(z) = \int_0^\infty C_{\mu_n}(zx)\zeta_\alpha(x)\,\mathrm{d}x \quad \text{and} \quad C_{\Upsilon^\alpha(\mu)}(z) = \int_0^\infty C_\mu(zx)\zeta_\alpha(x)\,\mathrm{d}x,$$

for all n in $\mathbb{N}$ and z in $\mathbb{R}$. According to [Sa99, Lemma 7.7],

$$C_{\mu_n}(y) \longrightarrow C_\mu(y), \quad \text{for all } y \text{ in } \mathbb{R},$$

so by the dominated convergence theorem, (3.67) follows, if, for each z in $\mathbb{R}$, we find a Borel function $h_z\colon [0,\infty[\to [0,\infty[$, such that

$$\forall n \in \mathbb{N}\ \forall x \in [0,\infty[\colon \left|C_{\mu_n}(zx)\zeta_\alpha(x)\right| \le h_z(x) \qquad \text{and} \qquad \int_0^\infty h_z(x)\,\mathrm{d}x < \infty. \tag{3.68}$$

Towards that end, let, for each n in $\mathbb{N}$, (γ_n, σ_n) denote the generating pair for μ_n. Since $\mu_n \xrightarrow{\text{w}} \mu$, Gnedenko's theorem (cf. [GnKo68, Theorem 1, p.87]) asserts that

$$S := \sup_{n\in\mathbb{N}} \sigma_n(\mathbb{R}) < \infty \quad \text{and} \quad G := \sup_{n\in\mathbb{N}} |\gamma_n| < \infty.$$

Now, by Corollary 3.42, for any n in $\mathbb{N}$, z in $\mathbb{R}$ and x in $[0,\infty[$ we have

$$\left|C_{\mu_n}(zx)\zeta_\alpha(x)\right| \le (G+5S)\max\{1, z^2x^2\}\zeta_\alpha(x),$$

and here, by formula (3.49),

$$\begin{aligned}\int_0^\infty (G+5S)\max\{1,z^2x^2\}\zeta_\alpha(x)\,\mathrm{d}x &\le (G+5S)\int_{\mathbb{R}} (1+z^2x^2)\zeta_\alpha(x)\,\mathrm{d}x \\ &= (G+5S) + (G+5S)z^2\tfrac{2}{\Gamma(2\alpha+1)} < \infty.\end{aligned}$$

Thus, for any z in $\mathbb{R}$, the Borel function

$$h_z(x) = (G+5S)\max\{1, z^2x^2\}\zeta_\alpha(x), \qquad (x \in [0,\infty[),$$

satisfies (3.68). This concludes the proof. □

We close this section by mentioning that a replacement of e^{-y} by $\zeta_\alpha(y)$ in the proof of Proposition 3.22 produces a proof of the following assertion:

$$\forall \mu \in \mathcal{ID}(*)\ \forall \alpha \in [0,1]\colon \mu \text{ has } p\text{'th moment} \iff \Upsilon^\alpha(\mu) \text{ has } p\text{'th moment}.$$

3.5 Stochastic Interpretation of Υ and Υ^α

The purpose of this section is to show that for any measure μ in $\mathcal{ID}(*)$, the measure $\Upsilon(\mu)$ can be realized as the distribution of a stochastic integral w.r.t. to the (classical) Lévy process corresponding to μ. We establish also a similar stochastic interpretation of $\Upsilon^\alpha(\mu)$ for any α in $]0,1[$. The main tool in this is Proposition 2.6.

Theorem 3.43. *Let μ be an arbitrary measure in $\mathcal{ID}(*)$, and let (X_t) be a (classical) Lévy process (in law), such that $L\{X_1\} = \mu$. Then the stochastic integral*

$$Z = \int_0^1 -\log(1-t)\,\mathrm{d}X_t$$

exists, as the limit in probability, of the stochastic integrals $\int_0^{1-1/n} -\log(1-t)\,\mathrm{d}X_t$, as $n \to \infty$. Furthermore, the distribution of Z is exactly $\Upsilon(\mu)$.

Proof. The existence of the stochastic integral $\int_0^1 -\log(1-t)\,\mathrm{d}X_t$ follows from Proposition 2.6, once we have verified that $\int_0^1 |C_\mu(-u\log(1-t))|\,\mathrm{d}t < \infty$, for any u in $\mathbb{R}$. Using the change of variable: $t = 1 - \mathrm{e}^{-x}$, $x \in \mathbb{R}$, we find that

$$\int_0^1 \big|C_\mu(-u\log(1-t))\big|\,\mathrm{d}t = \int_0^\infty \big|C_\mu(ux)\big|\mathrm{e}^{-x}\,\mathrm{d}x,$$

and here the right hand side is finite, according to Lemma 3.15.

Combining next Proposition 2.6 and Theorem 3.17 we find for any u in $\mathbb{R}$ that

$$C_{L\{Z\}}(u) = \int_0^1 C_\mu(-u\log(1-t))\,\mathrm{d}t = \int_0^\infty C_\mu(ux)\mathrm{e}^{-x}\,\mathrm{d}x = C_{\Upsilon(\mu)}(u),$$

which implies that $L\{Z\} = \Upsilon(\mu)$, as desired. □

Before proving the analog of Theorem 3.43 for Υ^α, recall that R_α denotes the inverse of the distribution function Z_α of the probability measure $\zeta_\alpha(x)\,\mathrm{d}x$.

Theorem 3.44. *Let μ be an arbitrary measure in $\mathcal{ID}(*)$, and let (X_t) be a (classical) Lévy process (in law), such that $L\{X_1\} = \mu$. For each $\alpha \in]0,1[$, the stochastic integral*

$$Y = \int_0^1 R_\alpha(s)\,\mathrm{d}X_s \tag{3.69}$$

exists, as a limit in probability, and the law of Y is $\Upsilon^\alpha(\mu)$.

Proof. It suffices to consider α in $]0,1[$. In order to ensure the existence of the stochastic integral in (3.69) , it suffices, by Proposition 2.6, to verify that $\int_0^1 |C_\mu(zR_\alpha(t))|\,\mathrm{d}t < \infty$ for all z in $\mathbb{R}$. Denoting by λ the Lebesgue measure on $[0,1]$, note that $Z_\alpha(\zeta_\alpha(x)\,\mathrm{d}x) = \lambda$, so that $R_\alpha(\lambda) = \zeta_\alpha(x)\,\mathrm{d}x$. Hence, we find that

$$\begin{aligned}\int_0^1 \big|C_\mu(zR_\alpha(t))\big|\,\mathrm{d}t &= \int_0^\infty \big|C_\mu(zu)\big|\,R_\alpha(\lambda)(\mathrm{d}u)\\ &= \int_0^\infty \big|C_\mu(zu)\big|\cdot\zeta_\alpha(u)\,\mathrm{d}u\\ &\le \int_0^\infty \big(|\gamma| + 5\nu(\mathbb{R})\big)\max\{1, z^2u^2\}\zeta_\alpha(u)\,\mathrm{d}u < \infty,\end{aligned}$$

where (γ,ν) is the generating pair for μ (cf. Corollary 3.42). Thus, by Proposition 2.6, the stochastic integral $Y = \int_0^1 R_\alpha(t)\,\mathrm{d}X_t$ makes sense, and the cumulant function of Y is given by

$$C\{z \ddagger Y\} = \int_0^1 C_\mu(zR_\alpha(t))\,\mathrm{d}t = \int_0^1 C_\mu(zu)\zeta_\alpha(u)\,\mathrm{d}u = C_{\Upsilon^\alpha(\mu)}(z),$$

where we have used Theorem 3.37. This completes the proof. $\square$

3.6 Mappings of Upsilon-Type: Further Results

We now summarize several pieces of recent work that extend some of the results presented in the previous part of the present section.

We start by considering a general concept of Upsilon transformations, that has the transformations Υ_0 and Υ_0^α as special cases. Another special case, denoted $\Upsilon_0^{(q)}$ $(q > -2)$ is briefly discussed; this is related to the tempered stable distributions. Further, extensions of the mappings Υ_0 and Υ_0^α to multivariate infinitely divisible distributions are discussed, and applications of these to the construction of Lévy copulas with desirable properties is indicated. Finally, a generalization of $\Upsilon_0^{(q)}$ to transformations of the class $\mathfrak{M}_L(\mathbb{M}_m^+)$ of Lévy measures on the cone of positive definite $m\times m$ matrices is mentioned.

General Upsilon Transformations

The collaborative work discussed in the subsequent parts of the present Section have led to taking up a systematic study of *generalized Upsilon transformations*. Here we mention some first results of this, based on unpublished notes by V. Pérez-Abreu, J. Rosinski, K. Sato and the authors. Detailed expositions will appear elsewhere.

Let ρ be a Lévy measure on $\mathbb{R}$, let τ be a measure on $\mathbb{R}_{>0}$ and introduce the measure ρ_τ on $\mathbb{R}$ by

$$\rho_\tau(\mathrm{d}x) = \int_0^\infty \rho(y^{-1}\mathrm{d}x)\tau(\mathrm{d}y). \tag{3.70}$$

Note here that if X is an infinitely divisible random variable with Lévy measure $\rho(\mathrm{d}x)$ then yX has Lévy measure $\rho(y^{-1}\mathrm{d}x)$.

Definition 3.45. *Given a measure τ on $\mathbb{R}_{>0}$ we define Υ_0^τ as the mapping $\Upsilon_0^\tau : \rho \mapsto \rho_\tau$ where ρ_τ is given by (3.70) and the domain of Υ_0^τ is*

$$\mathrm{dom}_L \Upsilon_0^\tau = \left\{ \rho \in \mathfrak{M}_L(\mathbb{R}) \;\middle|\; \rho_\tau \in \mathfrak{M}_L(\mathbb{R}) \right\}.$$

We have $\mathrm{dom}_L \Upsilon_0^\tau = \mathfrak{M}_L(\mathbb{R})$ if and only if

$$\int_0^\infty \left(1 + y^2\right) \tau(\mathrm{d}y) < \infty.$$

Furthermore, letting

$$\mathfrak{M}_0(\mathbb{R}) = \left\{ \rho \in \mathfrak{M}(\mathbb{R}) \;\middle|\; \int_0^\infty (1 + |t|)\, \rho(\mathrm{d}t) < \infty \right\}$$

(finite variation case) we have $\Upsilon_0^\tau : \mathfrak{M}_0(\mathbb{R}) \to \mathfrak{M}_0(\mathbb{R})$ if and only if

$$\int_0^\infty (1 + |y|)\, \tau(\mathrm{d}y) < \infty.$$

Mappings of type Υ_0^τ have the important property of being commutative under composition. Under rather weak conditions the mappings are one-to-one, and the image Lévy measures possess densities with respect to Lebesgue measure. This is true, in particular, of the examples considered below.

Now, suppose that τ has a density h that is a continuous function on $\mathbb{R}_{>0}$. Then writing ρ_h for ρ_τ we have

$$\rho_h(\mathrm{d}x) = \int_0^\infty \rho(y^{-1}\mathrm{d}x) h(y) \mathrm{d}y. \tag{3.71}$$

Clearly, the mappings Υ_0 and Υ_0^α are special instances of (3.71).

Example 3.46. Φ_0 transformation. The Υ_0^h transformation obtained by letting

$$h(y) = 1_{[-1,1]}(y) y^{-1}$$

is denoted by Φ_0. Its domain is

$$\mathrm{dom}_L \Phi_0 = \left\{ \rho \in \mathfrak{M}_L(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}\setminus[-1,1]} \log|y|\, \rho(\mathrm{d}y) < \infty \right\}.$$

As is well known, this transformation maps $\mathrm{dom}_L \Phi_0$ onto the class of selfdecomposable Lévy measures.

Example 3.47. $\Upsilon_0^{(q)}$ *transformations.* The special version of Υ_0^h obtained by taking

$$h(y) = y^q \mathrm{e}^{-y}$$

is denoted $\Upsilon_0^{(q)}$. For each $q > -1$, $\mathrm{dom}_L \Upsilon_0^{(q)} = \mathfrak{M}_L(\mathbb{R})$, for $q = -1$ the domain equals $\mathrm{dom}_L \Phi_0$, while, for $q \in (-2, -1)$, $\Upsilon_0^{(q)}$ has domain

$$\mathrm{dom}_L \Upsilon_0^{(q)} = \left\{ \rho \in \mathfrak{M}_L(\mathbb{R}) \;\Big|\; \int_{\mathbb{R}\setminus[-1,1]} |y|^{-q-1} \rho(\mathrm{d}y) < \infty \right\}.$$

These transformations are closely related to the tempered stable laws. In fact, let $\sigma(\mathrm{d}x) = c_{\pm} \alpha x^{-1-\alpha} k(x) \mathrm{d}x$ with

$$k(x) = \int_0^\infty \mathrm{e}^{-xc} \nu(\mathrm{d}c)$$

be the Lévy measure of an element in $\mathcal{R}(*)$. Then σ is the image under $\Upsilon_0^{(-1-\alpha)}$ of the Lévy measure

$$\rho(\mathrm{d}x) = x^{-\alpha} \underleftarrow{\nu}(\mathrm{d}x), \tag{3.72}$$

where $\underleftarrow{\nu}$ is the image of the measure ν under the mapping $x \mapsto x^{-1}$.

Interestingly, $\Upsilon_0 \Phi_0 = \Phi_0 \Upsilon_0 = \Upsilon_0^{(-1)}$. The transformations Υ_0^h may in wide generality be characterized in terms of stochastic integrals, as follows. Let

$$H(\xi) = \int_\xi^\infty h(y)\, \mathrm{d}y,$$

set $s = H(\xi)$ and let K, with derivative k, be the inverse function of H, so that $K(H(\xi)) = \xi$ and hence, by differentiation, $k(s)h(\xi) = 1$. Let ρ be an arbitrary element of $\mathfrak{M}_L(\mathbb{R})$ and let L be a Lévy process such that L_1 has Lévy measure ρ. Then, under mild regularity conditions, the integral

$$Y = \int_0^{H(0)} K(s)\, \mathrm{d}L_s \tag{3.73}$$

exists and the random variable Y is infinitely divisible with Lévy measure $\rho_h = \Upsilon_0^h(\rho)$.

Upsilon Transformations of $\mathcal{ID}^d(*)$

The present subsection is based on the paper [BaMaSa04] to which we refer for proofs, additional results, details and references.

We denote the class of infinitely divisible probability laws on $\mathbb{R}^d$ by $\mathcal{ID}^d(*)$. Let h be a function as in the previous subsection and let L be a d-dimensional Lévy process. Then, under a mild regularity condition on h, a d-dimensional random vector Y is determined by

$$Y = \int_0^{H(0)} K(s)\,\mathrm{d}L_s$$

cf. the previous subsection.

If h is the density determining Υ_0 then each of the components of Y belongs to class $\mathcal{B}(*)$ and Y is said to be of class $\mathcal{B}^d(*)$, the d-dimensional Goldie-Steutel-Bondesson class. Similarly, the d-dimensional Thorin class $\mathcal{T}^d(*)$ is defined by taking the components of L_1 to be in $\mathcal{L}(*)$. In [BaMaSa04], probabilistic characterizations of $\mathcal{B}^d(*)$ and $\mathcal{T}^d(*)$ are given, and relations to selfdecomposability and to iterations of Υ_0 and Φ_0 are studied in considerable detail.

Application to Lévy Copulas

We proceed to indicate some applications of Υ_0 and Φ_0 and of the abovementioned results to the construction of Lévy copulas for which the associated probability measures have prescribed marginals in the Goldie-Steutel-Bondesson or Thorin class or Lévy class (the class of selfdecomposable laws). For proofs and details, see [BaLi04].

The concept of copulas for multivariate probability distributions has an analogue for multivariate Lévy measures, termed Lévy copulas. Similar to probabilistic copulas, a Lévy copula describes the dependence structure of a multivariate Lévy measure. The Lévy measure, ρ say, is then completely characterized by knowledge of the Lévy copula and the m one-dimensional margins which are obtained as projections of ρ onto the coordinate axes. An advantage of modeling dependence via Lévy copulas rather that distributional copulas is that the resulting probability laws are automatically infinitely divisible.

For simplicity, we consider only Lévy measures and Lévy copulas living on $\mathbb{R}^m_{>0}$. Suppose that $\mu_1, \dots, \mu_m$ are one-dimensional infinitely divisible distributions, all of which are in the Goldie-Steutel-Bondesson class or the Thorin class or the Lévy class. Using any Lévy copula gives an infinitely divisible distribution μ with margins $\mu_1, \dots, \mu_m$. But μ itself does not necessarily belong to the Bondesson class or the Thorin class or the Lévy class, i.e. not every Lévy copula gives rise to such distributions. However, that can be achieved by the use of Upsilon transformations. For the Goldie-Steutel-Bondesson class and the Lévy class this is done with the help of the mappings Υ_0 and Φ_0, respectively, and combining the mappings Φ_0 and Υ_0 one can construct multivariate distributions in the Thorin class with prescribed margins in the Thorin class.

Upsilon Transformations for Matrix Subordinators

The present subsection is based on the paper [BaPA05] to which we refer for proofs, additional results, details and references.

An extension of Υ_0 to a one-to-one mapping of the class of d-dimensional Lévy measures into itself was considered in the previous subsection. Here we shall briefly discuss another type of generalization, to one-to-one mappings of $\mathcal{ID}_+^{m\times m}(*)$, the set of infinitely divisible positive semidefinite $m\times m$ matrices, into itself. This class of mappings constitutes an extension to the positive definite matrix setting of the class $\{\Upsilon_0^{(q)}\}_{-1<q<\infty}$ considered above, and we shall use the same notation $\Upsilon_0^{(q)}$ in the general matrix case.

We begin by reviewing several facts about infinitely divisible matrices with values in the cone $\overline{\mathbb{M}}_m^+$ of symmetric nonnegative definite $m \times m$ matrices.

Let $\mathbb{M}_{m\times m}$ denote the linear space of $m \times m$ real matrices, $\mathbb{M}_m$ the linear subspace of symmetric matrices, $\overline{\mathbb{M}}_m^+$ the closed cone of non-negative definite matrices in $\mathbb{M}_m$, $\mathbb{M}_m^+$ and $\{X > 0\}$ the open cone of positive definite matrices in $\mathbb{M}_m$.

For $X \in \mathbb{M}_{m\times m}$, $X^\top$ is the transpose of X and $\mathrm{tr}(X)$ the trace of X. For X in $\overline{\mathbb{M}}_m^+$, $X^{1/2}$ is the unique symmetric matrix in $\overline{\mathbb{M}}_m^+$ such that $X = X^{1/2}X^{1/2}$. Given a nonsingular matrix X in $\mathbb{M}_{m\times m}$, X^{-1} denotes its inverse, $|X|$ its determinant and $X^{-\top}$ the inverse of its transpose. When X is in $\mathbb{M}_m^+$ we simply write $X > 0$.

The cone $\overline{\mathbb{M}}_m^+$ is not a linear subspace of the linear space $\mathbb{M}_{m\times m}$ of $m \times m$ matrices and the theory of infinite divisibility on Euclidean spaces does not apply immediately to $\overline{\mathbb{M}}_m^+$. In general, the study of infinitely divisible random elements in closed cones requires separate work.

A random matrix M is *infinitely divisible in* $\overline{\mathbb{M}}_m^+$ if and only if for each integer $p \geq 1$ there exist p independent identically distributed random matrices $M_1, ..., M_p$ in $\overline{\mathbb{M}}_m^+$ such that $M \overset{\mathrm{d}}{=} M_1 + \cdots + M_p$. In this case, the Lévy-Khintchine representation has the following special form, which is obtained from [Sk91] p.156-157.

Proposition 3.48. *An infinitely divisible random matrix M is infinitely divisible in $\overline{\mathbb{M}}_m^+$ if and only if its cumulant transform is of the form*

$$\mathcal{C}(\Theta; M) = \mathrm{itr}(\Psi^0\Theta) + \int_{\overline{\mathbb{M}}_m^+} (\mathrm{e}^{\mathrm{itr}(X\Theta)} - 1)\rho(\mathrm{d}X), \quad \Theta \in \mathbb{M}_m^+, \tag{3.74}$$

where $\Psi^0 \in \overline{\mathbb{M}}_m^+$ and the Lévy measure ρ satisfies $\rho(\mathbb{M}_m \backslash \overline{\mathbb{M}}_m^+) = 0$ and has order of singularity

$$\int_{\overline{\mathbb{M}}_m^+} \min(1, \|X\|)\rho(\mathrm{d}X) < \infty. \tag{3.75}$$

Moreover, the Laplace transform of M is given by

$$\mathcal{L}_M(\Theta) = \exp\{-\mathcal{K}(\Theta; M)\}, \quad \Theta \in \mathbb{M}_m^+, \tag{3.76}$$

where $\mathcal{K}$ is the Laplace exponent

$$\mathcal{K}(\Theta; M) = \mathrm{tr}(\Psi^0\Theta) + \int_{\overline{\mathbb{M}}_m^+} (1 - \mathrm{e}^{-\mathrm{tr}(X\Theta)})\rho(\mathrm{d}X). \tag{3.77}$$

For ρ in $\mathfrak{M}_L(\mathbb{M}_m^+)$ and $q > -1$ consider the mapping $\Upsilon_0^{(q)} : \rho \mapsto \rho_q$ given by

$$\rho_q(\mathrm{d}Z) = \int_{X>0} \rho(\overline{X}^{-\top} \mathrm{d}Z\overline{X}^{-1})\, |X|^q\, \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X. \tag{3.78}$$

The measure ρ_q is a Lévy measure on $\overline{\mathbb{M}}_m^+$.

To establish that for each $q > -1$ the mapping $\Upsilon_0^{(q)}$ is one-to-one the following type of Laplace transform of elements $\rho \in \mathfrak{M}_L(\mathbb{M}_m^+)$ is introduced:

$$\mathcal{L}^p\rho(\Theta) = \int_{X>0} \mathrm{e}^{-\mathrm{tr}(X\Theta)}\, |X|^p\, \rho(\mathrm{d}X). \tag{3.79}$$

For any $p \geq 1$ and ρ in $\mathfrak{M}_L(\mathbb{M}_m^+)$, the transform (3.79) is finite for any $\Theta \in \mathbb{M}_m^+$, and the following theorem implies the bijectivity.

Theorem 3.49. *Let $p \geq 1$ and $p + q \geq 1$. Then*

$$\mathcal{L}^p\rho_q(\Theta) = |\Theta|^{-\frac{1}{2}(m+1)-(p+q)} \int_{V>0} \mathcal{L}^p\rho(\mathbf{V})\, |V|^{p+q}\, \mathrm{e}^{-\mathrm{tr}(\boldsymbol{\Theta}^{-1}V)} \mathrm{d}V. \tag{3.80}$$

for $\Theta \in \mathbb{M}_m^+$

As in the one-dimensional case, the transformed Lévy measure determined by the mapping $\Upsilon_0^{(q)}$ is absolutely continuous (with respect to Lebesgue measure on $\mathbb{M}_m^+$) and the density possesses an integral representation, showing in particular that the density is a completely monotone function on $\mathbb{M}_m^+$.

Theorem 3.50. *For each $q > -1$ the Lévy measure ρ_q is absolutely continuous with Lévy density r_q given by*

$$r_q(X) = |\mathbf{X}|^q \int_{Y>0} |Y|^{-\frac{1}{2}(m+1)-q}\, \mathrm{e}^{-\mathrm{tr}(\mathbf{X}\mathbf{Y}^{-1})} \rho(\mathrm{d}Y) \tag{3.81}$$

$$= |\mathbf{X}|^q \int_{Y>0} |Y|^{\frac{1}{2}(m+1)+q}\, \mathrm{e}^{-\mathrm{tr}(\mathbf{X}\mathbf{Y})}\, \underleftarrow{\rho}(\mathrm{d}Y). \tag{3.82}$$

4 Free Infinite Divisibility and Lévy Processes

Free probability is a subject in the theory of non-commutative probability. It was originated by Voiculescu in the Nineteen Eighties and has since been extensively studied, see e.g. [VoDyNi92], [Vo98] and [Bi03]. The present section provides an introduction to the area, somewhat in parallel to the exposition of the classical case in Section 2.5. Analogues of some of the subclasses of $\mathcal{ID}(*)$ discussed in that section are introduced. Finally, a discussion of free Lévy processes is given.

4.1 Non-Commutative Probability and Operator Theory

In classical probability, one might say that the basic objects of study are random variables, represented as measurable functions from a probability space $(\Omega, \mathcal{F}, P)$ into the real numbers $\mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}$. To any such random variable $X\colon \Omega \to \mathbb{R}$ the distribution μ_X of X is determined by the equation:

$$\int_{\mathbb{R}} f(t)\ \mu_X(\mathrm{d}t) = \mathbb{E}(f(X)),$$

for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$, and where $\mathbb{E}$ denotes expectation (or integration) w.r.t. P. We shall also use the notation $L\{X\}$ for μ_X.

In *non-commutative* probability, one replaces the random variables by (self-adjoint) operators on a Hilbert space $\mathcal{H}$. These operators are then referred to as "non–commutative random variables". The term non-commutative refers to the fact that, in this setting, the multiplication of "random variables" (i.e. composition of operators) is no longer commutative, as opposed to the usual multiplication of classical random variables. The non-commutative situation is often remarkably different from the classical one, and most often more complicated.

By $\mathcal{B}(\mathcal{H})$ we denote the vector space of all bounded operators on $\mathcal{H}$, i.e. linear mappings $a\colon \mathcal{H} \to \mathcal{H}$, which are continuous, or, equivalently, which satisfy that

$$\|a\| := \sup\{\|a\xi\| \mid \xi \in \mathcal{H},\ \|\xi\| \leq 1\} < \infty.$$

The mapping $a \mapsto \|a\|$ is a norm on $\mathcal{B}(\mathcal{H})$, called the operator norm, and $\mathcal{B}(\mathcal{H})$ is complete in the operator norm. Composition of operators form a (non-commutative) multiplication on $\mathcal{B}(\mathcal{H})$, which, together with the linear operations, turns $\mathcal{B}(\mathcal{H})$ into an algebra.

Recall next that $\mathcal{B}(\mathcal{H})$ is equipped with an involution (the adjoint operation) $a \mapsto a^*\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, which is given by:

$$\langle a\xi, \eta\rangle = \langle \xi, a^*\eta\rangle, \quad (a \in \mathcal{B}(\mathcal{H}),\ \xi, \eta \in \mathcal{H}).$$

Instead of working with the whole algebra $\mathcal{B}(\mathcal{H})$ as the set of "random variables" under consideration, it is, for most purposes, natural to restrict attention to certain subalgebras of $\mathcal{B}(\mathcal{H})$.

A (unital) *C^*-algebra* acting on a Hilbert space $\mathcal{H}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, which contains the multiplicative unit $\mathbf{1}$ of $\mathcal{B}(\mathcal{H})$ (i.e. $\mathbf{1}$ is the identity mapping on $\mathcal{H}$), and which is closed under the adjoint operation and topologically closed w.r.t. the operator norm.

A *von Neumann algebra*, acting on $\mathcal{H}$, is a unital C^*-algebra acting on $\mathcal{H}$, which is even closed in the weak operator topology on $\mathcal{B}(\mathcal{H})$ (i.e. the weak topology on $\mathcal{B}(\mathcal{H})$ induced by the linear functionals: $a \mapsto \langle a\xi, \eta\rangle,\ \ \xi, \eta \in \mathcal{H}$).

A *state* on the (unital) C^*-algebra $\mathcal{A}$ is a positive linear functional $\tau\colon \mathcal{A} \to \mathbb{C}$, taking the value 1 at the identity operator $\mathbf{1}$ on $\mathcal{H}$. If τ satisfies, in addition, the trace property:

$$\tau(ab) = \tau(ba), \quad (a, b \in \mathcal{A}),$$

then τ is called a *tracial state*[6]. A tracial state τ on a von Neumann algebra $\mathcal{A}$ is called *normal*, if its restriction to the unit ball of $\mathcal{A}$ (w.r.t. the operator norm) is continuous in the weak operator topology.

Definition 4.1. (i) *A C^*-probability space is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a unital C^*-algebra and τ is a faithful state on $\mathcal{A}$.*
(ii) *A W^*-probability space is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a von Neumann algebra and τ is a faithful, normal tracial state on $\mathcal{A}$.*

The assumed faithfulness of τ in Definition 4.1 means that τ does not annihilate any non-zero positive operator. It implies that $\mathcal{A}$ is finite in the sense of F. Murray and J. von Neumann.

In the following, we shall mostly be dealing with W^*-probability spaces. So suppose that $(\mathcal{A}, \tau)$ is a W^*-probability space and that a is a selfadjoint operator (i.e. $a^* = a$) in $\mathcal{A}$. Then, as in the classical case, we can associate a (spectral) distribution to a in a natural way: Indeed, by the Riesz representation theorem, there exists a unique probability measure μ_a on $(\mathbb{R}, \mathcal{B})$, satisfying that

$$\int_{\mathbb{R}} f(t)\ \mu_a(\mathrm{d}t) = \tau(f(a)), \tag{4.1}$$

for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$. In formula (4.1), $f(a)$ has the obvious meaning if f is a polynomial. For general Borel functions f, $f(a)$ is defined in terms of spectral theory (see e.g. [Ru91]).

The (spectral) distribution μ_a of a selfadjoint operator a in $\mathcal{A}$ is automatically concentrated on the spectrum $\mathrm{sp}(a)$, and is thus, in particular, compactly supported. If one wants to be able to consider any probability measure μ on $\mathbb{R}$ as the spectral distribution of some selfadjoint operator, then it is necessary to take unbounded (i.e. non-continuous) operators into account. Such an operator a is, generally, not defined on all of $\mathcal{H}$, but only on a subspace $\mathcal{D}(a)$ of $\mathcal{H}$, called the domain of a. We say then that a is an operator *in* $\mathcal{H}$ rather than *on* $\mathcal{H}$. For most of the interesting examples, $\mathcal{D}(a)$ is a dense subspace of $\mathcal{H}$, in which case a is said to be densely defined. We have included a detailed discussion on unbounded operators in the Appendix (Section A), from which we extract the following brief discussion.

If $(\mathcal{A}, \tau)$ is a W^*-probability space acting on $\mathcal{H}$ and a is an unbounded operator in $\mathcal{H}$, a cannot be an element of $\mathcal{A}$. The closest a can get to $\mathcal{A}$ is to be *affiliated* with $\mathcal{A}$, which means that a commutes with any unitary operator u, that commutes with all elements of $\mathcal{A}$. If a is selfadjoint, a is affiliated with $\mathcal{A}$ if and only if $f(a) \in \mathcal{A}$ for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$. In this case,

[6] In quantum physics, τ is of the form $\tau(a) = \mathrm{tr}(\rho a)$, where ρ is a trace class selfadjoint operator on $\mathcal{H}$ with trace 1, that expresses the state of a quantum system, and a would be an observable, i.e. a selfadjoint operator on $\mathcal{H}$, the mean value of the outcome of observing a being $\mathrm{tr}(\rho a)$.

(4.1) determines, again, a unique probability measure μ_a on $\mathbb{R}$, which we also refer to as the (spectral) distribution of a, and which generally has unbounded support. Furthermore, any probability measure on $\mathbb{R}$ can be realized as the (spectral) distribution of some selfadjoint operator affiliated with some W^*-probability space. In the following we shall also use the notation $L\{a\}$ for the distribution of a (possibly unbounded) operator a affiliated with $(\mathcal{A}, \tau)$. By $\overline{\mathcal{A}}$ we denote the set of operators in $\mathcal{H}$ which are affiliated with $\mathcal{A}$.

4.2 Free Independence

The key concept on relations between classical random variables X and Y is *independence*. One way of defining that X and Y (defined on the same probability space $(\Omega, \mathcal{F}, P)$) are independent is to ask that all compositions of X and Y with bounded Borel functions be uncorrelated:

$$\mathbb{E}\big\{[f(X) - \mathbb{E}\{f(X)\}] \cdot [g(Y) - \mathbb{E}\{g(Y)\}]\big\} = 0,$$

for any bounded Borel functions $f, g \colon \mathbb{R} \to \mathbb{R}$.

In the early 1980's, D.V. Voiculescu introduced the notion of *free independence* among non-commutative random variables:

Definition 4.2. *Let $a_1, a_2, \dots, a_r$ be selfadjoint operators affiliated with a W^*-probability space $(\mathcal{A}, \tau)$. We say then that $a_1, a_2, \dots, a_r$ are freely independent w.r.t. τ, if*

$$\tau\big\{[f_1(a_{i_1}) - \tau(f_1(a_{i_1}))][f_2(a_{i_2}) - \tau(f_2(a_{i_2}))] \cdots [f_p(a_{i_p}) - \tau(f_p(a_{i_p}))]\big\} = 0,$$

for any p in $\mathbb{N}$, any bounded Borel functions $f_1, f_2, \dots, f_p \colon \mathbb{R} \to \mathbb{R}$ and any indices $i_1, i_2, \dots, i_p$ in $\{1, 2, \dots, r\}$ satisfying that $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{p-1} \neq i_p$.

At a first glance, the definition of free independence looks, perhaps, quite similar to the definition of classical independence given above, and indeed, in many respects free independence is conceptually similar to classical independence. For example, if $a_1, a_2, \dots, a_r$ are freely independent selfadjoint operators affiliated with $(\mathcal{A}, \tau)$, then all numbers of the form $\tau\{f_1(a_{i_1}) f_2(a_{i_2}) \cdots f_p(a_{i_p})\}$ (where $i_1, i_2, \dots, i_p \in \{1, 2, \dots, r\}$ and $f_1, f_2, \dots, f_p \colon \mathbb{R} \to \mathbb{R}$ are bounded Borel functions), are uniquely determined by the distributions $L\{a_i\}$, $i = 1, 2, \dots, r$. On the other hand, free independence is a truly non-commutative notion, which can be seen, for instance, from the easily checked fact that two classical random variables are never freely independent, unless one of them is trivial, i.e. constant with probability one (see e.g. [Vo98]).

Voiculescu originally introduced free independence in connection with his deep studies of the von Neumann algebras associated to the free group factors (see [Vo85], [Vo91], [Vo90]). We prefer in these notes, however, to indicate the significance of free independence by explaining its connection with random matrices. In the 1950's, the phycicist E.P. Wigner showed that the spectral distribution of large selfadjoint random matrices with independent complex Gaussian entries is, approximately, the semi-circle distribution, i.e. the distribution on $\mathbb{R}$ with density $s \mapsto \sqrt{4-s^2} \cdot 1_{[-2,2]}(s)$ w.r.t. Lebesgue measure. More precisely, for each n in $\mathbb{N}$, let $X^{(n)}$ be a selfadjoint complex Gaussian random matrix of the kind considered by Wigner (and suitably normalized), and let tr_n denote the (usual) tracial state on the $n \times n$ matrices $M_n(\mathbb{C})$. Then for any positive integer p, Wigner showed that

$$\mathbb{E}\big\{\mathrm{tr}_n\big[(X^{(n)})^p\big]\big\} \underset{n\to\infty}{\longrightarrow} \int_{-2}^{2} s^p \sqrt{4-s^2}\,\mathrm{d}s.$$

In the late 1980's, Voiculescu generalized Wigner's result to families of independent selfadjoint Gaussian random matrices (cf. [Vo91]): For each n in $\mathbb{N}$, let $X_1^{(n)}, X_2^{(n)}, \dots, X_r^{(n)}$ be independent[7] random matrices of the kind considered by Wigner. Then for any indices $i_1, i_2, \dots, i_p$ in $\{1, 2, \dots, r\}$,

$$\mathbb{E}\big\{\mathrm{tr}_n\big[X_{i_1}^{(n)} X_{i_2}^{(n)} \cdots X_{i_p}^{(n)}\big]\big\} \underset{n\to\infty}{\longrightarrow} \tau\{x_{i_1} x_{i_2} \cdots x_{i_p}\},$$

where $x_1, x_2, \dots, x_r$ are freely independent selfadjoint operators in a W^*-probability space $(\mathcal{A}, \tau)$, and such that $L\{x_i\}$ is the semi-circle distribution for each i.

By Voiculescu's result, free independence describes what the assumed classical independence between the random matrices is turned into, as $n \to \infty$. Also, from a classical probabilistic point of view, free probability theory may be considered as (an aspect of) the probability theory of large random matrices.

Voiculescu's result reveals another general fact in free probability, namely that the role of the Gaussian distribution in classical probability is taken over by the semi-circle distribution in free probability. In particular, as also proved by Voiculescu, the limit distribution appearing in the free version of the central limit theorem is the semi-circle distribution (see e.g. [VoDyNi92]).

4.3 Free Independence and Convergence in Probability

In this section, we study the relationship between convergence in probability and free independence. The results will be used in the proof of the free Lévy-Itô decomposition in Section 6.5 below. We start by defining the notion of convergence in probability in the non-commutative setting:

Definition 4.3. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and let a and a_n, $n \in \mathbb{N}$, be operators in $\overline{\mathcal{A}}$. We say then that $a_n \to a$ in probability, as $n \to \infty$, if $|a_n - a| \to 0$ in distribution, i.e. if $L\{|a_n - a|\} \to \delta_0$ weakly.*

[7] in the classical sense; at the level of the entries.

Convergence in probability, as defined above, corresponds to the so-called *measure topology*, which is discussed in detail in the Appendix (Section A). As mentioned there, if we assume that the operators a_n and a are all selfadjoint, then convergence in probability is equivalent to the condition:

$$L\{a_n - a\} \xrightarrow{\mathrm{w}} \delta_0.$$

Lemma 4.4. *Let (b_n) be a sequence of (not necessarily selfadjoint) operators in a W^*-probability space $(\mathcal{A}, \tau)$, and assume that $\|b_n\| \le 1$ for all n. Assume, further, that $b_n \to b$ in probability as $n \to \infty$ for some operator b in $\mathcal{A}$. Then also $\|b\| \le 1$ and $\tau(b_n) \to \tau(b)$, as $n \to \infty$.*

Proof. To see that $\|b\| \le 1$, note first that $b_n^* b_n \to b^* b$ in probability as $n \to \infty$, since operator multiplication and the adjoint operation are both continuous operations in the measure topology. This implies that $b_n^* b_n \to b^* b$ in distribution, i.e. that $L\{b_n^* b_n\} \xrightarrow{\mathrm{w}} L\{b^* b\}$ as $n \to \infty$ (cf. Proposition A.9). Since $\mathrm{supp}(L\{b_n^* b_n\}) = \mathrm{sp}(b_n^* b_n) \subseteq [0,1]$ for all n (recall that τ is faithful), a standard argument shows that also $[0,1] \supseteq \mathrm{supp}(L\{b^* b\}) = \mathrm{sp}(b^* b)$, whence $\|b\| \le 1$.

To prove the second statement, consider, for each n in $\mathbb{N}$, $b_n' = \frac{1}{2}(b_n + b_n^*)$ and $b_n'' = \frac{1}{2\mathrm{i}}(b_n - b_n^*)$, and define b', b'' similarly from b. Then b_n', b_n'', b', b'' are all selfadjoint operators in $\mathcal{A}$ of norm less than or equal to 1. Since addition, scalar-multiplication and the adjoint operation are all continuous operations in the measure topology, it follows, furthermore, that $b_n' \to b'$ and $b_n'' \to b''$ in probability as $n \to \infty$. As above, this implies that $L\{b_n'\} \xrightarrow{\mathrm{w}} L\{b'\}$ and $L\{b_n''\} \xrightarrow{\mathrm{w}} L\{b''\}$ as $n \to \infty$.

Now, choose a continuous bounded function $f\colon \mathbb{R} \to \mathbb{R}$, such that $f(x) = x$ for all x in $[-1,1]$. Then, since $\mathrm{sp}(b_n'), \mathrm{sp}(b')$ are contained in $[-1,1]$, we find that

$$\begin{aligned}\tau(b_n') = \tau(f(b_n')) &= \int_{\mathbb{R}} f(x)\, L\{b_n'\}(\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} f(x)\, L\{b'\}(\mathrm{d}x) \\ &= \tau(f(b')) = \tau(b').\end{aligned}$$

Similarly, $\tau(b_n'') \to \tau(b'')$ as $n \to \infty$, and hence also $\tau(b_n) = \tau(b_n' + \mathrm{i}b_n'') \to \tau(b' + \mathrm{i}b'') = \tau(b)$, as $n \to \infty$. □

Lemma 4.5. *Let r be a positive integer, and let $(b_{1,n})_{n\in\mathbb{N}}, \dots, (b_{r,n})_{n\in\mathbb{N}}$ be sequences of bounded (not necessarily selfadjoint) operators in the W^*-probability space $(\mathcal{A}, \tau)$. Assume, for each j, that $\|b_{j,n}\| \le 1$ for all n and that $b_{j,n} \to b_j$ in probability as $n \to \infty$, for some operator b_j in $\mathcal{A}$. If $b_{1,n}, b_{2,n}, \dots, b_{r,n}$ are freely independent for each n, then the operators $b_1, b_2, \dots, b_r$ are also freely independent.*

Proof. Assume that $b_{1,n}, b_{2,n}, \dots, b_{r,n}$ are freely independent for all n, and let $i_1, i_2, \dots, i_p$ in $\{1, 2, \dots, r\}$ be given. Then there is a universal polynomial

$P_{i_1,\dots,i_p}$ in rp complex variables, depending only on $i_1,\dots,i_p$, such that for all n in $\mathbb{N}$,

$$\tau(b_{i_1,n}b_{i_2,n}\cdots b_{i_p,n}) = P_{i_1,\dots,i_p}\Big[\big\{\tau(b_{1,n}^{\ell})\big\}_{1\le\ell\le p},\dots,\big\{\tau(b_{r,n}^{\ell})\big\}_{1\le\ell\le p}\Big]. \tag{4.2}$$

Now, since operator multiplication is a continuous operation with respect to the measure topology, $b_{i_1,n}b_{i_2,n}\cdots b_{i_p,n} \to b_{i_1}b_{i_2}\cdots b_{i_p}$ in probability as $n\to\infty$. Furthermore, $\|b_{i_1,n}b_{i_2,n}\cdots b_{i_p,n}\|\le 1$ for all n, so by Lemma 4.4 we have

$$\tau\big(b_{i_1,n}b_{i_2,n}\cdots b_{i_p,n}\big) \underset{n\to\infty}{\longrightarrow} \tau\big(b_{i_1}b_{i_2}\cdots b_{i_p}\big).$$

Similarly,

$$\tau(b_{j,n}^{\ell}) \underset{n\to\infty}{\longrightarrow} \tau(b_j^{\ell}), \quad \text{for any } j \text{ in } \{1,2,\dots,r\} \text{ and } \ell \text{ in } \mathbb{N}.$$

Combining these observations with (4.2), we conclude that also

$$\tau(b_{i_1}b_{i_2}\cdots b_{i_p}) = P_{i_1,\dots,i_p}\Big[\big\{\tau(b_1^{\ell})\big\}_{1\le\ell\le p},\dots,\big\{\tau(b_r^{\ell})\big\}_{1\le\ell\le p}\Big],$$

and since this holds for arbitrary $i_1,\dots,i_p$ in $\{1,2,\dots,r\}$, it follows that $b_1,\dots,b_r$ are freely independent, as desired. □

For a selfadjoint operator a affiliated with a W^*-probability space $(\mathcal{A},\tau)$, we denote by $\kappa(a)$ the *Cayley transform* of a, i.e.

$$\kappa(a) = (a - \mathrm{i}\mathbf{1}_{\mathcal{A}})(a + \mathrm{i}\mathbf{1}_{\mathcal{A}})^{-1}.$$

Recall that even though a may be an unbounded operator, $\kappa(a)$ is a unitary operator in $\mathcal{A}$.

Lemma 4.6. *Let $a_1,a_2,\dots,a_r$ be selfadjoint operators affiliated with the W^*-probability space $(\mathcal{A},\tau)$. Then $a_1,a_2,\dots,a_r$ are freely independent if and only if $\kappa(a_1),\kappa(a_2),\dots,\kappa(a_r)$ are freely independent.*

Proof. This is an immediate consequence of the fact that a_j and $\kappa(a_j)$ generate the same von Neumann subalgebra of $\mathcal{A}$ for each j (cf. [Pe89, Lemma 5.2.8]). □

Proposition 4.7. *Suppose $r\in\mathbb{N}$ and that $(a_{1,n})_{n\in\mathbb{N}},\dots,(a_{r,n})_{n\in\mathbb{N}}$ are sequences of selfadjoint operators affiliated with the W^*-probability space $(\mathcal{A},\tau)$. Assume, further, that for each j in $\{1,2,\dots,r\}$, $a_{j,n}\to a_j$ in probability as $n\to\infty$, for some selfadjoint operator a_j affiliated with $(\mathcal{A},\tau)$. If the operators $a_{1,n},a_{2,n},\dots,a_{r,n}$ are freely independent for each n, then the operators $a_1,a_2,\dots,a_r$ are also freely independent.*

Proof. Assume that $a_{1,n}, a_{2,n}, \dots, a_{r,n}$ are freely independent for all n. Then, by Lemma 4.6, the unitaries $\kappa(a_{1,n}), \dots, \kappa(a_{r,n})$ are freely independent for each n in $\mathbb{N}$. Moreover, since the Cayley transform is continuous in the measure topology (cf. [St59, Lemma 5.3]), we have

$$\kappa(a_{j,n}) \underset{n\to\infty}{\longrightarrow} \kappa(a_j), \qquad \text{in probability,}$$

for each j. Hence, by Lemma 4.5, the unitaries $\kappa(a_1), \dots, \kappa(a_r)$ are freely independent, and, appealing once more to Lemma 4.6, this means that $a_1, \dots, a_r$ themselves are freely independent. $\square$

Remark 4.8. Let $\mathcal{B}$ and $\mathcal{C}$ be two freely independent von Neumann subalgebras of a W^*-probability space $(\mathcal{A}, \tau)$. Let, further, (b_n) and (c_n) be two sequences of selfadjoint operators, which are affiliated with $\mathcal{B}$ and $\mathcal{C}$, respectively, in the sense that $f(b_n) \in \mathcal{B}$ and $g(c_n) \in \mathcal{C}$ for any n in $\mathbb{N}$ and any bounded Borel functions $f, g \colon \mathbb{R} \to \mathbb{R}$. Assume that $b_n \to b$ and $c_n \to c$ in probability as $n \to \infty$. Then b and c are also freely independent. This follows, of course, from Proposition 4.7, but it is also an immediate consequence of the fact that the set $\overline{\mathcal{B}}$ of closed, densely defined operators, affiliated with $\mathcal{B}$, is complete (and hence closed) in the measure topology. Indeed, the restriction to $\overline{\mathcal{B}}$ of the measure topology on $\overline{\mathcal{A}}$ is the measure topology on $\overline{\mathcal{B}}$ (induced by $\tau_{|\mathcal{B}}$). Thus, b is affiliated with $\mathcal{B}$ and similarly c is affiliated with $\mathcal{C}$, so that, in particular, b and c are freely independent.

4.4 Free Additive Convolution

From a probabilistic point of view, free additive convolution may be considered merely as a new type of convolution on the set of probability measures on $\mathbb{R}$. Let a and b be selfadjoint operators in a W^*-probability space $(\mathcal{A}, \tau)$, and note that $a + b$ is selfadjoint too. Denote then the (spectral) distributions of a, b and $a + b$ by μ_a, μ_b and μ_{a+b}. If a and b are freely independent, it is not hard to see that the moments of μ_{a+b} (and hence μ_{a+b} itself) is uniquely determined by μ_a and μ_b. Hence we may write $\mu_a \boxplus \mu_b$ instead of μ_{a+b}, and we say that $\mu_a \boxplus \mu_b$ is the *free additive*[8] *convolution* of μ_a and μ_b.

Since the distribution μ_a of a selfadjoint operator a in $\mathcal{A}$ is a compactly supported probability measure on $\mathbb{R}$, the definition of free additive convolution, stated above, works at most for all compactly supported probability measures on $\mathbb{R}$. On the other hand, given any two compactly supported probability measures μ_1 and μ_2 on $\mathbb{R}$, it follows from a free product construction (see [VoDyNi92]), that it is always possible to find a W^*-probability space

[8]The reason for the term additive is that there exists another convolution operation called *free multiplicative convolution*, which arises naturally out of the non-commutative setting (i.e. the non-commutative multiplication of operators). In the present notes we do not consider free multiplicative convolution.

$(\mathcal{A}, \tau)$ and free selfadjoint operators a, b in $\mathcal{A}$, such that a and b have distributions μ_1 and μ_2 respectively. Thus, the operation $\boxplus$ introduced above is, in fact, defined on all compactly supported probability measures on $\mathbb{R}$. To extend this operation to all probability measures on $\mathbb{R}$, one needs, as indicated above, to consider unbounded selfadjoint operators in a Hilbert space, and then to proceed with a construction similar to that described above. We postpone a detailed discussion of this matter to the Appendix (see Remark A.3), since, for our present purposes, it is possible to study free additive convolution by virtue of the Voiculescu transform, which we introduce next.

By $\mathbb{C}^+$ (respectively $\mathbb{C}^-$) we denote the set of complex numbers with strictly positive (respectively strictly negative) imaginary part.

Let μ be a probability measure on $\mathbb{R}$, and consider its Cauchy (or Stieltjes) transform $G_\mu \colon \mathbb{C}^+ \to \mathbb{C}^-$ given by:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\, \mu(\mathrm{d}t), \quad (z \in \mathbb{C}^+).$$

Then define the mapping $F_\mu \colon \mathbb{C}^+ \to \mathbb{C}^+$ by:

$$F_\mu(z) = \frac{1}{G_\mu(z)}, \quad (z \in \mathbb{C}^+),$$

and note that F_μ is analytic on $\mathbb{C}^+$. It was proved by Bercovici and Voiculescu in [BeVo93, Proposition 5.4 and Corollary 5.5] that there exist positive numbers η and M, such that F_μ has an (analytic) right inverse F_μ^{-1} defined on the region

$$\Gamma_{\eta,M} := \{z \in \mathbb{C} \mid |\mathrm{Re}(z)| < \eta \mathrm{Im}(z),\ \mathrm{Im}(z) > M\}.$$

In other words, there exists an open subset $G_{\eta,M}$ of $\mathbb{C}^+$ such that F_μ is injective on $G_{\eta,M}$ and such that $F_\mu(G_{\eta,M}) = \Gamma_{\eta,M}$.

Now the *Voiculescu transform* ϕ_μ of μ is defined by

$$\phi_\mu(z) = F_\mu^{-1}(z) - z,$$

on any region of the form $\Gamma_{\eta,M}$, where F_μ^{-1} is defined. It follows from [BeVo93, Corollary 5.3] that $\mathrm{Im}(F_\mu^{-1}(z)) \le \mathrm{Im}(z)$ and hence $\mathrm{Im}(\phi_\mu(z)) \le 0$ for all z in $\Gamma_{\eta,M}$.

The Voiculescu transform ϕ_μ should be viewed as a modification of Voiculescu's $\mathcal{R}$-transform (see e.g. [VoDyNi92]), since we have the correspondence:

$$\phi_\mu(z) = \mathcal{R}_\mu(\tfrac{1}{z}).$$

A third variant, which we shall also make use of is the *free cumulant transform*, given by:

$$\mathcal{C}_\mu(z) = z\mathcal{R}_\mu(z) = z\phi_\mu(\tfrac{1}{z}). \tag{4.3}$$

The key property of the Voiculescu transform is the following important result, which shows that the Voiculescu transform (and its variants) can be viewed as the free analogue of the classical cumulant function (the logarithm of the characteristic function). The result was first proved by Voiculescu for probability measures μ with compact support, and then by Maassen in the case where μ has variance. Finally Bercovici and Voiculescu proved the general case.

Theorem 4.9 ([Vo86],[Ma92],[BeVo93]). *Let μ_1 and μ_2 be probability measures on $\mathbb{R}$, and consider their free additive convolution $\mu_1 \boxplus \mu_2$. Then*

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z),$$

for all z in any region $\Gamma_{\eta,M}$, where all three functions are defined.

Remark 4.10. We shall need the fact that a probability measure on $\mathbb{R}$ is uniquely determined by its Voiculescu transform. To see this, suppose μ and μ' are probability measures on $\mathbb{R}$, such that $\phi_\mu = \phi_{\mu'}$, on a region $\Gamma_{\eta,M}$. It follows then that also $F_\mu = F_{\mu'}$ on some open subset of $\mathbb{C}^+$, and hence (by analytic continuation), $F_\mu = F_{\mu'}$ on all of $\mathbb{C}^+$. Consequently μ and μ' have the same Cauchy (or Stieltjes) transform, and by the Stieltjes Inversion Formula (cf. e.g. [Ch78, page 90]), this means that $\mu = \mu'$.

In [BeVo93, Proposition 5.6], Bercovici and Voiculescu proved the following characterization of Voiculescu transforms:

Theorem 4.11 ([BeVo93]). *Let ϕ be an analytic function defined on a region $\Gamma_{\eta,M}$, for some positive numbers η and M. Then the following assertions are equivalent:*

(i) *There exists a probability measure μ on $\mathbb{R}$, such that $\phi(z) = \phi_\mu(z)$ for all z in a domain $\Gamma_{\eta,M'}$, where $M' \geq M$.*
(ii) *There exists a number M' greater than or equal to M, such that*
  (a) *$Im(\phi(z)) \leq 0$ for all z in $\Gamma_{\eta,M'}$.*
  (b) *$\phi(z)/z \to 0$, as $|z| \to \infty$, $z \in \Gamma_{\eta,M'}$.*
  (c) *For any positive integer n and any points $z_1, \dots, z_n$ in $\Gamma_{\eta,M'}$, the $n \times n$ matrix*
$$\left[\frac{z_j - \overline{z_k}}{z_j + \phi(z_j) - \overline{z_k} - \overline{\phi(z_k)}}\right]_{1 \leq j,k \leq n},$$
  is positive definite.

The relationship between weak convergence of probability measures and the Voiculescu transform was settled in [BeVo93, Proposition 5.7] and [BePa96, Proposition 1]:

Proposition 4.12 ([BeVo93],[BePa96]). *Let (μ_n) be a sequence of probability measures on $\mathbb{R}$. Then the following assertions are equivalent:*

(a) *The sequence (μ_n) converges weakly to a probability measure μ on $\mathbb{R}$.*

(b) *There exist positive numbers η and M, and a function ϕ, such that all the functions ϕ, ϕ_{μ_n} are defined on $\Gamma_{\eta,M}$, and such that*

(b1) $\phi_{\mu_n}(z) \to \phi(z)$, *as* $n \to \infty$, *uniformly on compact subsets of* $\Gamma_{\eta,M}$,

(b2) $\sup_{n\in\mathbb{N}} \Big|\frac{\phi_{\mu_n}(z)}{z}\Big| \to 0$, *as* $|z| \to \infty$, $z \in \Gamma_{\eta,M}$.

(c) *There exist positive numbers η and M, such that all the functions ϕ_{μ_n} are defined on $\Gamma_{\eta,M}$, and such that*

(c1) $\lim_{n\to\infty} \phi_{\mu_n}(\mathrm{i}y)$ *exists for all y in* $[M, \infty[$.

(c2) $\sup_{n\in\mathbb{N}} \Big|\frac{\phi_{\mu_n}(\mathrm{i}y)}{y}\Big| \to 0$, *as* $y \to \infty$.

If the conditions (a),(b) and (c) are satisfied, then $\phi = \phi_\mu$ on $\Gamma_{\eta,M}$.

Remark 4.13 (Cumulants I). Under the assumption of finite moments of all orders, both classical and free convolution can be handled completely by a combinatorial approach based on cumulants. Suppose, for simplicity, that μ is a compactly supported probability measure on $\mathbb{R}$. Then for n in $\mathbb{N}$, the classical cumulant c_n of μ may be defined as the n'th derivative at 0 of the cumulant transform $\log f_\mu$. In other words, we have the Taylor expansion:

$$\log f_\mu(z) = \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n.$$

Consider further the sequence $(m_n)_{n\in\mathbb{N}_0}$ of moments of μ. Then the sequence (m_n) is uniquely determined by the sequence (c_n) (and vice versa). The formulas determining m_n from (c_n) are generally quite complicated. However, by viewing the sequences (m_n) and (c_n) as multiplicative functions M and C on the lattice of all partitions of $\{1, 2, \dots, n\}$, $n \in \mathbb{N}$ (cf. e.g. [Sp97]), the relationship between (m_n) and (c_n) can be elegantly expressed by the formula:

$$C = M \star \mathrm{Moeb},$$

where Moeb denotes the Möbius transform and where $\star$ denotes *combinatorial convolution* of multiplicative functions on the lattice of all partitions (see [Sp97],[Ro64] or [BaCo89]).

The *free* cumulants (k_n) of μ were introduced by R. Speicher in [Sp94]. They may, similarly, be defined as the coefficients in the Taylor expansion of the free cumulant transform $\mathcal{C}_\mu$:

$$\mathcal{C}_\mu(z) = \sum_{n=1}^{\infty} k_n z^n,$$

(see (4.3)). Viewing then (k_n) and (m_n) as multiplicative functions k and m on the lattice of all *non-crossing* partitions of $\{1, 2, \dots, n\}$, $n \in \mathbb{N}$, the relationship between (k_n) and (m_n) is expressed by the exact same formula:

$$k = m \star \text{Moeb}, \tag{4.4}$$

where now $\star$ denotes combinatorial convolution of multiplicative functions on the lattice of all *non-crossing* partitions (see [Sp97]).

For a family $a_1, a_2, \dots, a_r$ of selfadjoint operators in a W^*-probability space $(\mathcal{A}, \tau)$ it is also possible to define generalized cumulants, which are related to the family of all mixed moments (w.r.t. τ) of $a_1, a_2, \dots, a_r$ by a formula similar to (4.4) (see e.g. [Sp97]). In terms of these multivariate cumulants, free independence of $a_1, a_2, \dots, a_r$ has a rather simple formulation, and using this formulation, R. Speicher gave a simple and completely combinatorial proof of the fact that the free cumulants (and hence the free cumulant transform) linearize free convolution (see [Sp94]). A treatment of the theory of classical multivariate cumulants can be found in [BaCo89].

4.5 Basic Results in Free Infinite Divisibility

In this section we recall the definition and some basic facts about infinite divisibility w.r.t. free additive convolution. In complete analogy with the classical case, a probability measure μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible, if for any n in $\mathbb{N}$ there exists a probability measure μ_n on $\mathbb{R}$, such that

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ terms}}.$$

It was proved in [Pa96] that the class $\mathcal{ID}(\boxplus)$ of $\boxplus$-infinitely divisible probability measures on $\mathbb{R}$ is closed w.r.t. weak convergence. For the corresponding classical result, see [GnKo68, §17, Theorem 3]. As in classical probability, $\boxplus$-infinitely divisible probability measures are characterized as those probability measures that have a (free) Lévy-Khintchine representation:

Theorem 4.14 ([Vo86],[Ma92],[BeVo93]).
Let μ be a probability measure on $\mathbb{R}$. Then μ is $\boxplus$-infinitely divisible, if and only if there exist a finite measure σ on $\mathbb{R}$ and a real constant γ, such that

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\,\sigma(\mathrm{d}t), \qquad (z \in \mathbb{C}). \tag{4.5}$$

Moreover, for a $\boxplus$-infinitely divisible probability measure μ on $\mathbb{R}$, the real constant γ and the finite measure σ, described above, are uniquely determined.

Proof. The equivalence between $\boxplus$-infinite divisibility and the existence of a representation in the form (4.5) was proved (in the general case) by Voiculescu and Bercovici in [BeVo93, Theorem 5.10]. They proved first that μ is $\boxplus$-infinitely divisible, if and only if ϕ_μ has an extension to a function of the form: $\phi\colon \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$, i.e. a Pick function multiplied by -1. Equation (4.5) (and its uniqueness) then follows from the existence (and uniqueness) of the integral representation of Pick functions (cf. [Do74, Chapter 2, Theorem I]). Compared to the general integral representation for Pick functions, just referred to, there is a linear term missing on the right hand side of (4.5), but this corresponds to the fact that $\frac{\phi(\mathrm{i}y)}{y} \to 0$ as $y \to \infty$, if ϕ is a Voiculescu transform (cf. Theorem 4.11 above). □

Definition 4.15. *Let μ be a $\boxplus$-infinitely divisible probability measure on $\mathbb{R}$, and let γ and σ be, respectively, the (uniquely determined) real constant and finite measure on $\mathbb{R}$ appearing in (4.5). We say then that the pair (γ, σ) is the free generating pair for μ.*

In terms of the free cumulant transform, the free Lévy-Khintchine representation resembles more closely the classical Lévy-Khintchine representation, as the following proposition shows.

Proposition 4.16. *A probability measure ν on $\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if there exist a non-negative number a, a real number η and a Lévy measure ρ, such that the free cumulant transform $\mathcal{C}_\nu$ has the representation:*

$$\mathcal{C}_\nu(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-tz} - 1 - tz1_{[-1,1]}(t)\Big)\,\rho(\mathrm{d}t), \quad (z \in \mathbb{C}^-). \tag{4.6}$$

In that case, the triplet (a, ρ, η) is uniquely determined and is called the free characteristic triplet for ν.

Proof. Let ν be a measure in $\mathcal{ID}(\boxplus)$ with free generating pair (γ, σ), and consider its free Lévy-Khintchine representation (in terms of the Voiculescu transform):

$$\phi_\nu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\,\sigma(\mathrm{d}t), \quad (z \in \mathbb{C}^+). \tag{4.7}$$

Then define the triplet (a, ρ, η) by (2.3), and note that

$$\sigma(\mathrm{d}t) = a\delta_0(\mathrm{d}t) + \frac{t^2}{1+t^2}\rho(\mathrm{d}t),$$

$$\gamma = \eta - \int_{\mathbb{R}} t\Big(1_{[-1,1]}(t) - \frac{1}{1+t^2}\Big)\,\rho(\mathrm{d}t).$$

Now, for z in $\mathbb{C}^-$, the corresponding free cumulant transform $\mathcal{C}_\nu$ is given by

$$\begin{aligned}
&\mathcal{C}_\nu(z) \\
&= z\phi_\nu(1/z) = z\Big(\gamma + \int_{\mathbb{R}} \frac{1+t(1/z)}{(1/z)-t}\,\sigma(\mathrm{d}t)\Big) \\
&= \gamma z + z\int_{\mathbb{R}} \frac{z+t}{1-tz}\,\sigma(\mathrm{d}t) = \gamma z + \int_{\mathbb{R}} \frac{z^2+tz}{1-tz}\,\sigma(\mathrm{d}t) \\
&= \eta z - \Big[\int_{\mathbb{R}} t\Big(1_{[-1,1]}(t) - \frac{1}{1+t^2}\Big)\,\rho(\mathrm{d}t)\Big]z + az^2 + \int_{\mathbb{R}} \frac{z^2+tz}{1-tz}\frac{t^2}{1+t^2}\,\rho(\mathrm{d}t).
\end{aligned}$$

Note here that

$$1_{[-1,1]}(t) - \frac{1}{1+t^2} = 1 - \frac{1}{1+t^2} - 1_{\mathbb{R}\setminus[-1,1]}(t) = \frac{t^2}{1+t^2} - 1_{\mathbb{R}\setminus[-1,1]}(t),$$

so that

$$\int_{\mathbb{R}} t\Big(1_{[-1,1]}(t) - \frac{1}{1+t^2}\Big)\ \rho(\mathrm{d}t) = \int_{\mathbb{R}} \Big(\frac{t}{1+t^2} - t^{-1}1_{\mathbb{R}\setminus[-1,1]}(t)\Big)\ t^2\rho(\mathrm{d}t).$$

Note also that

$$\frac{z^2+tz}{(1-tz)(1+t^2)} = \frac{z^2}{1-tz} + \frac{tz}{1+t^2}.$$

Therefore,

$$\begin{aligned}\mathcal{C}_\nu(z) = \eta z - \Big[\int_{\mathbb{R}} \Big(\frac{t}{1+t^2} - t^{-1}1_{\mathbb{R}\setminus[-1,1]}(t)\Big)\ t^2\rho(\mathrm{d}t)\Big]z + az^2 \\ + \int_{\mathbb{R}} \Big(\frac{z^2}{1-tz} + \frac{tz}{1+t^2}\Big)\ t^2\rho(\mathrm{d}t) \\ = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{z^2}{1-tz} + t^{-1}z1_{\mathbb{R}\setminus[-1,1]}(t)\Big)\ t^2\rho(\mathrm{d}t) \\ = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{(tz)^2}{1-tz} + tz1_{\mathbb{R}\setminus[-1,1]}(t)\Big)\ \rho(\mathrm{d}t).\end{aligned}$$

Further,

$$\begin{aligned}\frac{(tz)^2}{1-tz} + tz1_{\mathbb{R}\setminus[-1,1]}(t) &= \Big(\frac{(tz)^2}{1-tz} + tz\Big) - tz1_{[-1,1]}(t) \\ &= \frac{tz}{1-tz} - tz1_{[-1,1]}(t) \\ &= \frac{1}{1-tz} - 1 - tz1_{[-1,1]}(t).\end{aligned}$$

We conclude that

$$\mathcal{C}_\nu(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-tz} - 1 - tz1_{[-1,1]}(t)\Big)\ \rho(\mathrm{d}t). \qquad (4.8)$$

Clearly the above calculations may be reversed, so that (4.7) and (4.8) are equivalent. □

Apart from the striking similarity between (2.2) and (4.6), note that these particular representations clearly exhibit how μ (respectively ν) is always the convolution of a Gaussian distribution (respectively a semi-circle distribution) and a distribution of generalized Poisson (respectively free Poisson) type (cf. also the Lévy-Itô decomposition described in Section 6.5). In particular, the cumulant transform for the Gaussian distribution with mean η and variance a is: $u \mapsto i\eta u - \frac{1}{2}au^2$, and the free cumulant transform for the semi-circle distribution with mean η and variance a is $z \mapsto \eta z + az^2$ (see [VoDyNi92]).

The next result, due to Bercovici and Pata, is the free analogue of Khintchine's characterization of classically infinitely divisible probability measures. It plays an important role in Section 4.6.

Definition 4.17. *Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of positive integers, and let*

$$A = \{\mu_{nj} \mid n \in \mathbb{N},\ j \in \{1, 2, \dots, k_n\}\},$$

be an array of probability measures on $\mathbb{R}$. We say then that A is a null array, if the following condition is fulfilled:

$$\forall \epsilon > 0\colon\ \lim_{n\to\infty} \max_{1\le j\le k_n} \mu_{nj}(\mathbb{R} \setminus [-\epsilon, \epsilon]) = 0.$$

Theorem 4.18 ([BePa00]). *Let $\{\mu_{nj} \mid n \in \mathbb{N},\ j \in \{1, 2, \dots, k_n\}\}$ be a null-array of probability measures on $\mathbb{R}$, and let $(c_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. If the probability measures $\mu_n = \delta_{c_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ converge weakly, as $n \to \infty$, to a probability measure μ on $\mathbb{R}$, then μ has to be $\boxplus$-infinitely divisible.*

4.6 Classes of Freely Infinitely Divisible Probability Measures

In this section we study the free counterparts $\mathcal{S}(\boxplus)$ and $\mathcal{L}(\boxplus)$ to the classes $\mathcal{S}(*)$ and $\mathcal{L}(*)$ of stable and selfdecomposable distributions. We show in particular that we have the following hierarchy

$$\mathcal{G}(\boxplus) \subset \mathcal{S}(\boxplus) \subset \mathcal{L}(\boxplus) \subset \mathcal{ID}(\boxplus), \tag{4.9}$$

where $\mathcal{G}(\boxplus)$ denotes the class of semi-circle distributions. We start with the formal definitions of and $\mathcal{S}(\boxplus)$ and $\mathcal{L}(\boxplus)$.

Definition 4.19. (i) *A probability measure μ on $\mathbb{R}$ is called stable w.r.t. free convolution (or just $\boxplus$-stable), if the class*

$$\{\psi(\mu) \mid \psi\colon \mathbb{R} \to \mathbb{R} \text{ is an increasing affine transformation}\}$$

is closed under the operation $\boxplus$. By $\mathcal{S}(\boxplus)$ we denote the class of $\boxplus$-stable probability measures on $\mathbb{R}$.

(ii) *A probability measure μ on $\mathbb{R}$ is selfdecomposable w.r.t. free additive convolution (or just $\boxplus$-selfdecomposable), if for any c in $]0, 1[$ there exists a probability measure μ_c on $\mathbb{R}$, such that*

$$\mu = D_c\mu \boxplus \mu_c. \tag{4.10}$$

By $\mathcal{L}(\boxplus)$ we denote the class of $\boxplus$-selfdecomposable probability measures on $\mathbb{R}$.

Note that for a probability measure μ on $\mathbb{R}$ and a constant c in $]0,1[$, there can be only one probability measure μ_c, such that $\mu = D_c\mu \boxplus \mu_c$. Indeed, choose positive numbers η and M, such that all three Voiculescu transforms ϕ_μ, $\phi_{D_c\mu}$ and ϕ_{μ_c} are defined on the region $\Gamma_{\eta,M}$. Then by Theorem 4.9, ϕ_{μ_c} is uniquely determined on $\Gamma_{\eta,M}$, and hence, by Remark 4.10, μ_c is uniquely determined too.

In order to prove the inclusions in (4.9), we need the following technical result.

Lemma 4.20. *Let μ be a probability measure on $\mathbb{R}$, and let η and M be positive numbers such that the Voiculescu transform ϕ_μ is defined on $\Gamma_{\eta,M}$ (see Section 4.4). Then for any constant c in $\mathbb{R}\setminus\{0\}$, $\phi_{D_c\mu}$ is defined on $|c|\Gamma_{\eta,M} = \Gamma_{\eta,|c|M}$, and*

(i) *if $c>0$, then $\phi_{D_c\mu}(z) = c\phi_\mu(c^{-1}z)$ for all z in $c\Gamma_{\eta,M}$,*
(ii) *if $c<0$, then $\phi_{D_c\mu}(z) = \overline{c\phi_\mu(c^{-1}\overline{z})}$ for all z in $|c|\Gamma_{\eta,M}$.*

In particular, for a constant c in $[-1,1]$, the domain of $\phi_{D_c\mu}$ contains the domain of ϕ_μ.

Proof. (i) This is a special case of [BeVo93, Lemma 7.1].

(ii) Note first that by virtue of (i), it suffices to prove (ii) in the case $c=-1$.

We start by noting that the Cauchy transform G_μ (see Section 4.4) is actually well-defined for all z in $\mathbb{C}\setminus\mathbb{R}$ (even for all z outside $\mathrm{supp}(\mu)$), and that $G_\mu(\overline{z}) = \overline{G_\mu(z)}$, for all such z. Similarly, F_μ is defined for all z in $\mathbb{C}\setminus\mathbb{R}$, and $F_\mu(z) = \overline{F_\mu(\overline{z})}$, for such z.

Note next that for any z in $\mathbb{C}\setminus\mathbb{R}$, $G_{D_{-1}\mu}(z) = -G_\mu(-z)$, and consequently

$$F_{D_{-1}\mu}(z) = -F_\mu(-z) = -\overline{F_\mu(-\overline{z})}.$$

Now, since $-\overline{\Gamma_{\eta,M}} = \Gamma_{\eta,M}$, it follows from the equation above, that $F_{D_{-1}\mu}$ has a right inverse on $\Gamma_{\eta,M}$, given by $F^{-1}_{D_{-1}\mu}(z) = -\overline{F^{-1}_\mu(-\overline{z})}$, for all z in $\Gamma_{\eta,M}$. Consequently, for z in $\Gamma_{\eta,M}$, we have

$$\phi_{D_{-1}\mu}(z) = F^{-1}_{D_{-1}\mu}(z) - z = -\overline{F^{-1}_\mu(-\overline{z})} - z = -\overline{(F^{-1}_\mu(-\overline{z}) - (-\overline{z}))} = -\overline{\phi_\mu(-\overline{z})},$$

as desired. □

Remark 4.21. With respect to dilation the free cumulant transform behaves exactly as the classical cumulant function, i.e.

$$\mathcal{C}_{D_c\mu}(z) = \mathcal{C}_\mu(cz), \tag{4.11}$$

for any probability measure μ on $\mathbb{R}$ and any positive constant c. This follows easily from Lemma 4.20. As a consequence, it follows as in the classical case that a probability measure μ on $\mathbb{R}$ belongs to $\mathcal{S}(\boxplus)$, if and only if the following condition is satisfied (for z^{-1} in a region of the form $\Gamma(\eta, M)$)

$$\forall a, a' > 0 \, \forall b, b' \in \mathbb{R} \, \exists a'' > 0 \, \exists b'' \in \mathbb{R} \colon \mathcal{C}_\mu(az) + bz + \mathcal{C}_\mu(a'z) + b'z = \mathcal{C}_\mu(a''z) + b''z.$$

It is easy to see that the above condition is equivalent to the following

$$\forall a > 0 \; \exists a'' > 0 \; \exists b'' \in \mathbb{R} \colon \mathcal{C}_\mu(z) + \mathcal{C}_\mu(az) = \mathcal{C}_\mu(a''z) + b''z. \tag{4.12}$$

Similarly, a probability measure μ on $\mathbb{R}$ is $\boxplus$-selfdecomposable, if and only if there exists, for any c in $]0,1[$, a probability measure μ_c on $\mathbb{R}$, such that

$$\mathcal{C}_\mu(z) = \mathcal{C}_\mu(cz) + \mathcal{C}_{\mu_c}(z), \tag{4.13}$$

for z^{-1} in a region of the form $\Gamma(\eta, M)$. In terms of the Voiculescu transform ϕ_μ, formula (4.13) takes the equivalent form

$$\phi_\mu(z) = c\phi_\mu(c^{-1}z) + \phi_{\mu_c}(z),$$

for all z in a region $\Gamma_{\eta,M}$.

Proposition 4.22. (i) *Any semi-circle law is $\boxplus$-stable.*
(ii) *Let μ be a $\boxplus$-stable probability measure on $\mathbb{R}$. Then μ is necessarily $\boxplus$-selfdecomposable.*

Proof. (i) Let $\gamma_{0,2}$ denote the standard semi-circle distribution, i.e.

$$\gamma_{0,2}(\mathrm{d}x) = 1_{[-2,2]}(x)\sqrt{4 - x^2}\,\mathrm{d}x.$$

Then, by definition,

$$\mathcal{G}(\boxplus) = \{D_a\gamma_{0,2} \boxplus \delta_b \mid a \geq 0, \;\; b \in \mathbb{R}\}.$$

It is easy to see that $\mathcal{S}(\boxplus)$ is closed under the operations D_a $(a > 0)$, and under (free) convolution with δ_b $(b \in \mathbb{R})$. Therefore, it suffices to show that $\gamma_{0,2} \in \mathcal{S}(\boxplus)$. By [VoDyNi92, Example 3.4.4], the free cumulant transform of $\gamma_{0,2}$ is given by

$$\mathcal{C}_{\gamma_{0,2}}(z) = z^2, \qquad (z \in \mathbb{C}^+),$$

and clearly this function satisfies condition (4.12) above.

(ii) Let μ be a measure in $\mathcal{S}(\boxplus)$. The relationship between the constants a and a'' in (4.12) is of the form $a'' = f(a)$, where $f \colon]0,\infty[\,\to\,]1,\infty[$ is a continuous, strictly increasing function, satisfying that $f(t) \to 1$ as $t \to 0^+$ and $f(t) \to \infty$ as $t \to \infty$ (see the proof of [BeVo93, Lemma 7.4]). Now, given c in $]0,1[$, put $a = f^{-1}(1/c) \in \,]0,\infty[$, so that

$$\mathcal{C}_\mu(z) + \mathcal{C}_\mu(az) = \mathcal{C}_\mu(c^{-1}z) + bz,$$

for suitable b in $\mathbb{R}$. Putting $z = cw$, it follows that

$$\mathcal{C}_\mu(w) - \mathcal{C}_\mu(cw) = \mathcal{C}_\mu(acw) - bcw.$$

Based on Theorem 4.11 is is not hard to see that $z \mapsto \mathcal{C}_\mu(acw) - bcw$ is the free cumulant transform of some measure μ_c in $\mathcal{P}$. With this μ_c, condition (4.13) is satisfied. □

We turn next to the last inclusion in (4.9).

Lemma 4.23. *Let μ be a $\boxplus$-selfdecomposable probability measure on $\mathbb{R}$, let c be a number in $]0,1[$, and let μ_c be the probability measure on $\mathbb{R}$ determined by the equation:*

$$\mu = D_c\mu \boxplus \mu_c.$$

Let η and M be positive numbers, such that ϕ_μ is defined on $\Gamma_{\eta,M}$. Then ϕ_{μ_c} is defined on $\Gamma_{\eta,M}$ as well.

Proof. Choose positive numbers η' and M' such that $\Gamma_{\eta',M'} \subseteq \Gamma_{\eta,M}$ and such that ϕ_μ and ϕ_{μ_c} are both defined on $\Gamma_{\eta',M'}$. For z in $\Gamma_{\eta',M'}$, we then have (cf. Lemma 4.20):

$$\phi_\mu(z) = c\phi_\mu(c^{-1}z) + \phi_{\mu_c}(z).$$

Recalling the definition of the Voiculescu transform, the above equation means that

$$F_\mu^{-1}(z) - z = c\phi_\mu(c^{-1}z) + F_{\mu_c}^{-1}(z) - z, \quad (z \in \Gamma_{\eta',M'}),$$

so that

$$F_{\mu_c}^{-1}(z) = F_\mu^{-1}(z) - c\phi_\mu(c^{-1}z), \quad (z \in \Gamma_{\eta',M'}).$$

Now put $\psi(z) = F_\mu^{-1}(z) - c\phi_\mu(c^{-1}z)$ and note that ψ is defined and holomorphic on all of $\Gamma_{\eta,M}$ (cf. Lemma 4.20), and that

$$F_{\mu_c}(\psi(z)) = z, \quad (z \in \Gamma_{\eta',M'}). \tag{4.14}$$

We note next that ψ takes values in $\mathbb{C}^+$. Indeed, since F_μ is defined on $\mathbb{C}^+$, we have that $\mathrm{Im}(F_\mu^{-1}(z)) > 0$, for any z in $\Gamma_{\eta,M}$ and furthermore, for all such z, $\mathrm{Im}(\phi_\mu(c^{-1}z)) \leq 0$, as noted in Section 4.4.

Now, since F_{μ_c} is defined and holomorphic on all of $\mathbb{C}^+$, both sides of (4.14) are holomorphic on $\Gamma_{\eta,M}$. Since $\Gamma_{\eta',M'}$ has an accumulation point in $\Gamma_{\eta,M}$, it follows, by uniqueness of analytic continuation, that the equality in (4.14) actually holds for all z in $\Gamma_{\eta,M}$. Thus, F_{μ_c} has a right inverse on $\Gamma_{\eta,M}$, which means that ϕ_{μ_c} is defined on $\Gamma_{\eta,M}$, as desired. □

Lemma 4.24. *Let μ be a $\boxplus$-selfdecomposable probability measure on $\mathbb{R}$, and let (c_n) be a sequence of numbers in $]0,1[$. For each n, let μ_{c_n} be the probability measure on $\mathbb{R}$ satisfying*

$$\mu = D_{c_n}\mu \boxplus \mu_{c_n}.$$

Then, if $c_n \to 1$ as $n \to \infty$, we have $\mu_{c_n} \xrightarrow{\mathrm{w}} \delta_0$, as $n \to \infty$.

Proof. Choose positive numbers η and M, such that ϕ_μ is defined on $\Gamma_{\eta,M}$. Note then that, by Lemma 4.23, $\phi_{\mu_{c_n}}$ is also defined on $\Gamma_{\eta,M}$ for each n in $\mathbb{N}$ and, moreover,

$$\phi_{\mu_{c_n}}(z) = \phi_\mu(z) - c_n\phi_\mu(c_n^{-1}z), \quad (z \in \Gamma_{\eta,M},\ n \in \mathbb{N}). \tag{4.15}$$

Assume now that $c_n \to 1$ as $n \to \infty$. From (4.15) and continuity of ϕ_μ it is then straightforward that $\phi_{\mu_{c_n}}(z) \to 0 = \phi_{\delta_0}(z)$, as $n \to \infty$, uniformly on compact subsets of $\Gamma_{\eta,M}$. Note furthermore that

$$\sup_{n\in\mathbb{N}}\left|\frac{\phi_{\mu_{c_n}}(z)}{z}\right| = \sup_{n\in\mathbb{N}}\left|\frac{\phi_\mu(z)}{z} - \frac{\phi_\mu(c_n^{-1}z)}{c_n^{-1}z}\right| \to 0, \quad \text{as } |z| \to \infty,\ z \in \Gamma_{\eta,M},$$

since $\frac{\phi_\mu(z)}{z} \to 0$ as $|z| \to \infty$, $z \in \Gamma_{\eta,M}$, and since $c_n^{-1} \geq 1$ for all n. It follows thus from Proposition 4.12 that $\mu_{c_n} \xrightarrow{\text{w}} \delta_0$, for $n \to \infty$, as desired. □

Theorem 4.25. *Let μ be a probability measure on $\mathbb{R}$. If μ is $\boxplus$-selfdecomposable, then μ is $\boxplus$-infinitely divisible.*

Proof. Assume that μ is $\boxplus$-selfdecomposable. Then by successive applications of (4.10), we get for any c in $]0,1[$ and any n in $\mathbb{N}$ that

$$\mu = D_{c^n}\mu \boxplus D_{c^{n-1}}\mu_c \boxplus D_{c^{n-2}}\mu_c \boxplus \cdots \boxplus D_c\mu_c \boxplus \mu_c. \tag{4.16}$$

The idea now is to show that for a suitable choice of $c = c_n$, the probability measures:

$$D_{c_n^n}\mu, D_{c_n^{n-1}}\mu_{c_n}, D_{c_n^{n-2}}\mu_{c_n}, \dots, D_{c_n}\mu_{c_n}, \mu_{c_n}, \quad (n \in \mathbb{N}), \tag{4.17}$$

form a null-array (cf. Theorem 4.18). Note for this, that for any choice of c_n in $]0,1[$, we have that

$$D_{c_n^j}\mu_{c_n}(\mathbb{R} \setminus [-\epsilon,\epsilon]) \leq \mu_{c_n}(\mathbb{R} \setminus [-\epsilon,\epsilon]),$$

for any j in $\mathbb{N}$ and any ϵ in $]0,\infty[$. Therefore, in order that the probability measures in (4.17) form a null-array, it suffices to choose c_n in such a way that

$$D_{c_n^n}\mu \xrightarrow{\text{w}} \delta_0 \quad \text{and} \quad \mu_{c_n} \xrightarrow{\text{w}} \delta_0, \quad \text{as } n \to \infty.$$

We claim that this will be the case if we put (for example)

$$c_n = \mathrm{e}^{-\frac{1}{\sqrt{n}}}, \quad (n \in \mathbb{N}). \tag{4.18}$$

To see this, note that with the above choice of c_n, we have:

$$c_n \to 1 \quad \text{and} \quad c_n^n \to 0, \quad \text{as } n \to \infty.$$

Thus, it follows immediately from Lemma 4.24, that $\mu_{c_n} \xrightarrow{\text{w}} \delta_0$, as $n \to \infty$. Moreover, if we choose a (classical) real valued random variable X with distribution μ, then, for each n, $D_{c_n^n}\mu$ is the distribution of $c_n^n X$. Now, $c_n^n X \to 0$, *almost surely*, as $n \to \infty$, and this implies that $c_n^n X \to 0$, *in distribution*, as $n \to \infty$.

We have verified, that if we choose c_n according to (4.18), then the probability measures in (4.17) form a null-array. Hence by (4.16) (with $c = c_n$) and Theorem 4.18, μ is $\boxplus$-infinitely divisible. □

Proposition 4.26. *Let μ be a $\boxplus$-selfdecomposable probability measure on $\mathbb{R}$, let c be a number in $]0, 1[$ and let μ_c be the probability measure on $\mathbb{R}$ satisfying the condition:*

$$\mu = D_c\mu \boxplus \mu_c.$$

Then μ_c is $\boxplus$-infinitely divisible.

Proof. As noted in the proof of Theorem 4.25, for any d in $]0, 1[$ and any n in $\mathbb{N}$ we have

$$\mu = D_{d^n}\mu \boxplus D_{d^{n-1}}\mu_d \boxplus D_{d^{n-2}}\mu_d \boxplus \cdots \boxplus D_d\mu_d \boxplus \mu_d,$$

where μ_d is defined by the case $n = 1$. Using now the above equation with $d = c^{1/n}$, we get for each n in $\mathbb{N}$ that

$$D_c\mu \boxplus \mu_c = \mu = D_c\mu \boxplus D_{c^{(n-1)/n}}\mu_{c^{1/n}} \boxplus D_{c^{(n-2)/n}}\mu_{c^{1/n}} \boxplus \cdots \boxplus D_{c^{1/n}}\mu_{c^{1/n}} \boxplus \mu_{c^{1/n}}. \tag{4.19}$$

From this it follows that

$$\mu_c = D_{c^{(n-1)/n}}\mu_{c^{1/n}} \boxplus D_{c^{(n-2)/n}}\mu_{c^{1/n}} \boxplus \cdots \boxplus D_{c^{1/n}}\mu_{c^{1/n}} \boxplus \mu_{c^{1/n}}, \quad (n \in \mathbb{N}). \tag{4.20}$$

Indeed, by taking Voiculescu transforms in (4.19) and using Theorem 4.9, it follows that the Voiculescu transforms of the right and left hand sides of (4.20) coincide on some region $\Gamma_{\eta,M}$. By Remark 4.10, this implies the validity of (4.20).

By (4.20) and Theorem 4.18, it remains now to show that the probability measures:

$$D_{c^{(n-1)/n}}\mu_{c^{1/n}}, D_{c^{(n-2)/n}}\mu_{c^{1/n}}, \ldots, D_{c^{1/n}}\mu_{c^{1/n}}, \mu_{c^{1/n}},$$

form a null-array. Since $c^{j/n} \in]0, 1[$ for any j in $\{1, 2, \ldots, n-1\}$, this is the case if and only if $\mu_{c^{1/n}} \xrightarrow{\text{w}} \delta_0$, as $n \to \infty$. But since $c^{1/n} \to 1$, as $n \to \infty$, Lemma 4.24 guarantees the validity of the latter assertion. □

4.7 Free Lévy Processes

Let $(\mathcal{A}, \tau)$ be a W^*-probability space acting on a Hilbert space $\mathcal{H}$ (see Section 4.1 and the Appendix). By a (stochastic) process affiliated with $\mathcal{A}$, we shall simply mean a family $(Z_t)_{t\in[0,\infty[}$ of *selfadjoint* operators in $\overline{\mathcal{A}}$, which is indexed by the non-negative reals. For such a process (Z_t), we let μ_t denote the (spectral) distribution of Z_t, i.e. $\mu_t = L\{Z_t\}$. We refer to the family

(μ_t) of probability measures on $\mathbb{R}$ as the family of *marginal distributions* of (Z_t). Moreover, if $s,t \in [0,\infty[$, such that $s < t$, then $Z_t - Z_s$ is again a selfadjoint operator in $\overline{\mathcal{A}}$ (see the Appendix), and we may consider its distribution $\mu_{s,t} = L\{Z_t - Z_s\}$. We refer to the family $(\mu_{s,t})_{0\le s<t}$ as the family of *increment distributions* of (Z_t).

Definition 4.27. *A free Lévy process (in law), affiliated with a W^*-probability space $(\mathcal{A},\tau)$, is a process $(Z_t)_{t\ge 0}$ of selfadjoint operators in $\overline{\mathcal{A}}$, which satisfies the following conditions:*

(i) *whenever $n \in \mathbb{N}$ and $0 \le t_0 < t_1 < \cdots < t_n$, the increments*

$$Z_{t_0}, Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \dots, Z_{t_n} - Z_{t_{n-1}},$$

are freely independent random variables.

(ii) $Z_0 = 0$.

(iii) *for any s,t in $[0,\infty[$, the (spectral) distribution of $Z_{s+t} - Z_s$ does not depend on s.*

(iv) *for any s in $[0,\infty[$, $Z_{s+t} - Z_s \to 0$ in distribution, as $t \to 0$, i.e. the spectral distributions $L\{Z_{s+t} - Z_s\}$ converge weakly to δ_0, as $t \to 0$.*

Note that under the assumption of (ii) and (iii) in the definition above, condition (iv) is equivalent to saying that $Z_t \to 0$ in distribution, as $t \searrow 0$.

Remark 4.28. **(Free additive processes I)** A process (Z_t) of selfadjoint operators in $\overline{\mathcal{A}}$, which satisfies conditions (i), (ii) and (iv) of Definition 4.27, is called a *free additive process (in law)*. Given such a process (Z_t), let, as above, $\mu_s = L\{Z_s\}$ and $\mu_{s,t} = L\{Z_t - Z_s\}$, whenever $0 \le s < t$. It follows then that whenever $0 \le r < s < t$, we have

$$\mu_s = \mu_r \boxplus \mu_{r,s} \quad \text{and} \quad \mu_{r,t} = \mu_{r,s} \boxplus \mu_{s,t}, \tag{4.21}$$

and furthermore

$$\mu_{s+t,s} \xrightarrow{\text{w}} \delta_0, \quad \text{as} \quad t \to 0, \tag{4.22}$$

for any s in $[0,\infty[$.

Conversely, given any family $\{\mu_t \mid t \ge 0\} \cup \{\mu_{s,t} \mid 0 \le s < t\}$ of probability measures on $\mathbb{R}$, such that (4.21) and (4.22) are satisfied, there exists a free additive process (in law) (Z_t) affiliated with a W^*-probability space $(\mathcal{A},\tau)$, such that $\mu_s = L\{Z_s\}$ and $\mu_{s,t} = L\{Z_t - Z_s\}$, whenever $0 \le s < t$. In fact, for any families (μ_t) and $(\mu_{s,t})$ satisfying condition (4.21), there exists a process (Z_t) affiliated with some W^*-probability space $(\mathcal{A},\tau)$, such that conditions (i) and (ii) in Definition 4.27 are satisfied, and such that $\mu_s = L\{Z_s\}$ and $\mu_{s,t} = L\{Z_t - Z_s\}$. This was noted in [Bi98] and [Vo98] (see also Remark 6.29 below). Note that with the notation introduced above, the free Lévy processes (in law) are exactly those free additive processes (in law), for which $\mu_{s,t} = \mu_{t-s}$ for all s,t such that $0 \le s < t$. In this case the condition (4.21) simplifies to

$$\mu_t = \mu_s \boxplus \mu_{t-s}, \quad (0 \le s < t). \tag{4.23}$$

In particular, for any family (μ_t) of probability measures on $\mathbb{R}$, such that (4.23) is satisfied, and such that $\mu_t \xrightarrow{\text{w}} \delta_0$ as $t \searrow 0$, there exists a free Lévy process (in law) (Z_t), such that $\mu_t = L\{Z_t\}$ for all t.

Consider now a free Lévy process $(Z_t)_{t\ge 0}$, with marginal distributions (μ_t). As for (classical) Lévy processes, it follows then, that each μ_t is necessarily $\boxplus$-infinitely divisible. Indeed, for any n in $\mathbb{N}$ we have:

$$Z_t = \sum_{j=1}^{n} (Z_{jt/n} - Z_{(j-1)t/n}),$$

and thus, in view of conditions (i) and (iii) in Definition 4.27,

$$\mu_t = \mu_{t/n} \boxplus \cdots \boxplus \mu_{t/n} \quad (n \text{ terms}).$$

5 Connections between Free and Classical Infinite Divisibility

An important connection between free and classical infinite divisibility was established by Bercovici and Pata, in the form of a bijection Λ from the class of classical infinitely divisible laws to the class of free infinitely divisible laws. The mapping Υ of Section 3.2 embodies a direct version of the Bercovici-Pata bijection and shows rather surprisingly that, in a sense, the class of free infinitely divisible laws corresponds to a regular subset of the class of all classical infinitely divisible laws. The mapping Λ also give rise to a direct connection between the classical and the free Lévy processes, as discussed at the end of the section.

5.1 The Bercovici-Pata Bijection Λ

The bijection to be defined next was introduced by Bercovici and Pata in [BePa99].

Definition 5.1. *By the Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ we denote the mapping defined as follows: Let μ be a measure in $\mathcal{ID}(*)$, and consider its generating pair (γ, σ) (see formula (2.1)). Then $\Lambda(\mu)$ is the measure in $\mathcal{ID}(\boxplus)$ that has (γ, σ) as free generating pair (see Definition 4.15).*

Since the $*$-infinitely divisible (respectively $\boxplus$-infinitely divisible) probability measures on $\mathbb{R}$ are exactly those measures that have a (unique) Lévy-Khintchine representation (respectively free Lévy-Khintchine representation), it follows immediately that Λ is a (well-defined) bijection between $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$. In terms of characteristic triplets, the Bercovici-Pata bijection may be characterized as follows.

Proposition 5.2. *If μ is a measure in $\mathcal{ID}(*)$ with (classical) characteristic triplet (a, ρ, η), then $\Lambda(\mu)$ has free characteristic triplet (a, ρ, η) (cf. Proposition 4.16).*

Proof. Suppose $\mu \in \mathcal{ID}(*)$ with generating pair (γ, σ) and characteristic triplet (a, ρ, η), the relationship between which is given by (2.3). Then, by definition of Λ, $\Lambda(\mu)$ has free generating pair (γ, σ), and the calculations in the proof of Proposition 4.16 (with ν replaced by $\Lambda(\mu)$) show that $\Lambda(\mu)$ has free characteristic triplet (a, ρ, η). □

Example 5.3. (a) Let μ be the standard Gaussian distribution, i.e.

$$\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}x^2)\,\mathrm{d}x.$$

Then $\Lambda(\mu)$ is the semi-circle distribution, i.e.

$$\Lambda(\mu)(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4 - x^2} \cdot 1_{[-2,2]}(x)\,\mathrm{d}x.$$

(b) Let μ be the classical Poisson distribution $\mathrm{Poiss}^*(\lambda)$ with mean $\lambda > 0$, i.e.

$$\mu(\{n\}) = \mathrm{e}^{-\lambda}\frac{\lambda^n}{n!}, \qquad (n \in \mathbb{N}_0).$$

Then $\Lambda(\mu)$ is the free Poisson distribution $\mathrm{Poiss}^{\boxplus}(\lambda)$ with mean λ, i.e.

$$\Lambda(\mu)(\mathrm{d}x) = \begin{cases} (1-\lambda)\delta_0 + \frac{1}{2\pi x}\sqrt{(x-a)(b-x)} \cdot 1_{[a,b]}(x)\,\mathrm{d}x, & \text{if } 0 \le \lambda \le 1, \\ \frac{1}{2\pi x}\sqrt{(x-a)(b-x)} \cdot 1_{[a,b]}(x)\,\mathrm{d}x, & \text{if } \lambda > 1, \end{cases}$$

where $a = (1 - \sqrt{\lambda})^2$ and $b = (1 + \sqrt{\lambda})^2$.

Remark 5.4 (Cumulants II). Let μ be a compactly supported probability measure in $\mathcal{ID}(*)$, and consider its sequence (c_n) of classical cumulants (cf. Remark 4.13). Then the Bercovici-Pata bijection Λ may also be defined as the mapping that sends μ to the probability measure on $\mathbb{R}$ with free cumulants (c_n). In other words, the free cumulants for $\Lambda(\mu)$ are the classical cumulants for μ. This fact was noted by M. Anshelevich in [An01, Lemma 6.5]. In view of the theory of free cumulants for several variables (cf. Remark 4.13), this point of view might be used to generalize the Bercovici-Pata bijection to multidimensional probability measures.

5.2 Connection between Υ and Λ

The starting point of this section is the following observation that links the Bercovici-Pata bijection Λ to the Υ-transformation of Section 3.

Theorem 5.5. *For any $\mu \in \mathcal{ID}(*)$ we have*

$$C_{\Upsilon(\mu)}(\zeta) = \mathcal{C}_{\Lambda(\mu)}(\mathrm{i}\zeta) = \int_0^\infty C_\mu(\zeta x)\mathrm{e}^{-x}\,\mathrm{d}x, \qquad (\zeta \in \,]-\infty, 0[). \tag{5.1}$$

Proof. These identities follow immediately by combining Proposition 5.2, Proposition 4.16, Theorem 3.16 and Theorem 3.17. □

Remark 5.6. Theorem 5.5 shows, in particular, that any free cumulant function of an element in $\mathcal{ID}(\boxplus)$ is, in fact, identical to a classical cumulant function of an element of $\mathcal{ID}(*)$. The second equality in (5.1) provides an alternative, more direct, way of passing from the measure μ to its free counterpart, $\Lambda(\mu)$, without passing through the Lévy-Khintchine representations. This way is often quite effective, when it comes to calculating $\Lambda(\mu)$ for specific examples of μ. Taking Theorem 3.43 into account, we note that for any measure μ in $\mathcal{ID}(*)$, the free cumulant transform of the measure $\Lambda(\mu)$ is equal to the classical cumulant transform of the stochastic integral $\int_0^1 -\log(1-t)\,\mathrm{d}X_t$, where (X_t) is a classical Lévy process (in law), such that $L\{X_1\} = \mu$.

In analogy with the proof of Proposition 3.38, The second equality in (5.1) provides an easy proof of the following algebraic properties of Λ:

Theorem 5.7. *The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$, has the following (algebraic) properties:*

(i) *If $\mu_1, \mu_2 \in \mathcal{ID}(*)$, then $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$.*
(ii) *If $\mu \in \mathcal{ID}(*)$ and $c \in \mathbb{R}$, then $\Lambda(D_c\mu) = D_c\Lambda(\mu)$.*
(iii) *For any constant c in $\mathbb{R}$, we have $\Lambda(\delta_c) = \delta_c$.*

Proof. The proof is similar to that of Proposition 3.38. Indeed, property (ii), say, may be proved as follows: For μ in $\mathcal{ID}(*)$ and ζ in $]-\infty, 0[$, we have

$$\begin{aligned}\mathcal{C}_{\Lambda(D_c\mu)}(\mathrm{i}\zeta) &= \int_{\mathbb{R}} C_{D_c\mu}(\zeta x)\mathrm{e}^{-x}\,\mathrm{d}x = \int_{\mathbb{R}} C_\mu(c\zeta x)\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \mathcal{C}_{\Lambda(\mu)}(\mathrm{i}c\zeta) = \mathcal{C}_{D_c\Lambda(\mu)}(\mathrm{i}\zeta),\end{aligned}$$

and the result then follows from uniqueness of analytic continuation. □

Corollary 5.8. *The bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is invariant under affine transformations, i.e. if $\mu \in \mathcal{ID}(*)$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ is an affine transformation, then*

$$\Lambda(\psi(\mu)) = \psi(\Lambda(\mu)).$$

Proof. Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be an affine transformation, i.e. $\psi(t) = ct + d$, $(t \in \mathbb{R})$, for some constants c, d in $\mathbb{R}$. Then for a probability measure μ on $\mathbb{R}$, $\psi(\mu) = D_c\mu * \delta_d$, and also $\psi(\mu) = D_c\mu \boxplus \delta_d$. Assume now that $\mu \in \mathcal{ID}(*)$. Then by Theorem 5.7,

$$\Lambda(\psi(\mu)) = \Lambda(D_c\mu * \delta_d) = D_c\Lambda(\mu) \boxplus \Lambda(\delta_d) = D_c\Lambda(\mu) \boxplus \delta_d = \psi(\Lambda(\mu)),$$

as desired. □

As a consequence of the corollary above, we get a short proof of the following result, which was proved by Bercovici and Pata in [BePa99].

Corollary 5.9 ([BePa99]). *The bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ maps the $*$-stable probability measures on $\mathbb{R}$ onto the $\boxplus$-stable probability measures on $\mathbb{R}$.*

Proof. Assume that μ is a $*$-stable probability measure on $\mathbb{R}$, and let $\psi_1, \psi_2\colon \mathbb{R} \to \mathbb{R}$ be increasing affine transformations on $\mathbb{R}$. Then $\psi_1(\mu) * \psi_2(\mu) = \psi_3(\mu)$, for yet another increasing affine transformation $\psi_3\colon \mathbb{R} \to \mathbb{R}$. Now by Corollary 5.8 and Theorem 5.7(i),

$$\begin{aligned}\psi_1(\Lambda(\mu)) \boxplus \psi_2(\Lambda(\mu)) &= \Lambda(\psi_1(\mu)) \boxplus \Lambda(\psi_2(\mu)) = \Lambda(\psi_1(\mu) * \psi_2(\mu))\\ &= \Lambda(\psi_3(\mu)) = \psi_3(\Lambda(\mu)),\end{aligned}$$

which shows that $\Lambda(\mu)$ is $\boxplus$-stable.

The same line of argument shows that μ is $*$-stable, if $\Lambda(\mu)$ is $\boxplus$-stable. □

Corollary 5.10. *Let μ be a $*$-selfdecomposable probability measure on $\mathbb{R}$ and let $(\mu_c)_{c\in]0,1[}$ be the family of probability measures on $\mathbb{R}$ defined by the equation:*

$$\mu = D_c\mu * \mu_c.$$

Then, for any c in $]0,1[$, we have the decomposition:

$$\Lambda(\mu) = D_c\Lambda(\mu) \boxplus \Lambda(\mu_c). \tag{5.2}$$

Consequently, a probability measure μ on $\mathbb{R}$ is $*$-selfdecomposable, if and only if $\Lambda(\mu)$ is $\boxplus$-selfdecomposable, and thus the bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ maps the class $\mathcal{L}(*)$ of $*$-selfdecomposable probability measures onto the class $\mathcal{L}(\boxplus)$ of $\boxplus$-selfdecomposable probability measures.

Proof. For any c in $]0,1[$, the measures $D_c\mu$ and μ_c are both $*$-infinitely divisible (see Section 2.5), and hence, by (i) and (ii) of Theorem 5.7,

$$\Lambda(\mu) = \Lambda(D_c\mu * \mu_c) = D_c\Lambda(\mu) \boxplus \Lambda(\mu_c).$$

Since this holds for all c in $]0,1[$, it follows that $\Lambda(\mu)$ is $\boxplus$-selfdecomposable.

Assume conversely that μ' is a $\boxplus$-selfdecomposable probability measure on $\mathbb{R}$, and let $(\mu'_c)_{c\in]0,1[}$ be the family of probability measures on $\mathbb{R}$ defined by:

$$\mu' = D_c\mu' \boxplus \mu'_c.$$

By Theorem 4.25 and Proposition 4.26, $\mu', \mu'_c \in \mathcal{ID}(\boxplus)$, so we may consider the $*$-infinitely divisible probability measures $\mu := \Lambda^{-1}(\mu')$ and $\mu_c := \Lambda^{-1}(\mu'_c)$. Then by (i) and (ii) of Theorem 5.7,

$$\begin{aligned}\mu = \Lambda^{-1}(\mu') &= \Lambda^{-1}(D_c(\mu') \boxplus \mu'_c) = \Lambda^{-1}(D_c\Lambda(\mu) \boxplus \Lambda(\mu_c))\\ &= \Lambda^{-1}(\Lambda(D_c\mu * \mu_c)) = D_c\mu * \mu_c.\end{aligned}$$

Since this holds for any c in $]0,1[$, μ is $*$-selfdecomposable. □

To summarize, we note that the Bercovici-Pata bijection Λ maps each of the classes $\mathcal{G}(*), \mathcal{S}(*), \mathcal{L}(*), \mathcal{ID}(*)$ in the hierarchy (2.13) onto the corresponding free class in (4.9).

Remark 5.11. Above we have discussed the free analogues of the classical stable and selfdecomposable laws, defining the free versions via free convolution properties. Alternatively, one may define the classes of free stable and free selfdecomposable laws in terms of monotonicity properties of the associated Lévy measures, simply using the same characterizations as those holding in the classical case, see Section 2.5. The same approach leads to free analogues $\mathcal{R}(\boxplus)$, $\mathcal{T}(\boxplus)$ and $\mathcal{B}(\boxplus)$ of the classes $\mathcal{R}(*)$, $\mathcal{T}(*)$ and $\mathcal{B}(*)$. We shall however not study these latter analogues here.

Remark 5.12. We end this section by mentioning the possible connection between the mapping Υ^{α}, introduced in Section 3.4, and the notion of α-probability theory (usually denoted q-deformed probability). For each q in $[-1,1]$, the so called q-deformed probability theory has been developed by a number of authors (see e.g. [BoSp91] and [Ni95]). For $q=0$, this corresponds to Voiculescu's free probability and for $q=1$ to classical probability. Since the right hand side of (3.60) interpolates correspondingly between the free and classical Lévy-Khintchine representations, one may speculate whether the right hand side of (3.60) (for $\alpha=q$) might be interpreted as a kind of Lévy-Khintchine representation for the q-analogue of the cumulant transform (see [Ni95]).

5.3 Topological Properties of Λ

In this section, we study some topological properties of Λ. The key result is the following theorem, which is the free analogue of a result due to B.V. Gnedenko (cf. [GnKo68, §19, Theorem 1]).

Theorem 5.13. *Let μ be a measure in $\mathcal{ID}(\boxplus)$, and let (μ_n) be a sequence of measures in $\mathcal{ID}(\boxplus)$. For each n, let (γ_n, σ_n) be the free generating pair for μ_n, and let (γ, σ) be the free generating pair for μ. Then the following two conditions are equivalent:*

(i) $\mu_n \xrightarrow{\text{w}} \mu$, *as* $n \to \infty$.
(ii) $\gamma_n \to \gamma$ *and* $\sigma_n \xrightarrow{\text{w}} \sigma$, *as* $n \to \infty$.

Proof. (ii) $\Rightarrow$ (i): Assume that (ii) holds. By Theorem 4.12 it is sufficient to show that

(a) $\phi_{\mu_n}(\mathrm{i}y) \to \phi(\mathrm{i}y)$, as $n \to \infty$, for all y in $]0,\infty[$.
(b) $\displaystyle\sup_{n\in\mathbb{N}} \left|\frac{\phi_{\mu_n}(\mathrm{i}y)}{y}\right| \to 0$, as $y \to \infty$.

Regarding (a), note that for any y in $]0,\infty[$, the function $t \mapsto \frac{1+t\mathrm{i}y}{\mathrm{i}y-t}$, $t \in \mathbb{R}$, is continuous and bounded. Therefore, by the assumptions in (ii),

$$\phi_{\mu_n}(\mathrm{i}y) = \gamma_n + \int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{\mathrm{i}y-t}\ \sigma_n(\mathrm{d}t) \underset{n\to\infty}{\longrightarrow} \gamma + \int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{\mathrm{i}y-t}\ \sigma(\mathrm{d}t) = \phi_\mu(\mathrm{i}y).$$

Turning then to (b), note that for n in $\mathbb{N}$ and y in $]0,\infty[$,

$$\frac{\phi_{\mu_n}(\mathrm{i}y)}{y} = \frac{\gamma_n}{y} + \int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{y(\mathrm{i}y-t)}\ \sigma_n(\mathrm{d}t).$$

Since the sequence (γ_n) is, in particular, bounded, it suffices thus to show that

$$\sup_{n\in\mathbb{N}} \Big| \int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{y(\mathrm{i}y-t)}\ \sigma_n(\mathrm{d}t) \Big| \to 0, \quad \text{as } y \to \infty. \tag{5.3}$$

For this, note first that since $\sigma_n \xrightarrow{\mathrm{w}} \sigma$, as $n \to \infty$, and since $\sigma(\mathbb{R}) < \infty$, it follows by standard techniques that the family $\{\sigma_n \mid n \in \mathbb{N}\}$ is tight (cf. [Br92, Corollary 8.11]).

Note next, that for any t in $\mathbb{R}$ and any y in $]0,\infty[$,

$$\Big| \frac{1+t\mathrm{i}y}{y(\mathrm{i}y-t)} \Big| \le \frac{1}{y(y^2+t^2)^{1/2}} + \frac{|t|}{(y^2+t^2)^{1/2}}.$$

From this estimate it follows that

$$\sup_{y\in[1,\infty[,t\in\mathbb{R}} \Big| \frac{1+t\mathrm{i}y}{y(\mathrm{i}y-t)} \Big| \le 2,$$

and that for any N in $\mathbb{N}$ and y in $[1,\infty[$,

$$\sup_{t\in[-N,N]} \Big| \frac{1+t\mathrm{i}y}{y(\mathrm{i}y-t)} \Big| \le \frac{N+1}{y}.$$

From the two estimates above, it follows that for any N in $\mathbb{N}$, and any y in $[1,\infty[$, we have

$$\begin{aligned}\sup_{n\in\mathbb{N}} \Big| \int_{\mathbb{R}} \frac{1+t\mathrm{i}y}{y(\mathrm{i}y-t)}\ \sigma_n(\mathrm{d}t) \Big| &\le \frac{N+1}{y} \sup_{n\in\mathbb{N}} \sigma_n([-N,N]) + 2 \cdot \sup_{n\in\mathbb{N}} \sigma_n([-N,N]^c) \\ &\le \frac{N+1}{y} \sup_{n\in\mathbb{N}} \sigma_n(\mathbb{R}) + 2 \cdot \sup_{n\in\mathbb{N}} \sigma_n([-N,N]^c).\end{aligned} \tag{5.4}$$

Now, given ϵ in $]0,\infty[$ we may, since $\{\sigma_n \mid n \in \mathbb{N}\}$ is tight, choose N in $\mathbb{N}$, such that $\sup_{n\in\mathbb{N}} \sigma_n([-N,N]^c) \le \frac{\epsilon}{4}$. Moreover, since $\sigma_n \xrightarrow{\mathrm{w}} \sigma$ and $\sigma(\mathbb{R}) < \infty$, the sequence $\{\sigma_n(\mathbb{R}) \mid n \in \mathbb{N}\}$ is, in particular, bounded, and hence, for the chosen

N, we may subsequently choose y_0 in $[1,\infty[$, such that $\frac{N+1}{y_0}\sup_{n\in\mathbb{N}}\sigma_n(\mathbb{R})\le\frac{\epsilon}{2}$. Using then the estimate in (5.4), it follows that

$$\sup_{n\in\mathbb{N}}\Big|\int_{\mathbb{R}}\frac{1+tiy}{y(iy-t)}\;\sigma_n(dt)\Big|\le\epsilon,$$

whenever $y\ge y_0$. This verifies (5.3).

(i) $\Rightarrow$ (ii): Suppose that $\mu_n\xrightarrow{w}\mu$, as $n\to\infty$. Then by Theorem 4.12, there exists a number M in $]0,\infty[$, such that

(c) $\forall y\in[M,\infty[\colon\ \phi_{\mu_n}(iy)\to\phi_\mu(iy)$, as $n\to\infty$.

(d) $\displaystyle\sup_{n\in\mathbb{N}}\Big|\frac{\phi_{\mu_n}(iy)}{y}\Big|\to 0$, as $y\to\infty$.

We show first that the family $\{\sigma_n\mid n\in\mathbb{N}\}$ is conditionally compact w.r.t. weak convergence, i.e. that any subsequence $(\sigma_{n'})$ has a subsequence $(\sigma_{n''})$, which converges weakly to some finite measure σ^* on $\mathbb{R}$. By [GnKo68, §9, Theorem 3 bis], it suffices, for this, to show that $\{\sigma_n\mid n\in\mathbb{N}\}$ is tight, and that $\{\sigma_n(\mathbb{R})\mid n\in\mathbb{N}\}$ is bounded. The key step in the argument is the following observation: For any n in $\mathbb{N}$ and any y in $]0,\infty[$, we have,

$$\begin{aligned}-\mathrm{Im}\phi_{\mu_n}(iy)&=-\mathrm{Im}\Big(\gamma_n+\int_{\mathbb{R}}\frac{1+tiy}{iy-t}\;\sigma_n(dt)\Big)\\&=-\mathrm{Im}\Big(\int_{\mathbb{R}}\frac{1+tiy}{iy-t}\;\sigma_n(dt)\Big)=y\int_{\mathbb{R}}\frac{1+t^2}{y^2+t^2}\;\sigma_n(dt).\end{aligned}\tag{5.5}$$

We show now that $\{\sigma_n\mid n\in\mathbb{N}\}$ is tight. For fixed y in $]0,\infty[$, note that

$$\{t\in\mathbb{R}\mid |t|\ge y\}\subseteq\big\{t\in\mathbb{R}\mid \tfrac{1+t^2}{y^2+t^2}\ge\tfrac12\big\},$$

so that, for any n in $\mathbb{N}$,

$$\sigma_n(\{t\in\mathbb{R}\mid |t|\ge y\})\le 2\int_{\mathbb{R}}\frac{1+t^2}{y^2+t^2}\;\sigma_n(dt)=-2\mathrm{Im}\Big(\frac{\phi_{\mu_n}(iy)}{y}\Big)\le 2\Big|\frac{\phi_{\mu_n}(iy)}{y}\Big|.$$

Combining this estimate with (d), it follows immediately that $\{\sigma_n\mid n\in\mathbb{N}\}$ is tight.

We show next that the sequence $\{\sigma_n(\mathbb{R})\mid n\in\mathbb{N}\}$ is bounded. For this, note first that with M as in (c), there exists a constant c in $]0,\infty[$, such that

$$c\le\frac{M(1+t^2)}{M^2+t^2},\quad\text{for all } t \text{ in } \mathbb{R}.$$

It follows then, by (5.5), that for any n in $\mathbb{N}$,

$$c\sigma_n(\mathbb{R})\le\int_{\mathbb{R}}\frac{M(1+t^2)}{M^2+t^2}\;\sigma_n(dt)=-\mathrm{Im}\phi_{\mu_n}(iM),$$

and therefore by (c),

$$\limsup_{n\to\infty} \sigma_n(\mathbb{R}) \le \limsup_{n\to\infty} \big\{ -c^{-1} \cdot \mathrm{Im}\phi_{\mu_n}(iM) \big\} = -c^{-1} \cdot \mathrm{Im}\phi_{\mu}(iM) < \infty,$$

which shows that $\{\sigma_n(\mathbb{R}) \mid n \in \mathbb{N}\}$ is bounded.

Having established that the family $\{\sigma_n \mid n \in \mathbb{N}\}$ is conditionally compact, recall next from Remark 2.3, that in order to show that $\sigma_n \xrightarrow{\mathrm{w}} \sigma$, it suffices to show that any subsequence $(\sigma_{n'})$ has a subsequence, which converges weakly to σ. A similar argument works, of course, to show that $\gamma_n \to \gamma$. So consider any subsequence $(\gamma_{n'}, \sigma_{n'})$ of the sequence of generating pairs. Since $\{\sigma_n \mid n \in \mathbb{N}\}$ is conditionally compact, there is a subsequence (n'') of (n'), such that the sequence $(\sigma_{n''})$ is weakly convergent to some finite measure σ^* on $\mathbb{R}$. Since the function $t \mapsto \frac{1+tiy}{\mathrm{i}y-t}$ is continuous and bounded for any y in $]0,\infty[$, we know then that

$$\int_{\mathbb{R}} \frac{1+tiy}{\mathrm{i}y-t}\,\sigma_{n''}(\mathrm{d}t) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} \frac{1+tiy}{\mathrm{i}y-t}\,\sigma^*(\mathrm{d}t),$$

for any y in $]0,\infty[$. At the same time, we know from (c) that

$$\gamma_{n''} + \int_{\mathbb{R}} \frac{1+tiy}{\mathrm{i}y-t}\,\sigma_{n''}(\mathrm{d}t) = \phi_{\mu_{n''}}(\mathrm{i}y) \xrightarrow[n\to\infty]{} \phi_\mu(\mathrm{i}y) = \gamma + \int_{\mathbb{R}} \frac{1+tiy}{\mathrm{i}y-t}\,\sigma(\mathrm{d}t),$$

for any y in $[M,\infty[$. From these observations, it follows that the sequence $(\gamma_{n''})$ must converge to some real number γ^*, which then has to satisfy the identity:

$$\gamma^* + \int_{\mathbb{R}} \frac{1+tiy}{\mathrm{i}y-t}\,\sigma^*(\mathrm{d}t) = \phi_\mu(\mathrm{i}y) = \gamma + \int_{\mathbb{R}} \frac{1+tiy}{\mathrm{i}y-t}\,\sigma(\mathrm{d}t),$$

for all y in $[M,\infty[$. By uniqueness of the free Lévy-Khintchine representation (cf. Theorem 4.14) and uniqueness of analytic continuation, it follows that we must have $\sigma^* = \sigma$ and $\gamma^* = \gamma$. We have thus verified the existence of a subsequence $(\gamma_{n''}, \sigma_{n''})$ which converges (coordinate-wise) to (γ, σ), and that was our objective. □

As an immediate consequence of Theorem 5.13 and the corresponding result in classical probability, we get the following

Corollary 5.14. *The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is a homeomorphism w.r.t. weak convergence. In other words, if μ is a measure in $\mathcal{ID}(*)$ and (μ_n) is a sequence of measures in $\mathcal{ID}(*)$, then $\mu_n \xrightarrow{\mathrm{w}} \mu$, as $n \to \infty$, if and only if $\Lambda(\mu_n) \xrightarrow{\mathrm{w}} \Lambda(\mu)$, as $n \to \infty$.*

Proof. Let (γ, σ) be the generating pair for μ and, for each n, let (γ_n, σ_n) be the generating pair for μ_n.

Assume first that $\mu_n \xrightarrow{\mathrm{w}} \mu$. Then by [GnKo68, §19, Theorem 1], $\gamma_n \to \gamma$ and $\sigma_n \xrightarrow{\mathrm{w}} \sigma$. Since (γ_n, σ_n) (respectively (γ, σ)) is the free generating pair for $\Lambda(\mu_n)$ (respectively $\Lambda(\mu)$), it follows then from Theorem 5.13 that $\Lambda(\mu_n) \xrightarrow{\mathrm{w}} \Lambda(\mu)$.

The same argument applies to the converse implication. □

We end this section by presenting the announced proof of property (v) in Theorem 3.18. The proof follows easily by combining Theorem 5.5 and Theorem 5.13.

Proof of Theorem 3.18(v).

Let $\mu, \mu_1, \mu_2, \mu_3, \ldots,$ be probability measures in $\mathcal{ID}(*)$, such that $\mu_n \xrightarrow{\mathrm{w}} \mu$, as $n \to \infty$. We need to show that $\Upsilon(\mu_n) \xrightarrow{\mathrm{w}} \Upsilon(\mu)$ as $n \to \infty$. Since Λ is continuous w.r.t. weak convergence, $\Lambda(\mu_n) \xrightarrow{\mathrm{w}} \Lambda(\mu)$, as $n \to \infty$, and this implies that $\mathcal{C}_{\Lambda(\mu_n)}(\mathrm{i}\zeta) \to \mathcal{C}_{\Lambda(\mu)}(\mathrm{i}\zeta)$, as $n \to \infty$, for any ζ in $]-\infty, 0[$ (use e.g. Theorem 5.13). Thus,

$$C_{\Upsilon(\mu_n)}(\zeta) = \mathcal{C}_{\Lambda(\mu_n)}(\mathrm{i}\zeta) \underset{n\to\infty}{\longrightarrow} \mathcal{C}_{\Lambda(\mu)}(\mathrm{i}\zeta) = C_{\Upsilon(\mu)}(\zeta),$$

for any negative number ζ, and hence also $f_{\Upsilon(\mu_n)}(\zeta) = \exp(C_{\Upsilon(\mu_n)}(\zeta)) \to \exp(C_{\Upsilon(\mu)}(\zeta)) = f_{\Upsilon(\mu)}(\zeta)$, as $n \to \infty$, for such ζ. Applying now complex conjugation, it follows that $f_{\Upsilon(\mu_n)}(\zeta) \to f_{\Upsilon(\mu)}(\zeta)$, as $n \to \infty$, for any (non-zero) ζ, and this means that $\Upsilon(\mu_n) \xrightarrow{\mathrm{w}} \Upsilon(\mu)$, as $n \to \infty$. □

5.4 Classical vs. Free Lévy Processes

Consider now a free Lévy process $(Z_t)_{t\geq 0}$, with marginal distributions (μ_t). As for (classical) Lévy processes, it follows then, that each μ_t is necessarily $\boxplus$-infinitely divisible. Indeed, for any n in $\mathbb{N}$ we have: $Z_t = \sum_{j=1}^{n}(Z_{jt/n} - Z_{(j-1)t/n})$, and thus, in view of conditions (i) and (iii) in Definition 4.27, $\mu_t = \mu_{t/n} \boxplus \cdots \boxplus \mu_{t/n}$ (n terms). From the observation just made, it follows that the Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ gives rise to a correspondence between classical and free Lévy processes:

Proposition 5.15. *Let $(Z_t)_{t\geq 0}$ be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, and with marginal distributions (μ_t). Then there exists a (classical) Lévy process $(X_t)_{t\geq 0}$, with marginal distributions $(\Lambda^{-1}(\mu_t))$.*

Conversely, for any (classical) Lévy process (X_t) with marginal distributions (μ_t), there exists a free Lévy process (in law) (Z_t) with marginal distributions $(\Lambda(\mu_t))$.

Proof. Consider a free Lévy process (in law) (Z_t) with marginal distributions (μ_t). Then, as noted above, $\mu_t \in \mathcal{ID}(\boxplus)$ for all t, and hence we may define $\mu'_t = \Lambda^{-1}(\mu_t)$, $t \geq 0$. Then, whenever $0 \leq s < t$,

$$\mu'_t = \Lambda^{-1}(\mu_s \boxplus \mu_{t-s}) = \Lambda^{-1}(\mu_s) * \Lambda^{-1}(\mu_{t-s}) = \mu'_s * \mu'_{t-s}.$$

Hence, by the Kolmogorov Extension Theorem (cf. [Sa99, Theorem 1.8]), there exists a (classical) stochastic process (X_t) (defined on some probability space $(\Omega, \mathcal{F}, P)$), with marginal distributions (μ'_t), and which satisfies conditions (i)-(iii) of Definition 2.2. Regarding condition (iv), note that since (Z_t) is a free Lévy process, $\mu_t \xrightarrow{\mathrm{w}} \delta_0$ as $t \searrow 0$, and hence, by continuity of Λ^{-1} (cf. Corollary 5.14),

$$\mu_t' = \Lambda^{-1}(\mu_t) \xrightarrow{\mathrm{w}} \Lambda^{-1}(\delta_0) = \delta_0, \quad \text{as } t \searrow 0.$$

Thus, (X_t) is a (classical) Lévy process in law, and hence we can find a modification of (X_t) which is a genuine Lévy process.

The second statement of the proposition follows by a similar argument, using Λ rather than Λ^{-1}, and that the marginal distributions of a classical Lévy process are necessarily $*$-infinitely divisible. Furthermore, we have to call upon the existence statement for free Lévy processes (in law) in Remark 4.28. □

Example 5.16. The *free Brownian motion* is the free Lévy process (in law), $(W_t)_{t\geq 0}$, which corresponds to the classical Brownian motion, $(B_t)_{t\geq 0}$, via the correspondence described in Proposition 5.15. In particular (cf. Example 5.3),

$$L\{W_t\}(\mathrm{d}s) = \frac{1}{2\pi t}\sqrt{4t - s^2} \cdot 1_{[-\sqrt{4t},\sqrt{4t}]}(s)\,\mathrm{d}s, \qquad (t > 0).$$

Remark 5.17. **(Free additive processes II)** Though our main objectives in this section are free Lévy processes, we mention, for completeness, that the Bercovici-Pata bijection Λ also gives rise to a correspondence between classical and free additive processes (in law). Thus, to any classical additive process (in law), with corresponding marginal distributions (μ_t) and increment distributions $(\mu_{s,t})_{0\leq s<t}$, there corresponds a free additive process (in law), with marginal distributions $(\Lambda(\mu_t))$ and increment distributions $(\Lambda(\mu_{s,t}))_{0\leq s<t}$. And vice versa.

This follows by the same method as used in the proof of Proposition 5.15 above, once it has been established that for a free additive process (in law) (Z_t), the distributions $\mu_t = L\{Z_t\}$ and $\mu_{s,t} = L\{Z_t - Z_s\}$, $0 \leq s < t$, are necessarily $\boxplus$-infinitely divisible (for the corresponding classical result, see [Sa99, Theorem 9.1]). The key to this result is Theorem 4.18, together with the fact that (Z_t) is actually uniformly stochastically continuous on compact intervals, in the following sense: For any compact interval $[0, b]$ in $[0,\infty[$, and for any positive numbers ϵ, ρ, there exists a positive number δ such that $\mu_{s,t}(\mathbb{R} \setminus [-\epsilon, \epsilon]) < \rho$, for any s, t in $[0, b]$, for which $s < t < s + \delta$. As in the classical case, this follows from condition (iv) in Definition 4.27, by a standard compactness argument (see [Sa99, Lemma 9.6]). Now for any t in $[0,\infty[$ and any n in $\mathbb{N}$, we have (cf. (4.21)),

$$\mu_t = \mu_{0,t/n} \boxplus \mu_{t/n,2t/n} \boxplus \mu_{2t/n,3t/n} \boxplus \cdots \boxplus \mu_{(n-1)t/n,t}. \tag{5.6}$$

Since (Z_t) is uniformly stochastically continuous on $[0, t]$, it follows that the family $\{\mu_{(j-1)t/n,jt/n} \mid n \in \mathbb{N},\ 1 \leq j \leq n\}$ is a null-array, and hence, by Theorem 4.18, (5.6) implies that μ_t is $\boxplus$-infinitely divisible. Applying then this fact to the free additive process (in law) $(Z_t - Z_s)_{t \geq s}$, it follows that also $\mu_{s,t}$ is $\boxplus$-infinitely divisible whenever $0 \leq s < t$.

Remark 5.18. **(An alternative concept of free Lévy processes)** For a classical Lévy process (X_t), condition (iii) in Definition 2.2 is equivalent to the condition that whenever $0 \leq s < t$, the conditional distribution $\mathrm{Prob}(X_t \mid X_s)$ depends only on $t - s$. Conditional probabilities in free probability were studied by Biane in [Bi98], and he noted, in particular, that in the free case, the condition just stated *is not* equivalent to condition (iii) in Definition 4.27. Consequently, in free probability there are two classes of stochastic processes, that may naturally be called Lévy processes: The ones we defined in Definition 4.27 and the ones for which condition (iii) in Definition 4.27 is replaced by the condition on the conditional distributions, mentioned above. In [Bi98] these two types of processes were denoted FAL1 respectively FAL2. We should mention here that in [Bi98], the assumption of stochastic continuity (condition (iv) in Definition 4.27) was not included in the definitions of neither FAL1 nor FAL2. We have included that condition, primarily because it is crucial for the definition of the stochastic integral to be constructed in the next section.

6 Free Stochastic Integration

In the classical setting, stochastic integration with respect to Lévy processes and to Poisson random measures is of key importance. This Section establishes base elements of a similar theory of free stochastic integration. As applications, a representation of free selfdecomposable variates as stochastic integrals is given and free OU processes are introduced. Furthermore, the free Lévy-Itô decomposition is derived.

6.1 Stochastic Integrals w.r.t. free Lévy Processes

As mentioned in Section 2.3, if (X_t) is a classical Lévy process and $f\colon [A,B] \to \mathbb{R}$ is a continuous function defined on an interval $[A,B]$ in $[0,\infty[$, then the stochastic integral $\int_A^B f(t)\,\mathrm{d}X_t$ may be defined as the limit in probability of approximating Riemann sums. More precisely, for each n in $\mathbb{N}$, let $\mathcal{D}_n = \{t_{n,0}, t_{n,1}, \dots, t_{n,n}\}$ be a subdivision of $[A,B]$, i.e.

$$A = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = B.$$

Assume that

$$\lim_{n\to\infty} \max_{j=1,2,\dots,n} (t_{n,j} - t_{n,j-1}) = 0. \tag{6.1}$$

Moreover, for each n, choose intermediate points:

$$t_{n,j}^{\#} \in [t_{n,j-1}, t_{n,j}], \quad j = 1,2,\dots,n. \tag{6.2}$$

Then the Riemann sums

$$S_n = \sum_{j=1}^{n} f(t_{n,j}^{\#}) \cdot (X_{t_{n,j}} - X_{t_{n,j-1}}),$$

converge *in probability*, as $n \to \infty$, to a random variable S. Moreover, this random variable S does not depend on the choice of subdivisions $\mathcal{D}_n$ (satisfying (6.1)), nor on the choice of intermediate points $t_{n,j}^{\#}$. Hence, it makes sense to call S the stochastic integral of f over $[A, B]$ w.r.t. (X_t), and we denote S by $\int_A^B f(t)\, \mathrm{d}X_t$.

The construction just sketched depends, of course, heavily on the stochastic continuity of the Lévy process in law (X_t) (condition (iv) in Definition 2.2). A proof of the assertions made above can be found in [Lu75, Theorem 6.2.3]. We show next how the above construction carries over, via the Bercovici-Pata bijection, to a corresponding stochastic integral w.r.t. free Lévy processes (in law).

Theorem 6.1. *Let (Z_t) be a free Lévy process (in law), affiliated with a W^*-probability space $(\mathcal{A}, \tau)$. Then for any compact interval $[A, B]$ in $[0, \infty[$ and any continuous function $f\colon [A, B] \to \mathbb{R}$, the stochastic integral $\int_A^B f(t)\, \mathrm{d}Z_t$ exists as the limit in probability (see Definition 4.3) of approximating Riemann sums. More precisely, there exists a (unique) selfadjoint operator T affiliated with $(\mathcal{A}, \tau)$, such that for any sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of subdivisions of $[A, B]$, satisfying (6.1), and for any choice of intermediate points $t_{n,j}^{\#}$, as in (6.2), the corresponding Riemann sums*

$$T_n = \sum_{j=1}^{n} f(t_{n,j}^{\#}) \cdot (Z_{t_{n,j}} - Z_{t_{n,j-1}}),$$

converge in probability to T as $n \to \infty$. We call T the stochastic integral of f over $[A, B]$ w.r.t. (Z_t), and denote it by $\int_A^B f(t)\, \mathrm{d}Z_t$.

In the proof below, we shall use the notation:

$$*_{j=1}^{r} \mu_j := \mu_1 * \cdots * \mu_r \quad \text{and} \quad \boxplus_{j=1}^{r} \mu_j := \mu_1 \boxplus \cdots \boxplus \mu_r,$$

for probability measures $\mu_1, \ldots, \mu_r$ on $\mathbb{R}$.

Proof of Theorem 6.1. Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[A, B]$ satisfying (6.1), let $t_{n,j}^{\#}$ be a family of intermediate points as in (6.2), and consider, for each n, the corresponding Riemann sum:

$$T_n = \sum_{j=1}^{n} f(t_{n,j}^{\#}) \cdot (Z_{t_{n,j}} - Z_{t_{n,j-1}}) \in \overline{\mathcal{A}}.$$

We show that (T_n) is a Cauchy sequence w.r.t. convergence in probability or, equivalently, w.r.t. the measure topology (see the Appendix). Given any n, m in $\mathbb{N}$, we form the subdivision

$$A = s_0 < s_1 < \cdots < s_{p(n,m)} = B,$$

which consists of the points in $\mathcal{D}_n \cup \mathcal{D}_m$ (so that $p(n,m) \leq n+m$). Then, for each j in $\{1,2,\ldots,p(n,m)\}$, we choose (in the obvious way) $s_{n,j}^{\#}$ in $\{t_{n,k}^{\#} \mid k = 1,2,\ldots,n\}$ and $s_{m,j}^{\#}$ in $\{t_{m,k}^{\#} \mid k = 1,2,\ldots,m\}$ such that

$$T_n = \sum_{j=1}^{p(n,m)} f(s_{n,j}^{\#}) \cdot (Z_{s_j} - Z_{s_{j-1}}) \quad \text{and} \quad T_m = \sum_{j=1}^{p(n,m)} f(s_{m,j}^{\#}) \cdot (Z_{s_j} - Z_{s_{j-1}}).$$

It follows then that

$$T_n - T_m = \sum_{j=1}^{p(n,m)} \big(f(s_{n,j}^{\#}) - f(s_{m,j}^{\#})\big) \cdot (Z_{s_j} - Z_{s_{j-1}}).$$

Let (μ_t) denote the family of marginal distributions of (Z_t), and then consider a classical Lévy process (X_t) with marginal distributions $(\Lambda^{-1}(\mu_t))$ (cf. Proposition 5.15). For each n, form the Riemann sum

$$S_n = \sum_{j=1}^{n} f(t_{n,j}^{\#}) \cdot (X_{t_{n,j}} - X_{t_{n,j-1}}),$$

corresponding to the same $\mathcal{D}_n$ and $t_{n,j}^{\#}$ as above. Then for any n,m in $\mathbb{N}$, we have also that

$$S_n - S_m = \sum_{j=1}^{p(n,m)} \big(f(s_{n,j}^{\#}) - f(s_{m,j}^{\#})\big) \cdot (X_{s_j} - X_{s_{j-1}}).$$

From this expression, it follows that

$$\begin{aligned} L\{S_n - S_m\} &= *_{j=1}^{p(n,m)} D_{f(s_{n,j}^{\#}) - f(s_{m,j}^{\#})} L\{X_{s_j} - X_{s_{j-1}}\} \\ &= *_{j=1}^{p(n,m)} D_{f(s_{n,j}^{\#}) - f(s_{m,j}^{\#})} \Lambda^{-1}(\mu_{s_j - s_{j-1}}), \end{aligned}$$

so that (by Theorem 5.7),

$$\begin{aligned} \Lambda(L\{S_n - S_m\}) &= \boxplus_{j=1}^{p(n,m)} D_{f(s_{n,j}^{\#}) - f(s_{m,j}^{\#})} \mu_{s_j - s_{j-1}} \\ &= L\Big\{ \sum_{j=1}^{p(n,m)} \big(f(s_{n,j}^{\#}) - f(s_{m,j}^{\#})\big) \cdot (Z_{s_j} - Z_{s_{j-1}}) \Big\} \\ &= L\{T_n - T_m\}. \end{aligned}$$

We know from the classical theory (cf. [Lu75, Theorem 6.2.3]), that (S_n) is a Cauchy sequence w.r.t. convergence in probability, i.e. that $L\{S_n - S_m\} \xrightarrow{\text{w}} \delta_0$, as $n,m \to \infty$. By continuity of Λ, it follows thus that also

$$L\{T_n - T_m\} = \Lambda(L\{S_n - S_m\}) \xrightarrow{\text{w}} \Lambda(\delta_0) = \delta_0, \quad \text{as } n, m \to \infty.$$

By Proposition A.8, this means that (T_n) is a Cauchy sequence w.r.t. the measure topology, and since $\overline{\mathcal{A}}$ is complete in the measure topology (Proposition A.5), there exists an operator T in $\overline{\mathcal{A}}$, such that $T_n \to T$ in the measure topology, i.e. in probability. Since T_n is selfadjoint for each n (see the Appendix) and since the adjoint operation is continuous w.r.t. the measure topology (Proposition A.5), T is necessarily a selfadjoint operator.

It remains to show that the operator T, found above, does not depend on the choice of subdivisions $(\mathcal{D}_n)$ or intermediate points $t^{\#}_{n,j}$. Suppose thus that (T_n) and (T'_n) are two sequences of Riemann sums of the kind considered above. Then by the argument given above, there exist operators T and T' in $\overline{\mathcal{A}}$, such that $T_n \to T$ and $T'_n \to T'$ in probability. Furthermore, if we consider the "mixed sequence" $T_1, T'_2, T_3, T'_4, \ldots$, then the corresponding sequence of subdivisions also satisfies (6.1), and hence this mixed sequence also converges in probability to an operator T'' in $\overline{\mathcal{A}}$. Since the mixed sequence has subsequences converging, in probability, to T and T' respectively, and since the measure topology is a Hausdorff topology (cf. Proposition A.5), we may thus conclude that $T = T'' = T'$, as desired. □

The stochastic integral $\int_A^B f(t)\, \mathrm{d}Z_t$, introduced above, extends to continuous functions $f\colon [A,B] \to \mathbb{C}$ in the usual way (the result being non-selfadjoint in general). From the construction of $\int_A^B f(t)\, \mathrm{d}Z_t$ as the limit of approximating Riemann sums, it follows immediately that whenever $0 \le A < B < C$, we have

$$\int_A^C f(t)\, \mathrm{d}Z_t = \int_A^B f(t)\, \mathrm{d}Z_t + \int_B^C f(t)\, \mathrm{d}Z_t,$$

for any continuous function $f\colon [A,C] \to \mathbb{C}$. Another consequence of the construction, given in the proof above, is the following correspondence between stochastic integrals w.r.t. classical and free Lévy processes (in law).

Corollary 6.2. *Let (X_t) be a classical Lévy process with marginal distributions (μ_t), and let (Z_t) be a corresponding free Lévy process (in law) with marginal distributions $(\Lambda(\mu_t))$ (cf. Proposition 5.15). Then for any compact interval $[A,B]$ in $[0,\infty[$ and any continuous function $f\colon [A,B] \to \mathbb{R}$, the distributions $L\{\int_A^B f(t)\, \mathrm{d}X_t\}$ and $L\{\int_A^B f(t)\, \mathrm{d}Z_t\}$ are $*$-infinitely divisible respectively $\boxplus$-infinitely divisible and, moreover*

$$L\big\{\int_A^B f(t)\, \mathrm{d}Z_t\big\} = \Lambda\big[L\big\{\int_A^B f(t)\, \mathrm{d}X_t\big\}\big].$$

Proof. Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[A,B]$ satisfying (6.1), let $t^{\#}_{n,j}$ be a family of intermediate points as in (6.2), and consider, for each n, the corresponding Riemann sums:

$$S_n = \sum_{j=1}^{n} f(t^{\#}_{n,j}) \cdot (X_{t_{n,j}} - X_{t_{n,j-1}}) \quad \text{and} \quad T_n = \sum_{j=1}^{n} f(t^{\#}_{n,j}) \cdot (Z_{t_{n,j}} - Z_{t_{n,j-1}}).$$

Since convergence in probability implies convergence in distribution (Proposition A.9), it follows from [Lu75, Theorem 6.2.3] and Theorem 6.1 above, that $L\{S_n\} \xrightarrow{\text{w}} L\{\int_A^B f(t)\ \mathrm{d}X_t\}$ and $L\{T_n\} \xrightarrow{\text{w}} L\{\int_A^B f(t)\ \mathrm{d}Z_t\}$. Since $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$ are closed w.r.t. weak convergence (as noted in Section 4.5), it follows thus that $L\{\int_A^B f(t)\ \mathrm{d}X_t\} \in \mathcal{ID}(*)$ and $L\{\int_A^B f(t)\ \mathrm{d}Z_t\} \in \mathcal{ID}(\boxplus)$. Moreover, by Theorem 5.7, $L\{T_n\} = \Lambda(L\{S_n\})$, for each n in $\mathbb{N}$, and hence the last assertion follows by continuity of Λ. □

6.2 Integral Representation of Freely Selfdecomposable Variates

As mentioned in Section 2.5, a (classical) random variable Y has distribution in $\mathcal{L}(*)$ if and only if it has a representation in law of the form

$$Y \overset{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-t}\ \mathrm{d}X_t, \tag{6.3}$$

where $(X_t)_{t\geq 0}$ is a (classical) Lévy process, satisfying the condition $\mathbb{E}[\log(1+|X_1|)] < \infty$. The aim of this section is to establish a similar correspondence between selfadjoint operators with (spectral) distribution in $\mathcal{L}(\boxplus)$ and free Lévy processes (in law).

The stochastic integral appearing in (6.3) is the limit in probability, as $R \to \infty$, of the stochastic integrals $\int_0^R \mathrm{e}^{-t}\,\mathrm{d}X_t$, i.e. we have

$$\int_0^R \mathrm{e}^{-t}\ \mathrm{d}X_t \xrightarrow{\text{p}} \int_0^\infty \mathrm{e}^{-t}\ \mathrm{d}X_t, \quad \text{as } R \to \infty,$$

(the convergence actually holds almost surely; see Proposition 6.3 below). The stochastic integral $\int_0^R \mathrm{e}^{-t}\ \mathrm{d}X_t$ is, in turn, defined as the limit of approximating Riemann sums as described in Section 6.1

For a free Lévy process (Z_t), we determine next under which conditions the stochastic integral $\int_0^\infty \mathrm{e}^{-t}\ \mathrm{d}Z_t$ makes sense as the limit, for $R \to \infty$, of the stochastic integrals $\int_0^R \mathrm{e}^{-t}\,\mathrm{d}Z_t$, which are defined by virtue of Theorem 6.1. Again, the result we obtain is derived by applications of the mapping Λ and the following corresponding classical result:

Proposition 6.3 ([JuVe83]). *Let (X_t) be a classical Lévy process defined on some probability space $(\Omega, \mathcal{F}, P)$, and let (γ, σ) be the generating pair for the $*$-infinitely divisible probability measure $L\{X_1\}$. Then the following conditions are equivalent:*

(i) $\int_{\mathbb{R}\setminus]-1,1[} \log(1+|t|)\ \sigma(\mathrm{d}t) < \infty$.
(ii) $\int_0^R \mathrm{e}^{-t}\ \mathrm{d}X_t$ *converges almost surely, as* $R \to \infty$.
(iii) $\int_0^R \mathrm{e}^{-t}\ \mathrm{d}X_t$ *converges in distribution, as* $R \to \infty$.
(iv) $\mathbb{E}[\log(1+|X_1|)] < \infty$.

Proof. This was proved in [JuVe83, Theorem 3.6.6]. We note, though, that in [JuVe83], the measure σ in condition (i) is replaced by the Lévy measure ρ appearing in the alternative Lévy-Khintchine representation (2.2) for $L\{X_1\}$. However, since $\rho(\mathrm{d}t) = \frac{1+t^2}{t^2} \cdot 1_{\mathbb{R}\setminus\{0\}}(t)\ \sigma(\mathrm{d}t)$, it is clear that the integrals $\int_{\mathbb{R}\setminus]-1,1[} \log(1+|t|)\ \rho(\mathrm{d}t)$ and $\int_{\mathbb{R}\setminus]-1,1[} \log(1+|t|)\ \sigma(\mathrm{d}t)$ are finite simultaneously. □

Proposition 6.4. *Let (Z_t) be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A},\tau)$, and let (γ,σ) be the free generating pair for the $\boxplus$-infinitely divisible probability measure $L\{Z_1\}$. Then the following statements are equivalent:*

(i) $\int_{\mathbb{R}\setminus]-1,1[} \log(1+|t|)\ \sigma(\mathrm{d}t) < \infty$.
(ii) $\int_0^R \mathrm{e}^{-t}\ \mathrm{d}Z_t$ *converges in probability, as* $R \to \infty$.
(iii) $\int_0^R \mathrm{e}^{-t}\ \mathrm{d}Z_t$ *converges in distribution, as* $R \to \infty$.

Proof. Let (μ_t) be the family of marginal distributions of (Z_t) and consider then a classical Lévy process (X_t) with marginal distributions $(\Lambda^{-1}(\mu_t))$ (cf. Proposition 5.15). By the definition of Λ, it follows then that (γ,σ) is the generating pair for the $*$-infinitely divisible probability measure $L\{X_1\}$.

(i) $\Rightarrow$ (ii): Assume that (i) holds. Then condition (i) in Proposition 6.3 is satisfied for the classical Lévy process (X_t). Hence by (ii) of that proposition, $\int_0^R \mathrm{e}^{-t}\ \mathrm{d}X_t$ converges almost surely, and hence in probability, as $R \to \infty$. Consider now any increasing sequence (R_n) of positive numbers, such that $R_n \nearrow \infty$, as $n \to \infty$. Then for any m, n in $\mathbb{N}$ such that $m > n$, we have by Corollary 6.2

$$\begin{aligned} L\big\{ \textstyle\int_0^{R_m} \mathrm{e}^{-t}\ \mathrm{d}Z_t - \int_0^{R_n} \mathrm{e}^{-t}\ \mathrm{d}Z_t \big\} &= L\big\{ \textstyle\int_{R_n}^{R_m} \mathrm{e}^{-t}\ \mathrm{d}Z_t \big\} = \Lambda\big[L\big\{ \textstyle\int_{R_n}^{R_m} \mathrm{e}^{-t}\ \mathrm{d}X_t \big\}\big] \\ &= \Lambda\big[L\big\{ \textstyle\int_0^{R_m} \mathrm{e}^{-t}\ \mathrm{d}X_t - \int_0^{R_n} \mathrm{e}^{-t}\ \mathrm{d}X_t \big\}\big]. \end{aligned} \tag{6.4}$$

Since the sequence $(\int_0^{R_n} \mathrm{e}^{-t}\ \mathrm{d}X_t)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to convergence in probability, it follows thus, by continuity of Λ, that so is the sequence $(\int_0^{R_n} \mathrm{e}^{-t}\ \mathrm{d}Z_t)_{n\in\mathbb{N}}$. Hence, by Proposition A.5, there exists a selfadjoint operator W affiliated with $(\mathcal{A},\tau)$, such that $\int_0^{R_n} \mathrm{e}^{-t}\ \mathrm{d}Z_t \to W$ in probability. It remains to argue that W does not depend on the sequence (R_n). This follows, for example, as in the proof of Theorem 6.1, by considering, for two given sequences (R_n) and (R'_n), a third increasing sequence (R''_n), containing infinitely many elements from both of the original sequences.

(ii) $\Rightarrow$ (i): Assume that (ii) holds. It follows then by (6.4) and continuity of Λ^{-1} that for any increasing sequence (R_n), as above, $(\int_0^{R_n} \mathrm{e}^{-t}\ \mathrm{d}X_t)$ is a Cauchy sequence w.r.t. convergence in probability. We deduce that (iii) of Proposition 6.3 is satisfied for (X_t), and hence so is (i) of that proposition. By definition of (X_t), this means exactly that (i) of Proposition 6.4 is satisfied for (Z_t).

(ii) $\Rightarrow$ (iii): This follows from Proposition A.9.

(iii)$\Rightarrow$(i): Suppose (iii) holds, and note that the limit distribution is necessarily $\boxplus$-infinitely divisible. Now by Corollary 6.2 and continuity of Λ^{-1}, condition (iii) of Proposition 6.3 is satisfied for (X_t), and hence so is (i) of that proposition. This means, again, that (i) in Proposition 6.4 is satisfied for (Z_t). $\square$

If (Z_t) is a free Lévy process (in law) affiliated with $(\mathcal{A}, \tau)$, such that (i) of Proposition 6.4 is satisfied, then we denote by $\int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}Z_t$ the selfadjoint operator affiliated with $(\mathcal{A}, \tau)$, to which $\int_0^R \mathrm{e}^{-t}\, \mathrm{d}Z_t$ converges, in probability, as $R \to \infty$. We note that $L\{\int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}Z_t\}$ is $\boxplus$-infinitely divisible, and that Corollary 6.2 and Proposition A.9 yield the following relation:

$$L\big\{ \textstyle\int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}Z_t \big\} = \Lambda\big[L\big\{ \textstyle\int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}X_t \big\}\big], \tag{6.5}$$

where (X_t) is a classical Lévy process corresponding to (Z_t) as in Proposition 5.15.

Theorem 6.5. *Let y be a selfadjoint operator affiliated with a W^*-probability space $(\mathcal{A}, \tau)$. Then the distribution of y is $\boxplus$-selfdecomposable if and only if y has a representation in law in the form:*

$$y \overset{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}Z_t, \tag{6.6}$$

for some free Lévy process (in law) (Z_t) affiliated with some W^*-probability space $(\mathcal{B}, \psi)$, and satisfying condition (i) of Proposition 6.4.

Proof. Put $\mu = L\{y\}$. Suppose first that μ is $\boxplus$-selfdecomposable and put $\mu' = \Lambda^{-1}(\mu)$. Then, by Corollary 5.10, μ' is $*$-selfdecomposable, and hence by the classical version of this theorem (cf. [JuVe83, Theorem 3.2]), there exists a classical Lévy process (X_t) defined on some probability space $(\Omega, \mathcal{F}, P)$, such that condition (i) in Proposition 6.3 is satisfied, and such that $\Lambda^{-1}(\mu) = L\{\int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}X_t\}$. Let (Z_t) be a free Lévy process (in law) affiliated with some W^*-probability space $(\mathcal{B}, \psi)$, and corresponding to (X_t) as in Proposition 5.15. Then, by definition of Λ, condition (i) in Proposition 6.4 is satisfied for (Z_t) and, by formula (6.5), $L\{\int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}Z_t\} = \mu$.

Assume, conversely, that there exists a free Lévy process (in law) (Z_t) affiliated with some W^*-probability space $(\mathcal{B}, \psi)$, such that condition (i) of Proposition 6.4 is satisfied, and such that $\mu = L\{\int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}Z_t\}$. Then consider a classical Lévy process (X_t) defined on some probability space $(\Omega, \mathcal{F}, P)$, and corresponding to (Z_t) as in Proposition 5.15. Condition (i) in Proposition 6.3 is then satisfied for (X_t) and, by (6.5), $\Lambda^{-1}(\mu) = L\{\int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}X_t\}$. Thus, by the classical version of this theorem, $\Lambda^{-1}(\mu)$ is $*$-selfdecomposable, and hence μ is $\boxplus$-selfdecomposable. $\square$

Remark 6.6 (Free OU processes). Let y be a selfadjoint operator affiliated with some W^*-probability space $(\mathcal{A}, \tau)$, and assume that there exists a free Lévy process (in law) (Z_t) affiliated with some W^*-probability space $(\mathcal{B}, \psi)$, such that condition (i) of Proposition 6.4 is satisfied, and such that $y \stackrel{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}Z_t$. Note then, that for any positive numbers s, λ, we have

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}Z_t &= \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathrm{d}Z_{\lambda t} = \int_s^\infty \mathrm{e}^{-\lambda t}\, \mathrm{d}Z_{\lambda t} + \int_0^s \mathrm{e}^{-\lambda t}\, \mathrm{d}Z_{\lambda t} \\ &= \mathrm{e}^{-\lambda s} \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathrm{d}Z_{\lambda(s+t)} + \int_0^{\lambda s} \mathrm{e}^{-t}\, \mathrm{d}Z_t, \end{aligned} \tag{6.7}$$

where we have introduced integration w.r.t. the processes $V_t = Z_{\lambda t}$ and $W_t = Z_{\lambda(s+t)}$, $t \geq 0$. The rules of transformation for stochastic integrals, used above, are easily verified by considering the integrals as limits of Riemann sums. That same point of view, together with the fact that (Z_t) has freely independent stationary increments (conditions (i) and (iii) in Definition 4.27), implies, furthermore, that $\int_0^\infty \mathrm{e}^{-\lambda t}\, \mathrm{d}Z_{\lambda(s+t)} \stackrel{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathrm{d}Z_{\lambda t} \stackrel{\mathrm{d}}{=} y$. Note also that the two terms in the last expression of (6.7) are freely independent. Thus, (6.7) shows, that for any positive numbers s, λ, we have a decomposition in the form: $y \stackrel{\mathrm{d}}{=} \mathrm{e}^{-\lambda s} y(\lambda, s) + u(\lambda, s)$, where $y(\lambda, s)$ and $u(\lambda, s)$ are freely independent, and where $y(\lambda, s) \stackrel{\mathrm{d}}{=} y$. In particular, we have verified, directly, that $L\{y\}$ is $\boxplus$-selfdecomposable. Moreover, if we choose a selfadjoint operator Y_0 affiliated with $(\mathcal{B}, \psi)$, which is freely independent of (Z_t), and such that $L\{Y_0\} = L\{y\}$ (extend $(\mathcal{B}, \psi)$ if necessary), then the expression:

$$Y_s = \mathrm{e}^{-\lambda s} Y_0 + \int_0^{\lambda s} \mathrm{e}^{-t}\, \mathrm{d}Z_t, \quad (s \geq 0),$$

defines an operator valued stochastic process (Y_s) affiliated with $(\mathcal{B}, \psi)$, satisfying that $Y_s \stackrel{\mathrm{d}}{=} y$ for all s. If we replace (Z_t) above by a classical Lévy process (X_t), satisfying condition (i) in Proposition 6.3, and let Y_0 be a (classical) random variable, which is independent of (X_t), then the corresponding process (Y_s) is a solution to the stochastic differential equation:

$$\mathrm{d}Y_s = -\lambda Y_s\, \mathrm{d}s + \mathrm{d}X_{\lambda s},$$

and (Y_s) is said to be a process of *Ornstein-Uhlenbeck type* or an *OU process*, for short (cf. [BaSh01a],[BaSh01b] and references given there).

6.3 Free Poisson Random Measures

In this section, we introduce free Poisson random measures and prove their existence. We mention in passing the related notions of free stochastic measures (cf. [An00]) and free white noise (cf. [Sp90]). We mention also that the existence of free Poisson random measures was established by Voiculescu in [Vo98] in a different way than the one presented below. Recall, that for any number λ in $[0,\infty[$, we denote by $\mathrm{Poiss}^{\boxplus}(\lambda)$ the free Poisson distribution with mean λ (cf. Example 5.3).

Definition 6.7. *Let $(\Theta,\mathcal{E},\nu)$ be a measure space, and put*

$$\mathcal{E}_0 = \{E \in \mathcal{E} \mid \nu(E) < \infty\}.$$

Let further $(\mathcal{A},\tau)$ be a W^*-probability space, and let $\mathcal{A}_+$ denote the cone of positive operators in $\mathcal{A}$. Then a free Poisson random measure on $(\Theta,\mathcal{E},\nu)$ with values in $(\mathcal{A},\tau)$, is a mapping $M\colon \mathcal{E}_0 \to \mathcal{A}_+$, with the following properties:

(i) *For any set E in $\mathcal{E}_0$, $L\{M(E)\} = \mathrm{Poiss}^{\boxplus}(\nu(E))$.*
(ii) *If $r \in \mathbb{N}$ and $E_1,\dots,E_r$ are disjoint sets from $\mathcal{E}_0$, then $M(E_1),\dots,M(E_r)$ are freely independent operators.*
(iii) *If $r \in \mathbb{N}$ and $E_1,\dots,E_r$ are disjoint sets from $\mathcal{E}_0$, then $M(\cup_{j=1}^r E_j) = \sum_{j=1}^r M(E_j)$.*

In the setting of Definition 6.7, the measure ν is called the *intensity measure* for the free Poisson random measure M. Note, in particular, that $M(E)$ is a *bounded* positive operator for all E in $\mathcal{E}_0$. The definition above might seem a little "poor"compared to that of a classical Poisson random measure. The following remark might offer a bit of consolation.

Remark 6.8. Suppose M is a free Poisson random measure on the measure space $(\Theta,\mathcal{E},\nu)$ with values in the W^*-probability space $(\mathcal{A},\tau)$. Let further (E_n) be a sequence of disjoint sets from $\mathcal{E}_0$. If we assume, in addition, that $\cup_{j\in\mathbb{N}} E_j \in \mathcal{E}_0$, then we also have that

$$M\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) = \sum_{j=1}^{\infty} M(E_j),$$

where the right hand side should be understood as the limit *in probability* (see Definition 4.3) of $\sum_{j=1}^n M(E_j)$ as $n \to \infty$.

Indeed, put $E = \cup_{j\in\mathbb{N}} E_j$, and assume that $E \in \mathcal{E}_0$. Then for any n in $\mathbb{N}$,

$$M(E) - \sum_{j=1}^{n} M(E_j) = M(E) - M(\cup_{j=1}^n E_j) = M(\cup_{j=n+1}^{\infty} E_j),$$

so that

$$\begin{aligned} L\Big\{M(E) - \sum_{j=1}^{n} M(E_j)\Big\} &= \mathrm{Poiss}^{\boxplus}\big(\nu(\cup_{j=n+1}^{\infty} E_j)\big) \\ &= \mathrm{Poiss}^{\boxplus}\big(\textstyle\sum_{j=n+1}^{\infty} \nu(E_j)\big) \xrightarrow{\text{w}} \delta_0, \end{aligned}$$

as $n \to \infty$, since $\sum_{j=n+1}^{\infty} \nu(E_j) \to 0$ as $n \to \infty$, because $\sum_{j=1}^{\infty} \nu(E_j) = \nu(E) < \infty$.

The main purpose of the section is to prove the general existence of free Poisson random measures.

Theorem 6.9. *Let $(\Theta, \mathcal{E}, \nu)$ be a measure space. Then there exists a W^*-probability space $(\mathcal{A}, \tau)$ and a free Poisson random measure M on $(\Theta, \mathcal{E}, \nu)$ with values in $(\mathcal{A}, \tau)$.*

The proof of Theorem 6.9 is given in a series of lemmas. First of all, though, we introduce some notation:

If $\mu_1, \mu_2, \dots, \mu_r$ are probability measures on $\mathbb{R}$, we put (as in Section 6.1)

$$\mathop{*}_{h=1}^{r} \mu_h = \mu_1 * \mu_2 * \cdots * \mu_r \quad \text{and} \quad \mathop{\boxplus}_{h=1}^{r} \mu_h = \mu_1 \boxplus \mu_2 \boxplus \cdots \boxplus \mu_r.$$

In the remaining part of this section, we consider the measure space $(\Theta, \mathcal{E}, \nu)$ appearing in Theorem 6.9. Consider then the set

$$\mathcal{I} = \bigcup_{k \in \mathbb{N}} \{(E_1, \dots, E_k) \mid E_1, \dots, E_k \in \mathcal{E}_0 \setminus \{\emptyset\} \text{ and } E_1, \dots, E_k \text{ are disjoint}\},$$

where we think of $(E_1, \dots, E_k)$ merely as a collection of sets from $\mathcal{E}_0$. In particular, we identify $(E_1, \dots, E_k)$ with $(E_{\pi(1)}, \dots, E_{\pi(k)})$ for any permutation π of $\{1, 2, \dots, k\}$. We introduce, furthermore, a partial order $\leq$ on $\mathcal{I}$ by the convention:

$$(E_1, \dots, E_k) \leq (F_1, \dots, F_l) \iff \text{each } E_i \text{ is a union of some of the } F_j\text{'s}.$$

Lemma 6.10. *Given a tuple $S = (E_1, \dots, E_k)$ from $\mathcal{I}$, there exists a W^*-probability space $(\mathcal{A}_S, \tau_S)$, which is generated by freely independent positive operators $M_S(E_1), \dots, M_S(E_k)$ from $\mathcal{A}_S$, satisfying that*

$$L\{M_S(E_i)\} = \text{Poiss}^{\boxplus}(\nu(E_i)), \qquad (i = 1, \dots, k).$$

Proof. This is an immediate consequence of Voiculescu's theory of (reduced) free products of von Neumann algebras (cf. [VoDyNi92]). Indeed, we may take $(\mathcal{A}_S, \tau_S)$ to be the (reduced) von Neumann algebra free product of the Abelian W^*-probability spaces $(L^\infty(\mathbb{R}, \mu_i), \mathbb{E}_{\mu_i})$, $i = 1, \dots, k$, where $\mu_i = \text{Poiss}^{\boxplus}(\nu(E_i))$ and $\mathbb{E}_{\mu_i}$ denotes expectation with respect to μ_i. □

Lemma 6.11. *Consider two elements $S = (E_1, \dots, E_k)$ and $T = (F_1, \dots, F_l)$ of $\mathcal{I}$, and suppose that $S \leq T$. Consider the W^*-probability spaces $(\mathcal{A}_S, \tau_S)$ and $(\mathcal{A}_T, \tau_T)$ given by Lemma 6.10. Then there exists an injective, unital, normal $*$-homomorphism $\iota_{S,T} \colon \mathcal{A}_S \to \mathcal{A}_T$, such that $\tau_S = \tau_T \circ \iota_{S,T}$.*

Proof. We adapt the notation from Lemma 6.10. For any fixed i in $\{1, \dots, k\}$, we have that $E_i = F_{j(i,1)} \cup \cdots \cup F_{j(i,l_i)}$, for suitable (distinct) $j(i,1), \dots, j(i, l_i)$ from $\{1, 2, \dots, l\}$. Note then that

$$\begin{aligned}
L\{M_T(F_{j(i,1)})+\cdots+M_T(F_{j(i,l_i)})\} &= \mathop{\boxplus}_{h=1}^{l_i} \mathrm{Poiss}^{\boxplus}(\nu(F_{j(i,h)})) \\
&= \mathrm{Poiss}^{\boxplus}\big(\nu(F_{j(i,1)})+\cdots+\nu(F_{j(i,l_i)})\big) \\
&= \mathrm{Poiss}^{\boxplus}\big(\nu(F_{j(i,1)}\cup\cdots\cup F_{j(i,l_i)})\big) \\
&= \mathrm{Poiss}^{\boxplus}(\nu(E_i)) = L\{M_S(E_i)\}.
\end{aligned}$$

In addition, $M_S(E_1),\ldots,M_S(E_k)$ are freely independent selfadjoint operators, and, similarly, the operators $\sum_{h=1}^{l_i} M_T(F_{j(i,h)})$, $i=1,\ldots,k$ are freely independent and selfadjoint. Combining these observations with [Vo90, Remark 1.8], it follows that there exists an injective, unital, normal $*$-homomorphism $\iota_{S,T}\colon \mathcal{A}_S\to\mathcal{A}_T$, such that

$$\iota_{S,T}(M_S(E_i)) = M_T(F_{j(i,1)})+\cdots+M_T(F_{j(i,l_i)}), \qquad (i=1,2,\ldots,r), \tag{6.8}$$

and such that $\tau_S=\tau_T\circ\iota_{S,T}$. □

Lemma 6.12. *Adapting the notation from Lemmas 6.10-6.11, the system*

$$(\mathcal{A}_S,\tau_S)_{S\in\mathcal{I}}, \quad \{\iota_{S,T}\mid S,T\in\mathcal{I},\ S\le T\}, \tag{6.9}$$

is a directed system of W^*-algebras and injective, unital, normal $*$-homomorphisms (cf. [KaRi83, Section 11.4]).

Proof. Suppose that $R=(D_1,\ldots,D_m)$, $S=(E_1,\ldots,E_k)$ and $T=(F_1,\ldots,F_l)$ are elements of $\mathcal{I}$, such that $R\le S\le T$. We have to show that $\iota_{R,T}=\iota_{S,T}\circ\iota_{R,S}$. We may write (unambiguously),

$$\begin{aligned}
D_h &= E_{i(h,1)}\cup\cdots\cup E_{i(h,k_h)}, \qquad (h=1,\ldots,m),\\
E_i &= F_{j(i,1)}\cup\cdots\cup E_{j(i,l_i)}, \qquad (i=1,\ldots,k),
\end{aligned}$$

for suitable $i(h,1),\ldots,i(h,k_h)$ in $\{1,2,\ldots,k\}$ and $j(i,1),\ldots,j(i,l_i)$ in $\{1,2,\ldots,l\}$. Then for any h in $\{1,\ldots,m\}$, we have

$$D_h = E_{i(h,1)}\cup\cdots\cup E_{i(h,k_h)} = \Big(\bigcup_{r=1}^{l_{i(h,1)}} F_{j(i(h,1),r)}\Big)\cup\cdots\cup\Big(\bigcup_{r=1}^{l_{i(h,k_h)}} F_{j(i(h,k_h),r)}\Big)$$

so that, by definition of $\iota_{R,T}$, $\iota_{R,S}$ and $\iota_{S,T}$ (cf. (6.8)),

$$\begin{aligned}
\iota_{R,T}(D_h) &= \sum_{r=1}^{l_{i(h,1)}} M_T(F_{j(i(h,1),r)})+\cdots+\sum_{r=1}^{l_{i(h,k_h)}} M_T(F_{j(i(h,k_h),r)}) \\
&= \iota_{S,T}\big[M_S(E_{i(h,1)})\big]+\cdots+\iota_{S,T}\big[M_S(E_{i(h,k_h)})\big] \\
&= \iota_{S,T}\big[M_S(E_{i(h,1)})+\cdots+M_S(E_{i(h,k_h)})\big] \\
&= \iota_{S,T}\big[\iota_{R,S}(D_h)\big].
\end{aligned}$$

Since $\mathcal{A}_R$ is generated, as a von Neumann algebra, by the operators

$$M_R(D_1), \dots, M_R(D_m),$$

and since $\iota_{R,T}$ and $\iota_{S,T} \circ \iota_{R,S}$ are both normal $*$-homomorphisms, it follows by Kaplansky's density theorem (cf. [KaRi83, Theorem 5.3.5]) and the calculation above that $\iota_{R,T} = \iota_{S,T} \circ \iota_{R,S}$, as desired. □

Lemma 6.13. *Let $\mathcal{A}^0$ denote the C^*-inductive limit of the directed system (6.9) and let $\iota_S \colon \mathcal{A}_S \to \mathcal{A}^0$ denote the canonical embedding of $\mathcal{A}_S$ into $\mathcal{A}^0$ (cf. [KaRi83, Proposition 11.4.1]). Then there is a unique tracial state τ^0 on $\mathcal{A}^0$, satisfying that*

$$\tau_S = \tau^0 \circ \iota_S, \qquad \textit{for all } S \textit{ in } \mathcal{I}. \tag{6.10}$$

Proof. Recall that the canonical embeddings $\iota_S \colon \mathcal{A}_S \to \mathcal{A}^0$ $(S \in \mathcal{I})$ satisfy the condition:

$$\iota_R = \iota_S \circ \iota_{R,S}, \qquad \text{whenever } R, S \in \mathcal{I} \text{ and } R \le S.$$

We note first that (6.10) gives rise to a well-defined mapping τ^0 on the set $\mathcal{A}^{00} = \cup_{S\in\mathcal{I}} \iota_S(\mathcal{A}_S)$. Indeed, suppose that $\iota_S(a') = \iota_T(a'')$ for some S, T in $\mathcal{I}$ and $a' \in \mathcal{A}_S$, $a'' \in \mathcal{A}_T$. We need to show that $\tau_S(a') = \tau_T(a'')$. Let $S \vee T$ denote the tuple in $\mathcal{I}$ consisting of all non-empty sets of the form $E \cap F$, where $E \in S$ and $F \in T$. Note that $S, T \le S \vee T$. Since $\iota_S = \iota_{S\vee T} \circ \iota_{S,S\vee T}$ and $\iota_T = \iota_{S\vee T} \circ \iota_{T,S\vee T}$, it follows, by injectivity of $\iota_{S\vee T}$, that $\iota_{S,S\vee T}(a') = \iota_{T,S\vee T}(a'')$. Hence, by Lemma 6.11,

$$\tau_S(a') = \tau_{S\vee T} \circ \iota_{S,S\vee T}(a') = \tau_{S\vee T} \circ \iota_{T,S\vee T}(a'') = \tau_T(a''),$$

as desired. Now, given a, b in $\mathcal{A}^{00}$, we can find S from $\mathcal{I}$, such that a, b are both in $\iota_S(\mathcal{A}_S)$, and hence it follows immediately that τ^0 is a linear tracial functional on the vector space $\mathcal{A}^{00}$. Furthermore, if $a = \iota_S(a')$ for some a' in $\mathcal{A}_S$, then

$$|\tau^0(a)| = |\tau_S(a')| \le \|a'\| = \|\iota_S(a')\| = \|a\|,$$

so that τ^0 is norm decreasing. Since $\mathcal{A}^{00}$ is norm dense in $\mathcal{A}^0$ (cf. [KaRi83, Proposition 11.4.1]), if follows then that τ^0 has a unique extension to a mapping $\tau^0 \colon \mathcal{A}^0 \to \mathbb{C}$, which is automatically linear, tracial and norm-decreasing. In addition, $\tau^0(\mathbf{1}_{\mathcal{A}^0}) = 1 = \|\tau^0\|$, so, altogether, it follows that τ^0 is a tracial state on $\mathcal{A}^0$, satisfying (6.10). □

Lemma 6.14. *Let $(\mathcal{A}^0, \tau^0)$ be as in Lemma 6.13. There exists a mapping $M^0 \colon \mathcal{E}_0 \to \mathcal{A}^0_+$, which satisfies conditions (i)-(iii) of Definition 6.7.*

Proof. We define M^0 by the equation:

$$M^0(E) = \iota_{\{E\}}(M_{\{E\}}(E)), \qquad (E \in \mathcal{E}_0).$$

Then $M^0(E)$ is positive for each E in $\mathcal{E}_0$, since $\iota_{\{E\}}$ is a $*$-homomorphism. Note also that if $E \in \mathcal{E}_0$ and $S \in \mathcal{I}$ such that $E \in S$, then $\{E\} \leq S$ and

$$M^0(E) = \iota_{\{E\}}(M_{\{E\}}(E)) = \iota_S \circ \iota_{\{E\},S}(M_{\{E\}}(E)) = \iota_S(M_S(E)). \tag{6.11}$$

We now have

(i) For each E in $\mathcal{E}_0$, we have that $\tau_{\{E\}} = \tau^0 \circ \iota_{\{E\}}$, and hence, since $\iota_{\{E\}}$ is a $*$-homomorphism, $M_{\{E\}}(E)$ and $M^0(E)$ have the same moments with respect to $\tau_{\{E\}}$ and τ^0, respectively. Since both operators are bounded, this implies that $L\{M^0(E)\} = L\{M_{\{E\}}(E)\} = \mathrm{Poiss}^{\boxplus}(\nu(E))$.

(ii) Let $E_1, \ldots, E_k$ be disjoint sets from $\mathcal{E}_0$ and consider the tuple $S = (E_1, \ldots, E_k) \in \mathcal{I}$. Then, since $\tau_S = \tau^0 \circ \iota_S$ and ι_S is a $*$-homomorphism, we find, using (6.11),

$$\tau^0\big(M^0(E_{i_1})M^0(E_{i_2}) \cdots M^0(E_{i_p})\big) = \tau_S\big(M_S(E_{i_1})M_S(E_{i_2}) \cdots M_S(E_{i_p})\big),$$

for any $i_1, \ldots, i_p$ in $\{1, 2, \ldots, k\}$. Since $M_S(E_1), \ldots, M_S(E_k)$ are freely independent, this implies that so are $M^0(E_1), \ldots, M^0(E_k)$.

(iii) Let $E_1, \ldots, E_k$ be disjoint sets from $\mathcal{E}_0$, put $E = \cup_{i=1}^k E_i$ and consider the tuple $S = (E_1, \ldots, E_k) \in \mathcal{I}$. Then, by definition of $\iota_{\{E\},S}$, we have

$$\begin{aligned} M^0(E) &= \iota_{\{E\}}(M_{\{E\}}(E)) = \iota_S \circ \iota_{\{E\},S}(M_{\{E\}}(E)) \\ &= \iota_S\big(M_S(E_1) + \cdots + M_S(E_k)\big) \\ &= \iota_S(M_S(E_1)) + \cdots + \iota_S(M_S(E_k)) \\ &= M^0(E_1) + \cdots + M^0(E_k). \end{aligned}$$

This concludes the proof. ◻

Lemma 6.15. *Let $(\mathcal{A}^0, \tau^0)$ be as in Lemma 6.13, let $\Phi^0 \colon \mathcal{A}^0 \to \mathcal{B}(\mathcal{H}^0)$ denote the GNS representation*[9] *of $\mathcal{A}^0$ associated to τ^0, and let $\mathcal{A}$ be the closure of $\Phi^0(\mathcal{A}^0)$ in $\mathcal{B}(\mathcal{H}^0)$ with respect to the weak operator topology. Let, further, ξ^0 denote the unit vector in $\mathcal{H}^0$, which corresponds to the unit $\mathbf{1}_{\mathcal{A}^0}$ via the GNS-construction, and let τ denote the vector state on $\mathcal{A}$ given by ξ^0. Then $(\mathcal{A}, \tau)$ is a W^*-probability space, and $\tau^0 = \tau \circ \Phi^0$.*

Proof. It follows immediately from the GNS-construction that

$$\tau^0 = \tau \circ \Phi^0, \tag{6.12}$$

so we only have to prove that τ is a faithful trace on $\mathcal{A}$. To see that τ is a trace, note that since τ^0 is a trace, it follows from (6.12) that τ is a trace on the weakly dense C^*-subalgebra $\Phi^0(\mathcal{A}^0)$ of $\mathcal{A}$. Since the multiplication of operators

[9] GNS stands for Gelfand-Naimark-Segal; see [KaRi83, Theorem 4.5.2].

is separately continuous in each variable in the weak operator topology, and since τ is a vector state, we may subsequently conclude that $\tau(ab) = \tau(ba)$ whenever, say, $a \in \mathcal{A}$ and $b \in \Phi^0(\mathcal{A}^0)$. Repeating the argument just given, it follows that τ is a trace on all of $\mathcal{A}$. This means, furthermore, that ξ^0 is a generating trace vector for $\mathcal{A}$, and hence, by [KaRi83, Lemma 7.2.14], it is also a generating trace vector for the commutant $\mathcal{A}' \subseteq \mathcal{B}(\mathcal{H}^0)$. This implies, in particular, that ξ^0 is separating for $\mathcal{A}$ (cf. [KaRi83, Corollary 5.5.12]), which, in turn, implies that τ is faithful on $\mathcal{A}$. $\square$

Proof of Theorem 6.9. Let Φ^0 and $(\mathcal{A}, \tau)$ be as in Lemma 6.15. We then define the mapping $M \colon \mathcal{E}_0 \to \mathcal{A}_+$ by setting

$$M(E) = \Phi^0(M^0(E)), \qquad (E \in \mathcal{E}_0).$$

Now, Φ^0 is a $*$-homomorphism and $\tau^0 = \tau \circ \Phi^0$, so Φ^0 preserves all (mixed) moments of the elements $M^0(E)$, $E \in \mathcal{E}_0$. Since M^0 satisfies conditions (i)-(iii) of Definition 6.7, it follows thus, using the same line of argumentation as in the proof of Lemma 6.14, that M satisfies conditions (i)-(iii) too. Consequently, M is a free Poisson random measure on $(\Theta, \mathcal{E}, \nu)$ with values in $(\mathcal{A}, \tau)$. $\square$

6.4 Integration with Respect to Free Poisson Random Measures

Throughout this section, we consider a free Poisson random measure M on the σ-finite measure space $(\Theta, \mathcal{E}, \nu)$ and with values in the W^*-probability space $(\mathcal{A}, \tau)$. We consider also a classical Poisson random measure N on $(\Theta, \mathcal{E}, \nu)$ defined on a classical probability space $(\Omega, \mathcal{F}, P)$. The aim of this section is to establish a theory of integration with respect to M, making sense, thus, to the integral $\int_\Theta f \, \mathrm{d}M$ for any function f in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$. As in most theories of integration, we start by defining integration for simple ν-integrable functions.

Definition 6.16. *Let s be a real-valued simple function in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$, i.e. s can be written, unambiguously, in the form*

$$s = \sum_{j=1}^{r} a_j 1_{E_j},$$

where $r \in \mathbb{N}$, $a_1, \ldots, a_r$ are distinct numbers in $\mathbb{R} \setminus \{0\}$ and $E_1, \ldots, E_r$ are disjoint sets from $\mathcal{E}_0$ (since s is ν-integrable). We then define the integral $\int_\Theta s \, \mathrm{d}M$ of s with respect to M as follows:

$$\int_\Theta s \, \mathrm{d}M = \sum_{j=1}^{r} a_j M(E_j) \in \mathcal{A}.$$

Remark 6.17. (a) Since $M(E) \in \mathcal{A}_+$ for any E in $\mathcal{E}_0$, it follows immediately from Definition 6.16 that $\int_\Theta s \, \mathrm{d}M$ is a selfadjoint operator in $\mathcal{A}$ for any real-valued simple function s in $\mathcal{L}^1(\Theta, \mathcal{E}, \mu)$.

(b) Suppose s and t are real-valued simple functions in $\mathcal{L}^1(\Theta,\mathcal{E},\nu)$ and that $c\in\mathbb{R}$. Then $s+t$ and $c\cdot s$ are clearly simple functions too, and, using standard arguments, it is not hard to see that

$$\int_\Theta (s+t)\,\mathrm{d}M=\int_\Theta s\,\mathrm{d}M+\int_\Theta t\,\mathrm{d}M,\quad\text{and}\quad \int_\Theta c\cdot s\,\mathrm{d}M=c\int_\Theta s\,\mathrm{d}M.$$

(c) Consider now, in addition, the classical Poisson random measure N on $(\Theta,\mathcal{E},\nu)$, defined on $(\Omega,\mathcal{F},P)$. Let, further, s be a real-valued simple function in $\mathcal{L}^1(\Theta,\mathcal{E},\nu)$. Then $L\{\int_\Theta s\,\mathrm{d}N\}\in\mathcal{ID}(*)$, $L\{\int_\Theta s\,\mathrm{d}M\}\in\mathcal{ID}(\boxplus)$, and

$$\Lambda\Big(L\Big\{\int_\Theta s\,\mathrm{d}N\Big\}\Big)=L\Big\{\int_\Theta s\,\mathrm{d}M\Big\},$$

where Λ is the Bercovici-Pata bijection. Indeed, we may write s in the form $s=\sum_{j=1}^r a_j 1_{E_j}$, where $r\in\mathbb{N}$, $a_1,\dots,a_r$ are distinct numbers in $\mathbb{R}\setminus\{0\}$ and $E_1,\dots,E_r$ are disjoint sets from $\mathcal{E}_0$. Then, using the properties of Λ, we find that

$$\begin{aligned}L\Big\{\int_\Theta s\,\mathrm{d}M\Big\}&=L\Big\{\sum_{j=1}^r a_jM(E_j)\Big\}=\boxplus_{j=1}^r D_{a_j}\mathrm{Poiss}^{\boxplus}(\nu(E_j))\\&=\boxplus_{j=1}^r D_{a_j}\Lambda\big[\mathrm{Poiss}^*(\nu(E_j))\big]=\Lambda\Big[\mathop{*}_{j=1}^r D_{a_j}\mathrm{Poiss}^*(\nu(E_j))\Big]\\&=\Lambda\Big[L\Big\{\sum_{j=1}^r a_jN(E_j)\Big\}\Big]=\Lambda\Big[L\Big\{\int_\Theta s\,\mathrm{d}N\Big\}\Big].\end{aligned}$$

By $\mathcal{L}^1(\Theta,\mathcal{E},\nu)_+$, we denote the set of positive functions from $\mathcal{L}^1(\Theta,\mathcal{E},\nu)$.

Proposition 6.18. *Let f be a real-valued function in $\mathcal{L}^1(\Theta,\mathcal{E},\nu)$, and choose a sequence (s_n) of real-valued simple $\mathcal{E}$-measurable functions, satisfying the conditions:*

$$\exists h\in\mathcal{L}^1(\Theta,\mathcal{E},\nu)_+\ \forall\theta\in\Theta\ \forall n\in\mathbb{N}\colon |s_n(\theta)|\le h(\theta),\tag{6.13}$$

and

$$\lim_{n\to\infty}s_n(\theta)=f(\theta),\qquad(\theta\in\Theta).\tag{6.14}$$

Then $s_n\in\mathcal{L}^1(\Theta,\mathcal{E},\nu)$ for all n, and the integrals $\int_\Theta s_n\,\mathrm{d}M$ converge in probability to a selfadjoint (possibly unbounded) operator $I(f)$ affiliated with $\mathcal{A}$.

Furthermore, the limit $I(f)$ is independent of the choice of approximating sequence (s_n) of simple functions (subject to conditions (6.13) and (6.14)).

In condition (6.13), we might have taken $h=|f|$, but it is convenient to allow for more general dominators.

Proof of Proposition 6.18. Let f, (s_n) and h be as set out in the proposition. Then, for any n in $\mathbb{N}$, $\int_\Theta |s_n|\,\mathrm{d}\nu\le\int_\Theta h\,\mathrm{d}\nu<\infty$, so that $s_n\in\mathcal{L}^1(\Theta,\mathcal{E},\nu)$ and

$\int_\Theta s_n \,\mathrm{d}M$ is well-defined. Note further that for any n, m in $\mathbb{N}$, $s_n - s_m$ is again a simple function in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$, and, using Remark 6.17(c),(d), it follows that

$$\begin{aligned} L\Big\{ \int_\Theta s_n \,\mathrm{d}M - \int_\Theta s_m \,\mathrm{d}M \Big\} &= L\Big\{ \int_\Theta (s_n - s_m) \,\mathrm{d}M \Big\} \\ &= \Lambda\Big[L\Big\{ \int_\Theta (s_n - s_m) \,\mathrm{d}N \Big\} \Big], \end{aligned} \tag{6.15}$$

with N the classical Poisson random measure introduced before. Since $h \in \mathcal{L}^1(\Theta, \mathcal{E}, \nu)$, it follows from Proposition 2.8 that $h \in \mathcal{L}^1(\Theta, \mathcal{E}, N(\cdot, \omega))$ for almost all ω in Ω. Hence, by Lebesgue's theorem on dominated convergence, we have that

$$\int_\Theta s_n(\theta) \, N(\mathrm{d}\theta, \omega) \longrightarrow \int_\Theta f(\theta) \, N(\mathrm{d}\theta, \omega), \quad \text{as } n \to \infty,$$

for almost all ω in Ω. In other words, $\int_\Theta s_n \,\mathrm{d}N \to \int_\Theta f \,\mathrm{d}N$, almost surely, as $n \to \infty$. In particular $\int_\Theta s_n \,\mathrm{d}N \to \int_\Theta f \,\mathrm{d}N$, in probability as $n \to \infty$, so the sequence $(\int_\Theta s_n \,\mathrm{d}N)_{n \in \mathbb{N}}$ is a Cauchy sequence in probability, i.e.

$$L\Big\{ \int_\Theta (s_n - s_m) \,\mathrm{d}N \Big\} \xrightarrow{\text{w}} \delta_0, \quad \text{as } n, m \to \infty.$$

Combining this with (6.15) and the continuity of Λ (cf. Corollary 5.14), it follows that $(\int_\Theta s_n \,\mathrm{d}M)_{n \in \mathbb{N}}$ is also a Cauchy sequence in probability, i.e. with respect to the measure topology. Since $\overline{\mathcal{A}}$ is complete in the measure topology (cf. Proposition A.5), there exists, thus, an operator $I(f)$ in $\overline{\mathcal{A}}$, such that $\int_\Theta s_n \,\mathrm{d}M \to I(f)$, in probability as $n \to \infty$. Since $\int_\Theta s_n \,\mathrm{d}M$ is selfadjoint for each n, and since the adjoint operation is continuous in the measure topology, $I(f)$ is a selfadjoint operator in $\overline{\mathcal{A}}$.

Suppose, finally, that (t_n) is another sequence of simple real-valued $\mathcal{E}$-measurable functions satisfying conditions (6.13) and (6.14) (with s_n replaced by t_n). Then, by the argument given above, $\int_\Theta t_n \,\mathrm{d}M \to I'(f)$, in probability as $n \to \infty$, for some selfadjoint operator $I'(f)$ in $\overline{\mathcal{A}}$. Consider now the mixed sequence (u_n) of simple real-valued $\mathcal{E}$-measurable functions given by:

$$u_1 = s_1, u_2 = t_1, u_3 = s_2, u_4 = t_2, \ldots,$$

and note that this sequence satisfies (6.13) and (6.14) too, so that $\int_\Theta u_n \,\mathrm{d}M \to I''(f)$, in probability as $n \to \infty$, for some selfadjoint operator $I''(f)$ in $\overline{\mathcal{A}}$. Now the subsequence (u_{2n-1}) converges in probability to both $I''(f)$ and $I(f)$ as $n \to \infty$, and the subsequence (u_{2n}) converges in probability to both $I''(f)$ and $I'(f)$ as $n \to \infty$. Since the measure topology is a Hausdorff topology, we may conclude, thus, that $I(f) = I''(f) = I'(f)$. This completes the proof. □

Definition 6.19. *Let f be a real-valued function in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$, and let $I(f)$ be the selfadjoint operator in $\overline{\mathcal{A}}$ described in Proposition 6.18. We call $I(f)$ the integral of f with respect to M and denote it by $\int_\Theta f \,\mathrm{d}M$.*

Corollary 6.20. *Let M and N be the free and classical Poisson random measures on $(\Theta, \mathcal{E}, \nu)$ introduced above. Then for any f in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$, we have $L\{\int_\Theta f\,\mathrm{d}N\} \in \mathcal{ID}(*)$, $L\{\int_\Theta f\,\mathrm{d}M\} \in \mathcal{ID}(\boxplus)$ and*

$$\Lambda\Big(L\Big\{\int_\Theta f\,\mathrm{d}N\Big\}\Big) = L\Big\{\int_\Theta f\,\mathrm{d}M\Big\}.$$

Proof. Choose a sequence (s_n) of real-valued simple $\mathcal{E}$-measurable functions satisfying conditions (6.13) and (6.14) of Proposition 6.18. Then, by Remark 6.17, $L\{\int_\Theta s_n\,\mathrm{d}N\} \in \mathcal{ID}(*)$, $L\{\int_\Theta s_n\,\mathrm{d}M\} \in \mathcal{ID}(\boxplus)$ and $\Lambda(L\{\int_\Theta s_n \mathrm{d}N\}) = L\{\int_\Theta s_n\,\mathrm{d}M\}$ for all n in $\mathbb{N}$. Furthermore,

$$\int_\Theta s_n\,\mathrm{d}N \xrightarrow{\text{a.s.}} \int_\Theta f\,\mathrm{d}N \quad \text{and} \quad \int_\Theta s_n\,\mathrm{d}M \xrightarrow{\text{p}} \int_\Theta f\,\mathrm{d}M, \quad \text{as } n \to \infty.$$

In particular (cf. Proposition A.9),

$$L\Big\{\int_\Theta s_n\,\mathrm{d}N\Big\} \xrightarrow{\text{w}} L\Big\{\int_\Theta f\,\mathrm{d}N\Big\} \quad \text{and} \quad L\Big\{\int_\Theta s_n\,\mathrm{d}M\Big\} \xrightarrow{\text{w}} L\Big\{\int_\Theta f\,\mathrm{d}M\Big\},$$

as $n \to \infty$. Since $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$ are both closed with respect to weak convergence (see Section 4.5), this implies that $L\{\int_\Theta f\,\mathrm{d}N\} \in \mathcal{ID}(*)$ and $L\{\int_\Theta f\,\mathrm{d}M\} \in \mathcal{ID}(\boxplus)$. Furthermore, by continuity of Λ, $\Lambda(L\{\int_\Theta f\,\mathrm{d}N\}) = L\{\int_\Theta f\,\mathrm{d}M\}$. □

Proposition 6.21. *For any real-valued functions f, g in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$ and any real number c, we have that*

$$\int_\Theta (f+g)\,\mathrm{d}M = \int_\Theta f\,\mathrm{d}M + \int_\Theta g\,\mathrm{d}M \quad \textit{and} \quad \int_\Theta c\cdot f\,\mathrm{d}M = c\int_\Theta f\,\mathrm{d}M.$$

Proof. If f and g are simple functions, this was noted in Remark 6.17. The general case follows by approximating f and g by simple functions as in Proposition 6.18 and using that addition and scalar-multiplication are continuous operations in the measure topology (cf. Proposition A.5). □

Proposition 6.22. *Let M be a free Poisson random measure on the σ-finite measure space $(\Theta, \mathcal{E}, \nu)$ with values in the W^*-probability space $(\mathcal{A}, \tau)$. Let, further, $f_1, f_2, \ldots, f_r$ be real-valued functions in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$ and let $\Theta_1, \Theta_2, \ldots, \Theta_r$ be disjoint $\mathcal{E}$-measurable subsets of Θ. Then the integrals*

$$\int_{\Theta_1} f_1\,\mathrm{d}M, \int_{\Theta_2} f_2\,\mathrm{d}M, \ldots, \int_{\Theta_r} f_r\,\mathrm{d}M,$$

are freely independent selfadjoint operators affiliated with $(\mathcal{A}, \tau)$.

Proof. For each j in $\{1, 2, \dots, r\}$, let $(s_{j,n})_{n\in\mathbb{N}}$ be a sequence of real valued simple $\mathcal{E}$-measurable functions, such that

$$|s_{j,n}(\theta)| \le |f_j(\theta)|, \qquad (\theta \in \Theta,\ n \in \mathbb{N}),$$

and

$$\lim_{n\to\infty} s_{j,n}(\theta) = f_j(\theta), \qquad (\theta \in \Theta).$$

Then, for each j in $\{1, 2, \dots, r\}$ and each n in $\mathbb{N}$, we may write $s_{j,n} \cdot 1_{\Theta_j}$ in the form:

$$s_{j,n} \cdot 1_{\Theta_j} = \sum_{l=1}^{k_{j,n}} \alpha(l, j, n) 1_{A(l,j,n)},$$

where $\alpha(1, j, n), \dots, \alpha(k_{j,n}, j, n) \in \mathbb{R} \setminus \{0\}$ and $A(1, j, n), \dots, A(k_{j,n}, j, n)$ are disjoint sets from $\mathcal{E}_0$, such that $A(l, j, n) \subseteq \Theta_j$ for all l. Now,

$$\int_\Theta s_{j,n} \cdot 1_{\Theta_j} \, \mathrm{d}M = \sum_{l=1}^{k_{j,n}} \alpha(l, j, n) M((A(l, j, n)), \qquad (j = 1, 2, \dots, r,\ n \in \mathbb{N}),$$

so by the properties of free Poisson random measures, the integrals

$$\int_\Theta s_{1,n} \cdot 1_{\Theta_1} \, \mathrm{d}M, \dots, \int_\Theta s_{r,n} \cdot 1_{\Theta_r} \, \mathrm{d}M,$$

are freely independent for each n in $\mathbb{N}$. Finally, for each j in $\{1, 2, \dots, r\}$ we have (cf. Proposition 6.18)

$$\int_{\Theta_j} f_j \, \mathrm{d}M = \int_\Theta f_j \cdot 1_{\Theta_j} \, \mathrm{d}M = \lim_{n\to\infty} \int_\Theta s_{j,n} \cdot 1_{\Theta_j} \, \mathrm{d}M,$$

where the limit is taken in probability. Taking now Proposition 4.7 into account, we obtain the desired conclusion. □

6.5 The Free Lévy-Itô Decomposition

In this section we derive the free version of the Lévy-Itô decomposition. We mention in passing the related decomposition of free white noises, which was established in [GlScSp92].

Throughout this section we put

$$\mathsf{H} =]0, \infty[\times \mathbb{R} \subseteq \mathbb{R}^2,$$

and we denote by $\mathsf{B}(\mathsf{H})$ the set of all Borel subsets of H. Furthermore, for any ϵ, t in $]0, \infty[$, such that $\epsilon < t$, we put

$$D(\epsilon, \infty) = \{s \in \mathbb{R} \mid \epsilon < |s| < \infty\} = \mathbb{R} \setminus [-\epsilon, \epsilon],$$

$$D(\epsilon, t) = \{s \in \mathbb{R} \mid \epsilon < |s| \le t\} = [-t, t] \setminus [-\epsilon, \epsilon].$$

We shall need the following well-known result about classical Poisson random measures.

Lemma 6.23. *Let ν be a Lévy measure on $\mathbb{R}$ and consider the σ-finite measure* $\mathrm{Leb}\otimes\nu$ *on* H*. Consider further a (classical) Poisson random measure N on* $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb}\otimes\nu)$*, defined on some probability space $(\Omega, \mathcal{F}, P)$.*

Then there is a subset Ω_0 of Ω, such that $\Omega_0 \in \mathcal{F}$, $P(\Omega_0) = 1$ and such that the following holds for any ω in Ω_0: For any ϵ, t in $]0, \infty[$, the restriction $[N(\cdot, \omega)]_{]0,t]\times D(\epsilon,\infty)}$ of the measure $N(\cdot, \omega)$ to the set $]0, t] \times D(\epsilon, \infty)$ is supported on a finite number of points, each of which has mass 1.

Proof. See [Sa99, Lemma 20.1] $\square$

Lemma 6.24. *Let ν and N be as in Lemma 6.23, and consider a positive Borel function $\varphi\colon \mathbb{R} \to [0, \infty[$.*

(i) *For almost all ω in Ω, the following holds:*

$$\forall\epsilon > 0\ \forall 0 \le s < t\colon \int_{]s,t]\times D(\epsilon,\infty)} \varphi(x)\, N(\mathrm{d}u, \mathrm{d}x, \omega) < \infty.$$

(ii) *If $\int_{[-1,1]} \varphi(x)\, \nu(\mathrm{d}x) < \infty$, then for almost all ω in Ω, the following holds:*

$$\forall 0 \le s < t\colon \int_{]s,t]\times\mathbb{R}} \varphi(x)\, N(\mathrm{d}u, \mathrm{d}x, \omega) < \infty.$$

Proof. Since φ is positive, it suffices to consider the case $s = 0$ in (i) and (ii). Moreover, since φ only takes finite values, statement (i) follows immediately from Lemma 6.23.

To prove (ii), assume that $\int_{[-1,1]} \varphi(x)\, \nu(\mathrm{d}x) < \infty$. By virtue of (i), it suffices then to prove, for instance, that for almost all ω in Ω, the following holds:

$$\forall t > 0\colon \int_{]0,t]\times[-1,1]} \varphi(x)\, N(\mathrm{d}u, \mathrm{d}x, \omega) < \infty. \tag{6.16}$$

Since the integrals in (6.16) increase with t, it suffices to prove that for any fixed t in $]0, \infty[$,

$$\int_{]0,t]\times[-1,1]} \varphi(x)\, N(\mathrm{d}u, \mathrm{d}x, \omega) < \infty, \quad \text{for almost all } \omega.$$

This, in turn, follows immediately from the following calculation:

$$\begin{aligned}\mathbb{E}\Big\{\int_{]0,t]\times[-1,1]} \varphi(x)\, N(\mathrm{d}u, \mathrm{d}x)\Big\} &= \int_{]0,t]\times[-1,1]} \varphi(x)\, \mathrm{Leb}\otimes\nu(\mathrm{d}u, \mathrm{d}x) \\ &= t\int_{[-1,1]} \varphi(x)\, \nu(\mathrm{d}x) < \infty,\end{aligned}$$

where we have used Proposition 2.8. $\square$

Lemma 6.25. *Let ν be a Lévy measure on $\mathbb{R}$, and let M be a Free Poisson random measure on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb} \otimes \nu)$ with values in the W^*-probability space $(\mathcal{A}, \tau)$. Let, further, N be a (classical) Poisson random measure on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb} \otimes \nu)$, defined on a classical probability space $(\Omega, \mathcal{F}, P)$.*

(i) *For any ϵ, s, t in $[0, \infty[$, such that $s < t$ and $\epsilon > 0$, the integrals*

$$\int_{]s,t] \times D(\epsilon,n)} x \, M(\mathrm{d}u, \mathrm{d}x), \qquad (n \in \mathbb{N}),$$

converge in probability, as $n \to \infty$, to some (possibly unbounded) selfadjoint operator affiliated with $\mathcal{A}$, which we denote by $\int_{]s,t] \times D(\epsilon,\infty)} x \, M(\mathrm{d}u, \mathrm{d}x)$. Furthermore (cf. Lemma 6.24), $L\{\int_{]s,t] \times D(\epsilon,\infty)} x \, N(\mathrm{d}u, \mathrm{d}x)\} \in \mathcal{ID}(*)$, $L\{\int_{]s,t] \times D(\epsilon,\infty)} x \, M(\mathrm{d}u, \mathrm{d}x)\} \in \mathcal{ID}(\boxplus)$ and

$$L\Big\{ \int_{]s,t] \times D(\epsilon,\infty)} x \, M(\mathrm{d}u, \mathrm{d}x) \Big\} = \Lambda\Big(L\Big\{ \int_{]s,t] \times D(\epsilon,\infty)} x \, N(\mathrm{d}s, \mathrm{d}x) \Big\} \Big). \tag{6.17}$$

(ii) *If $\int_{[-1,1]} |x| \, \nu(\mathrm{d}x) < \infty$, then for any s, t in $[0, \infty[$, such that $s < t$, the integrals*

$$\int_{]s,t] \times [-n,n]} x \, M(\mathrm{d}u, \mathrm{d}x), \qquad (n \in \mathbb{N}),$$

converge in probability, as $n \to \infty$, to some (possibly unbounded) selfadjoint operator affiliated with $\mathcal{A}$, which we denote by $\int_{]s,t] \times \mathbb{R}} x \, M(\mathrm{d}u, \mathrm{d}x)$. Furthermore (cf. Lemma 6.24),

$$L\Big\{ \int_{]s,t] \times \mathbb{R}} x \, N(\mathrm{d}u, \mathrm{d}x) \Big\} \in \mathcal{ID}(*), \quad L\Big\{ \int_{]s,t] \times \mathbb{R}} x \, M(\mathrm{d}u, \mathrm{d}x) \Big\} \in \mathcal{ID}(\boxplus)$$

and

$$L\Big\{ \int_{]s,t] \times \mathbb{R}} x \, M(\mathrm{d}u, \mathrm{d}x) \Big\} = \Lambda\Big(L\Big\{ \int_{]s,t] \times \mathbb{R}} x \, N(\mathrm{d}s, \mathrm{d}x) \Big\} \Big).$$

Proof. (i) Note first that for any n in $\mathbb{N}$ and any ϵ, s, t in $[0, \infty[$, such that $s < t$ and $\epsilon > 0$, we have that

$$\int_{]s,t] \times D(\epsilon,n)} |x| \, \mathrm{Leb} \otimes \nu(\mathrm{d}u, \mathrm{d}x) = (t - s) \int_{D(\epsilon,n)} |x| \, \nu(\mathrm{d}x) < \infty,$$

since ν is a Lévy measure. Hence, by application of Proposition 6.18, the integral $\int_{]s,t] \times D(\epsilon,n)} x \, M(\mathrm{d}u, \mathrm{d}x)$ is well-defined and furthermore, by Corollary 6.20,

$$L\Big\{ \int_{]s,t] \times D(\epsilon,n)} x \, M(\mathrm{d}u, \mathrm{d}x) \Big\} = \Lambda\Big(L\Big\{ \int_{]s,t] \times D(\epsilon,n)} x \, N(\mathrm{d}u, \mathrm{d}x) \Big\} \Big). \tag{6.18}$$

Note now that by Lemma 6.24(i) there is a subset Ω_0 of Ω, such that $\Omega_0 \in \mathcal{F}$, $P(\Omega_0) = 1$ and

$$\int_{]s,t]\times D(\epsilon,\infty)} |x|\, N(\mathrm{d}u, \mathrm{d}x, \omega) < \infty, \quad \text{for all } \omega \text{ in } \Omega_0.$$

Then $\int_{]s,t]\times D(\epsilon,\infty)} x\, N(\mathrm{d}u, \mathrm{d}x, \omega)$ is well-defined for all ω in Ω_0 and by Lebesgue's theorem on dominated convergence,

$$\int_{]s,t]\times D(\epsilon,n)} x\, N(\mathrm{d}u, \mathrm{d}x, \omega) \underset{n\to\infty}{\longrightarrow} \int_{]s,t]\times D(\epsilon,\infty)} x\, N(\mathrm{d}u, \mathrm{d}x, \omega),$$

for all ω in Ω_0, i.e. almost surely. In particular

$$\int_{]s,t]\times D(\epsilon,n)} x\, N(\mathrm{d}u, \mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_{]s,t]\times D(\epsilon,\infty)} x\, N(\mathrm{d}u, \mathrm{d}x), \quad \text{in probability,}$$

and hence $(\int_{]s,t]\times D(\epsilon,n)} x\, N(\mathrm{d}u, \mathrm{d}x))_{n\in\mathbb{N}}$ is a Cauchy sequence in probability. Now, for any n, m in $\mathbb{N}$, such that $n \le m$, we have, by Proposition 6.21 and Corollary 6.20,

$$\begin{aligned} &L\Big\{ \int_{]s,t]\times D(\epsilon,m)} x\, M(\mathrm{d}u, \mathrm{d}x) - \int_{]s,t]\times D(\epsilon,n)} x\, M(\mathrm{d}u, \mathrm{d}x)\Big\} \\ &\quad = L\Big\{ \int_{]s,t]\times D(n,m)} x\, M(\mathrm{d}u, \mathrm{d}x)\Big\} \\ &\quad = \Lambda\Big(L\Big\{ \int_{]s,t]\times D(n,m)} x\, N(\mathrm{d}u, \mathrm{d}x)\Big\}\Big) \\ &\quad = \Lambda\Big(L\Big\{ \int_{]s,t]\times D(\epsilon,m)} x\, N(\mathrm{d}u, \mathrm{d}x) - \int_{]s,t]\times D(\epsilon,n)} x\, N(\mathrm{d}u, \mathrm{d}x)\Big\}\Big). \end{aligned}$$

By continuity of Λ, this shows that $(\int_{]s,t]\times D(\epsilon,n)} x\, M(\mathrm{d}u, \mathrm{d}x))_{n\in\mathbb{N}}$ is a Cauchy sequence in probability, and hence, by completeness of $\overline{\mathcal{A}}$ in the measure topology,

$$\int_{]s,t]\times D(\epsilon,\infty)} x\, M(\mathrm{d}u, \mathrm{d}x) := \lim_{n\to\infty} \int_{]s,t]\times D(\epsilon,n)} x\, M(\mathrm{d}u, \mathrm{d}x),$$

exists in $\overline{\mathcal{A}}$ as the limit in probability.

Finally, since $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$ are closed with respect to weak convergence, we have that

$$L\Big\{ \int_{]s,t]\times D(\epsilon,\infty)} x\, N(\mathrm{d}u, \mathrm{d}x)\Big\} \in \mathcal{ID}(*)$$

and

$$L\Big\{\int_{]s,t]\times D(\epsilon,\infty)} x\,M(\mathrm{d}u,\mathrm{d}x)\Big\} \in \mathcal{ID}(\boxplus).$$

Moreover, since convergence in probability implies convergence in distribution (cf. Proposition A.9), it follows from (6.18) and continuity of Λ that (6.17) holds.

(ii) Suppose $\int_{[-1,1]} |x|\,\nu(\mathrm{d}x) < \infty$. Then for any n in $\mathbb{N}$ and any s,t in $[0,\infty[$, such that $s<t$, we have that

$$\begin{aligned}\int_{]s,t]\times[-n,n]} |x|\,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x) &= (t-s)\int_{[-n,n]} |x|\,\nu(\mathrm{d}x)\\ &= (t-s)\Big(\int_{[-1,1]} |x|\,\nu(\mathrm{d}x) + \int_{D(1,n)} |x|\,\nu(\mathrm{d}x)\Big)\\ &< \infty,\end{aligned}$$

since ν is a Lévy measure. Hence, by application of Proposition 6.18, the integral $\int_{]s,t]\times[-n,n]} x\,M(\mathrm{d}u,\mathrm{d}x)$ is well-defined and, by Corollary 6.20,

$$L\Big\{\int_{]s,t]\times[-n,n]} x\,M(\mathrm{d}u,\mathrm{d}x)\Big\} = \Lambda\Big(L\Big\{\int_{]s,t]\times[-n,n]} x\,N(\mathrm{d}u,\mathrm{d}x)\Big\}\Big).$$

From this point on, the proof is exactly the same as that of (i) given above; the only difference being that the application of Lemma 6.24(i) above must be replaced by an application of Lemma 6.24(ii). □

We are now ready to give a proof of the Lévy-Itô decomposition for free Lévy processes (in law). As is customary in the classical case (cf. [Sa99]), we divide the general formulation into two parts.

Theorem 6.26 (Free Lévy-Itô Decomposition I). *Let (Z_t) be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A},\tau)$, let ν be the Lévy measure appearing in the free generating triplet for $L\{Z_1\}$ and assume that $\int_{-1}^{1} |x|\ \nu(\mathrm{d}x) < \infty$. Then (Z_t) has a representation in the form:*

$$Z_t \overset{\mathrm{d}}{=} \gamma t\mathbf{1}_{\mathcal{A}^0} + \sqrt{a}W_t + \int_{]0,t]\times\mathbb{R}} x\ M(\mathrm{d}u,\mathrm{d}x), \qquad (t\geq 0), \tag{6.19}$$

where $\gamma\in\mathbb{R}$, $a\geq 0$, (W_t) is a free Brownian motion in some W^*-probability space $(\mathcal{A}^0,\tau^0)$ (see Example 5.16) and M is a free Poisson random measure on $(\mathsf{H},\mathsf{B}(\mathsf{H}),\mathrm{Leb}\otimes\nu)$ with values in $(\mathcal{A}^0,\tau^0)$. Furthermore, the process

$$U_t := \int_{]0,t]\times\mathbb{R}} x\ M(\mathrm{d}u,\mathrm{d}x), \qquad (t\geq 0),$$

is a free Lévy process (in law), which is freely independent of (W_t), and the right hand side of (6.19), as a whole, is a free Lévy process (in law).

As the symbol $\overset{\mathrm{d}}{=}$ appearing in (6.19) just means that the two operators have the same (spectral) distribution, it does not follow directly from (6.19) that the right hand side is a free Lévy process (in law) (contrary to the situation in the classical Lévy-Itô decomposition).

Proof of Theorem 6.26. By Proposition 5.15, we may choose a classical Lévy process (X_t), defined on some probability space $(\Omega, \mathcal{F}, P)$, such that $\Lambda(L\{X_t\}) = L\{Z_t\}$ for all t in $[0, \infty[$. Then ν is the Lévy measure for $L\{X_1\}$, so by the classical Lévy-Itô Theorem (cf. Theorem 2.9), (X_t) has a representation in the form:

$$X_t \overset{\text{a.s.}}{=} \gamma t + \sqrt{a} B_t + \int_{]0,t]\times\mathbb{R}} x N(\mathrm{d}u, \mathrm{d}x), \qquad (t \geq 0),$$

where (B_t) is a (classical) Brownian motion on $(\Omega, \mathcal{F}, P)$, N is a (classical) Poisson random measure on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb} \otimes \nu)$, defined on $(\Omega, \mathcal{F}, P)$ and (B_t) and N are independent. Put

$$Y_t := \int_{]0,t]\times\mathbb{R}} x \, N(\mathrm{d}u, \mathrm{d}x), \qquad (t \geq 0).$$

Now choose a free Brownian motion (W_t) in some W^*-probability space $(\mathcal{A}^1, \tau^1)$, and recall that $L\{W_t\} = \Lambda(L\{B_t\})$ for all t. Choose, further, a free Poisson random measure M on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb} \otimes \nu)$ with values in some W^*-probability space $(\mathcal{A}^2, \tau^2)$. Next, let $(\mathcal{A}^0, \tau^0)$ be the (reduced) free product of the two W^*-probability spaces $(\mathcal{A}^1, \tau^1)$ and $(\mathcal{A}^2, \tau^2)$ (cf. [VoDyNi92, Definition 1.6.1]). We may then consider $\mathcal{A}^1$ and $\mathcal{A}^2$ as two freely independent unital W^*-subalgebras of $\mathcal{A}^0$, such that $\tau^0_{|\mathcal{A}^1} = \tau^1$ and $\tau^0_{|\mathcal{A}^2} = \tau^2$. In particular, (W_t) and M are freely independent in $(\mathcal{A}^0, \tau^0)$.

Since $\int_{[-1,1]} |x| \, \nu(\mathrm{d}x) < \infty$, it follows from Lemma 6.25(ii) that for any t in $]0, \infty[$, the integral $U_t = \int_{]0,t]\times\mathbb{R}} x \, M(\mathrm{d}u, \mathrm{d}x)$ is well-defined, and $L\{U_t\} = \Lambda(L\{Y_t\})$. Furthermore, it follows immediately from Definition 6.16, Proposition 6.18 and Lemma 6.25 that for any t in $[0, t[$, $U_t = \int_{]0,t]\times\mathbb{R}} x \, M(\mathrm{d}u, \mathrm{d}x)$ is in the closure of $\mathcal{A}^2$ with respect to the measure topology. As noted in Remark 4.8, the set $\overline{\mathcal{A}^2}$ of closed, densely defined operators affiliated with $\mathcal{A}^2$ is complete (and hence closed) in the measure topology, and therefore U_t is affiliated with $\mathcal{A}^2$ for all t. This implies, in particular, that the two processes (W_t) and (U_t) are freely independent.

Now, for any t in $]0, \infty[$, we have

$$\begin{aligned}
L\big\{\gamma t\mathbf{1}_{\mathcal{A}^0} + \sqrt{a}W_t + U_t\big\} &= \delta_{\gamma t} \boxplus D_{\sqrt{a}}L\{W_t\} \boxplus L\{U_t\} \\
&= \Lambda(\delta_{\gamma t}) \boxplus D_{\sqrt{a}}\Lambda(L\{B_t\}) \boxplus \Lambda\big(L\{Y_t\}\big) \\
&= \Lambda\big(\delta_{\gamma t} * D_{\sqrt{a}}L\{B_t\} * L\{Y_t\}\big) \\
&= \Lambda\big(L\{\gamma t + \sqrt{a}B_t + Y_t\}\big) \\
&= \Lambda(L\{X_t\}) \\
&= L\{Z_t\},
\end{aligned}$$

and this proves (6.19). We prove next that the process (U_t) is a free Lévy process (in law). For this, recall that (Y_t) is a (classical) Lévy process defined on $(\Omega, \mathcal{F}, P)$ (cf. [Sa99, Theorem 19.3]), and such that $L\{U_t\} = \Lambda(L\{Y_t\})$ for all t. Since (Y_t) has stationary increments, we find for any s, t in $[0,\infty[$ that

$$\begin{aligned}
L\{U_{s+t} - U_s\} &= L\Big\{\int_{]s,s+t]\times\mathbb{R}} x\, M(\mathrm{d}u, \mathrm{d}x)\Big\} = \Lambda\Big(L\Big\{\int_{]s,s+t]\times\mathbb{R}} x\, N(\mathrm{d}u, \mathrm{d}x)\Big\}\Big) \\
&= \Lambda\big(L\{Y_{s+t} - Y_s\}\big) = \Lambda(L\{Y_t\}) = L\{U_t\},
\end{aligned}$$

where we have used Lemma 6.25(ii). Thus, (U_t) has stationary increments too. Furthermore, by continuity of Λ,

$$L\{U_t\} = \Lambda\big(L\{Y_t\}\big) \xrightarrow{\mathrm{w}} \Lambda(\delta_0) = \delta_0, \quad \text{as } t \searrow 0,$$

so that (U_t) is stochastically continuous. Finally, to prove that (U_t) has freely independent increments, consider r in $\mathbb{N}$ and $t_0, t_1, \dots, t_r$ in $[0,\infty[$, such that $0 = t_0 < t_1 < \cdots < t_r$. Then for any j in $\{1, 2, \dots, r\}$ we have (cf. Lemma 6.25) that

$$U_{t_j} - U_{t_{j-1}} = \int_{]t_{j-1},t_j]\times\mathbb{R}} x\, M(\mathrm{d}u, \mathrm{d}x) = \lim_{n\to\infty} \int_{]t_{j-1},t_j]\times[-n,n]} x\, M(\mathrm{d}u, \mathrm{d}x),$$

where the limit is taken in probability. Since

$$\int_{]t_{j-1},t_j]\times[-n,n]} |x|\, \mathrm{Leb}\otimes\nu(\mathrm{d}u, \mathrm{d}x) < \infty$$

for any n in $\mathbb{N}$ and any j in $\{1, 2, \dots, r\}$, it follows from Proposition 6.22 that for any n in $\mathbb{N}$, the integrals

$$\int_{]t_{j-1},t_j]\times[-n,n]} x\, M(\mathrm{d}u, \mathrm{d}x), \qquad j = 1, 2, \dots, r,$$

are freely independent operators. Hence, by Proposition 4.7, the increments

$$U_{t_1}, U_{t_2} - U_{t_1}, \dots, U_{t_r} - U_{t_{r-1}}$$

are also freely independent.

It remains to note that the right hand side of (6.19) is a free Lévy process (in law). This follows immediately from the fact that the sum of two freely independent free Lévy processes (in law) is again a free Lévy process (in law). Indeed, the stochastic continuity condition follows from the fact that addition is a continuous operation in the measure topology, and the remaining conditions are immediate consequences of basic properties of free independence. This concludes the proof. □

Theorem 6.27 (Free Lévy-Itô Decomposition II). *Let (Z_t) be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A}, \tau)$ and let ν be the Lévy measure appearing in the free characteristic triplet for $L\{Z_1\}$. Then (Z_t) has a representation in the form:*

$$Z_t \overset{\mathrm{d}}{=} \eta t \mathbf{1}_{\mathcal{A}^0} + \sqrt{a} W_t + V_t, \qquad (t \geq 0), \tag{6.20}$$

where

$\eta \in \mathbb{R}$, $a \geq 0$ and (W_t) is a free Brownian motion in a W^*-probability space $(\mathcal{A}^0, \tau^0)$.
(V_t) is a free Lévy process (in law) given by

$$V_t := \lim_{\epsilon \searrow 0} \Big[\int_{]0,t] \times D(\epsilon,\infty)} x \; M(\mathrm{d}u, \mathrm{d}x) - \Big(\int_{]0,t] \times D(\epsilon,1)} x \, \mathrm{Leb} \otimes \nu(\mathrm{d}u, \mathrm{d}x) \Big) \mathbf{1}_{\mathcal{A}^0} \Big],$$

where M is a free Poisson random measure on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb} \otimes \nu)$ with values in $(\mathcal{A}^0, \tau^0)$, and the limit is taken in probability.
(W_t) and (V_t) are freely independent processes.

Furthermore, the right hand side of (6.20), as a whole, is a free Lévy process (in law).

Proof. The proof proceeds along the same lines as that of Theorem 6.26, and we shall not repeat all the arguments. Let (X_t) be a classical Lévy process defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $L\{Z_t\} = \Lambda(L\{X_t\})$ for all t. In particular, the Lévy measure for $L\{X_1\}$ is ν. Hence, by Theorem 2.9(ii), (X_t) has a representation in the form

$$X_t \overset{\mathrm{a.s.}}{=} \eta t + \sqrt{a} B_t + Y_t, \qquad (t \geq 0),$$

where

$\eta \in \mathbb{R}$, $a \geq 0$ and (B_t) is a (classical) Brownian motion on $(\Omega, \mathcal{F}, P)$.
(Y_t) is a classical Lévy process given by

$$Y_t := \lim_{\epsilon \searrow 0} \Big[\int_{]0,t] \times D(\epsilon,\infty)} x \; N(\mathrm{d}u, \mathrm{d}x) - \int_{]0,t] \times D(\epsilon,1)} x \, \mathrm{Leb} \otimes \nu(\mathrm{d}u, \mathrm{d}x) \Big],$$

where N is a (classical) Poisson random measure on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb} \otimes \nu)$, defined on $(\Omega, \mathcal{F}, P)$, and the limit is almost surely.

(B_t) and (Y_t) are independent processes.

For all ϵ, t in $]0,\infty[$, we put:

$$Y_{\epsilon,t} = \int_{]0,t]\times D(\epsilon,\infty)} x\ N(\mathrm{d}u,\mathrm{d}x) - \int_{]0,t]\times D(\epsilon,1)} x\,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x),$$

so that $Y_t = \lim_{\epsilon\searrow 0} Y_{t,\epsilon}$ almost surely, for each t.

As in the proof of Theorem 6.26 above, we choose, next, a W^*-probability space $(\mathcal{A}^0,\tau^0)$, which contains a free Brownian motion (W_t) and a free Poisson random measure M on $(\mathsf{H},\mathsf{B}(\mathsf{H}),\mathrm{Leb}\otimes\nu)$, which generate freely independent W^*-subalgebras. For any ϵ in $]0,\infty[$, we put (cf. Lemma 6.25(i)),

$$V_{\epsilon,t} = \int_{]0,t]\times D(\epsilon,\infty)} x\ M(\mathrm{d}u,\mathrm{d}x) - \Big(\int_{]0,t]\times D(\epsilon,1)} x\,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x)\Big)\mathbf{1}_{\mathcal{A}^0}.$$

Then for any t in $]0,\infty[$ and any ϵ_1,ϵ_2 in $]0,1[$, such that $\epsilon_1 > \epsilon_2$, we have that

$$V_{\epsilon_2,t}-V_{\epsilon_1,t} = \int_{]0,t]\times D(\epsilon_2,\epsilon_1)} x\ M(\mathrm{d}u,\mathrm{d}x) - \Big(\int_{]0,t]\times D(\epsilon_2,\epsilon_1)} x\,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x)\Big)\mathbf{1}_{\mathcal{A}^0}.$$

Making the same calculation for $Y_{\epsilon_2,t} - Y_{\epsilon_1,t}$ and taking Corollary 6.20 into account, it follows that $L\{V_{\epsilon_2,t} - V_{\epsilon_1,t}\} = \Lambda(L\{Y_{\epsilon_2,t} - Y_{\epsilon_1,t}\})$. Hence, by continuity of Λ and completeness of the measure topology, we may conclude that the limit $V_t := \lim_{\epsilon\searrow 0} V_{\epsilon,t}$ exists in probability, and that $L\{V_t\} = \Lambda(L\{Y_t\})$. Moreover, as in the proof of Theorem 6.26, it follows that (W_t) and (V_t) are freely independent processes.

Now for any t in $]0,\infty[$, we have:

$$\begin{aligned} L\big\{\eta t\mathbf{1}_{\mathcal{A}^0} + \sqrt{a}W_t + V_t\big\} &= \delta_{\eta t}\boxplus D_{\sqrt{a}}L\{W_t\}\boxplus L\{V_t\} \\ &= \Lambda\big(\delta_{\eta t} * D_{\sqrt{a}}L\{B_t\} * L\{Y_t\}\big) = \Lambda\big(L\{X_t\}\big) = L\{Z_t\}. \end{aligned}$$

It remains to prove that (V_t) is a free Lévy process (in law). For this, note first that whenever $s,t\ge 0$, we have (cf. Lemma 6.25(i)),

$$\begin{aligned} &V_{s+t} - V_s \\ &= \lim_{\epsilon\searrow 0}\big(V_{\epsilon,s+t} - V_{\epsilon,s}\big) \\ &= \lim_{\epsilon\searrow 0}\Big[\int_{]s,s+t]\times D(\epsilon,\infty)} x\ M(\mathrm{d}u,\mathrm{d}x) - \Big(\int_{]s,s+t]\times D(\epsilon,1)} x\,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x)\Big)\mathbf{1}_{\mathcal{A}^0}\Big]. \end{aligned}$$

Making the same calculation for $Y_{s+t} - Y_s$, and taking Lemma 6.25(i) as well as the continuity of Λ into account, it follows that

$$L\{V_{s+t} - V_s\} = \Lambda(L\{Y_{s+t} - Y_s\}) = \Lambda(L\{Y_t\}) = L\{V_t\},$$

so that (V_t) has stationary increments. The stochastic continuity of (V_t) follows exactly as in the proof of Theorem 6.26. To see, finally, that (V_t) has freely independent increments, assume that $0 = t_0 < t_1 < t_2 < \cdots < t_r$, and consider ϵ in $]0,\infty[$. Then for any j in $\{1,2,\ldots,r\}$,

$$\begin{aligned} V_{\epsilon,t_j} - V_{\epsilon,t_{j-1}} = \lim_{n\to\infty} \Big[\int_{]t_{j-1},t_j]\times D(\epsilon,n)} x \; M(\mathrm{d}u,\mathrm{d}x) \\ - \Big(\int_{]t_{j-1},t_j]\times D(\epsilon,1)} x \, \mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x) \Big) \mathbf{1}_{\mathcal{A}^0} \Big]. \end{aligned}$$

Hence, by Proposition 6.22 and Proposition 4.7, the increments $V_{\epsilon,t_j} - V_{\epsilon,t_{j-1}}$, $j = 1,2,\ldots,r$ are freely independent, for any fixed positive ϵ. Yet another application of Proposition 4.7 then yields that the increments

$$V_{t_j} - V_{t_{j-1}} = \lim_{\epsilon\searrow 0} \big(V_{\epsilon,t_j} - V_{\epsilon,t_{j-1}} \big), \qquad (j = 1,2,\ldots,r),$$

are freely independent too. □

Remark 6.28. Let (Z_t) be a free Lévy process in law, such that $L\{Z_1\}$ has Lévy measure ν. If $\int_{[-1,1]} |x|\,\nu(\mathrm{d}x) < \infty$, then Theorems 6.26 and 6.27 provide two different "Lévy-Itô decompositions" of (Z_t). The relationship between the two representations, however, is simply that

$$\eta = \gamma + \int_{[-1,1]} x\,\nu(\mathrm{d}x) \quad \text{and} \quad V_t = U_t - t\Big(\int_{[-1,1]} x\,\nu(\mathrm{d}x) \Big) \mathbf{1}_{\mathcal{A}^0}, \qquad (t \geq 0).$$

Remark 6.29. The proof of the general free Lévy-Itô decomposition, Theorem 6.27, also provides a proof of the general existence of free Lévy processes (in law). Indeed, the conclusion of the proof of Theorem 6.27 might also be formulated in the following way: For any classical Lévy process (X_t), there exists a W^*-probability space $(\mathcal{A}^0,\tau^0)$ containing a free Brownian motion (W_t) and a free Poisson random measure M on $(\mathsf{H},\mathsf{B}(\mathsf{H}),\mathrm{Leb}\otimes\nu)$, which are freely independent, and such that

$$\begin{aligned} &\Lambda(L\{X_t\}) = \\ &L\Big\{ \eta t \mathbf{1}_{\mathcal{A}^0} + \sqrt{a} W_t + \\ &\lim_{\epsilon\searrow 0} \Big[\int_{]0,t]\times D(\epsilon,\infty)} x \; M(\mathrm{d}u,\mathrm{d}x) - \Big(\int_{]0,t]\times D(\epsilon,1)} x \, \mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x) \Big) \mathbf{1}_{\mathcal{A}^0} \Big] \Big\}, \end{aligned} \tag{6.21}$$

for suitable constants η in $\mathbb{R}$ and a in $]0,\infty[$. In addition, the process appearing in the right hand side of (6.21) is a free Lévy process (in law) affiliated with $(\mathcal{A}^0,\tau^0)$.

Assume now that $(\nu_t)_{t\geq 0}$ is a family of distributions in $\mathcal{ID}(\boxplus)$, satisfying the two conditions

$$\nu_t = \nu_s \boxplus \nu_{t-s}, \qquad (0 \leq s < t),$$

and

$$\nu_t \xrightarrow{\text{w}} \delta_0, \quad \text{as } t \searrow 0.$$

Then put $\mu_t = \Lambda^{-1}(\nu_t)$ for all t, and note that the family (μ_t) satisfies the corresponding conditions:

$$\mu_t = \mu_s * \mu_{t-s}, \qquad (0 \leq s < t),$$

and

$$\mu_t \xrightarrow{\text{w}} \delta_0, \quad \text{as } t \searrow 0,$$

by the properties of Λ^{-1}. Hence, by the well-known existence result for classical Lévy processes, there exists a classical Lévy process (X_t), such that $L\{X_t\} = \mu_t$ and hence $\Lambda(L\{X_t\}) = \nu_t$ for all t. Therefore, the right hand side of (6.21) is a free Lévy process (in law), (Z_t), such that $L\{Z_t\} = \nu_t$ for all t.

The above argument for the existence of free Lévy processes (in law) is, of course, based on the existence of free Poisson random measures proved in Theorem 6.9. The existence of free Lévy processes (in law) can also, as noted in [Bi98] and [Vo98], be proved directly by a construction similar to that given in the proof of Theorem 6.9. The latter approach, however, is somewhat more complicated than the construction given in the proof of Theorem 6.9, since, in the general case, one has to deal with unbounded operators throughout the construction, whereas free Poisson random measures only involve bounded operators.

A Unbounded Operators Affiliated with a W^*-Probability Space

In this appendix we give a brief account on the theory of closed, densely defined operators affiliated with a finite von Neumann algebra[10]. We start by introducing von Neumann algebras. For a detailed introduction to von Neumann algebras, we refer to [KaRi83], but also the paper [Ne74], referred to below, has a nice short introduction to that subject. For background material on unbounded operators, see [Ru91].

Let $\mathcal{H}$ be a Hilbert space, and consider the vector space $\mathcal{B}(\mathcal{H})$ of bounded (or continuous) linear mappings (or operators) $a\colon \mathcal{H} \to \mathcal{H}$. Recall that composition of operators constitutes a multiplication on $\mathcal{B}(\mathcal{H})$, and that the adjoint operation $a \mapsto a^*$ is an involution on $\mathcal{B}(\mathcal{H})$ (i.e. $(a^*)^* = a$). Altogether $\mathcal{B}(\mathcal{H})$ is a $*$-algebra[11]. For any subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$, we denote by $\mathcal{S}'$ the *commutant*

[10] To make the appendix appear in self-contained form, some of the definitions that already appeared in Section 4.1 will be repeated below.

[11] Throughout this appendix, the $*$ refers to the adjoint operation and not to classical convolution.

of $\mathcal{S}$, i.e.

$$\mathcal{S}' = \{b \in \mathcal{B}(\mathcal{H}) \mid by = yb \text{ for all } y \text{ in } \mathcal{S}\}.$$

A *von Neumann algebra* acting on $\mathcal{H}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, which contains the multiplicative unit $\mathbf{1}$ of $\mathcal{B}(\mathcal{H})$, and which is closed under the adjoint operation and closed in the weak operator topology (see [KaRi83, Definition 5.1.1]). By von Neumann's fundamental double commutant theorem, a von Neumann algebra may also be characterized as a subset $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, which is closed under the adjoint operation and equals the commutant of its commutant: $\mathcal{A}'' = \mathcal{A}$.

A trace (or tracial state) on a von Neumann algebra $\mathcal{A}$ is a positive linear functional $\tau\colon \mathcal{A} \to \mathbb{C}$, satisfying that $\tau(\mathbf{1}) = 1$ and that $\tau(ab) = \tau(ba)$ for all a, b in $\mathcal{A}$. We say that τ is a normal trace on $\mathcal{A}$, if, in addition, τ is continuous on the unit ball of $\mathcal{A}$ w.r.t. the weak operator topology. We say that τ is faithful, if $\tau(a^*a) > 0$ for any non-zero operator a in $\mathcal{A}$.

We shall use the terminology W^**-probability space* for a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and $\tau\colon \mathcal{A} \to \mathbb{C}$ is a faithful, normal tracial state on $\mathcal{A}$. In the remaining part of this appendix, $(\mathcal{A}, \tau)$ denotes a W^*-probability space acting on the Hilbert space $\mathcal{H}$.

By a linear operator *in* $\mathcal{H}$, we shall mean a (not necessarily bounded) linear operator $a\colon \mathcal{D}(a) \to \mathcal{H}$, defined on a subspace $\mathcal{D}(a)$ of $\mathcal{H}$. For an operator a in $\mathcal{H}$, we say that

- a is *densely defined*, if $\mathcal{D}(a)$ is dense in $\mathcal{H}$,
- a is *closed*, if the graph $\mathcal{G}(a) = \{(h, ah) \mid h \in \mathcal{D}(a)\}$ of a is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$,
- a is *preclosed*, if the norm closure $\overline{\mathcal{G}(a)}$ is the graph of a (uniquely determined) operator, denoted $[a]$, in $\mathcal{H}$,
- a is *affiliated with* $\mathcal{A}$, if $au = ua$ for any unitary operator u in the commutant $\mathcal{A}'$.

For a densely defined operator a in $\mathcal{H}$, the *adjoint operator* a^* has domain

$$\mathcal{D}(a^*) = \Big\{\eta \in \mathcal{H} \;\Big|\; \sup\{|\langle a\xi, \eta\rangle| \mid \xi \in \mathcal{D}(a),\ \|\xi\| \le 1\} < \infty\Big\},$$

and is given by

$$\langle a\xi, \eta\rangle = \langle \xi, a^*\eta\rangle, \qquad (\xi \in \mathcal{D}(a),\ \eta \in \mathcal{D}(a^*)).$$

We say that a is *selfadjoint* if $a = a^*$ (in particular this requires that $\mathcal{D}(a^*) = \mathcal{D}(a)$).

If a is bounded, a is affiliated with $\mathcal{A}$ if and only if $a \in \mathcal{A}$. In general, a selfadjoint operator a in $\mathcal{H}$ is affiliated with $\mathcal{A}$, if and only if $f(a) \in \mathcal{A}$ for any *bounded Borel* function $f\colon \mathbb{R} \to \mathbb{C}$ (here $f(a)$ is defined in terms of spectral theory). As in the bounded case, if a is a selfadjoint operator affiliated with $\mathcal{A}$, there exists a unique probability measure μ_a on $\mathbb{R}$, concentrated on the spectrum $\mathrm{sp}(a)$, and satisfying that

$$\int_{\mathbb{R}} f(t)\ \mu_a(\mathrm{d}t) = \tau(f(a)),$$

for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{C}$. We call μ_a the (spectral) distribution of a, and we shall denote it also by $L\{a\}$. Unless a is bounded, $\mathrm{sp}(a)$ is an unbounded subset of $\mathbb{R}$ and, in general, μ_a is not compactly supported.

By $\overline{\mathcal{A}}$ we denote the set of closed, densely defined operators in $\mathcal{H}$, which are affiliated with $\mathcal{A}$. In general, dealing with unbounded operators is somewhat unpleasant, compared to the bounded case, since one needs constantly to take the domains into account. However, the following two important propositions allow us to deal with operators in $\overline{\mathcal{A}}$ in a quite relaxed manner.

Proposition A.1 (cf. [Ne74]). *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. If $a, b \in \overline{\mathcal{A}}$, then $a+b$ and ab are densely defined, preclosed operators affiliated with $\mathcal{A}$, and their closures $[a+b]$ and $[ab]$ belong to $\overline{\mathcal{A}}$. Furthermore, $a^* \in \overline{\mathcal{A}}$.*

By virtue of the proposition above, the adjoint operation may be restricted to an involution on $\overline{\mathcal{A}}$, and we may define operations, the *strong sum* and the *strong product*, on $\overline{\mathcal{A}}$, as follows:

$$(a, b) \mapsto [a+b], \quad \text{and} \quad (a, b) \mapsto [ab], \quad (a, b \in \overline{\mathcal{A}}).$$

Proposition A.2 (cf. [Ne74]). *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. Equipped with the adjoint operation and the strong sum and product, $\overline{\mathcal{A}}$ is a $*$-algebra.*

The effect of the above proposition is, that w.r.t. the adjoint operation and the strong sum and product, we can manipulate with operators in $\overline{\mathcal{A}}$, without worrying about domains etc. So, for example, we have rules like

$$[[a+b]c] = [[ac]+[bc]], \quad [a+b]^* = [a^*+b^*], \quad [ab]^* = [b^*a^*],$$

for operators a, b, c in $\overline{\mathcal{A}}$. Note, in particular, that the strong sum of two selfadjoint operators in $\overline{\mathcal{A}}$ is again a selfadjoint operator. In the following, we shall omit the brackets in the notation for the strong sum and product, and it will be understood that all sums and products are formed in the strong sense.

Remark A.3. If $a_1, a_2 \dots, a_r$ are selfadjoint operators in $\overline{\mathcal{A}}$, we say that they are *freely independent* if, for any bounded Borel functions $f_1, f_2, \dots, f_r\colon \mathbb{R} \to \mathbb{R}$, the bounded operators $f_1(a_1), f_2(a_2), \dots, f_r(a_r)$ in $\mathcal{A}$ are freely independent in the sense of Section 4. Given any two probability measures μ_1 and μ_2 on $\mathbb{R}$, it follows from a free product construction (see [VoDyNi92]), that one can always find a W^*-probability space $(\mathcal{A}, \tau)$ and selfadjoint operators a and b affiliated with $\mathcal{A}$, such that $\mu_1 = L\{a\}$ and $\mu_2 = L\{b\}$. As noted above, for such operators $a+b$ is again a selfadjoint operator in $\overline{\mathcal{A}}$, and, as was proved in [BeVo93, Theorem 4.6], the (spectral) distribution $L\{a+b\}$ depends only on μ_1 and μ_2. We may thus define the free additive convolution $\mu_1 \boxplus \mu_2$ of μ_1 and μ_2 to be $L\{a+b\}$.

Next, we shall equip $\overline{\mathcal{A}}$ with a topology; the so called measure topology, which was introduced by Segal in [Se53] and later studied by Nelson in [Ne74]. For any positive numbers ϵ, δ, we denote by $N(\epsilon, \delta)$ the set of operators a in $\overline{\mathcal{A}}$, for which there exists an orthogonal projection p in $\mathcal{A}$, satisfying that

$$p(\mathcal{H}) \subseteq \mathcal{D}(a), \quad \|ap\| \leq \epsilon \quad \text{and} \quad \tau(p) \geq 1 - \delta. \tag{A.1}$$

Definition A.4. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. The measure topology on $\overline{\mathcal{A}}$ is the vector space topology on $\overline{\mathcal{A}}$ for which the sets $N(\epsilon, \delta)$, $\epsilon, \delta > 0$, form a neighbourhood basis for 0.*

It is clear from the definition of the sets $N(\epsilon, \delta)$ that the measure topology satisfies the first axiom of countability. In particular, all convergence statements can be expressed in terms of sequences rather than nets.

Proposition A.5 (cf. [Ne74]). *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and consider the $*$-algebra $\overline{\mathcal{A}}$. We then have*

(i) *Scalar-multiplication, the adjoint operation and strong sum and product are all continuous operations w.r.t. the measure topology. Thus, $\overline{\mathcal{A}}$ is a topological $*$-algebra w.r.t. the measure topology.*
(ii) *The measure topology on $\overline{\mathcal{A}}$ is a complete Hausdorff topology.*

We shall note, next, that the measure topology on $\overline{\mathcal{A}}$ is, in fact, the topology for convergence in probability. Recall first, that for a closed, densely defined operator a in $\mathcal{H}$, we put $|a| = (a^*a)^{1/2}$. In particular, if $a \in \overline{\mathcal{A}}$, then $|a|$ is a selfadjoint operator in $\overline{\mathcal{A}}$ (see [KaRi83, Theorem 6.1.11]), and we may consider the probability measure $L\{|a|\}$ on $\mathbb{R}$.

Definition A.6. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and let a and a_n, $n \in \mathbb{N}$, be operators in $\overline{\mathcal{A}}$. We say then that $a_n \to a$ in probability, as $n \to \infty$, if $|a_n - a| \to 0$ in distribution, i.e. if $L\{|a_n - a|\} \to \delta_0$ weakly.*

If a and a_n, $n \in \mathbb{N}$, are *selfadjoint* operators in $\overline{\mathcal{A}}$, then, as noted above, $a_n - a$ is selfadjoint for each n, and $L\{|a_n - a|\}$ is the transformation of $L\{a_n - a\}$ by the mapping $t \mapsto |t|$, $t \in \mathbb{R}$. In this case, it follows thus that $a_n \to a$ in probability, if and only if $a_n - a \to 0$ in distribution, i.e. if and only if $L\{a_n - a\} \to \delta_0$ weakly.

From the definition of $L\{|a_n - a|\}$, it follows immediately that we have the following characterization of convergence in probability:

Lemma A.7. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and let a and a_n, $n \in \mathbb{N}$, be operators in $\overline{\mathcal{A}}$. Then $a_n \to a$ in probability, if and only if*

$$\forall \epsilon > 0\colon \tau\big[1_{]\epsilon,\infty[}(|a_n - a|)\big] \to 0, \quad \textit{as } n \to \infty.$$

Proposition A.8 (cf. [Te81]). *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. Then for any positive numbers ϵ, δ, we have*

$$N(\epsilon, \delta) = \big\{a \in \overline{\mathcal{A}} \bigm| \tau\big[1_{]\epsilon,\infty[}(|a|)\big] \le \delta\big\}, \tag{A.2}$$

where $N(\epsilon, \delta)$ is defined via (A.1). In particular, a sequence a_n in $\overline{\mathcal{A}}$ converges, in the measure topology, to an operator a in $\overline{\mathcal{A}}$, if and only if $a_n \to a$ in probability.

Proof. The last statement of the proposition follows immediately from formula (A.2) and Lemma A.7. To prove (A.2), note first that by considering the polar decomposition of an operator a in $\overline{\mathcal{A}}$ (cf. [KaRi83, Theorem 6.1.11]), it follows that $N(\epsilon, \delta) = \{a \in \overline{\mathcal{A}} \mid |a| \in N(\epsilon, \delta)\}$. From this, the inclusion $\supseteq$ in (A.2) follows easily. Regarding the reverse inclusion, suppose $a \in N(\epsilon, \delta)$, and let p be a projection in $\mathcal{A}$, such that (A.1) is satisfied with a replaced by $|a|$. Then, using spectral theory, it can be shown that the ranges of the projections p and $1_{]\epsilon,\infty[}(|a|)$ only have 0 in common. This implies that $\tau[1_{]\epsilon,\infty[}(|a|)] \le \tau(\mathbf{1} - p) \le \delta$. We refer to [Te81] for further details. □

Finally, we shall need the fact that convergence in probability implies convergence in distribution, also in the non-commutative setting. The key point in the proof given below is that weak convergence can be expressed in terms of the Cauchy transform (cf. [Ma92, Theorem 2.5]).

Proposition A.9. *Let (a_n) be a sequence of selfadjoint operators affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, and assume that a_n converges in probability, as $n \to \infty$, to a selfadjoint operator a affiliated with $(\mathcal{A}, \tau)$. Then $a_n \to a$ in distribution too, i.e. $L\{a_n\} \xrightarrow{\text{w}} L\{a\}$, as $n \to \infty$.*

Proof. Let x, y be real numbers such that $y > 0$, and put $z = x + \mathrm{i}y$. Then define the function $f_z \colon \mathbb{R} \to \mathbb{C}$ by

$$f_z(t) = \frac{1}{t - z} = \frac{1}{(t - x) - \mathrm{i}y}, \qquad (t \in \mathbb{R}),$$

and note that f_z is continuous and bounded with $\sup_{t\in\mathbb{R}} |f_z(t)| = y^{-1}$. Thus, we may consider the bounded operators $f_z(a_n), f_z(a) \in \mathcal{A}$. Note then that (using strong products and sums),

$$\begin{aligned} f_z(a_n) - f_z(a) &= (a_n - z\mathbf{1})^{-1} - (a - z\mathbf{1})^{-1} \\ &= (a_n - z\mathbf{1})^{-1}\big((a - z\mathbf{1}) - (a_n - z\mathbf{1})\big)(a - z\mathbf{1})^{-1} \\ &= (a_n - z\mathbf{1})^{-1}(a - a_n)(a - z\mathbf{1})^{-1}. \end{aligned} \tag{A.3}$$

Now, given any positive numbers ϵ, δ, we may choose N in $\mathbb{N}$, such that $a_n - a \in N(\epsilon, \delta)$, whenever $n \ge N$. Moreover, since $\|f_z(a_n)\|, \|f_z(a)\| \le y^{-1}$, we have that $f_z(a_n), f_z(a) \in N(y^{-1}, 0)$. Using then the rule: $N(\epsilon_1, \delta_1)N(\epsilon_2, \delta_2) \subseteq N(\epsilon_1\epsilon_2, \delta_1 + \delta_2)$, which holds for all ϵ_1, ϵ_2 in $]0, \infty[$ and δ_1, δ_2 in $[0, \infty[$ (see

[Ne74, Formula 17']), it follows from (A.3) that $f_z(a_n) - f_z(a) \in N(\epsilon y^{-2}, \delta)$, whenever $n \geq N$. We may thus conclude that $f_z(a_n) \to f_z(a)$ in the measure topology, i.e. that $L\{|f_z(a_n) - f_z(a)|\} \xrightarrow{\text{w}} \delta_0$, as $n \to \infty$. Using now the Cauchy-Schwarz inequality for τ, it follows that

$$\begin{aligned} \left|\tau(f_z(a_n) - f_z(a))\right|^2 &\leq \tau(|f_z(a_n) - f_z(a)|^2) \cdot \tau(\mathbf{1}) \\ &= \int_0^\infty t^2 \; L\{|f_z(a_n) - f_z(a)|\}(\mathrm{d}t) \longrightarrow 0, \end{aligned}$$

as $n \to \infty$, since $\mathrm{supp}(L\{|f_z(a_n) - f_z(a)|\}) \subseteq [0, 2y^{-1}]$ for all n, and since $t \mapsto t^2$ is a continuous bounded function on $[0, 2y^{-1}]$.

Finally, let G_n and G denote the Cauchy transforms for $L\{a_n\}$ and $L\{a\}$ respectively. From what we have established above, it follows then that

$$G_n(z) = -\tau(f_z(a_n)) \longrightarrow -\tau(f_z(a)) = G(z), \quad \text{as } n \to \infty,$$

for any complex number $z = x + \mathrm{i}y$ for which $y > 0$. By [Ma92, Theorem 2.5], this means that $L\{a_n\} \xrightarrow{\text{w}} L\{a\}$, as desired. □

References

[An00] M. Anchelevich, *Free stochastic measures via noncrossing partitions*, Adv. Math. **155** (2000), 154-179. 130

[An01] M. Anshelevich, *Partition-dependent Stochastic Measures and q-deformed Cumulants*, Doc. Math. **6** (2001), 343–384. 114

[An02] M. Anshelevich, *Itô Formula for Free Stochastic Integrals*, J. Funct. Anal. **188** (2002), 292–315.

[BaCo89] O.E. Barndorff-Nielsen and D.R. Cox, *Asymptotic Techniques for Use in Statistics*, Monographs on Statistics and Applied Probability, Chapman and Hall (1989). 102, 103

[Ba98] O.E. Barndorff-Nielsen, *Processes of normal inverse Gaussian type*, Finance and Stochastics **2** (1998), 41-68. 45

[BaMiRe01] O.E. Barndorff-Nielsen, T. Mikosch and S. Resnick (Eds.), *Lévy Processes - Theory and Applications*, Boston: Birkhäuser (2001). 35, 45

[BaPeSa01] O.E. Barndorff-Nielsen, J. Pedersen and k. Sato, *Multivariate subordination, selfdecomposability and stability*, Adv. Appl. Prob. **33** (2001), 160-187.

[BaSh01a] O.E. Barndorff-Nielsen and N. Shephard, *Non-Gaussian OU based models and some of their uses in financial economics (with Discussion)*, J. R. Statist. Soc. B **63** (2001), 167-241. 45, 130

[BaSh01b] O.E. Barndorff-Nielsen and N. Shephard, *Modelling by Lévy processes for financial econometrics*, in O.E. Barndorff-Nielsen, T. Mikosch and S. Resnick (Eds.): *Lévy Processes - Theory and Applications*, Boston: Birkhäuser (2001), 283-318. 45, 130

[BaLi04] O.E. Barndorff-Nielsen and A. Lindner, *Some aspects of Lévy copulas.* (2004) (Submitted.). 90

[BaMaSa04] O.E. Barndorff-Nielsen, M. Maejima and K. Sato, *Some classes of multivariate infinitely divisible distributions admitting stochastic integral representation*. Bernoulli (To appear). 61, 89, 90

[BaPA05] O.E. Barndorff-Nielsen and V. Pérez-Abreu, *Matrix subordinators and related Upsilon transformations.* (In preparation). 90

[BaTh02a] O.E. Barndorff-Nielsen and S. Thorbjørnsen, *Selfdecomposability and Lévy processes in free probability*, Bernoulli **8** (2002), 323-366. 35

[BaTh02b] O.E. Barndorff-Nielsen and S. Thorbjørnsen, *Lévy laws in free probability*, Proc. Nat. Acad. Sci., vol. 99, no. 26 (2002), 16568-16575. 35

[BaTh02c] O.E. Barndorff-Nielsen and S. Thorbjørnsen, *Lévy processes in free probability*, Proc. Nat. Acad. Sci., vol. 99, no. 26 (2002), 16576-16580. 35

[BaTh04a] O.E. Barndorff-Nielsen and S. Thorbjørnsen, *A connection between free and classical infinite divisibility*, Inf. Dim. Anal. Quant. Prob. **7** (2004), 573-590. 35

[BaTh04b] O.E. Barndorff-Nielsen and S. Thorbjørnsen, *Regularising mappings of Lévy measures*, Stoch. Proc. Appl. (To appear). 35

[BaTh04c] O.E. Barndorff-Nielsen and S. Thorbjørnsen, *Bicontinuity of the Upsilon transformations*, MaPhysto Research Report 2004-**25**, University of Aarhus (Submitted). 59, 83

[BaTh05] O.E. Barndorff-Nielsen and S. Thorbjørnsen, *The Lévy-Itô Decomposition in Free Probability*, Prob. Theory and Rel. Fields, **131** (2005), 197-228. 35

[BeVo93] H. Bercovici and D.V. Voiculescu, *Free Convolution of Measures with Unbounded Support*, Indiana Univ. Math. J. **42** (1993), 733-773. 100, 101, 103, 107, 108, 152

[BePa96] H. Bercovici and V. Pata, *The Law of Large Numbers for Free Identically Distributed Random Variables*, Ann. Probability **24** (1996), 453-465. 101

[BePa99] H. Bercovici and V. Pata, *Stable Laws and Domains of Attraction in Free Probability Theory*, Ann. Math. **149** (1999), 1023-1060. 113, 116

[BePa00] H. Bercovici and V. Pata, *A Free Analogue of Hincin's Characterization of Infinite Divisibility*, Proc. AMS. **128** (2000), 1011-1015. 106

[Be96] J. Bertoin, *Lévy Processes*, Cambridge University Press (1996). 35

[Be97] J. Bertoin, *Subordinators: Examples and Applications*, in P. Bernard (Ed.): *Lectures on Probability Theory and Statistics*, Ecole d'Éte de St-Flour XXVII, Berlin: Springer-Verlag (1997), 4-91. 35

[Be00] J. Bertoin, *Subordinators, Lévy processes with no negative jumps and branching processes*, MaPhySto Lecture Notes Series (2000-**8**), (Århus University).

[Bi98] P. Biane, *Processes with free increments*, Math. Zeitschrift **227** (1998), 143-174. 112, 123, 150

[Bi03] P. Biane, *Free probability for probabilists*, Quantum probability communications, Vol. XI (Grenoble, 1998), 55–71, QP-PQ, XI, World Sci. Publishing (2003). 92

[BiSp98] P. Biane and R. Speicher, *Stochastic calculus with respect to free Brownian motion and analysis on Wigner space*, Probab. Theory Related Fields **112** (1998), 373-409.

[Bo92] L. Bondesson, *Generalized Gamma Convolutions and Related Classes of Distributions and Densities*, Lecture Notes in Statistics **76**, Berlin: Springer-Verlag (1992). 45

[BoSp91] M. Bożejko and R. Speicher, *An example of a generalized Brownian motion*, Comm. Math. Phys. **137** (1991), 519-531. 117

[Br92] L. Breiman, *Probability*, Classics In Applied Mathematics 7, SIAM (1992). 118

[BrReTw82] P.J. Brockwell, S.I. Resnick and R.L. Tweedie *Storage processes with general release rule and additive inputs.*, Adv. Appl. Prob. **14** (1982), 392-433. 45

[ChYo03] L. Chaumont and M. Yor, *Exercises in Probability*. Cambridge University Press (2003). 74

[ChSh02] A.S. Cherny and A.N. Shirayev, *On Stochastic Integrals up to infinity and Predictable Criteria for integrability*, Notes from a MaPhySto Summerschool, August 2002. 37

[Ch78] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, Science Publishers (1978). 101

[Do74] W.F. Donoghue, Jr., *Monotone Matrix Functions and Analytic Continuation*, Grundlehren der mathematichen Wissenschaften **207**, Springer-Verlag (1974). 103

[Fe71] W. Feller, *An Introduction to Probability Theory and its Applications, volume II*, Wiley (1971). 45, 51, 74, 75, 80

[Ge80] S. Geman, *A limit theorem for the norm of random matrices*, Annals of Probability **8** (1980), 252-261.

[GlScSp92] P. Glockner, M. Schürmann and R. Speicher, *Realization of free white noises*, Arch. Math. **58** (1992), 407-416. 140

[GnKo68] B.V. Gnedenko and A.N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley Publishing Company, Inc. (1968). 41, 85, 103, 117, 119, 120

[GrRoVaYo99] M. Gradinaru, B. Roynette, P. Vallois and M. Yor, *Abel transform and integrals of Bessel local time*, Ann. Inst. Henri Poincaré **35**, 531-572. 74

[HiPe00] F. Hiai and D. Petz, *The Semicircle Law, Free Random Variables and Entropy*, Mathematical Surveys and Monographs, Vol. 77. Providence: American Mathematical Society (2000).

[JuVe83] Z.J. Jurek and W. Verwaat, *An integral representation for selfdecomposable Banach space valued random variables*, Z. Wahrscheinlichkeitstheorie verw. Geb. **62** (1983), 247-262. 45, 127, 128, 129

[JuMa93] Z.J. Jurek and J.D. Mason, *Operator-Limit Distributions in Probability Theory*, New York: Wiley (1993). 45

[KaRi83] R.V. Kadison and J.R. Ringrose, *Fundamentals of the theory of operator algebras, vol. I-II*, Academic Press (1983, 1986). 133, 134, 135, 136, 150, 151, 153, 154

[LG99] J.-F. Le Gall, *Spatial Branching Processes, Random Snakes and Partial Differential Equations*, Basel: Birkhäuser (1999).

[Lu75] E. Lukacs, *Stochastic Convergence (second edition)*, Academic Press (1975). 39, 124, 125, 127

[Ma92] H. Maassen, *Addition of freely independent random variables*, J. Funct. Anal. **106**, (1992), 409-438. 101, 103, 154, 155

[Ne74] E. Nelson, *Notes on Non-commutative Integration*, J. Funct. Anal. **15** (1974), 103-116. 150, 152, 153, 155

[Ni95] A. Nica, *A one-parameter family of transforms, linearizing convolution laws for probability distributions*, Comm. Math. Phys. **168** (1995), 187-207. 117

[MoOs69] S.A. Molchanov and E. Ostrovskii, *Symmetric stable processes as traces of degenerate diffusion processes*, Teor. Verojatnost. Primen. **14** (1969), 127-130. 74

[Pa96] V. Pata, *Domains of partial attraction in non-commutative probability*, Pacific J. Math. **176** (1996), 235-248. 103

[Pe89] G.K. Pedersen, *Analysis Now*, Graduate Texts in Mathematics **118**, Springer Verlag (1989). 98

[PeSa04] J. Pedersen and K. Sato, *Semigroups and processes with parameter in a cone*, Abstract and applied analysis, 499-513, World Sci. Publ., River Edge, NJ, (2004).

[Ro64] G.-C. Rota, *On the foundations of combinatorial theory I: Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie Verw. Geb. **2** (1964), 340-368. 102

[Ros02] J. Rosinski, *Tempered stable processes*, In O.E. Barndorff-Nielsen (Ed.), *Second MaPhySto Conference on Lévy Processes: Theory and Applications*, Aarhus: MaPhySto (2002), 215-220.

[Ros04] J. Rosinski, *Tempering stable processes*, Preprint (2004). 45

[Ru91] W. Rudin, *Functional Analysis (second edition)*, McGraw-Hill Inc. (1991). 94, 150

[Sa99] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge studies in advanced math. **68** (1999). 35, 37, 39, 43, 45, 47, 62, 85, 121, 122, 141, 144, 146

[Sa00] K. Sato, *Subordination and selfdecomposability*, MaPhySto Research Report (2000-**40**), (Århus University). 37, 46

[Se53] I.E. Segal, *A non-commutative extension of abstract integration*, Ann. Math. **57** (1953), 401-457; correction **58** (1953), 595-596. 153

[SaTa94] G. Samorodnitsky and M.S. Taqqu, *Stable Non-Gaussian Random Processes*, New York: Chapman and Hall (1994). 45

[Sk91] A.V. Skorohod, *Random Processes with Independent Increments*, Kluwer Academic Publisher (1991), Dordrecht, Netherlands (Russian original 1986). 91

[Sp90] R. Speicher, *A new example of 'independence' and 'white noise'*, Probab. Th. Rel. Fields **84** (1990), 141-159. 130

[Sp94] R. Speicher, *Multiplicative functions on the lattice of non-crossing partitions and free convolution*, Math. Ann. **298** (1994), 611-628. 102, 103

[Sp97] R. Speicher, *Free Probability Theory and Non-crossing Partitions*, Sém. Lothar. Combin. **39** (1997), Article B39c (`electronic`). 102, 103

[St59] W.F. Stinespring, *Integration theory for gages and duality for unimodular groups*, Transactions of the AMS. **90** (1959), 15-56. 99

[Te81] M. Terp, *L^p Spaces associated with von Neumann Algebras*, Lecture notes, University of Copenhagen (1981). 154

[Th77] O. Thorin, *On the infinite divisibility of the Pareto distribution*, Scand. Actuarial J. (1977), 31-40. 45

[Th78] O. Thorin, *An extension of the notion of a generalized Γ-convolution*, Scand. Actuarial J. (1978), 141-149.

[Vo85] D.V. Voiculescu, *Symmetries of some reduced free product C^*-algebras*, Operator Algebras and their Connections with Topology and Ergodic Theory, Lecture Notes in Math. **1132** (1985), Springer Verlag, 556-588. 95

[Vo86] D.V. Voiculescu, *Addition of certain non-commuting random variables*, J. Funct. Anal. **66**, (1986), 323-346. 101, 103

[Vo90] D.V. Voiculescu, *Circular and semicircular systems and free product factors*, in "Operator Algebras, Unitary Representations, Enveloping Algebras and Invariant Theory", *Progress in Mathematics* **92**, Birkhäuser (1990), 45-60. 95, 133

[Vo91] D.V. Voiculescu, *Limit laws for random matrices and free products*, Invent. Math. **104** (1991), 201-220. 95, 96

[VoDyNi92] D.V. Voiculescu, K.J. Dykema and A. Nica, *Free Random Variables*, CRM Monographs Series, vol. 1, A.M.S. (1992). 92, 96, 99, 100, 106, 108, 132, 145, 152

[Vo98] D.V. Voiculescu, *Lectures on Free Probability*, Lecture notes from the 1998 Saint-Fluor Summer School on Probability Theory. 92, 95, 112, 131, 150

[Wo82] S.J. Wolfe, *On a Continuous Analogue of the Stochastic Difference Equation* $X_n = \rho X_{n-1} + B_n$, Stochastic Process. Appl. **12** (1982), 301-312. 45