5

Dualisability and algebraic constructions

We show that there are many natural algebraic constructions under which dualisability is not always preserved. In particular, we find two dualisable algebras whose product is not dualisable.

Dualisability is such a natural algebraic property that it is tempting to suppose that it might interact well with natural algebraic constructions. This idea is supported by what is known about dualisability and one-point extensions. Davey and Knox [25] have shown that the one-point extensions of certain dualisable non-unary algebras are also dualisable. In this chapter, we shall prove that the one-point extension of a dualisable unary algebra is also dualisable.

Following this line of investigation, it is natural to ask whether each subalgebra of a dualisable algebra must be dualisable, whether each homomorphic image of a dualisable algebra must be dualisable, and whether a product of dualisable algebras must be dualisable. The subalgebra question was answered in the negative in Theorem 2.1.4: every non-dualisable unary algebra is a subalgebra of a dualisable algebra. In relation to the product question, it is known that each finite power of a dualisable algebra is dualisable. By the Independence Theorem, 1.4.1, any two finite algebras that generate the same quasi-variety must either both be dualisable or both be non-dualisable. The quasi-varieties $\mathbb{ISP}(\underline{\mathbf{M}})$ and $\mathbb{ISP}(\underline{\mathbf{M}}^n)$ coincide for each finite algebra $\underline{\mathbf{M}}$ and each $n \in \omega\backslash\{0\}$. So it follows that every finite power of a dualisable algebra is also dualisable.

The range of examples of dualisable and non-dualisable algebras that are useful for testing conjectures is growing. But there are presently not many naturally occurring varieties, containing both dualisable and non-dualisable algebras, for which there is a complete characterisation of dualisability. There are many varieties in which *every* finite algebra is known to be dualisable.

Every finite lattice is dualisable [29]. Similarly, each finite abelian group, Boolean algebra and semilattice is dualisable [17]. We also have complete descriptions of dualisability for some classes that are not varieties: for example, the class of graph algebras [23], and the class of three-element unary algebras (Theorem 3.0.1).

One variety of algebras for which there is an interesting characterisation of dualisability is that of commutative rings with identity. Consider an arbitrary finite commutative ring with identity $\underline{\mathbf{R}} = \langle R; +, \cdot, ^{-}, 0, 1\rangle$. An element $r \in R$ is said to be nilpotent if $r^n = 0$, for some $n \in \omega\backslash\{0\}$. The set J of all nilpotent elements of $\underline{\mathbf{R}}$ coincides with the Jacobson radical of $\underline{\mathbf{R}}$. We say that J is self annihilating if $rs = 0$, for all $r, s \in J$. Clark, Idziak, Sabourin, Szabó and Willard [14] have proven that the ring $\underline{\mathbf{R}}$ is dualisable if and only if its Jacobson radical J is self annihilating. It is now easy to check that the class of all dualisable commutative rings with identity is closed under taking subalgebras and finite products. It is also possible, though not quite so easy, to prove that every homomorphic image of a dualisable commutative ring with identity is dualisable.

Quackenbush and Szabó [59, 60] have studied dualisability within the variety of groups. A finite group is dualisable if all of its Sylow subgroups are cyclic. A finite group is non-dualisable if it has a non-abelian Sylow subgroup. These two results do not provide us with any examples of non-dualisable products of dualisable groups, or of dualisable groups with non-dualisable homomorphic images.

Within any congruence-distributive variety, a finite algebra is dualisable if and only if it has a near-unanimity term [29, 22]. However, within most naturally occurring congruence-distributive varieties, either all algebras have a near-unanimity term (for example, lattice-based varieties), or no non-trivial algebra has a near-unanimity term (for example, implication algebras [49]). There may be congruence-distributive varieties in which dualisability is not always preserved by taking products. In this chapter, we choose to focus our efforts instead on unary algebras and p-semilattices.

A **p-semilattice** is a bounded meet semilattice $\mathbf{P} = \langle P; \wedge, ^*, 0, 1\rangle$ with a unary operation * such that

$$a^* = \max\{\, b \in P \mid a \wedge b = 0\,\},$$

for every $a \in P$. The class of p-semilattices forms a variety: a finite equational basis was found by Balbes and Horn [2]. A p-semilattice is said to be **boolean** if it satisfies the equation $x^{**} \approx x$. We will prove that a finite p-semilattice is dualisable if and only if it is boolean. So the class of dual-

construction	*preserves dualisability?*	*references*
subalgebra	✗	2.1.4, 5.3.2, 7.1.2
homomorphic image	✗	5.3.3, 5.3.4
retract	✗	5.3.3, 5.3.4
term retract	✓	5.2.1
finite power	✓	[17, 61, 30]
finite product	✗	5.3.8
finite coproduct	✗	5.3.8
one-point extension	?	[25], 5.1.11

Table 5.1

construction on unary algebras	*preserves dualisability?*	*references*
finite disjoint union	✗	5.3.8
one-point extension	✓	5.1.11
pointed one-point extension	✓	6.3.2
disjoint union with self	✓	5.1.10
disjoint union with finite algebra in q-v	✓	5.1.10
finite distant union	?	6.1.1

Table 5.2

isable p-semilattices is closed under taking subalgebras, homomorphic images and finite products. The class of non-dualisable p-semilattices is closed under taking finite products. But, as the following example shows, it is not closed under taking non-trivial subalgebras or non-trivial homomorphic images. The two-element p-semilattice $\underline{\mathbf{P}}_2 = \langle\{0,1\}; \wedge, {}^*, 0, 1\rangle$ is dualisable, and the three-element p-semilattice $\underline{\mathbf{P}}_3 = \langle\{0, \frac{1}{2}, 1\}; \wedge, {}^*, 0, 1\rangle$ is not dualisable. However, there is a retraction $\gamma : \underline{\mathbf{P}}_3 \to \underline{\mathbf{P}}_2$, given by $\gamma(a) := a^{**}$.

In general, the dualisability of an algebra depends on the structure of the quasi-variety it generates. If $\underline{\mathbf{M}}$ and $\underline{\mathbf{N}}$ are arbitrary algebras of the same type, then the quasi-variety $\mathbb{ISP}(\underline{\mathbf{M}} \times \underline{\mathbf{N}})$ may be quite different from $\mathbb{ISP}(\underline{\mathbf{M}})$ and $\mathbb{ISP}(\underline{\mathbf{N}})$. So the algebra $\underline{\mathbf{M}} \times \underline{\mathbf{N}}$ might be non-dualisable, even if both $\underline{\mathbf{M}}$ and $\underline{\mathbf{N}}$ are dualisable. In this chapter, we find two dualisable unary algebras whose product is not dualisable. We also determine whether or not dualisability is preserved by various other algebraic constructions. The results are summarised in Tables 5.1 and 5.2, which also list some relevant results from other chapters. During our investigations, we demonstrate that algebras which generate the same variety do not have to share dualisability or non-dualisability.

Section 5.2 is based on a paper by both authors [26]. Section 5.3 is based on part of a paper by the first author [51], and Section 5.1 is an extension of another part of this paper.

5.1 Coproducts of unary algebras

In this chapter, we will show that the coproduct of two dualisable algebras can be non-dualisable. Before doing this, we need to clarify what we mean by the coproduct of two algebras of the same type. Let $\mathbf{A}$ and $\mathbf{B}$ be algebras of type F. There are three natural classes of algebras within which we can look for a coproduct of $\mathbf{A}$ and $\mathbf{B}$:

- the class of all algebras of type F;
- the variety generated by $\mathbf{A}$ and $\mathbf{B}$;
- the quasi-variety generated by $\mathbf{A}$ and $\mathbf{B}$.

We shall show how to construct coproducts of unary algebras within varieties. The following two general lemmas are part of the folklore of algebra.

5.1.1 Lemma *Let $\mathcal{A}$ be a quasi-variety of algebras, and let $\mathbf{A}$ be any algebra of the same type. There is a largest homomorphic image of $\mathbf{A}$ in $\mathcal{A}$.*

5.1.2 Lemma *Let $\mathcal{C}$ be a class of algebras of the same type, let $\mathcal{A}$ be a quasi-variety contained in $\mathcal{C}$ and let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$. Assume that $\mathbf{C}$ is the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}$. Then the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{A}$ exists and it is the largest homomorphic image of $\mathbf{C}$ that belongs to $\mathcal{A}$.*

In fact, the previous lemma is a special case of a very general result from category theory: left adjoints preserve colimits [46].

Now assume that $\mathbf{A}$ and $\mathbf{B}$ are unary algebras of type F. The **disjoint union** $\mathbf{A} \dot\cup \mathbf{B}$ is obtained by putting $\mathbf{A}$ and $\mathbf{B}$ next to each other:

$$\mathbf{A} \dot\cup \mathbf{B} := \big\langle (A \times \{0\}) \cup (B \times \{1\});\ F^{\mathbf{A} \dot\cup \mathbf{B}} \big\rangle,$$

where

$$u^{\mathbf{A} \dot\cup \mathbf{B}}\big((a, 0)\big) := (u^{\mathbf{A}}(a), 0) \quad \text{and} \quad u^{\mathbf{A} \dot\cup \mathbf{B}}\big((b, 1)\big) := (u^{\mathbf{B}}(b), 1),$$

for all $u \in F$, $a \in A$ and $b \in B$.

Next, let $\mathbf{A}' \leqslant \mathbf{A}$ and $\mathbf{B}' \leqslant \mathbf{B}$, and let $\alpha : \mathbf{A}' \twoheadrightarrow \mathbf{D}$ and $\beta : \mathbf{B}' \twoheadrightarrow \mathbf{D}$ be surjective homomorphisms. The **amalgamated union** $\mathbf{A} \cup_{\alpha\beta} \mathbf{B}$ is obtained by pasting $\mathbf{A}$ and $\mathbf{B}$ together on $\mathbf{D}$:

$$\mathbf{A} \cup_{\alpha\beta} \mathbf{B} := (\mathbf{A} \mathbin{\dot\cup} \mathbf{B})/\theta_{\alpha\beta},$$

where $\theta_{\alpha\beta}$ is the congruence on $\mathbf{A} \mathbin{\dot\cup} \mathbf{B}$ whose non-trivial blocks are precisely those of the form

$$\big(\alpha^{-1}(d) \times \{0\}\big) \cup \big(\beta^{-1}(d) \times \{1\}\big),$$

for some $d \in D$.

The algebra built most freely from $\mathbf{A}$ and $\mathbf{B}$ is the disjoint union $\mathbf{A} \mathbin{\dot\cup} \mathbf{B}$. Clearly, $\mathbf{A} \mathbin{\dot\cup} \mathbf{B}$ is the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in the class of all algebras of type F. Now let $\mathcal{V}$ be any variety containing $\mathbf{A}$ and $\mathbf{B}$. We have to be a little more careful when constructing the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{V}$. The disjoint union $\mathbf{A} \mathbin{\dot\cup} \mathbf{B}$ will not belong to $\mathcal{V}$ if there is a constant equation $\tau(x) \approx \tau(y)$ in the theory of $\mathcal{V}$, for some unary term τ of type F. We shall see that the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{V}$ is generally an amalgamated union of $\mathbf{A}$ and $\mathbf{B}$.

The variety $\mathcal{V}$ is determined by the one-variable equations $\sigma(x) \approx \tau(x)$ and the constant equations $\tau(x) \approx \tau(y)$ in its theory. So the equational theory of $\mathcal{V}$ is the same as the equational theory of the free algebra $\mathbf{F}_{\mathcal{V}}(2)$. Define $F_{\mathcal{V}}(0)$ to be the set of all $k \in F_{\mathcal{V}}(2)$ such that k is the value of a constant term function of $\mathbf{F}_{\mathcal{V}}(2)$. Then $F_{\mathcal{V}}(0)$ is a subuniverse of $\mathbf{F}_{\mathcal{V}}(2)$. If $F_{\mathcal{V}}(0)$ is non-empty, then the algebra $\mathbf{F}_{\mathcal{V}}(0)$ is an initial object in the category $\mathcal{V}$: for all $\mathbf{C} \in \mathcal{V}$, there is a unique homomorphism $\imath_{\mathbf{C}} : \mathbf{F}_{\mathcal{V}}(0) \to \mathbf{C}$.

5.1.3 Lemma *Let $\mathcal{V}$ be a variety of unary algebras and let $\mathbf{A}, \mathbf{B} \in \mathcal{V}$.*

(i) *Assume that $F_{\mathcal{V}}(0)$ is empty. Then the disjoint union $\mathbf{A} \mathbin{\dot\cup} \mathbf{B}$ is the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{V}$.*

(ii) *Assume that $F_{\mathcal{V}}(0)$ is non-empty. Define the congruence θ on the algebra $\mathbf{F}_{\mathcal{V}}(0)$ by $\theta := \ker(\imath_{\mathbf{A}}) \vee \ker(\imath_{\mathbf{B}})$. Let $\alpha : \imath_{\mathbf{A}}\big(\mathbf{F}_{\mathcal{V}}(0)\big) \twoheadrightarrow \mathbf{F}_{\mathcal{V}}(0)/\theta$ and $\beta : \imath_{\mathbf{B}}\big(\mathbf{F}_{\mathcal{V}}(0)\big) \twoheadrightarrow \mathbf{F}_{\mathcal{V}}(0)/\theta$ be the natural homomorphisms. Then the amalgamated union $\mathbf{A} \cup_{\alpha\beta} \mathbf{B}$ is the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{V}$.*

Proof Define $\mathrm{Th}(\mathcal{V})$ to be the equational theory of $\mathcal{V}$. We will be using two easy facts. One, the variety $\mathcal{V}$ is determined by the one-variable equations and the constant equations in $\mathrm{Th}(\mathcal{V})$. Two, an algebra satisfies the one-variable equations in $\mathrm{Th}(\mathcal{V})$ if and only if each of its one-generated subalgebras belongs to $\mathcal{V}$.

First assume that $F_{\mathcal{V}}(0)$ is empty. Since $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, every one-generated subalgebra of $\mathbf{A} \mathbin{\dot\cup} \mathbf{B}$ belongs to $\mathcal{V}$. So $\mathbf{A} \mathbin{\dot\cup} \mathbf{B}$ satisfies all the one-variable equations in $\mathrm{Th}(\mathcal{V})$. As there are no constant equations in $\mathrm{Th}(\mathcal{V})$, it follows that $\mathbf{A} \,\dot\cup\, \mathbf{B}$ is a member of $\mathcal{V}$. So $\mathbf{A} \,\dot\cup\, \mathbf{B}$ is the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{V}$.

Now assume that $F_{\mathcal{V}}(0)$ is non-empty. First we want to check that the amalgamated union $\mathbf{C} := \mathbf{A} \cup_{\alpha\beta} \mathbf{B}$ belongs to $\mathcal{V}$. Every one-generated subalgebra of $\mathbf{C}$ is a subalgebra of a homomorphic image of $\mathbf{A}$ or $\mathbf{B}$. So $\mathbf{C}$ satisfies all the one-variable equations in $\mathrm{Th}(\mathcal{V})$.

Now choose a unary term τ such that $\tau(x) \approx \tau(y)$ belongs to $\mathrm{Th}(\mathcal{V})$. We want to show that $\tau^{\mathbf{C}}$ is a constant term function of $\mathbf{C}$. There is some $k \in F_{\mathcal{V}}(0)$ such that k is the value of the constant term function $\tau^{\mathbf{F}_{\mathcal{V}}(2)}$ of $\mathbf{F}_{\mathcal{V}}(2)$. Since $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, we know that $\tau^{\mathbf{A}}$ and $\tau^{\mathbf{B}}$ are constant, with values $\imath_{\mathbf{A}}(k)$ and $\imath_{\mathbf{B}}(k)$, respectively. To see that $\tau^{\mathbf{C}}$ is constant, it is enough to show that $\imath_{\mathbf{A}}(k)$ and $\imath_{\mathbf{B}}(k)$ are identified in $\mathbf{C} = \mathbf{A} \cup_{\alpha\beta} \mathbf{B}$. More precisely, we want to show that $(\imath_{\mathbf{A}}(k), 0)\ \theta_{\alpha\beta}\ (\imath_{\mathbf{B}}(k), 1)$. But this holds, since $\mathbf{F}_{\mathcal{V}}(0)/\theta \in \mathcal{V}$, and therefore

$$\begin{aligned}
\alpha\big(\imath_{\mathbf{A}}(k)\big) &= \alpha\big(\tau^{\mathbf{A}}(\imath_{\mathbf{A}}(k))\big) = \tau^{\mathbf{F}_{\mathcal{V}}(0)/\theta}\big(\alpha(\imath_{\mathbf{A}}(k))\big)\\
&= \tau^{\mathbf{F}_{\mathcal{V}}(0)/\theta}\big(\beta(\imath_{\mathbf{B}}(k))\big) = \beta\big(\tau^{\mathbf{B}}(\imath_{\mathbf{B}}(k))\big)\\
&= \beta\big(\imath_{\mathbf{B}}(k)\big).
\end{aligned}$$

So $\tau^{\mathbf{C}}$ is a constant term function of $\mathbf{C}$, whence $\mathbf{C}$ satisfies all the constant equations in $\mathrm{Th}(\mathcal{V})$. We have shown that $\mathbf{C}$ belongs to $\mathcal{V}$.

We shall now show that $\mathbf{C}$ is the largest homomorphic image of $\mathbf{A} \,\dot\cup\, \mathbf{B}$ that belongs to $\mathcal{V}$. By Lemma 5.1.2, it will then follow that $\mathbf{C}$ is the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{V}$. Let $\varphi : \mathbf{A} \,\dot\cup\, \mathbf{B} \twoheadrightarrow \mathbf{D}$ be a surjective homomorphism such that $\mathbf{D} \in \mathcal{V}$. We just need to check that $\theta_{\alpha\beta} \leqslant \ker(\varphi)$ in the congruence lattice $\mathbf{Con}(\mathbf{A} \,\dot\cup\, \mathbf{B})$.

Let $\eta_{\mathbf{A}} : \mathbf{A} \hookrightarrow \mathbf{A} \,\dot\cup\, \mathbf{B}$ and $\eta_{\mathbf{B}} : \mathbf{B} \hookrightarrow \mathbf{A} \,\dot\cup\, \mathbf{B}$ be the natural embeddings. Since $\varphi \circ \eta_{\mathbf{A}} \circ \imath_{\mathbf{A}} : \mathbf{F}_{\mathcal{V}}(0) \to \mathbf{D}$ is a homomorphism and $\mathbf{D} \in \mathcal{V}$, we have $\imath_{\mathbf{D}} = \varphi \circ \eta_{\mathbf{A}} \circ \imath_{\mathbf{A}}$. This implies that $\ker(\imath_{\mathbf{A}}) \leqslant \ker(\imath_{\mathbf{D}})$ and, using a similar argument, $\ker(\imath_{\mathbf{B}}) \leqslant \ker(\imath_{\mathbf{D}})$. So we have

$$\theta = \ker(\imath_{\mathbf{A}}) \vee \ker(\imath_{\mathbf{B}}) \leqslant \ker(\imath_{\mathbf{D}}).$$

To prove that $\theta_{\alpha\beta} \leqslant \ker(\varphi)$, we will show that every non-trivial block of $\theta_{\alpha\beta}$ is contained in a block of $\ker(\varphi)$. Let S be a non-trivial block of $\theta_{\alpha\beta}$. Then

$$S = \big(\alpha^{-1}(k/\theta) \times \{0\}\big) \cup \big(\beta^{-1}(k/\theta) \times \{1\}\big),$$

for some $k \in F_{\mathcal{V}}(0)$. Now let $a \in \alpha^{-1}(k/\theta)$. There exists $\ell \in F_{\mathcal{V}}(0)$ such that $a = \imath_{\mathbf{A}}(\ell)$. Since $\alpha : \imath_{\mathbf{A}}\big(\mathbf{F}_{\mathcal{V}}(0)\big) \twoheadrightarrow \mathbf{F}_{\mathcal{V}}(0)/\theta$ is the natural homomorphism, we have

$$\ell/\theta = \alpha(\imath_{\mathbf{A}}(\ell)) = \alpha(a) = k/\theta.$$

We have already shown that $\theta \leqslant \ker(\imath_{\mathbf{D}})$. So this implies that $\imath_{\mathbf{D}}(\ell) = \imath_{\mathbf{D}}(k)$. Since $\varphi \circ \eta_{\mathbf{A}} \circ \imath_{\mathbf{A}} = \imath_{\mathbf{D}}$, it follows that

$$\varphi\big((a,0)\big) = \varphi \circ \eta_{\mathbf{A}} \circ \imath_{\mathbf{A}}(\ell) = \imath_{\mathbf{D}}(\ell) = \imath_{\mathbf{D}}(k).$$

So we obtain

$$\varphi\big(\alpha^{-1}(k/\theta) \times \{0\}\big) = \{\imath_{\mathbf{D}}(k)\} \quad \text{and} \quad \varphi\big(\beta^{-1}(k/\theta) \times \{1\}\big) = \{\imath_{\mathbf{D}}(k)\},$$

using symmetry. Thus $|\varphi(S)| = 1$, and therefore $\theta_{\alpha\beta} \leqslant \ker(\varphi)$. Hence the algebra $\mathbf{C} = \mathbf{A} \cup_{\alpha\beta} \mathbf{B}$ is the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{V}$. ∎

In Section 5.3, we shall find a pair of dualisable unary algebras $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$, with $\underline{\mathbf{L}} \in \mathbb{HSP}(\underline{\mathbf{K}})$, such that the disjoint union $\underline{\mathbf{K}} \dot\cup \underline{\mathbf{L}}$ is non-dualisable. To finish this section, we give an example of a similar coproduct construction on unary algebras that *does* preserve dualisability.

Consider a finite unary algebra $\underline{\mathbf{M}}$, and let $\underline{\mathbf{N}}$ be a finite algebra in $\mathbb{ISP}(\underline{\mathbf{M}})$. The quasi-varieties $\mathbb{ISP}(\underline{\mathbf{M}})$ and $\mathbb{ISP}(\underline{\mathbf{M}} \dot\cup \underline{\mathbf{N}})$ are not necessarily the same. Indeed, they must be different if $\underline{\mathbf{M}}$ has a constant term function. Nevertheless, we shall prove that, if $\underline{\mathbf{M}}$ is dualisable, then the disjoint union $\underline{\mathbf{M}} \dot\cup \underline{\mathbf{N}}$ is also dualisable. This is true even if $\underline{\mathbf{N}}$ is non-dualisable.

One important special case of this result is when $|N| = 1$. Let $\underline{\mathbf{1}}$ be a one-element algebra of the same type as $\underline{\mathbf{M}}$. Then the **one-point extension of** $\underline{\mathbf{M}}$ is the disjoint union $\underline{\mathbf{M}} \dot\cup \underline{\mathbf{1}}$. So it will follow that the one-point extension of a dualisable unary algebra is also dualisable. A similar result, that the one-point extension of a finitely dualisable unary algebra is also finitely dualisable, was proved directly by Clark, Davey and Pitkethly [13].

We begin by investigating the precise difference between the quasi-varieties $\mathbb{ISP}(\underline{\mathbf{M}})$ and $\mathbb{ISP}(\underline{\mathbf{M}} \dot\cup \underline{\mathbf{N}})$.

5.1.4 Lemma *Let $\underline{\mathbf{M}}$ and $\underline{\mathbf{N}}$ be finite unary algebras, with $\underline{\mathbf{N}} \in \mathbb{ISP}(\underline{\mathbf{M}})$. Then every connected algebra in $\mathbb{ISP}(\underline{\mathbf{M}} \dot\cup \underline{\mathbf{N}})$ belongs to $\mathbb{ISP}(\underline{\mathbf{M}})$.*

Proof Assume that $M \cap N = \varnothing$. Then we can work with the union $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$, rather than the disjoint union. Let $\mathbf{C} \in \mathbb{ISP}(\underline{\mathbf{M}} \cup \underline{\mathbf{N}})$, with $\mathbf{C}$ connected. We want to show that $\mathbf{C}$ is separated by homomorphisms into $\underline{\mathbf{M}}$. So assume that $a, b \in C$ with $a \neq b$. Since $\mathbf{C}$ is separated by homomorphisms into $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$, there is a homomorphism $x : \mathbf{C} \to \underline{\mathbf{M}} \cup \underline{\mathbf{N}}$ such that $x(a) \neq x(b)$. As $\mathbf{C}$ is connected, we must have $x(C) \subseteq M$ or $x(C) \subseteq N$. We can assume that $x(C) \subseteq N$. Since $\underline{\mathbf{N}} \in \mathbb{ISP}(\underline{\mathbf{M}})$, there exists a homomorphism $y : \underline{\mathbf{N}} \to \underline{\mathbf{M}}$ with $y(x(a)) \neq y(x(b))$. Thus $y \circ x : \mathbf{C} \to \underline{\mathbf{M}}$ separates a and b. Using the $\mathbb{ISP}$ Theorem, 1.1.1, it now follows that $\mathbf{C} \in \mathbb{ISP}(\underline{\mathbf{M}})$. ∎

5.1.5 Corollary *Let $\underline{\mathbf{M}}$ be a finite unary algebra and let $\underline{\mathbf{1}}$ be a one-element algebra of the same type as $\underline{\mathbf{M}}$.*

(i) *An algebra $\mathbf{A}$ of the same type as $\underline{\mathbf{M}}$ belongs to $\mathbb{ISP}((\underline{\mathbf{M}} \dot\cup \underline{\mathbf{1}}) \dot\cup \underline{\mathbf{1}})$ if and only if every connected component of $\mathbf{A}$ belongs to $\mathbb{ISP}(\underline{\mathbf{M}})$.*

(ii) *The class $\mathbb{ISP}((\underline{\mathbf{M}} \dot\cup \underline{\mathbf{1}}) \dot\cup \underline{\mathbf{1}})$ is the largest quasi-variety that is generated by a disjoint union of algebras from $\mathbb{ISP}(\underline{\mathbf{M}})$.*

Proof Let $\mathbf{A}$ be of the same type as $\underline{\mathbf{M}}$. Assume that $\mathbf{A} \in \mathbb{ISP}((\underline{\mathbf{M}} \dot\cup \underline{\mathbf{1}}) \dot\cup \underline{\mathbf{1}})$. Using the previous lemma twice, every connected component of $\mathbf{A}$ belongs to $\mathbb{ISP}(\underline{\mathbf{M}})$. Conversely, if every connected component of $\mathbf{A}$ belongs to $\mathbb{ISP}(\underline{\mathbf{M}})$, then $\mathbf{A}$ is separated by homomorphisms into $(\underline{\mathbf{M}} \dot\cup \underline{\mathbf{1}}) \dot\cup \underline{\mathbf{1}}$. So (i) holds.

For (ii), let $\mathbf{B}$ be a disjoint union of algebras from $\mathbb{ISP}(\underline{\mathbf{M}})$. Then $\mathbf{B}$ belongs to $\mathbb{ISP}((\underline{\mathbf{M}} \dot\cup \underline{\mathbf{1}}) \dot\cup \underline{\mathbf{1}})$, by (i). So the quasi-variety generated by $\mathbf{B}$ is contained in the quasi-variety $\mathbb{ISP}((\underline{\mathbf{M}} \dot\cup \underline{\mathbf{1}}) \dot\cup \underline{\mathbf{1}})$. ∎

The following lemma compares quasi-varieties that are generated by disjoint unions of $\underline{\mathbf{M}}$'s and $\underline{\mathbf{1}}$'s.

5.1.6 Lemma *Let $\underline{\mathbf{M}}$ be a finite unary algebra and let $\underline{\mathbf{1}}$ be a one-element algebra of the same type as $\underline{\mathbf{M}}$. Then*

$$\mathbb{ISP}(\underline{\mathbf{M}}) \subseteq \mathbb{ISP}(\underline{\mathbf{M}} \dot\cup \underline{\mathbf{M}}) \subseteq \mathbb{ISP}(\underline{\mathbf{M}} \dot\cup \underline{\mathbf{1}}) \subseteq \mathbb{ISP}((\underline{\mathbf{M}} \dot\cup \underline{\mathbf{1}}) \dot\cup \underline{\mathbf{1}}), \quad \textit{where}$$

(i) *the first inclusion is an equality if and only if there is no element of M that is fixed by every endomorphism of $\underline{\mathbf{M}}$,*

(ii) *each of the second and third inclusions is an equality if and only if $\underline{\mathbf{M}}$ has a one-element subalgebra,*

(iii) *all four quasi-varieties are equal if and only if $\underline{\mathbf{M}}$ has at least two one-element subalgebras.*

Proof This lemma is easy to prove using the following consequence of the $\mathbb{ISP}$ Theorem, 1.1.1: for all finite algebras $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbb{ISP}(\mathbf{A}) \subseteq \mathbb{ISP}(\mathbf{B})$ if and only if $\mathbf{A}$ is separated by homomorphisms into $\mathbf{B}$. ∎

The next easy lemma slots $\mathbb{ISP}(\underline{\mathbf{M}} \dot\cup \underline{\mathbf{N}})$ into the chain of quasi-varieties given in the preceding lemma, in the case that $|N| > 1$.

5.1.7 Lemma *Let $\underline{\mathbf{M}}$ be a finite unary algebra and let $\underline{\mathbf{N}}$ be a finite non-trivial algebra in $\mathbb{ISP}(\underline{\mathbf{M}})$. Then $\mathbb{ISP}(\underline{\mathbf{M}}) \subseteq \mathbb{ISP}(\underline{\mathbf{M}} \dot\cup \underline{\mathbf{N}}) \subseteq \mathbb{ISP}(\underline{\mathbf{M}} \dot\cup \underline{\mathbf{M}})$.*

To illustrate the previous collection of results, we consider a particular three-element unary algebra.

5.1.8 Example Define the unary algebra

$$\underline{\mathbf{M}} := \langle \{0, 1, 2\}; 001, 111 \rangle,$$

and let $\underline{\mathbf{N}}$ be the subalgebra of $\underline{\mathbf{M}}$ on the set $N := \{0, 1\}$. Then there is no homomorphism from $\underline{\mathbf{M}}$ into $\underline{\mathbf{N}}$; indeed, the only endomorphism of $\underline{\mathbf{M}}$ is the identity. With the help of the previous two lemmas, it is now easy to check that

$$\begin{aligned} \mathbb{ISP}(\underline{\mathbf{M}}) \subset \mathbb{ISP}(\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{N}}) \subset \mathbb{ISP}(\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{M}}) \\ \subset \mathbb{ISP}(\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{1}}) \subset \mathbb{ISP}((\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{1}}) \mathbin{\dot\cup} \underline{\mathbf{1}}), \end{aligned}$$

where every inclusion is proper. To see the differences between these quasi-varieties, consider an algebra $\mathbf{A}$ of the same type as $\underline{\mathbf{M}}$. By Corollary 5.1.5, we know that $\mathbf{A} \in \mathbb{ISP}((\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{1}}) \mathbin{\dot\cup} \underline{\mathbf{1}})$ if and only if every connected component of $\mathbf{A}$ belongs to $\mathbb{ISP}(\underline{\mathbf{M}})$. Now assume that $\mathbf{A} \in \mathbb{ISP}((\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{1}}) \mathbin{\dot\cup} \underline{\mathbf{1}})$. Then, as explained below, we have:

(i) $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{1}})$ if and only if at most one connected component of $\mathbf{A}$ has only one element;

(ii) $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{M}})$ if and only if $\mathbf{A}$ is trivial or $\mathbf{A}$ has no one-element connected components;

(iii) $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{N}})$ if and only if $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{M}})$ and at most one connected component of $\mathbf{A}$ has a subalgebra isomorphic to $\underline{\mathbf{M}}$;

(iv) $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$ if and only if $\mathbf{A}$ is connected.

Claims (i) and (ii) follow since $\underline{\mathbf{M}}$ has no one-element subalgebras. Claim (iv) follows since $\underline{\mathbf{M}}$ has a constant term function. Claim (iii) requires some knowledge of the structure of the algebras in $\mathbb{ISP}(\underline{\mathbf{M}})$. In particular, claim (iii) uses the fact: for every non-trivial algebra $\mathbf{C} \in \mathbb{ISP}(\underline{\mathbf{M}})$, if $\mathbf{C}$ does not have a subalgebra isomorphic to $\underline{\mathbf{M}}$, then there exists a homomorphism from $\mathbf{C}$ into $\underline{\mathbf{N}}$.

We now turn to proving that the finite unary algebra $\underline{\mathbf{M}} \mathbin{\dot\cup} \underline{\mathbf{N}}$ is dualisable, whenever $\underline{\mathbf{M}}$ is dualisable and $\underline{\mathbf{N}} \in \mathbb{ISP}(\underline{\mathbf{M}})$. The following preliminary lemma is similar to Lemma 3.1.5, which was used to obtain the Petal Duality Lemma, 3.1.6.

5.1.9 Lemma *Let $\underline{\mathbf{M}}$ be a finite unary algebra and let $\mathbf{A}$ belong to the quasi-variety $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}})$. Assume that $\alpha : \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ has a finite support and that α agrees with an evaluation on each subset of $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ with at most four elements. Then there is a connected component $\mathbf{C}$ of $\mathbf{A}$ such that C is a support for α.*

Proof There is a finite non-empty support S for α. Let $\mathbf{C}_1, \dots, \mathbf{C}_n$ be the connected components of $\mathbf{A}$ that contain a member of S, where $n \in \omega\backslash\{0\}$, and define the subuniverse B of $\mathbf{A}$ by $B := C_1 \cup \dots \cup C_n$.

We can assume that the map α is not constant. So there are $y, z \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ such that $\alpha(y) \neq \alpha(z)$. Define the sequence $y_0, \dots, y_n$ of homomorphisms in $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ by

$$y_0 := y \quad \text{and} \quad y_{i+1} := y_i{\restriction}_{A\backslash C_{i+1}} \cup z{\restriction}_{C_{i+1}},$$

for all $i \in \{0, \dots, n-1\}$. Then $y_n = y{\restriction}_{A\backslash B} \cup z{\restriction}_B$. As $S \subseteq B$ and S is a support for α, we have

$$\alpha(y_0) = \alpha(y) \neq \alpha(z) = \alpha(y_n).$$

This implies that $\alpha(y_j) \neq \alpha(y_{j+1})$, for some $j \in \{0, \dots, n-1\}$.

To prove that C_{j+1} is a support for α, let $w, x \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ such that $w{\restriction}_{C_{j+1}} = x{\restriction}_{C_{j+1}}$. There is some $a \in A$ such that α is given by evaluation at a on $\{w, x, y_j, y_{j+1}\}$. So

$$y_j(a) = \alpha(y_j) \neq \alpha(y_{j+1}) = y_{j+1}(a),$$

and therefore $a \in C_{j+1}$. Thus

$$\alpha(w) = w(a) = x(a) = \alpha(x),$$

whence C_{j+1} is a support for α. ∎

5.1.10 Theorem *Let $\underline{\mathbf{M}}$ and $\underline{\mathbf{N}}$ be finite unary algebras, with $\underline{\mathbf{N}} \in \mathbb{ISP}(\underline{\mathbf{M}})$. If $\underline{\mathbf{M}}$ is dualisable, then $\underline{\mathbf{M}} \mathbin{\dot{\cup}} \underline{\mathbf{N}}$ is dualisable.*

Proof Assume that $M \cap N = \varnothing$ and that $\underline{\mathbf{M}}$ is dualisable. We shall show that the union $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$ is dualisable. Define the two quasi-varieties

$$\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}}) \quad \text{and} \quad \mathcal{B} := \mathbb{ISP}(\underline{\mathbf{M}} \cup \underline{\mathbf{N}}).$$

Then $\mathcal{A}$ is contained in $\mathcal{B}$.

Let $\mathcal{A}(\underline{\mathbf{N}}, \underline{\mathbf{M}}) = \{e_1, \dots, e_k\}$, where $k \in \omega$. For every $i \in \{1, \dots, k\}$, define the endomorphism

$$\overline{e}_i : \underline{\mathbf{M}} \cup \underline{\mathbf{N}} \to \underline{\mathbf{M}} \cup \underline{\mathbf{N}} \quad \text{by} \quad \overline{e}_i := \mathrm{id}_M \cup e_i.$$

Then $\overline{e}_i(M \cup N) \subseteq M$, for each $i \in \{1, \dots, k\}$. Since $\underline{\mathbf{N}} \in \mathbb{ISP}(\underline{\mathbf{M}})$, the endomorphisms $\overline{e}_1, \dots, \overline{e}_k$ of $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$ separate the elements of N. So, in the case that $|N| > 1$, we have $k > 0$.

Now let $\mathbf{B} \in \mathcal{B}$ and let $\beta : \mathcal{B}(\mathbf{B}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}}) \to M \cup N$ be a brute-force morphism. Then β preserves every endomorphism of $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$, since the graph of an endomorphism is an algebraic relation on $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$. Using the Brute Force Lemma, 1.4.5, we know that β has a finite support and is locally an evaluation.

We will know that $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$ is dualisable once we have proved that β is an evaluation. The proof splits up into three cases.

Case 1: β is constant and $|N| = 1$. Let 0 denote the unique element of $\underline{\mathbf{N}}$, and let $z : \mathbf{B} \to \underline{\mathbf{M}} \cup \underline{\mathbf{N}}$ denote the constant homomorphism with value 0. The brute-force morphism β must preserve the unary algebraic relation $N = \{0\}$ on $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$. So $\beta(z) = 0$, and therefore 0 is the value of the constant map β.

The map β preserves the unary algebraic relation M on $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$. Since the value of β is $0 \notin M$, this implies that $x(B) \nsubseteq M$, for all $x \in \mathcal{B}(\mathbf{B}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$. So there are no homomorphisms from $\mathbf{B}$ into $\underline{\mathbf{M}}$. But we know that every connected component of $\mathbf{B}$ belongs to $\mathbb{ISP}(\underline{\mathbf{M}})$, by Lemma 5.1.4. It follows that there is some $b \in B$ that determines a one-element connected component of $\mathbf{B}$, and that $\underline{\mathbf{M}}$ has no one-element subalgebras. For all $x \in \mathcal{B}(\mathbf{B}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$, we have $\beta(x) = 0 = x(b)$. Thus β is an evaluation.

Case 2: β is constant and $|N| > 1$. We must have $k > 0$. Let m denote the value of β in $M \cup N$. For any $x \in \mathcal{B}(\mathbf{B}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$, we have

$$m = \beta(\bar{e}_1 \circ x) = \bar{e}_1(\beta(x)) \in M,$$

since β preserves the endomorphism $\bar{e}_1$ of $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$. Thus the value m of β belongs to M.

Define the set B_N to be the union of all the connected components of $\mathbf{B}$ that have at least one homomorphism into $\underline{\mathbf{N}}$. If B_N is non-empty, then it forms a subalgebra $\mathbf{B}_N$ of $\mathbf{B}$. In this case, there is a homomorphism $z_N : \mathbf{B}_N \to \underline{\mathbf{N}}$. If B_N is empty, we will just define $z_N : B_N \to N$ to be the empty map. As β preserves the unary relation N and has constant value $m \in M$, there is no homomorphism from $\mathbf{B}$ into $\underline{\mathbf{N}}$. So $B_N \neq B$.

Now define $B_M := B \backslash B_N$. Then B_M is the union of all the connected components of $\mathbf{B}$ that do not have any homomorphisms into $\underline{\mathbf{N}}$. Since $B_N \neq B$, the set B_M is the universe of a subalgebra $\mathbf{B}_M$ of $\mathbf{B}$. As $\mathbf{B}$ is separated by homomorphisms into $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$, it follows that $\mathbf{B}_M$ is separated by homomorphisms into $\underline{\mathbf{M}}$. So $\mathbf{B}_M \in \mathbb{ISP}(\underline{\mathbf{M}}) = \mathcal{A}$, by the $\mathbb{ISP}$ Theorem, 1.1.1.

Now define the constant map $\alpha : \mathcal{A}(\mathbf{B}_M, \underline{\mathbf{M}}) \to M$ with value m. We shall prove that α is a brute-force morphism. Since $\underline{\mathbf{M}}$ is dualisable, it will then follow that α is given by evaluation at some element of $B_M \subseteq B$. We shall then show that β is given by evaluation at the same element.

Clearly, the constant map α has a finite support. To prove that α is locally an evaluation, let Y be a finite non-empty subset of $\mathcal{A}(\mathbf{B}_M, \underline{\mathbf{M}})$. Define

$$Y^+ := \{\, y \cup z_N \mid y \in Y \} \subseteq \mathcal{B}(\mathbf{B}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}}).$$

Since β is locally an evaluation, there is some $b \in B$ such that β is given by evaluation at b on Y^+. For all $y \in Y$, we have

$$(y \cup z_N)(b) = \beta(y \cup z_N) = m \in M.$$

Since $z_N(B_N) \subseteq N$, this tells us that $b \in B_M$. Now, to see that α is given by evaluation at b on Y, let $y \in Y$. Then

$$\alpha(y) = m = \beta(y \cup z_N) = (y \cup z_N)(b) = y(b).$$

So α is locally an evaluation. Thus α is a brute-force morphism, by the Brute Force Lemma.

As $\underline{\mathbf{M}}$ is dualisable, it now follows that $\alpha : \mathcal{A}(\mathbf{B}_M, \underline{\mathbf{M}}) \to M$ is given by evaluation at some $a \in B_M \subseteq B$. There are no homomorphisms from any of the connected components of $\mathbf{B}_M$ into $\underline{\mathbf{N}}$. So, for all $x \in \mathcal{B}(\mathbf{B}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$, we have $x{\restriction}_{B_M} \in \mathcal{A}(\mathbf{B}_M, \underline{\mathbf{M}})$ and therefore

$$\beta(x) = m = \alpha(x{\restriction}_{B_M}) = x{\restriction}_{B_M}(a) = x(a).$$

So β is an evaluation.

Case 3: β is not constant. There are $v, w \in \mathcal{B}(\mathbf{B}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$ with $\beta(v) \neq \beta(w)$. By Lemma 5.1.9, there is a connected component $\mathbf{C}$ of $\mathbf{B}$ such that C is a support for β. Define the map

$$\gamma : \mathcal{B}(\mathbf{C}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}}) \to M \cup N \quad \text{by} \quad \gamma(y) := \beta(y \cup w{\restriction}_{B\setminus C}).$$

Then γ has a finite support, since β does.

To check that γ is locally an evaluation, let Y be a finite subset of the hom-set $\mathcal{B}(\mathbf{C}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$. As β is locally an evaluation, there is some $b \in B$ such that β is given by evaluation at b on the finite set

$$\{\, y \cup w{\restriction}_{B\setminus C} \mid y \in Y \,\} \cup \{ v{\restriction}_C \cup w{\restriction}_{B\setminus C}, w \} \subseteq \mathcal{B}(\mathbf{B}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}}).$$

Since C is a support for β, we have

$$(v{\restriction}_C \cup w{\restriction}_{B\setminus C})(b) = \beta(v{\restriction}_C \cup w{\restriction}_{B\setminus C}) = \beta(v) \neq \beta(w) = w(b),$$

and therefore $b \in C$. For all $y \in Y$, we now have

$$\gamma(y) = \beta(y \cup w{\restriction}_{B\setminus C}) = (y \cup w{\restriction}_{B\setminus C})(b) = y(b).$$

So γ is given by evaluation at b on Y. We have shown that the map γ has a finite support and is locally an evaluation.

The connected algebra $\mathbf{C}$ belongs to $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$, by Lemma 5.1.4. For all $z \in \mathcal{A}(\mathbf{C}, \underline{\mathbf{M}})$, we have $z \in \mathcal{B}(\mathbf{C}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$ and $\gamma(z) \in M$, since γ agrees with an evaluation on the set $\{z\}$. We can now define

$$\gamma' : \mathcal{A}(\mathbf{C}, \underline{\mathbf{M}}) \to M \quad \text{by} \quad \gamma' := \gamma{\restriction}_{\mathcal{A}(\mathbf{C}, \underline{\mathbf{M}})}.$$

The map γ' has a finite support and is locally an evaluation. Therefore γ' is a brute-force morphism, by the Brute Force Lemma. As $\underline{\mathbf{M}}$ is dualisable, the map γ' is given by evaluation at some $c \in C \subseteq B$.

We shall complete the proof that $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$ is dualisable by showing that β is also given by evaluation at c. To this end, choose any $x \in \mathcal{B}(\mathbf{B}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$. Since the algebra $\mathbf{C}$ is connected, we have either $x(C) \subseteq M$ or $x(C) \subseteq N$.

Case 3.1: $x(C) \subseteq M$. We must have

$$\beta(x) = \beta(x{\restriction}_C \cup w{\restriction}_{B\backslash C}) = \gamma(x{\restriction}_C) = \gamma'(x{\restriction}_C) = x(c),$$

as C is a support for β. So β is given by evaluation at c.

Case 3.2: $x(C) \subseteq N$. First we will check that $\beta(x) \in N$. Since β is locally an evaluation, the map β is given by evaluation at some $a \in B$ on the finite set $\{x, w, v{\restriction}_C \cup w{\restriction}_{B\backslash C}\}$. We know that $a \in C$, because

$$(v{\restriction}_C \cup w{\restriction}_{B\backslash C})(a) = \beta(v{\restriction}_C \cup w{\restriction}_{B\backslash C}) = \beta(v) \neq \beta(w) = w(a).$$

Therefore $\beta(x) = x(a) \in N$, as we are assuming that $x(C) \subseteq N$.

We now have $\beta(x) \in N$ and $x(c) \in N$. So, if $|N| = 1$, then $\beta(x) = x(c)$. We can now further assume that $|N| > 1$, giving $k > 0$. Let $i \in \{1, \dots, k\}$. The homomorphism $\overline{e}_i \circ x \in \mathcal{B}(\mathbf{B}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$ satisfies $\overline{e}_i \circ x(C) \subseteq M$. Thus $\beta(\overline{e}_i \circ x) = \overline{e}_i \circ x(c)$, since Case 3.1 applies to $\overline{e}_i \circ x$. This gives us

$$\overline{e}_i(\beta(x)) = \beta(\overline{e}_i \circ x) = \overline{e}_i \circ x(c) = \overline{e}_i(x(c)),$$

as β preserves the endomorphism $\overline{e}_i$ of $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$. Since $\beta(x), x(c) \in N$ and $\overline{e}_1, \dots, \overline{e}_k$ separate the elements of N, it follows that $\beta(x) = x(c)$. Thus β is given by evaluation at c. ∎

5.1.11 Corollary *Let $\underline{\mathbf{M}}$ be a finite unary algebra. If $\underline{\mathbf{M}}$ is dualisable, then the one-point extension of $\underline{\mathbf{M}}$ is also dualisable.*

The following result gives us a partial converse for the previous theorem.

5.1.12 Theorem *Let* $\underline{\mathbf{M}}$ *and* $\underline{\mathbf{N}}$ *be finite unary algebras, with* $\underline{\mathbf{N}} \in \mathbb{ISP}(\underline{\mathbf{M}})$. *If* $\underline{\mathbf{M}} \dot{\cup} \underline{\mathbf{N}}$ *is finitely dualisable, then* $\underline{\mathbf{M}}$ *is finitely dualisable.*

Proof Assume that $M \cap N = \varnothing$ and that $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$ is finitely dualisable. We want to show that $\underline{\mathbf{M}}$ is finitely dualisable. Define

$$\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}}) \quad \text{and} \quad \mathcal{B} := \mathbb{ISP}(\underline{\mathbf{M}} \cup \underline{\mathbf{N}}).$$

As in the proof of the previous theorem, there is a collection of endomorphisms $\overline{e}_1, \dots, \overline{e}_k$ of $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$, for some $k \in \omega$, such that $\overline{e}_1, \dots, \overline{e}_k$ each map into M and together separate the elements of N.

Since $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$ is finitely dualisable, there exists some $n \in \omega \backslash \{0\}$ such that $R_n(\underline{\mathbf{M}} \cup \underline{\mathbf{N}})$, the set of all n-ary algebraic relations on $\underline{\mathbf{M}} \cup \underline{\mathbf{N}}$, yields a duality on each algebra in $\mathcal{B}$. Define $m := \max(n, nk)$. We shall show that $R_m(\underline{\mathbf{M}})$ yields a duality on each finite connected algebra in $\mathcal{A}$. Each petal of $\mathcal{A}$ is connected. So it will then follow, by the Petal Duality Lemma, 3.1.6, that $\underline{\mathbf{M}}$ is finitely dualisable.

Let $\mathbf{C}$ be a finite connected algebra in $\mathcal{A}$, and let $\alpha : \mathcal{A}(\mathbf{C}, \underline{\mathbf{M}}) \to M$ preserve $R_m(\underline{\mathbf{M}})$. We wish to prove that α is an evaluation. Since $\mathcal{A} \subseteq \mathcal{B}$, we know that $\mathbf{C} \in \mathcal{B}$. We want to define $\beta : \mathcal{B}(\mathbf{C}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}}) \to M \cup N$ by

$$\beta(x) = \begin{cases} \alpha(x) & \text{if } x(C) \subseteq M, \\ x(c) & \text{if } x(C) \subseteq N, \text{ where } c \text{ is any element of } C \text{ such that} \\ & \alpha \text{ is given by evaluation at } c \text{ on } \{\overline{e}_1 \circ x, \dots, \overline{e}_k \circ x\}, \end{cases}$$

for all $x \in \mathcal{B}(\mathbf{C}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$. In the following two claims, we establish that β is a well-defined map that preserves $R_n(\underline{\mathbf{M}} \cup \underline{\mathbf{N}})$.

Claim 1 The map β is well defined.

For each $x \in \mathcal{B}(\mathbf{C}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$, we have $x(C) \subseteq M$ or $x(C) \subseteq N$, as $\mathbf{C}$ is connected. Now let $x \in \mathcal{B}(\mathbf{C}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$ with $x(C) \subseteq N$. Since α preserves $R_m(\underline{\mathbf{M}})$, the Preservation Lemma, 1.4.4, tells us that the map α agrees with an evaluation on every subset of $\mathcal{A}(\mathbf{C}, \underline{\mathbf{M}})$ with at most m elements. Therefore α agrees with an evaluation on $\{\overline{e}_1 \circ x, \dots, \overline{e}_k \circ x\} \subseteq \mathcal{A}(\mathbf{C}, \underline{\mathbf{M}})$, as $k \leqslant m$.

To see that β is well defined, let $c, d \in C$ such that α is given by evaluation at both c and d on $\{\overline{e}_1 \circ x, \dots, \overline{e}_k \circ x\}$. Then, for all $i \in \{1, \dots, k\}$, we have

$$\overline{e}_i(x(c)) = \overline{e}_i \circ x(c) = \alpha(\overline{e}_i \circ x) = \overline{e}_i \circ x(d) = \overline{e}_i(x(d)).$$

Since $x(c), x(d) \in x(C) \subseteq N$ and $\overline{e}_1, \dots, \overline{e}_k$ separate N, this implies that $x(c) = x(d)$. Thus β is well defined.

Claim 2 The map β preserves $R_n(\underline{\mathbf{M}} \cup \underline{\mathbf{N}})$.

We will use the Preservation Lemma. Let X be a subset of $\mathcal{B}(\mathbf{C}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$ with at most n elements. Define the subset X' of $\mathcal{A}(\mathbf{C}, \underline{\mathbf{M}})$ by

$$X' := \{\, x \in X \mid x(C) \subseteq M \,\} \\ \cup \{\, \bar{e}_i \circ x \mid i \in \{1, \dots, k\} \text{ and } x \in X \text{ with } x(C) \subseteq N \,\}.$$

The size of X' is at most $\max(n, nk) = m$. Since α preserves $R_m(\underline{\mathbf{M}})$, we know that α is given by evaluation at some $a \in C$ on X'. We shall show that β is given by evaluation at a on X.

Let $x \in X$. If $x(C) \subseteq M$, then $x \in X'$ and so $\beta(x) = \alpha(x) = x(a)$. So we can assume that $x(C) \subseteq N$. For each $i \in \{1, \dots, k\}$, we have $\bar{e}_i \circ x \in X'$ and therefore $\alpha(\bar{e}_i \circ x) = \bar{e}_i \circ x(a)$. Thus α is given by evaluation at a on $\{\bar{e}_1 \circ x, \dots, \bar{e}_k \circ x\}$. This implies that $\beta(x) = x(a)$, by the definition of β. Thus β is given by evaluation at a on X.

We have proved that $\beta : \mathcal{B}(\mathbf{C}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}}) \to M \cup N$ preserves $R_n(\underline{\mathbf{M}} \cup \underline{\mathbf{N}})$. As $R_n(\underline{\mathbf{M}} \cup \underline{\mathbf{N}})$ yields a duality on $\mathcal{B}$, there exists $b \in C$ such that β is given by evaluation at b. For every $y \in \mathcal{A}(\mathbf{C}, \underline{\mathbf{M}})$, we have $y \in \mathcal{B}(\mathbf{C}, \underline{\mathbf{M}} \cup \underline{\mathbf{N}})$ with $y(C) \subseteq M$, and therefore

$$\alpha(y) = \beta(y) = y(b).$$

Hence α is also given by evaluation at b, whence $R_m(\underline{\mathbf{M}})$ yields a duality on every finite connected algebra in $\mathcal{A}$. ∎

The corollary below was proved directly by Clark, Davey and Pitkethly [13].

5.1.13 Corollary *Let $\underline{\mathbf{M}}$ be a finite unary algebra. If the one-point extension of $\underline{\mathbf{M}}$ is finitely dualisable, then $\underline{\mathbf{M}}$ is also finitely dualisable.*

We do not know at present whether or not there is a non-dualisable unary algebra $\underline{\mathbf{M}}$ whose one-point extension $\underline{\mathbf{M}} \dot\cup \underline{\mathbf{1}}$ is dualisable (but not finitely dualisable). Finding such a pathological unary algebra $\underline{\mathbf{M}}$ would also solve the Finite Type Problem.

5.2 Term retractions and p-semilattices

In Chapter 2, we used term retractions to help lift some dualities for small algebras up to dualities for bigger algebras. In this section, we study the general relationship between term retractions and dualisability. We shall prove that a term retract of a dualisable algebra must also be dualisable, but that a term retract of a non-dualisable algebra is not necessarily non-dualisable.

The proof that term retractions preserve dualisability is very easy.

5.2.1 Theorem *A term retract of a dualisable algebra is also dualisable.*

Proof Let $\underline{\mathbf{N}}$ be a dualisable algebra and let $\gamma : \underline{\mathbf{N}} \twoheadrightarrow \underline{\mathbf{M}}$ be a term retraction. We can assume that γ fixes each element of M, by Lemma 2.3.1. Define the two quasi-varieties $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}})$ and $\mathcal{B} := \mathbb{ISP}(\underline{\mathbf{N}})$. Then $\mathcal{A}$ is contained in $\mathcal{B}$. Let $\mathbf{A} \in \mathcal{A}$ and let $\alpha : \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ be a brute-force morphism. In order to prove that $\underline{\mathbf{M}}$ is dualisable, we must show that α is an evaluation.

As $\mathbf{A} \in \mathcal{B}$ and $\gamma : \underline{\mathbf{N}} \to \underline{\mathbf{M}}$ is a homomorphism, we can define the map $\beta : \mathcal{B}(\mathbf{A}, \underline{\mathbf{N}}) \to N$ by $\beta(x) := \alpha(\gamma \circ x)$. We want to prove that β is a brute-force morphism. Since α is a brute-force morphism, the Brute Force Lemma, 1.4.5, tells us that α has a finite support and is locally an evaluation.

Let S be a finite support for α. To prove that S is also a support for β, let $x, y \in \mathcal{B}(\mathbf{A}, \underline{\mathbf{N}})$ such that $x{\restriction}_S = y{\restriction}_S$. Then $(\gamma \circ x){\restriction}_S = (\gamma \circ y){\restriction}_S$, and therefore

$$\beta(x) = \alpha(\gamma \circ x) = \alpha(\gamma \circ y) = \beta(y).$$

So S is a finite support for β.

To see that β is locally an evaluation, let X be a finite subset of $\mathcal{B}(\mathbf{A}, \underline{\mathbf{N}})$. As α is locally an evaluation, the map α is given by evaluation at some $a \in A$ on the finite subset $\{\, \gamma \circ x \mid x \in X \,\}$ of $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$. As $\mathbf{A} \in \mathcal{B} = \mathbb{ISP}(\underline{\mathbf{N}})$, there is a term function $\gamma^{\mathbf{A}} : A \to A$ of $\mathbf{A}$ corresponding to the term function $\gamma : N \to N$ of $\underline{\mathbf{N}}$. For all $x \in X$, we have

$$\beta(x) = \alpha(\gamma \circ x) = (\gamma \circ x)(a) = x(\gamma^{\mathbf{A}}(a)).$$

Thus β is given by evaluation at $\gamma^{\mathbf{A}}(a)$ on X.

We have shown that β has a finite support and is locally an evaluation. So, by the Brute Force Lemma, the map $\beta : \mathcal{B}(\mathbf{A}, \underline{\mathbf{N}}) \to N$ is a brute-force morphism. Since $\underline{\mathbf{N}}$ is dualisable, the map β is given by evaluation at some $b \in A$. For all $z \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \subseteq \mathcal{B}(\mathbf{A}, \underline{\mathbf{N}})$, we have

$$\alpha(z) = \alpha(\gamma \circ z) = \beta(z) = z(b),$$

as γ fixes M. Hence α is an evaluation. ∎

The remainder of this section is devoted to finding examples of non-dualisable algebras that have dualisable term retracts. Our examples come from the variety of p-semilattices, which was defined in the introduction to this chapter. We will need only a few basic p-semilattice facts, all of which follow relatively easily from the definition. A thorough introduction to p-semilattices can be found in O. Frink's foundational paper [32].

Recall that a p-semilattice is boolean if it satisfies the equation $x^{**} \approx x$. As the name suggests, every boolean p-semilattice is term equivalent to a Boolean algebra [32]. Every finite Boolean algebra is strongly dualisable, by the NU Strong Duality Theorem [6, 8]. Thus every finite boolean p-semilattice is strongly dualisable.

We shall finish the characterisation of dualisability for p-semilattices by proving that every finite non-boolean p-semilattice is non-dualisable. Our proof of this result illustrates the power and simplicity of the ghost-element method for establishing non-dualisability. The basic ghost-element method is described in the Ghost Element Theorem, 1.4.6. This method is adapted straight from the definition of the dualisability of an algebra, and is often easy to apply. In addition, some ghost-element proofs can be extended directly to a proof of a much stronger condition than non-dualisability.

Recall that a finite algebra $\underline{\mathbf{M}}$ is inherently non-dualisable if each finite algebra that has $\underline{\mathbf{M}}$ as a subalgebra is non-dualisable. Inherent non-dualisability was introduced by Davey, Idziak, Lampe and McNulty [23]. They used the following theorem to find inherently non-dualisable graph algebras. (Later we will prove a stronger result, Lemma 7.1.3.)

5.2.2 Inherent Non-dualisability Theorem [23, 8] *Let $\underline{\mathbf{M}}$ be a finite algebra and let $f : \omega \to \omega$. Assume that there is a subalgebra $\mathbf{A}$ of $\underline{\mathbf{M}}^S$, for some set S, and an infinite subset A_0 of A such that*

(i) *for every $n \in \omega$ and every congruence θ on $\mathbf{A}$ of index at most n, the equivalence relation $\theta{\restriction}_{A_0}$ has a unique block of size greater than $f(n)$,*

(ii) *the algebra $\mathbf{A}$ does not contain the element g of M^S that is defined by $g(s) := \rho_s(a_s)$, where a_s is any element of the unique block of $\ker(\rho_s){\restriction}_{A_0}$ of size greater than $f(|M|)$.*

Then $\underline{\mathbf{M}}$ is inherently non-dualisable.

We shall apply this theorem in its simplest form, taking the bounding function f to be constant with value 1. The theorem has been used in the literature with a non-constant bounding function [43].

The Inherent Non-dualisability Theorem enables us to complete the characterisation of dualisability for p-semilattices. An alternative method was used in the text by Clark and Davey [8] to prove that every finite subdirectly irreducible p-semilattice is non-dualisable, except for the two-element one.

5.2.3 Theorem *If a finite p-semilattice is non-boolean, then it is inherently non-dualisable.*

Proof Let $\underline{\mathbf{M}} = \langle M; \wedge, {}^*, 0, 1\rangle$ be a finite p-semilattice. Then $\underline{\mathbf{M}}$ satisfies the equations

$$x^* \wedge x \approx 0, \quad x^{**} \wedge x \approx x, \quad x^{***} \approx x^*, \quad 0^* \approx 1 \quad \text{and} \quad 1^* \approx 0.$$

Now assume that $\underline{\mathbf{M}}$ is non-boolean. There is some $a \in M$ such that $a \neq a^{**}$.

We will represent sequences in M^ω using the notation introduced on page 81. Define $\mathbf{A}$ to be the subalgebra of $\underline{\mathbf{M}}^\omega$ generated by the set A_0, where

$$A_0 := \{\, a_n^0 \mid n \in \omega\backslash\{0\} \,\}.$$

We will show that, if θ is a congruence on $\mathbf{A}$ of finite index, then $\theta{\restriction}_{A_0}$ has a unique non-trivial block.

Let θ be a congruence on $\mathbf{A}$ of finite index. Assume that $k, \ell, m, n \in \omega\backslash\{0\}$, with $k \neq \ell$ and $m \neq n$, such that $a_k^0 \equiv_\theta a_\ell^0$ and $a_m^0 \equiv_\theta a_n^0$. Then

$$(a_k^0)^* = (a_k^0)^* \wedge (a_k^0)^* \equiv_\theta (a_k^0)^* \wedge (a_\ell^0)^* = \widehat{a}^*.$$

By symmetry, we also have $(a_m^0)^* \equiv_\theta \widehat{a}^*$. So $(a_k^0)^* \equiv_\theta (a_m^0)^*$, which gives us

$$a_k^0 = (a_k^0)^{**} \wedge a_k^0 \equiv_\theta (a_m^0)^{**} \wedge a_k^0 = a_{mk}^{0\,0}.$$

By symmetry once again, we have $a_m^0 \equiv_\theta a_{km}^{0\,0}$. Thus $a_k^0 \equiv_\theta a_m^0$, whence $\theta{\restriction}_{A_0}$ has at most one non-trivial block. The equivalence relation $\theta{\restriction}_{A_0}$ has at least one non-trivial block, since A_0 is infinite and θ is of finite index.

Now define $g \in M^\omega$ by $g(n) := \rho_n(c_n)$, where c_n is any element of the non-trivial block of $\ker(\rho_n){\restriction}_{A_0}$. Then g is the constant sequence $\widehat{a}$. It remains to prove that $g \notin A$. Using the equations given at the beginning of this proof, it is easy to see that the set $\{0, 1, a, a^*, a^{**}\}$ forms a subalgebra of $\underline{\mathbf{M}}$. Since $a \neq a^{**}$, it also follows that $a \notin \{0, 1, a^*, a^{**}\}$. Define the subset C of M^ω by

$$C := \{\, c \in \{0, 1, a, a^*, a^{**}\}^\omega \mid c(0) \neq a \text{ or } 0 \in c(\omega) \,\}.$$

As 0 is the least element of $\underline{\mathbf{M}}$ and a is meet-irreducible in $\{0, 1, a, a^*, a^{**}\}$, the set C is closed under $\wedge$. Since $b^* \neq a$, for all $b \in \{0, 1, a, a^*, a^{**}\}$, the set C is closed under *. So C is a subuniverse of $\underline{\mathbf{M}}^\omega$. Thus $g \notin A$, as $A_0 \subseteq C$ and $g = \widehat{a} \notin C$. Hence $\underline{\mathbf{M}}$ is inherently non-dualisable, by the Inherent Non-dualisability Theorem, 5.2.2. ∎

Since every finite boolean p-semilattice is strongly dualisable, we now have a characterisation of dualisability for p-semilattices.

5.2.4 Theorem *A finite p-semilattice is dualisable if and only if it is boolean. Moreover, every dualisable p-semilattice is strongly dualisable, and every non-dualisable p-semilattice is inherently non-dualisable.*

We can use the previous theorem to find a plethora of non-dualisable algebras that have dualisable term retracts.

5.2.5 Example *Every finite non-boolean p-semilattice is non-dualisable yet has a non-trivial dualisable p-semilattice as a term retract.*

Proof Let $\underline{\mathbf{P}}$ be a finite p-semilattice. Then $\underline{\mathbf{P}}$ satisfies the equations

$$0^{**} \approx 0, \quad 1^{**} \approx 1, \quad x^{**} \wedge y^{**} \approx (x \wedge y)^{**} \quad \text{and} \quad x^{***} \approx x^{*}.$$

So we can define the homomorphism $\gamma : \underline{\mathbf{P}} \to \underline{\mathbf{P}}$ by $\gamma(a) = a^{**}$. Let $\underline{\mathbf{Q}}$ be the image of γ. For all $a \in P$, we have

$$\gamma(a)^{**} = a^{****} = a^{**} = \gamma(a).$$

This implies that $\underline{\mathbf{Q}}$ is boolean, and therefore dualisable. Moreover, every element of Q is fixed by γ. Thus $\gamma : \underline{\mathbf{P}} \twoheadrightarrow \underline{\mathbf{Q}}$ is a term retraction. If we assume that $\underline{\mathbf{P}}$ is non-boolean, then $\underline{\mathbf{P}}$ must be non-trivial and therefore $\underline{\mathbf{Q}}$ is also non-trivial. ∎

We have seen that the ghost-element method can be very easy to use. We can now illustrate another advantage of this method: ghost-element proofs can often be extended to encompass more examples. Using the following lemma, we will be able to explain one way in which this can happen.

5.2.6 Lemma *Let $\underline{\mathbf{M}}$ be a term reduct of a finite algebra $\underline{\mathbf{M}}^{\sharp}$. Define the quasi-varieties $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}})$ and $\mathcal{A}^{\sharp} := \mathbb{ISP}(\underline{\mathbf{M}}^{\sharp})$. Let $\mathbf{A}$ be a subalgebra of $\underline{\mathbf{M}}^{S}$, for some set S, and assume that $\alpha : \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ is a brute-force morphism. Define the algebra $\mathbf{B} := \mathrm{sg}_{(\underline{\mathbf{M}}^{\sharp})^S}(A)$, and define the map $\beta : \mathcal{A}^{\sharp}(\mathbf{B}, \underline{\mathbf{M}}^{\sharp}) \to M$ by $\beta(x) := \alpha(x{\restriction_A})$. Then β is a brute-force morphism, and $g_\beta = g_\alpha$.*

Proof This result is a simple corollary of the Brute Force Lemma, 1.4.5. The brute-force morphism α has a finite support and is locally an evaluation. It follows easily that β has a finite support and is locally an evaluation. So β is a brute-force morphism. For all $s \in S$, we have

$$g_\beta(s) = \beta(\pi_s{\restriction_B}) = \alpha(\pi_s{\restriction_A}) = g_\alpha(s).$$

Thus $g_\beta = g_\alpha$. ∎

Say we have a ghost-element proof that the algebra $\underline{\mathbf{M}}$ is non-dualisable. Then there is a brute-force morphism $\alpha : \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \to M$, for some set S and $\mathbf{A} \leqslant \underline{\mathbf{M}}^{S}$, such that $g_\alpha \notin A$. Now let $\underline{\mathbf{M}}^{\sharp}$ be a finite algebra that has $\underline{\mathbf{M}}$ as a term reduct. To show that $\underline{\mathbf{M}}^\sharp$ is non-dualisable, it is enough to check that the ghost element is not generated, that is, that $g_\alpha \notin \mathrm{sg}_{(\underline{\mathbf{M}}^\sharp)^S}(A)$. This feature of the ghost-element method was used in the development of some of the results from Chapter 3. For example, the proof of Theorem 3.4.4 was adapted from a proof that the single algebra $\langle\{0,1,2\};010,001,002,110\rangle$ is non-dualisable.

Ghost-element proofs can be extended not only by adding extra operations, but also by giving weaker conditions on the behaviour of the existing operations. In Chapter 3, the proof of Theorem 3.4.2 was adapted from a proof that the algebra $\langle\{0,1,2\};001,010\rangle$ is non-dualisable. To finish this section, we give an explicit illustration of this method of extending ghost-element proofs.

The following technical lemma, adapted from Theorem 5.2.3, will be used to construct various examples of inherently non-dualisable algebras.

5.2.7 Lemma *Let $\underline{\mathbf{M}} = \langle M; F \cup \{\wedge\}\rangle$ be a finite algebra such that $\wedge$ is a meet-semilattice operation on M and F is a set of unary operations on M. Assume that there exists $^* \in F$ and a pair of distinct elements $0, a \in M$ for which*

(i) $a \leqslant a^{**}$, $a^* \leqslant 0^*$ *and* $0^{**} \wedge a = 0$,

(ii) 0 *is the least element of* $\underline{\mathbf{M}}$,

(iii) a *is meet-irreducible in* $\mathrm{sg}_{\underline{\mathbf{M}}}(a)$,

(iv) $a \neq u(b)$, *for all* $u \in F$ *and all* $b \in \mathrm{sg}_{\underline{\mathbf{M}}}(a)$.

Then $\underline{\mathbf{M}}$ is inherently non-dualisable.

Proof This proof is basically the same as the proof of Theorem 5.2.3. Define $\mathbf{A}$ to be the subalgebra of $\underline{\mathbf{M}}^\omega$ generated by $A_0 := \{\, a_n^0 \mid n \in \omega\backslash\{0\} \,\}$. Using (i) and (ii), the third paragraph of the proof of Theorem 5.2.3 shows that, for each congruence θ on $\mathbf{A}$ of finite index, the equivalence relation $\theta{\restriction}_{A_0}$ has a unique non-trivial block.

To prove that the algebra $\underline{\mathbf{M}}$ is inherently non-dualisable, using the Inherent Non-dualisability Theorem, 5.2.2, it remains to check that the element $g := \widehat{a}$ of M^ω does not belong to A. Define

$$C := \{\, c \in M^\omega \mid c(0) \in \mathrm{sg}_{\underline{\mathbf{M}}}(a) \text{ and } \big(c(0) \neq a \text{ or } 0 \in c(\omega)\big) \,\}.$$

By (ii), (iii) and (iv), the set C forms a subalgebra of $\underline{\mathbf{M}}^\omega$. Since $A_0 \subseteq C$ and $g \notin C$, we have $g \notin A$. ∎

Figure 5.1 gives some examples of semilattices with added unary operations that satisfy the conditions of the previous lemma and are therefore inherently non-dualisable. In contrast, Davey, Jackson and Talukder [24] have proved that a finite semilattice with added algebraic operations must be dualisable. So, for

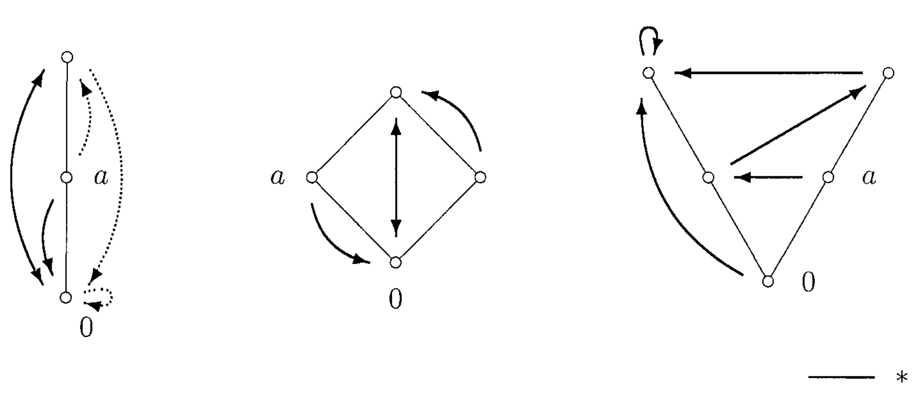

Figure 5.1 Some inherently non-dualisable algebras

example, for each finite p-semilattice $\langle M; \wedge, ^*, 0, 1\rangle$, the term reduct $\langle M; \wedge, ^{**}\rangle$ is dualisable.

5.3 Building non-dualisable algebras from dualisable ones

In this section, we find examples to show that non-dualisable algebras can be created from dualisable algebras using natural algebraic constructions.

5.3.1 Definition We shall begin by considering the two unary algebras

$$\underline{\mathbf{P}} := \langle\{0,1,2,3\}; 0011, 0101\rangle \quad \text{and} \quad \underline{\mathbf{Q}} := \langle\{0,1,2\}; 001, 010\rangle,$$

illustrated in Figure 5.2. Both the algebras $\underline{\mathbf{P}}$ and $\underline{\mathbf{Q}}$ are of type $\{u, v\}$, where

$$u^{\underline{\mathbf{P}}} := 0011, \quad v^{\underline{\mathbf{P}}} := 0101, \quad u^{\underline{\mathbf{Q}}} := 001 \quad \text{and} \quad v^{\underline{\mathbf{Q}}} := 010.$$

The algebra $\underline{\mathbf{Q}}$ is a subalgebra of $\underline{\mathbf{P}}$, and so $\underline{\mathbf{Q}} \in \mathbb{ISP}(\underline{\mathbf{P}})$. Note that $\underline{\mathbf{P}}$ and $\underline{\mathbf{Q}}$ do not generate the same quasi-variety, since $\underline{\mathbf{Q}}$ satisfies the quasi-equation $u(x) \approx v(x) \implies x \approx u(x)$ but $\underline{\mathbf{P}}$ does not.

5.3.2 Example *Define the two unary algebras* $\underline{\mathbf{P}}$ *and* $\underline{\mathbf{Q}}$ *as in 5.3.1.*

(i) *The dualisable algebra* $\underline{\mathbf{P}}$ *has a non-dualisable subalgebra* $\underline{\mathbf{Q}}$.

(ii) *The dualisable algebra* $\underline{\mathbf{P}}$ *and the non-dualisable algebra* $\underline{\mathbf{Q}}$ *generate the same variety.*

Proof We have already proved that $\underline{\mathbf{Q}}$ is not dualisable; see Theorem 3.0.1. The operations of $\underline{\mathbf{P}}$ are endomorphisms of the lattice $\mathbf{P}_0$ illustrated in Figure 5.2. So $\underline{\mathbf{P}}$ is dualisable, by the Lattice Endomorphism Theorem, 2.1.2.

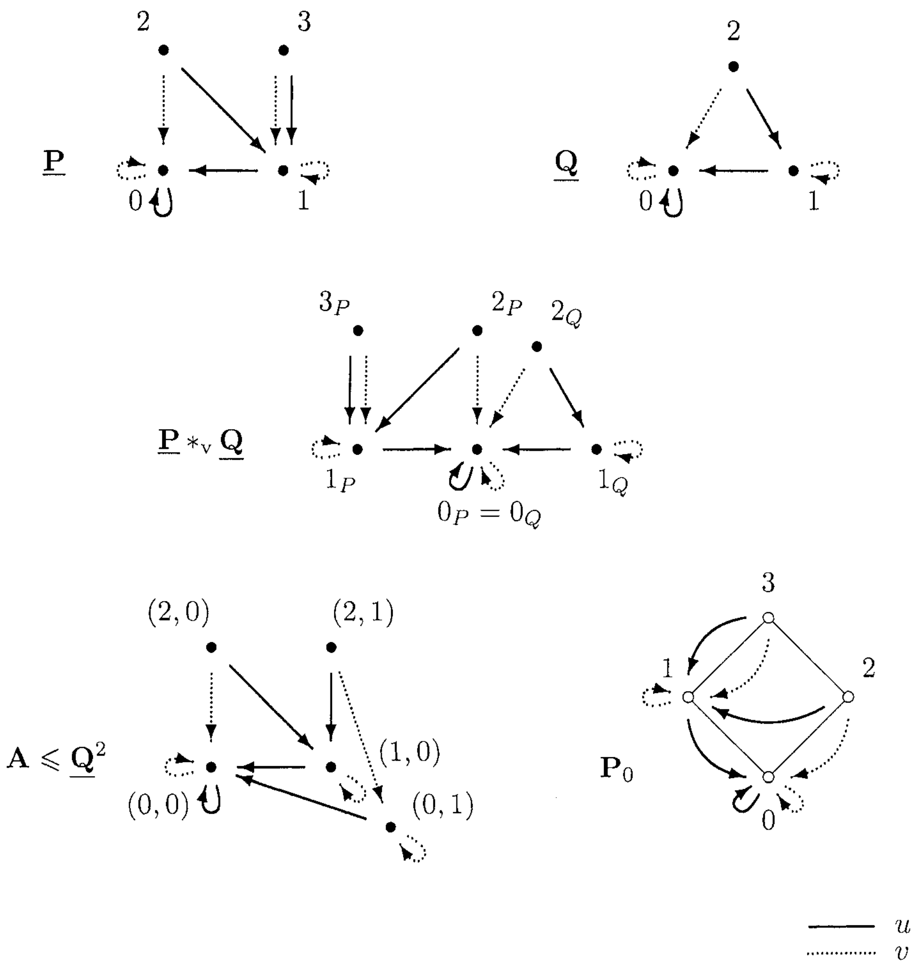

Figure 5.2

It remains to show that $\underline{\mathbf{P}}$ and $\underline{\mathbf{Q}}$ generate the same variety. We know that $\underline{\mathbf{Q}}$ belongs to $\mathrm{Var}(\underline{\mathbf{P}})$. To see that $\underline{\mathbf{P}}$ belongs to $\mathrm{Var}(\underline{\mathbf{Q}})$, let $\mathbf{A}$ be the subalgebra of $\underline{\mathbf{Q}}^2$ drawn in Figure 5.2. There is a surjective homomorphism $x : \mathbf{A} \twoheadrightarrow \underline{\mathbf{P}}$, given by $x\big((a,b)\big) := a + b$, for all $(a,b) \in A$. So $\mathrm{Var}(\underline{\mathbf{P}}) = \mathrm{Var}(\underline{\mathbf{Q}})$. ∎

In general, it is not possible to determine whether or not a finite unary algebra is dualisable simply by studying its abstract monoid of unary term functions. The monoid of unary term functions of an algebra is isomorphic to the monoid of unary term functions of the one-generated free algebra in the variety it generates.

Since $\mathrm{Var}(\underline{\mathbf{P}}) = \mathrm{Var}(\underline{\mathbf{Q}})$, in the previous example, the dualisable algebra $\underline{\mathbf{P}}$ and the non-dualisable algebra $\underline{\mathbf{Q}}$ have isomorphic monoids of unary term functions.

For all finite unary algebras $\mathbf{A}$ and $\mathbf{B}$ of the same type, let $\mathbf{A} *_{\mathrm{v}} \mathbf{B}$ denote the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in the variety $\mathrm{Var}(\mathbf{A}, \mathbf{B})$, and let $\mathbf{A} *_{\mathrm{q}} \mathbf{B}$ denote the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in the quasi-variety $\mathbb{ISP}(\mathbf{A}, \mathbf{B})$. These coproducts must exist, by Lemmas 5.1.2 and 5.1.3.

5.3.3 Example *Define the two unary algebras $\underline{\mathbf{P}}$ and $\underline{\mathbf{Q}}$ as in 5.3.1. Then the non-dualisable algebra $\underline{\mathbf{Q}}$ is a retract of the dualisable algebra $\underline{\mathbf{P}} *_{\mathrm{v}} \underline{\mathbf{Q}}$.*

Proof The only element of $\underline{\mathbf{P}}$ that is the value of a constant term function is 0. So we have $|F_{\mathrm{Var}(\underline{\mathbf{P}})}(0)| = 1$. Since $\mathrm{Var}(\underline{\mathbf{P}}) = \mathrm{Var}(\underline{\mathbf{Q}})$, it follows by Lemma 5.1.3 that $\underline{\mathbf{M}} := \underline{\mathbf{P}} *_{\mathrm{v}} \underline{\mathbf{Q}}$ is as drawn in Figure 5.2. Let $-_P : \underline{\mathbf{P}} \hookrightarrow \underline{\mathbf{M}}$ and $-_Q : \underline{\mathbf{Q}} \hookrightarrow \underline{\mathbf{M}}$ be the natural embeddings. We can define a retraction $x : \underline{\mathbf{M}} \twoheadrightarrow \underline{\mathbf{Q}}$ by

$$x(a_P) = 0, \text{ for all } a \in P, \text{ and } x(b_Q) = b, \text{ for all } b \in Q.$$

Define the homomorphism $y : \underline{\mathbf{M}} \to \underline{\mathbf{P}}$ by

$$y(a_P) = a, \text{ for all } a \in P, \text{ and } y(b_Q) = 0, \text{ for all } b \in Q.$$

Then x and y separate the elements of M. Since $\underline{\mathbf{Q}} \leqslant \underline{\mathbf{P}}$, this tells us that $\underline{\mathbf{M}} \in \mathbb{ISP}(\underline{\mathbf{P}})$, using the $\mathbb{ISP}$ Theorem, 1.1.1. As $\underline{\mathbf{P}}$ embeds into $\underline{\mathbf{M}}$, it follows that $\mathbb{ISP}(\underline{\mathbf{M}}) = \mathbb{ISP}(\underline{\mathbf{P}})$. We know that $\underline{\mathbf{P}}$ is dualisable. So $\underline{\mathbf{M}} = \underline{\mathbf{P}} *_{\mathrm{v}} \underline{\mathbf{Q}}$ is dualisable, by Independence Theorem, 1.4.1. ∎

The previous example is a special case of the following more general result.

5.3.4 Example *Every finite unary algebra with a one-element subalgebra is a retract of a dualisable algebra.*

Proof The proof of this result is almost identical to the proof of the previous example. Consider a finite unary algebra $\underline{\mathbf{M}}$, and assume there is some $m \in M$ that determines a one-element subalgebra of $\underline{\mathbf{M}}$. We know that $\underline{\mathbf{M}}$ is a subalgebra of a dualisable algebra $\underline{\mathbf{N}}$, by Theorem 2.1.4. Construct the new unary algebra $\underline{\mathbf{M}} \cup_m \underline{\mathbf{N}}$ by taking the disjoint union of $\underline{\mathbf{M}}$ and $\underline{\mathbf{N}}$, and identifying $(m, 0)$ in the copy of $\underline{\mathbf{M}}$ with $(m, 1)$ in the copy of $\underline{\mathbf{N}}$. Then $\underline{\mathbf{M}}$ is a retract of $\underline{\mathbf{M}} \cup_m \underline{\mathbf{N}}$. It is straightforward to check that $\underline{\mathbf{M}} \cup_m \underline{\mathbf{N}}$ is separated by homomorphisms into $\underline{\mathbf{N}}$. So the algebras $\underline{\mathbf{M}} \cup_m \underline{\mathbf{N}}$ and $\underline{\mathbf{N}}$ generate the same quasi-variety. Therefore $\underline{\mathbf{M}} \cup_m \underline{\mathbf{N}}$ is dualisable, by the Independence Theorem, 1.4.1. ∎

5.3.5 Definition Now define the unary algebras

$$\underline{\mathbf{K}} := \langle \{0,1,2,3\}; 0010, 0001 \rangle \quad \text{and} \quad \underline{\mathbf{L}} := \langle \{0,1,2\}; 001, 001 \rangle,$$

shown in Figure 5.3. We will prove that both $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$ are dualisable, but that the product $\underline{\mathbf{K}} \times \underline{\mathbf{L}}$, the coproducts $\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}$ and $\underline{\mathbf{K}} *_{\mathrm{q}} \underline{\mathbf{L}}$, and the disjoint union $\underline{\mathbf{K}} \dot{\cup} \underline{\mathbf{L}}$ are all non-dualisable. The algebras $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$ are of type $\{u, v\}$, where

$$u^{\underline{\mathbf{K}}} := 0010, \quad v^{\underline{\mathbf{K}}} := 0001, \quad u^{\underline{\mathbf{L}}} := 001 \quad \text{and} \quad v^{\underline{\mathbf{L}}} := 001.$$

The odd-looking algebra $\underline{\mathbf{L}}$ actually belongs to the variety $\mathrm{Var}(\underline{\mathbf{K}})$: there is a subalgebra $\mathbf{B}$ of $\underline{\mathbf{K}}^2$, drawn in Figure 5.3, that has $\underline{\mathbf{L}}$ as a homomorphic image.

The only element of $\underline{\mathbf{K}}$ that is the value of a constant term function is 0. This implies that $|F_{\mathrm{Var}(\underline{\mathbf{K}})}(0)| = 1$. Using Lemma 5.1.3, it is easy to see that the coproduct $\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}$ is as depicted in Figure 5.3. The elements of $\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}$ are separated by homomorphisms into $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$. So $\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}} \in \mathbb{ISP}(\underline{\mathbf{K}}, \underline{\mathbf{L}})$. It follows by Lemma 5.1.2 that $\underline{\mathbf{K}} *_{\mathrm{q}} \underline{\mathbf{L}} = \underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}$.

5.3.6 Lemma *Define the two unary algebras $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$ as in 5.3.5. Then both $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$ are dualisable.*

Proof The dualisability of $\underline{\mathbf{L}}$ follows easily from the Lattice Endomorphism Theorem, 2.1.2. To prove that $\underline{\mathbf{K}}$ is dualisable, we use Theorem 2.2.9. The element $0 \in K$ is the value of a constant term function of $\underline{\mathbf{K}}$.

Both the fundamental operations of $\underline{\mathbf{K}}$ are endomorphisms of the meet semilattice $\mathbf{K}_0 = \langle K; \wedge_0 \rangle$ drawn in Figure 5.3. So $\wedge_0 : \underline{\mathbf{K}}^2 \to \underline{\mathbf{K}}$ is a binary homomorphism of $\underline{\mathbf{K}}$. We now want to show that $g : K^2 \to K$, given by

$$g(a,b) = \begin{cases} 1 & \text{if } a = 1 \text{ and } b = 0, \\ 2 & \text{if } a = 2 \text{ and } b \neq 2, \\ 3 & \text{if } a = 3 \text{ and } b \neq 3, \\ 0 & \text{otherwise,} \end{cases}$$

is a binary homomorphism of $\underline{\mathbf{K}}$. To do this, let $a, b \in K$. Then

$$\begin{aligned} u(g(a,b)) = 1 &\iff g(a,b) = 2 \\ &\iff a = 2 \ \& \ b \neq 2 \\ &\iff u(a) = 1 \ \& \ u(b) = 0 \\ &\iff g(u(a), u(b)) = 1. \end{aligned}$$

Since $u(a) \in \{0,1\}$, we must have $g(u(a), u(b)) \in \{0,1\}$. As we also have $u(g(a,b)) \in \{0,1\}$, it now follows that $u(g(a,b)) = g(u(a), u(b))$. Thus g preserves u and, by symmetry, it also preserves v. Therefore g is a binary homomorphism of $\underline{\mathbf{K}}$.

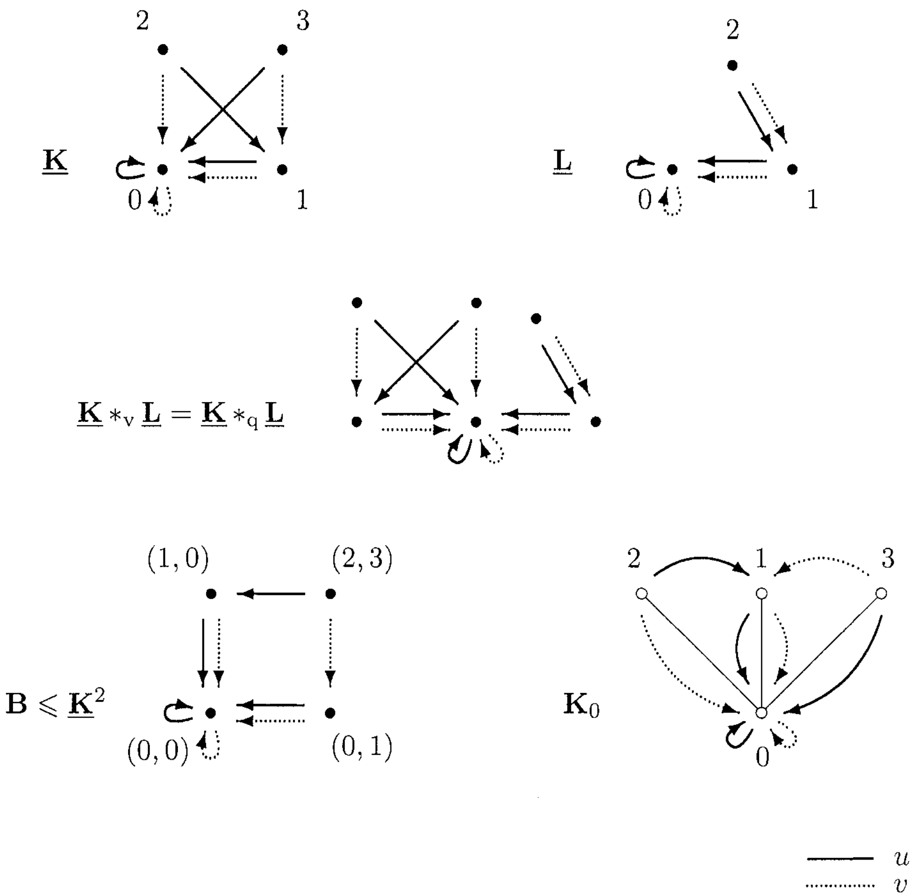

Figure 5.3

Define the set $G := \{\wedge_0, g\}$ of binary homomorphisms of $\underline{\mathbf{K}}$ and the subset $S := \{1, 2, 3\}$ of K. Every element of S is a strong idempotent of $\wedge_0$. Now let $k \in K\backslash\{0\}$. As $\underline{\mathbf{K}}$ satisfies $u(v(x)) \approx v(u(x))$, the operations $u^{\underline{\mathbf{K}}} = 0010$ and $v^{\underline{\mathbf{K}}} = 0001$ are endomorphisms of $\underline{\mathbf{K}}$. So $1 \in S \cap \mathrm{End}(\underline{\mathbf{K}})(k)$. We have

$$g(1, m) = 1 \iff m = 0,$$

for all $m \in K$. Thus G and $S \cap \mathrm{End}(\underline{\mathbf{K}})(k)$ distinguish 0 within K. By Theorem 2.2.9, the algebra $\underline{\mathbf{K}}$ is dualisable. ∎

The non-dualisability of $\underline{\mathbf{K}} \times \underline{\mathbf{L}}$ and $\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}$ will follow once we have established that $\underline{\mathbf{K}} \,\dot{\cup}\, \underline{\mathbf{L}}$ is non-dualisable.

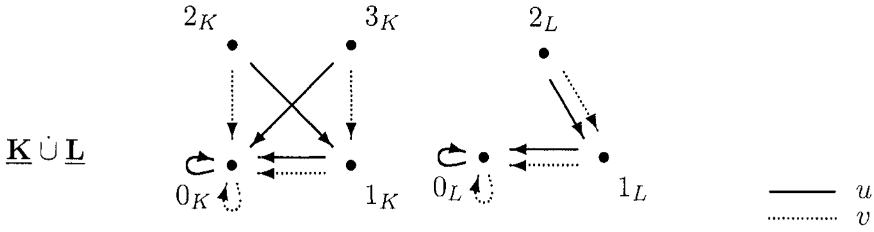

Figure 5.4 A non-dualisable disjoint union of two dualisable algebras

5.3.7 Lemma *Define the two unary algebras $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$ as in 5.3.5. Then the disjoint union $\underline{\mathbf{K}} \dot\cup \underline{\mathbf{L}}$ is not dualisable.*

Proof Define the algebra $\underline{\mathbf{M}} := \underline{\mathbf{K}} \dot\cup \underline{\mathbf{L}}$. Then there are natural embeddings $-_K : \underline{\mathbf{K}} \hookrightarrow \underline{\mathbf{M}}$ and $-_L : \underline{\mathbf{L}} \hookrightarrow \underline{\mathbf{M}}$; see Figure 5.4. We shall prove that $\underline{\mathbf{M}}$ is not dualisable by applying the Non-dualisability Lemma, 3.4.1.

For each $n \in \omega\backslash\{0\}$, define $a_n \in M^\omega$ by

$$a_n(i) = \begin{cases} 1_L & \text{if } i = 0, \\ 1_K & \text{if } i = n, \\ 0_K & \text{otherwise.} \end{cases}$$

For all $m, n \in \omega\backslash\{0\}$ such that $m \neq n$, define $b_{mn} \in M^\omega$ by

$$b_{mn}(i) = \begin{cases} 2_L & \text{if } i = 0, \\ 2_K & \text{if } i = m, \\ 3_K & \text{if } i = n, \\ 0_K & \text{otherwise.} \end{cases}$$

Now define two subsets of M^ω by

$$A_0 := \{\, a_n \mid n \in \omega\backslash\{0\} \,\} \quad \text{and} \quad B := \{\, b_{mn} \mid m, n \in \omega\backslash\{0\} \text{ and } m \neq n \,\}.$$

Let $\mathbf{A}$ denote the subalgebra of $\underline{\mathbf{M}}^\omega$ generated by $A_0 \cup B$.

Let $x : \mathbf{A} \to \underline{\mathbf{M}}$ be a homomorphism. We want to show that $\ker(x{\restriction}_{A_0})$ has a unique non-trivial block. For each $n \in \omega\backslash\{0\}$, we have

$$x(a_n) = x(u(b_{n\,n+1})) = u(x(b_{n\,n+1})).$$

Therefore $x(A_0) \subseteq u(M) = \{0_K, 1_K, 0_L, 1_L\}$. Since we want to prove that $\ker(x{\restriction}_{A_0})$ has a unique non-trivial block, we can assume that $x(A_0) \neq \{0_K\}$. So one of the following three cases must apply.

Case 1: $1_K \in x(A_0)$. There is some $m \in \omega\backslash\{0\}$ such that $x(a_m) = 1_K$. Let $n \in \omega\backslash\{0\}$ with $m \neq n$. Then

$$a_m \xleftarrow{u} b_{mn} \xrightarrow{v} a_n$$

in $\mathbf{A}$. Under the homomorphism x, this gives us

$$\boxed{1_K} \xleftarrow{u} 2_K \xrightarrow{v} 0_K$$

in $\underline{\mathbf{M}}$. So $x(a_n) = 0_K$, and therefore $A_0\backslash\{a_m\}$ is the unique non-trivial block of $\ker(x{\restriction}_{A_0})$.

Case 2: $0_L \in x(A_0)$. There is some $m \in \omega\backslash\{0\}$ for which $x(a_m) = 0_L$. Choose some $n \in \omega\backslash\{0\}$ such that $m \neq n$. Then

$$a_m \xleftarrow{u} b_{mn} \xrightarrow{v} a_n \quad \stackrel{x}{\Longrightarrow} \quad \boxed{0_L} \xleftarrow{u} 0_L, 1_L \xrightarrow{v} 0_L.$$

This implies that $x(a_n) = 0_L$. So A_0 is the only block of $\ker(x{\restriction}_{A_0})$.

Case 3: $1_L \in x(A_0)$. There is some $m \in \omega\backslash\{0\}$ such that $x(a_m) = 1_L$. Let $n \in \omega\backslash\{0\}$ with $m \neq n$. Then

$$a_m \xleftarrow{u} b_{mn} \xrightarrow{v} a_n \quad \stackrel{x}{\Longrightarrow} \quad \boxed{1_L} \xleftarrow{u} 2_L \xrightarrow{v} 1_L,$$

and therefore $x(a_n) = 1_L$. Thus A_0 is the only block of $\ker(x{\restriction}_{A_0})$.

Now define $g \in M^\omega$ by $g(i) := \rho_i(a_{n_i})$, where a_{n_i} is any element of the unique non-trivial block of $\ker(\rho_i{\restriction}_{A_0})$. Then

$$g(i) = \begin{cases} 1_L & \text{if } i = 0, \\ 0_K & \text{otherwise,} \end{cases}$$

for all $i \in \omega$. To show that $\underline{\mathbf{M}}$ is non-dualisable, it suffices, by the Non-dualisability Lemma, 3.4.1, to prove that $g \notin A$. Define $c \in M^\omega$ by

$$c(i) = \begin{cases} 0_L & \text{if } i = 0, \\ 0_K & \text{otherwise.} \end{cases}$$

We shall show that $C := \{c\} \cup A_0 \cup B$ forms a subalgebra of $\underline{\mathbf{M}}^\omega$. We have $u(c) = c = v(c)$ and, for each $n \in \omega\backslash\{0\}$, we have $u(a_n) = c = v(a_n)$. Lastly, for all $m, n \in \omega\backslash\{0\}$ with $m \neq n$, we know that $u(b_{mn}) = a_m \in A_0$ and $v(b_{mn}) = a_n \in A_0$. So C forms a subalgebra of $\underline{\mathbf{M}}^\omega$. Since $A \subseteq C$, we get $g \notin A$. Thus $\underline{\mathbf{M}}$ is not dualisable. ∎

construction	*preserves non-dualisability?*	*references*
non-trivial subalgebra	✗	5.2.5, 7.1.2, 7.1.6
non-trivial homomorphic image	✗	5.2.5
non-trivial retract	✗	5.2.5
non-trivial term retract	✗	5.2.5, 2.3.2
finite power	✓	[61, 30]
finite product	?	
finite coproduct	?	
one-point extension	?	5.1.13

Table 5.3

5.3.8 Example *Define the two unary algebras $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$ as in 5.3.5. Then both $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$ are dualisable, but the product $\underline{\mathbf{K}} \times \underline{\mathbf{L}}$, the coproducts $\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}$ and $\underline{\mathbf{K}} *_{\mathrm{q}} \underline{\mathbf{L}}$, and the disjoint union $\underline{\mathbf{K}} \dot{\cup} \underline{\mathbf{L}}$ are all non-dualisable.*

Proof We have just proved that $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$ are dualisable and that $\underline{\mathbf{K}} \dot{\cup} \underline{\mathbf{L}}$ is not dualisable. Let $\underline{\mathbf{1}}$ be a one-element algebra of the same type as $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$. It is straightforward to check that the disjoint union $\underline{\mathbf{K}} \dot{\cup} \underline{\mathbf{L}}$ is separated by homomorphisms into $(\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}) \dot{\cup} \underline{\mathbf{1}}$, and that $(\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}) \dot{\cup} \underline{\mathbf{1}}$ is separated by homomorphisms into $\underline{\mathbf{K}} \dot{\cup} \underline{\mathbf{L}}$. So

$$\mathbb{ISP}(\underline{\mathbf{K}} \dot{\cup} \underline{\mathbf{L}}) = \mathbb{ISP}((\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}) \dot{\cup} \underline{\mathbf{1}}).$$

Using the Independence Theorem, 1.4.1, and Corollary 5.1.11, it follows that $\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}} = \underline{\mathbf{K}} *_{\mathrm{q}} \underline{\mathbf{L}}$ must be non-dualisable.

The algebra $\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}$ is isomorphic to the subalgebra of $\underline{\mathbf{K}} \times \underline{\mathbf{L}}$ with the underlying set $\big(K \times \{0\}\big) \cup \big(\{0\} \times L\big)$. Therefore $\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}} \in \mathbb{ISP}(\underline{\mathbf{K}} \times \underline{\mathbf{L}})$. As both $\underline{\mathbf{K}}$ and $\underline{\mathbf{L}}$ are isomorphic to a subalgebra of $\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}}$, we must have $\underline{\mathbf{K}} \times \underline{\mathbf{L}} \in \mathbb{ISP}(\underline{\mathbf{K}} *_{\mathrm{v}} \underline{\mathbf{L}})$. Thus the product $\underline{\mathbf{K}} \times \underline{\mathbf{L}}$ is not dualisable, by the Independence Theorem. ∎

It is also reasonable to ask how the property of non-dualisability interacts with natural algebraic constructions. We know that non-dualisability is not always preserved by taking non-trivial subalgebras or non-trivial homomorphic images; see Example 5.2.5. The current state of our knowledge about non-dualisability and algebraic constructions is summarised in Table 5.3. This table reveals several open problems: for example, find a pair of non-dualisable algebras whose product is dualisable.