# A probabilistic approach to quantum Bayesian games of incomplete information 

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#### Abstract

A Bayesian game is a game of incomplete information in which the rules of the game are not fully known to all players. We consider the Bayesian game of Battle of Sexes that has several Bayesian Nash equilibria and investigate its outcome when the underlying probability set is obtained from generalized Einstein-Podolsky-Rosen experiments. We find that this probability set, which may become non-factorizable, results in a unique Bayesian Nash equilibrium of the game.


Keywords Quantum games • Bayesian Nash equilibria • EPR experiments • Quantum probability

## 1 Introduction

The standard approach to constructing quantum games [1-57] naturally uses the formalism of quantum mechanics in Hilbert space [58]. In recent years, however, a prob-

[^0]abilistic approach to this research area $[28,39,48-50,52-54]$ has been proposed that uses sets of non-factorizable quantum mechanical probabilities, i.e., without using the quantum mechanical concepts of state vectors, self-adjoint operators, and quantum measurement. This is with a view of making the area of quantum games more accessible to wider mathematical application, as the methods and range of solution concepts of game theory [59-62] are used and exploited, without any real need for invoking the Hilbert space formalism of quantum mechanics.

The probabilistic approach for a two-player two-strategy game directly uses sets of quantum probabilities corresponding to the measurement outcomes on a two-qubit quantum system. As is known, the setting of generalized Einstein-Podolsky-Rosen (EPR) experiments [63-71] performed on this system leads to the consideration of a set of 16 quantum probabilities. Properties of this probability set have been investigated by Cereceda [71], and it has been pointed out that the CHSH form of Bell's inequality [70] can be re-expressed in terms of constraints on the elements from this set. It is observed that only a non-factorizable probability set, as is defined later, can lead to the violation of Bell's inequality and that not every non-factorizable probability set violates Bell's inequality.

A Bayesian game is a game of incomplete information in which the rules of the game are not fully known to all players. In this paper, we study a Bayesian game that is a variant of the well-known Battle of Sexes game, also studied $[12,36]$ in the quantum game literature. In an earlier study [36], using the quantization protocol based on Schmidt decomposition [58], a Bayesian game of incomplete information [62] has been investigated in relation to the violation of Bell's inequality [58,65-68]. The present paper, however, adopts a different approach in that, without referring to the Hilbert space formalism, it finds the outcome of a Bayesian game when the considered probability set can be non-factorizable-this can arise, in an experimental situation, from a setup involving quantum entanglement, such as an EPR-type experiment, where the CHSH form of Bell's inequality is violated.

In its normal form representation, the game matrix of the Bayesian game has the same number of entries as the 16 elements $\varepsilon_{i}$ in the probability set that corresponds to the generalized EPR experiments [71]. We find that the richer structure of the Bayesian game permits a natural embedding of the classical factorizable game within the quantum game. We show that, whereas the classical factorizable Bayesian game of imperfect information has several Nash equilibria, its non-factorizable quantum version obtained from a set of quantum probabilities corresponding to generalized EPR experiments has a unique Nash equilibrium.

The suggested probabilistic approach to obtaining quantum games thus re-expresses players' payoff relations in terms of a set of probabilities that can also arise in a quantum mechanical experiment. As the approach is based on probabilities only, it does not refer to the formalism of quantum mechanics using state vectors, unitary transformations, and quantum measurements. As game theory is a broad area, with applications ranging from trade, politics, sociology, biology, engineering, etc., most researchers in this area are naturally not familiar with the mathematical formalism of quantum mechanics. This paper thus fills in that gap and demonstrates how an unusual game-theoretic outcome for a Bayesian game results when probability sets that are obtained in quantum mechanical experiments are the underlying probabilities
of a Bayesian game. In this approach, the quantum game reduces itself to the classical game when the considered probability set becomes factorizable.

The rest of this paper is organized as follows. Sections 2 and 3 present a review of the classical theory of a Bayesian game that is a variant of the game of Battle of Sexes. Section 4 describes quantum probabilities in generalized EPR experiments, their constraints, and how within the quantum game the players' payoff relations are defined in terms of these probabilities. Section 5 analyses the outcome of the Bayesian game of Battle of Sexes with EPR probabilities, and Sect. 6 discusses the results.

## 2 The Bayesian game of Battle of Sexes and its variant

The game of Battle of Sexes (BoS) describes [62] the following situation. Two people Alice and Bob wish to go out together, and two concerts are available: one with music by Bach, and one with music by Stravinsky. One person prefers Bach, and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy listening to the music of either composer. The situation is represented by the following matrix

> | Bob |
| :---: |
| Alice |
| $\mathcal{B}$ |
| $\mathcal{S}\left(\begin{array}{cc}(2,1) & (0,0) \\ (0,0) & (1,2)\end{array}\right)$, |

where Bach and Stravinsky are represented by symbols $\mathcal{B}$ and $\mathcal{S}$, respectively. For this game, a Nash equilibrium (NE) is a pair of strategies such that each player's strategy is the best reply to the strategic choice of the other players. In other words, unilateral deviation from a Nash equilibrium by a player in the form of a different choice of strategy will produce a payoff that is less than or equal to what a Nash equilibrium strategy will give to that player. Analysis shows [62] that this game has three mixedstrategy Nash equilibria $(0,0),\left(\frac{2}{3}, \frac{1}{3}\right),(1,1)$, where the numbers in parentheses are the Alice's and Bob's probabilities of choosing the strategy $\mathcal{B}$.

An interesting variant of this game [62] is the one in which Alice is unsure whether Bob prefers to join her or prefers to avoid her, whereas Bob knows Alice's preferences. Assume that Alice thinks that with probability $\frac{1}{2}$ Bob wants to go out with her, and with probability $\frac{1}{2}$ Bob wants to avoid her,

> | Alice playing against Bob’s two types |  |  |  |
| :---: | :---: | :---: | :---: |
| probability $\frac{1}{2}$ | probability $\frac{1}{2}$ |  |  |
| $\mathcal{B}$ | $\mathcal{S}$ |  |  |$) \mathcal{B} \quad \mathcal{S}, ~ \mathcal{B}\left(\begin{array}{cc}(2,0) & (0,2) \\ \mathcal{B}\left(\begin{array}{cc}(2,1) & (0,0) \\ (0,0) & (1,2)\end{array}\right) & (1,0)\end{array}\right)$,

Bob's first type Bob's second type
that is, from Alice's perspective, Bob has two possible types, the first is shown on the left and the second is on the right in (2).

Alice does not know Bob's type and is thus faced with the situation of choosing her rational action that is based on her belief about the action of Bob of each type. Given these beliefs, and her belief about the likelihood of each type, she can calculate her expected payoff in each case. For instance, given that Alice plays $\mathcal{B}$, and Bob of first type (who wishes to meet Alice) plays $\mathcal{B}$, whereas Bob of second type (who wishes to avoid Alice) plays $\mathcal{S}$, then Alice's expected payoff is $\frac{1}{2}(2)+\frac{1}{2}(0)=1$.

There are 4 possible pairs of actions of Bob's two types given as $(\mathcal{B}, \mathcal{B}),(\mathcal{B}, \mathcal{S})$, $(\mathcal{S}, \mathcal{B})$, and $(\mathcal{S}, \mathcal{S})$. Here, for instance, $(\mathcal{S}, \mathcal{B})$ describes that Bob's first type plays $\mathcal{S}$ and Bob's second type plays $\mathcal{B}$. Players' expected payoffs are then obtained as given below.

| Bob's two types |  |  |  |  |
| ---: | :---: | :---: | :---: | :---: |
|  | $(\mathcal{B}, \mathcal{B})$ | $(\mathcal{B}, \mathcal{S})$ | $(\mathcal{S}, \mathcal{B})$ | $(\mathcal{S}, \mathcal{S})$ |
| Alice $\mathcal{B}$ | $\left(2, \frac{1}{2}\right)$ | $\left(1, \frac{3}{2}\right)$ | $(1,0)$ | $(0,1)$. |
| $\mathcal{S}$ | $\left(0, \frac{1}{2}\right)$ | $\left(\frac{1}{2}, 0\right)$ | $\left(\frac{1}{2}, \frac{3}{2}\right)$ | $(1,1)$ |

One can then show [62] that the triplet $(\mathcal{B},(\mathcal{B}, \mathcal{S}))$, where the first entry in the bracket is Alice's action $\mathcal{B}$ and $(\mathcal{B}, \mathcal{S})$ is the pair of actions of the two types of Bob, constitutes a Nash equilibrium. This is a pure strategy Nash equilibrium consisting of three actions, one for Alice and one for each of the two types of Bob, with the property that (a) Alice's action is optimal given the actions of the two types of Bob, (b) the action of each type of Bob is optimal given the action of Alice.

## 3 Battle of Sexes with imperfect information

We now consider a situation in which neither player knows whether the other wants to meet or not [62]. As before, Alice assumes that Bob will prefer to join her with probability $\frac{1}{2}$ and will prefer to avoid her with probability $\frac{1}{2}$. Moreover, Bob expects with probability $\omega$, Alice will prefer to join him and with probability $(1-\omega)$ that she will prefer to avoid him. It is assumed that both Alice and Bob know their own preferences. The game can be represented as shown in Fig. 1 [62].

Consider the payoffs of Alice of type 1. She believes that with probability $\frac{1}{2}$, she faces Bob of type 1 and with probability $\frac{1}{2}$ she faces Bob of type 2. Assume that Bob of type 1 chooses $\mathcal{B}$ and Bob of type 2 chooses $\mathcal{S}$. Then, if Alice of type 1 chooses $\mathcal{B}$, her expected payoff is $\frac{1}{2}(2)+\frac{1}{2}(0)=1$, and if she chooses $\mathcal{S}$, her expected payoff is $\frac{1}{2}(0)+\frac{1}{2}(1)=\frac{1}{2}$.

We represent the action of the two types of Bob by the pairs $(\mathcal{B}, \mathcal{B}),(\mathcal{B}, \mathcal{S}),(\mathcal{S}, \mathcal{B})$, $(\mathcal{S}, \mathcal{S})$, where the first entry in the bracket is the action of Bob of type 1 and the second entry is the action of Bob of type 2. The expected payoff to Alice of type 1 when she chooses $\mathcal{B}$ or $\mathcal{S}$, against the 4 pairs of actions of the two types of Bob are

Bob's two types

|  | $(\mathcal{B}, \mathcal{B})$ | $(\mathcal{B}, \mathcal{S})$ | $(\mathcal{S}, \mathcal{B})$ | $(\mathcal{S}, \mathcal{S})$ |
| ---: | :---: | :---: | :---: | :---: |
| Alice of type $1 \mathcal{B}$ | 2 | 1 | 1 | 0 |
| $\mathcal{S}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |,



Fig. 1 Game of Battle of Sexes when each player is unsure of other player's preferences [62]
and similarly the payoff to Alice of type 2 is found as
Bob's two types

|  | $(\mathcal{B}, \mathcal{B})$ | $(\mathcal{B}, \mathcal{S})$ | $(\mathcal{S}, \mathcal{B})$ | $(\mathcal{S}, \mathcal{S})$ |
| ---: | :---: | :---: | :---: | :---: |
| Alice of type $2 \mathcal{B}$ | 0 | 1 | 1 | 2 |
| $\mathcal{S}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |.

Consider the case when Alice of type 1 plays $\mathcal{B}$, Alice of type 2 plays $\mathcal{S}$, Bob of type 1 plays $\mathcal{S}$, and Bob of type 2 plays $\mathcal{B}$. We represent this case by the quadruple $(\mathcal{B}, \mathcal{S}),(\mathcal{S}, \mathcal{B})$ where the entries in the first pair are chosen by Alice's two types, respectively, and the entries in the second pair are chosen by Bob's two types, respectively. For this quadruple, the payoffs to Alice's two types can be found from $(4,5)$ at the entries located at the intersection of the same column with entry $(\mathcal{S}, \mathcal{B})$ and the two row entries at $\mathcal{B}$ and $\mathcal{S}$ corresponding to Alice of type 1 and type 2, respectively:

$$
\begin{equation*}
\Pi_{\mathrm{A}_{1}}\{(\mathcal{B}, \mathcal{S}),(\mathcal{S}, \mathcal{B})\}=1, \Pi_{\mathrm{A}_{2}}\{(\mathcal{B}, \mathcal{S}),(\mathcal{S}, \mathcal{B})\}=\frac{1}{2} \tag{6}
\end{equation*}
$$

where the subscripts 1 and 2 for A or B give the type of that player.
The following table gives the expected payoffs to Alice's two types against the pairs of actions $(\mathcal{B}, \mathcal{B}),(\mathcal{B}, \mathcal{S}),(\mathcal{S}, \mathcal{B}),(\mathcal{S}, \mathcal{S})$ by Bob's two types, respectively,

| Bob's two types |  |  |  |  |
| ---: | :---: | :---: | :---: | :---: |
|  | $(\mathcal{B}, \mathcal{B})$ | $(\mathcal{B}, \mathcal{S})$ | $(\mathcal{S}, \mathcal{B})$ | $(\mathcal{S}, \mathcal{S})$ |
| Alice's two types $(\mathcal{B}, \mathcal{B})$ | $(2,0)$ | $(1,1)$ | $(1,1)$ | $(0,2)$ |
| $(\mathcal{S})$ | $(2,1)$ | $\left(1, \frac{1}{2}\right)$ | $\left(1, \frac{1}{2}\right)$ | $(0,0)$, |
| $(\mathcal{B})$ | $(0,0)$ | $\left(\frac{1}{2}, 1\right)$ | $\left(\frac{1}{2}, 1\right)$ | $(1,2)$ |
| $(\mathcal{S})$ | $(0,1)$ | $\left(\frac{1}{2}, \frac{1}{2}\right)$ | $\left(\frac{1}{2}, \frac{1}{2}\right)$ | $(1,0)$ |

where in the column on the left, for instance, $(\mathcal{B}, \mathcal{S})$ means that Alice of type 1 chooses $\mathcal{B}$ and Alice of type 2 chooses $\mathcal{S}$. The two payoff entries in brackets are the expected payoffs to Alice of type 1 and type 2, respectively. In (7), consider the entry $\left(\frac{1}{2}, 1\right)$ at the intersection of 3rd row $(\mathcal{S}, \mathcal{B})$ and 3 rd column $(\mathcal{S}, \mathcal{B})$. This means that Alice of the types 1 and 2 chooses $\mathcal{S}$ and $\mathcal{B}$, respectively, and Bob of the types 1 and 2 also chooses $\mathcal{S}$ and $\mathcal{B}$, respectively. The payoff to Alice of type 1 and type 2 are $\frac{1}{2}$ and 1 , respectively.

Similarly, the payoffs to Bob's two types can be found as

$$
\begin{align*}
& \Pi_{\mathrm{B}_{1}}\{(\mathcal{B}, \mathcal{S}),(\mathcal{S}, \mathcal{B})\}=\omega(0)+(1-\omega)(2)=2(1-\omega),  \tag{8}\\
& \Pi_{\mathrm{B}_{2}}\{(\mathcal{B}, \mathcal{S}),(\mathcal{S}, \mathcal{B})\}=\omega(0)+(1-\omega)(1)=1-\omega, \tag{9}
\end{align*}
$$

and one can write the expected payoffs to Bob's first type as

|  | Alice's two types |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $(\mathcal{B}, \mathcal{B})$ | $(\mathcal{B}, \mathcal{S})$ | $(\mathcal{S}, \mathcal{B})$ | $(\mathcal{S}, \mathcal{S})$ |  |
| Bob's type $1 \mathcal{B}$ | 1 | $\omega$ | $1-\omega$ | 0 |  |
| $\mathcal{S}$ | 0 | $2(1-\omega)$ | $2 \omega$ | 2 |  |,

and, similarly, the expected payoffs to Bob's second type are obtained as

| Alice's two types |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | $(\mathcal{B}, \mathcal{B})$ | $(\mathcal{B}, \mathcal{S})$ | $(\mathcal{S}, \mathcal{B})$ | $(\mathcal{S}, \mathcal{S})$ |
| Bob's type $2 \mathcal{B}$ | 0 | $1-\omega$ | $\omega$ | 1 |
| $\mathcal{S}$ | 2 | $2 \omega$ | $2(1-\omega)$ | 0 |.

As it is the case for payoffs to Alice's two types above, (10) and (11) can be joined together to obtain the payoffs to Bob's two types as

Bob's two types

|  | $(\mathcal{B}, \mathcal{B})$ | $(\mathcal{B}, \mathcal{S})$ | $(\mathcal{S}, \mathcal{B})$ | $(\mathcal{S}, \mathcal{S})$ |
| :---: | :---: | :---: | :---: | :---: |
| $(\mathcal{B}, \mathcal{B})$ | $(1,0)$ | $(1,2)$ | $(0,0)$ | $(0,2)$ |
| Alice's two types $(\mathcal{B}, \mathcal{S})$ | $(\omega, 1-\omega)$ | $(\omega, 2 \omega)$ | $(2(1-\omega),(1-\omega))$ | $(2(1-\omega), 2 \omega)$, |
| $(\mathcal{S}, \mathcal{B})((1-\omega), \omega)$ | $((1-\omega), 2(1-\omega))$ | $(2 \omega, \omega)$ | $(2 \omega, 2(1-\omega))$ |  |
| $(\mathcal{S}, \mathcal{S})$ | $(0,1)$ | $(0,0)$ | $(2,1)$ | $(2,0)$ |

where the entries in brackets are the expected payoffs to Bob of type 1 and type 2 , respectively. Now, (7) and (12) can be joined together to obtain

|  | Bob |  | $(\mathcal{S}, \mathcal{B})$ | $(\mathcal{B}, \mathcal{S})$ |
| ---: | :---: | :---: | :---: | :---: |
| $(\mathcal{B}, \mathcal{B})$ | $(2,0),(1,0)$ | $(1,1),(1,2)$ | $(1,1),(0,0)$ | $(0,2),(0,2)$ |
| Alice $(\mathcal{B}, \mathcal{S})$ | $(2,1),(\omega, 1-\omega)$ | $\left(1, \frac{1}{2}\right),(\omega, 2 \omega)$ | $\left(1, \frac{1}{2}\right),(2(1-\omega),(1-\omega))$ | $(0,0),(2(1-\omega), 2 \omega)$ |
| $(\mathcal{S}, \mathcal{B})$ | $(0,0),((1-\omega), \omega)$ | $\left(\frac{1}{2}, 1\right),((1-\omega), 2(1-\omega))$ | $\left(\frac{1}{2}, 1\right),(2 \omega, \omega)$ | $(1,2),(2 \omega, 2(1-\omega))$ |
| $(\mathcal{S}, \mathcal{S})$ | $(0,1),(0,1)$ | $\left(\frac{1}{2}, \frac{1}{2}\right),(0,0)$ | $\left(\frac{1}{2}, \frac{1}{2}\right),(2,1)$ | $(1,0),(2,0)$ |

where, for the two pairs of payoff entries, the first pair is for Alice's two types and the second payoff pair is for Bob's two types. It can be seen that when $\omega=\frac{2}{3}$, for instance, the strategy quadruples $\{(\mathcal{S}, \mathcal{B}),(\mathcal{S}, \mathcal{S})\}$ and $\{(\mathcal{B}, \mathcal{B}),(\mathcal{B}, \mathcal{S})\}$ corresponding to the payoff quadruples $(1,2),\left(\frac{4}{3}, \frac{2}{3}\right)$, and $(1,1),(1,2)$, respectively, are the pure strategy Bayesian Nash equilibria $[61,62]$.

### 3.1 Mixed-strategy version

Now, consider the mixed-strategy version of the game in which the players' probabilities of selecting $\mathcal{B}$ from the pure strategies $\mathcal{B}$ and $\mathcal{S}$ are given by the numbers $p, q, p^{\prime}$, and $q^{\prime} \in[0,1]$ for Alice of type 1 , Alice of type 2 , Bob of type 1 , and for Bob of type 2 , respectively. The mixed-strategy payoffs for Alice's and Bob's two types can then be found from Fig. 1 as

$$
\begin{aligned}
\Pi_{\mathrm{A}_{1}}\left(p ; p^{\prime}, q^{\prime}\right)= & \frac{1}{2}\binom{p}{1-p}^{T}\left(\begin{array}{ll}
2 & 0 \\
0 & 1
\end{array}\right)\binom{p^{\prime}}{1-p^{\prime}} \\
& +\frac{1}{2}\binom{p}{1-p}^{T}\left(\begin{array}{ll}
2 & 0 \\
0 & 1
\end{array}\right)\binom{q^{\prime}}{1-q^{\prime}} \\
\Pi_{\mathrm{A}_{2}}\left(q ; p^{\prime}, q^{\prime}\right)= & \frac{1}{2}\binom{q}{1-q}^{T}\left(\begin{array}{ll}
0 & 2 \\
1 & 0
\end{array}\right)\binom{p^{\prime}}{1-p^{\prime}} \\
& +\frac{1}{2}\binom{q}{1-q}^{T}\left(\begin{array}{ll}
0 & 2 \\
1 & 0
\end{array}\right)\binom{q^{\prime}}{1-q^{\prime}}
\end{aligned}
$$

$$
\begin{align*}
\Pi_{\mathrm{B}_{1}}\left(p^{\prime} ; p, q\right)= & \omega\binom{p}{1-p}^{T}\left(\begin{array}{cc}
1 & 0 \\
0 & 2
\end{array}\right)\binom{p^{\prime}}{1-p^{\prime}} \\
& +(1-\omega)\binom{q}{1-q}^{T}\left(\begin{array}{ll}
1 & 0 \\
0 & 2
\end{array}\right)\binom{p^{\prime}}{1-p^{\prime}}, \\
\Pi_{\mathrm{B}_{2}}\left(q^{\prime} ; p, q\right)= & \omega\binom{p}{1-p}^{T}\left(\begin{array}{ll}
0 & 2 \\
1 & 0
\end{array}\right)\binom{q^{\prime}}{1-q^{\prime}} \\
& +(1-\omega)\binom{q}{1-q}^{T}\left(\begin{array}{ll}
0 & 2 \\
1 & 0
\end{array}\right)\binom{q^{\prime}}{1-q^{\prime}}, \tag{14}
\end{align*}
$$

where $T$ is for transpose, and the subscripts 1 and 2 under A and B refer to the respective player's type. Also, a semicolon is used to separate Alice's and Bob's variables. The following Nash inequalities are then obtained

$$
\begin{align*}
\Pi_{\mathrm{A}_{1}}\left(p^{*} ; p^{\prime *}, q^{\prime *}\right)-\Pi_{\mathrm{A}_{1}}\left(p, p^{\prime *}, q^{* *}\right)= & \left.\frac{\partial \Pi_{\mathrm{A}_{1}}}{\partial p}\right|_{*}\left(p^{*}-p\right) \\
= & \left\{\frac{3}{2}\left(p^{*}+q^{* *}\right)-1\right\}\left(p^{*}-p\right) \geq 0, \\
\Pi_{\mathrm{A}_{2}}\left(q^{*} ; p^{\prime *}, q^{\prime *}\right)-\Pi_{\mathrm{A}_{2}}\left(q, p^{\prime *}, q^{\prime *}\right)= & \left.\frac{\partial \Pi_{\mathrm{A}_{2}}}{\partial q}\right|_{*}\left(q^{*}-q\right) \\
= & 2\left\{1-\left(p^{\prime *}+q^{\prime *}\right)\right\}\left(q^{*}-p\right) \geq 0, \\
\Pi_{\mathrm{B}_{1}}\left(p^{\prime *} ; p^{*}, q^{*}\right)-\Pi_{\mathrm{B}_{1}}\left(p^{\prime} ; p^{*}, q^{*}\right)= & \left.\frac{\partial \Pi_{\mathrm{B}_{1}}}{\partial p^{\prime}}\right|_{*}\left(p^{\prime *}-p^{\prime}\right) \\
= & \left\{3 \omega p^{*}+3(1-\omega) q^{*}-2\right\}\left(p^{* *}-p^{\prime}\right) \geq 0, \\
\Pi_{\mathrm{B}_{2}}\left(q^{\prime *} ; p^{*}, q^{*}\right)-\Pi_{\mathrm{B}_{2}}\left(q^{\prime} ; p^{*}, q^{*}\right)= & \left.\frac{\partial \Pi_{\mathrm{B}_{2}}}{\partial q^{\prime}}\right|_{*}\left(q^{\prime *}-q^{\prime}\right) \\
= & -\left\{3 \omega p^{*}+3(1-\omega) q^{*}\right. \\
& \times\left(q^{\prime *}-q^{\prime}\right) \geq 0, \tag{15}
\end{align*}
$$

where the quadruple $\left\{\left(p^{*}, q^{*}\right),\left(p^{\prime *}, q^{\prime *}\right)\right\}$ is the Nash equilibrium strategy set, which is indicated by the use of the asterisk label. From the inequalities (15), at $\omega=\frac{2}{3}$, for instance, the pure Bayesian Nash equilibria quadruples can be identified as $\{(0,1),(0,0)\}$ and $\{(1,1),(1,0)\}$, which correspond to the strategy quadruples $\{(\mathcal{S}, \mathcal{B}),(\mathcal{S}, \mathcal{S})\}$ and $\{(\mathcal{B}, \mathcal{B}),(\mathcal{B}, \mathcal{S})\}$ as can be observed above. Also, it is observed that $\left\{\left(\frac{1}{2}, 1\right),\left(\frac{2}{3}, 0\right)\right\}$, for instance, is a mixed-strategy Bayesian Nash equilibrium at which the players' payoffs are obtained from Eq. (14) as

$$
\begin{align*}
& \Pi_{\mathrm{A}_{1}}\left(\frac{1}{2} ; \frac{2}{3}, 0\right)=\frac{2}{3}, \quad \Pi_{\mathrm{A}_{2}}\left(1 ; \frac{2}{3}, 0\right)=\frac{4}{3}, \\
& \Pi_{\mathrm{B}_{1}}\left(\frac{2}{3} ; \frac{1}{2}, 1\right)=\frac{2}{3}, \quad \Pi_{\mathrm{B}_{2}}\left(0 ; \frac{1}{2}, 1\right)=2-\omega \tag{16}
\end{align*}
$$

## 4 Quantum probabilities in generalized Einstein-Podolsky-Rosen experiments

The above analysis of mixed-strategy Bayesian Nash equilibria assumes the underlying probabilities to be factorizable. We now would like to know whether the outcome of the Bayesian game from Fig. 1 is affected when the probabilities become non-factorizable. For this, we would consider the set of non-factorizable quantum probabilities obtained from generalized EPR experiments. The standard setting of such experiments [58,6371] involves a large number of runs. Two halves of an EPR pair originate from the same source travelling in opposite directions. One-half is received by observer 1, whereas observer 2 receives the other half. The two observers are space like separated and are unable to communicate.

As Fig. 2 shows, the two directions refer to two possible directions along which measurements can be taken, and in a run, the spin or polarization of the received half is measured. We call $D_{1}$ and $D_{2}$ observer 1's two directions and $D_{1}^{\prime}$ and $D_{2}^{\prime}$ observer 2's two directions. In a run, each observer selects one direction and thus a directional pair from $\left(D_{1}, D_{2}^{\prime}\right),\left(D_{1}, D_{1}^{\prime}\right),\left(D_{2}, D_{1}^{\prime}\right),\left(D_{2}, D_{2}^{\prime}\right)$ is selected by the observers in that run. The Stern-Gerlach type detectors are now rotated along these selected directions to perform the quantum measurement. Independent of which directional pair is chosen by the two observers, the outcome of the quantum measurement is either +1 or -1 along a measurement direction.

The relevant 16 probabilities are given [71] in the following,

$$
\begin{array}{cc} 
& \\
&  \tag{17}\\
& \\
\text { Observer 1 } & \\
& \\
& D_{1} \\
& +1 \\
& D_{2}^{\prime} \\
+1 & -1 \\
-1
\end{array}\left(\begin{array}{cc|cc}
\varepsilon_{1} & \varepsilon_{2} & \varepsilon_{5} & \varepsilon_{6}^{\prime} \\
\varepsilon_{3} & \varepsilon_{4} & \varepsilon_{7} & \varepsilon_{8} \\
\hline & & \\
\hline \varepsilon_{9} & \varepsilon_{10} & \varepsilon_{13} & \varepsilon_{14} \\
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{15} & \varepsilon_{16}
\end{array}\right),
$$

where, for instance, when the observer 1 selects the direction $D_{2}$ and observer 2 selects the direction $D_{1}^{\prime}$, and the Stern-Gerlach detectors are rotated along these directions, the probability that both experimental outcomes are -1 is $\varepsilon_{12}$, and the probability that the observer 1's experimental outcome is +1 and observer 2's experimental outcome is -1 is given by $\varepsilon_{10}$. "Appendix" described how the probabilities $\varepsilon_{j}$ are obtained from a pure state of two qubits.

We note that there are 16 probabilities in the above setting of generalized EPR experiments, and that the Fig. 1 giving a normal form representation of a Bayesian game of incomplete information also has the same number of entries in it. This naturally leads us to consider the situation in which the EPR probabilities are taken as the underlying probabilities of the strategy pairs in the Bayesian game in Fig. 1.

Now, the EPR probabilities can become non-factorizable. As we see in the following, this consideration motivates us to investigate how the outcome of the Bayesian


Fig. 2 Setting for generalized Einstein-Podolsky-Rosen experiments. We associate Alice's two directions to the two types of Alice, i.e., $D_{1} \sim$ Alice's type 1 and $D_{2} \sim$ Alice's type 2 . Similarly, $D_{1}^{\prime} \sim$ Bob's type 1 and $D_{2}^{\prime} \sim$ Bob's type 2
game is affected when the underlying probabilities of this game are obtained from EPR experiments and can thus become non-factorizable.

### 4.1 Constraints on EPR probabilities and defining the payoff relations

The elements of the probability set $\varepsilon_{j}(1 \leq j \leq 16)$ are known to satisfy certain other constraints that are described as follows. Note that when the directional pair ( $D_{1}, D_{2}^{\prime}$ ) is chosen for all runs of the experiment, the only possible outcomes are $(+1,+1)$, $(+1,-1),(-1,+1),(-1,-1)$. The same is true for other directional pairs $\left(D_{1}, D_{1}^{\prime}\right)$, $\left(D_{2}, D_{1}^{\prime}\right),\left(D_{2}, D_{2}^{\prime}\right)$. This leads to the normalization constraint

$$
\begin{equation*}
\sum_{j=1}^{4} \varepsilon_{j}=1=\sum_{j=5}^{8} \varepsilon_{j}, \quad \sum_{j=9}^{12} \varepsilon_{j}=1=\sum_{j=13}^{16} \varepsilon_{j} \tag{18}
\end{equation*}
$$

Also, in a particular run of the EPR experiment, the outcome of +1 or -1 (obtained along the direction $D_{1}$ or direction $D_{2}$ ) is independent of whether the direction $D_{1}^{\prime}$ or the direction $D_{2}^{\prime}$ is chosen in that run. Similarly, the outcome of +1 or -1 (obtained along $D_{1}^{\prime}$ or $D_{2}^{\prime}$ ) is independent of whether the direction $D_{1}$ or the direction $D_{2}$ is chosen in that run. These requirements, when translated in terms of the probability set $\varepsilon_{j}$, are expressed as

$$
\begin{align*}
& \varepsilon_{1}+\varepsilon_{2}=\varepsilon_{5}+\varepsilon_{6}, \quad \varepsilon_{1}+\varepsilon_{3}=\varepsilon_{9}+\varepsilon_{11}, \quad \varepsilon_{9}+\varepsilon_{10}=\varepsilon_{13}+\varepsilon_{14}, \varepsilon_{5}+\varepsilon_{7}=\varepsilon_{13}+\varepsilon_{15} \\
& \varepsilon_{3}+\varepsilon_{4}=\varepsilon_{7}+\varepsilon_{8}, \varepsilon_{11}+\varepsilon_{12}=\varepsilon_{15}+\varepsilon_{16}, \quad \varepsilon_{2}+\varepsilon_{4}=\varepsilon_{10}+\varepsilon_{12}, \quad \varepsilon_{6}+\varepsilon_{8}=\varepsilon_{14}+\varepsilon_{16} \tag{19}
\end{align*}
$$

A convenient solution of the system $(18,19)$ is reported by Cereceda [71] to be the one for which the set of probabilities $v=\left\{\varepsilon_{2}, \varepsilon_{3}, \varepsilon_{6}, \varepsilon_{7}, \varepsilon_{10}, \varepsilon_{11}, \varepsilon_{13}, \varepsilon_{16}\right\}$ is expressed in terms of the remaining set of probabilities $\mu=\left\{\varepsilon_{1}, \varepsilon_{4}, \varepsilon_{5}, \varepsilon_{8}, \varepsilon_{9}, \varepsilon_{12}\right.$, $\left.\varepsilon_{14}, \varepsilon_{15}\right\}$ that is given as

$$
\begin{align*}
\varepsilon_{2} & =\left(1-\varepsilon_{1}-\varepsilon_{4}+\varepsilon_{5}-\varepsilon_{8}-\varepsilon_{9}+\varepsilon_{12}+\varepsilon_{14}-\varepsilon_{15}\right) / 2, \\
\varepsilon_{3} & =\left(1-\varepsilon_{1}-\varepsilon_{4}-\varepsilon_{5}+\varepsilon_{8}+\varepsilon_{9}-\varepsilon_{12}-\varepsilon_{14}+\varepsilon_{15}\right) / 2, \\
\varepsilon_{6} & =\left(1+\varepsilon_{1}-\varepsilon_{4}-\varepsilon_{5}-\varepsilon_{8}-\varepsilon_{9}+\varepsilon_{12}+\varepsilon_{14}-\varepsilon_{15}\right) / 2, \\
\varepsilon_{7} & =\left(1-\varepsilon_{1}+\varepsilon_{4}-\varepsilon_{5}-\varepsilon_{8}+\varepsilon_{9}-\varepsilon_{12}-\varepsilon_{14}+\varepsilon_{15}\right) / 2, \\
\varepsilon_{10} & =\left(1-\varepsilon_{1}+\varepsilon_{4}+\varepsilon_{5}-\varepsilon_{8}-\varepsilon_{9}-\varepsilon_{12}+\varepsilon_{14}-\varepsilon_{15}\right) / 2,  \tag{20}\\
\varepsilon_{11} & =\left(1+\varepsilon_{1}-\varepsilon_{4}-\varepsilon_{5}+\varepsilon_{8}-\varepsilon_{9}-\varepsilon_{12}-\varepsilon_{14}+\varepsilon_{15}\right) / 2, \\
\varepsilon_{13} & =\left(1-\varepsilon_{1}+\varepsilon_{4}+\varepsilon_{5}-\varepsilon_{8}+\varepsilon_{9}-\varepsilon_{12}-\varepsilon_{14}-\varepsilon_{15}\right) / 2, \\
\varepsilon_{16} & =\left(1+\varepsilon_{1}-\varepsilon_{4}-\varepsilon_{5}+\varepsilon_{8}-\varepsilon_{9}+\varepsilon_{12}-\varepsilon_{14}-\varepsilon_{15}\right) / 2 .
\end{align*}
$$

This allows us to consider the elements of the set $\mu$ as independent variables.
In order to use the EPR setting to play the Bayesian game in Fig. 1, we call the observers 1 and 2 the players Alice and Bob, respectively, of the Bayesian game. We then associate one-half of the EPR pair to the player Alice and the other half to the player Bob. As Alice and Bob have two directions each, we associate Alice's two directions to the two types of Alice, that is, $D_{1} \sim$ Alice's type 1 and $D_{2} \sim$ Alice's type 2. Similarly, we associate Bob's two directions to the two types of Bob, that is, $D_{1}^{\prime} \sim$ Bob's type 1 and $D_{2}^{\prime} \sim$ Bob's type 2.

With these associations, and in view of the game in Fig. 1, in a run, each of the two directions $D_{1}$ and $D_{2}$ is chosen with probability $\frac{1}{2}$. Similarly, the directions $D_{1}^{\prime}$ and $D_{2}^{\prime}$ are chosen with probabilities $\omega$ and $(1-\omega)$, respectively. In view of the payoff relations (14) in the factorizable game, the players' payoff relations in the game with EPR probabilities can now be expressed as

$$
\begin{align*}
\Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)= & \frac{1}{2}\left\{(2) \varepsilon_{1}+(0) \varepsilon_{2}+(0) \varepsilon_{3}+(1) \varepsilon_{4}\right\} \\
& +\frac{1}{2}\left\{(2) \varepsilon_{5}+(0) \varepsilon_{6}+(0) \varepsilon_{7}+(1) \varepsilon_{8}\right\} \\
\Pi_{\mathrm{A}_{2}}\left(\varepsilon_{j}\right)= & \frac{1}{2}\left\{(0) \varepsilon_{9}+(2) \varepsilon_{10}+(1) \varepsilon_{11}+(0) \varepsilon_{12}\right\} \\
& +\frac{1}{2}\left\{(0) \varepsilon_{13}+(2) \varepsilon_{14}+(1) \varepsilon_{15}+(0) \varepsilon_{16}\right\}, \\
\Pi_{\mathrm{B}_{1}}\left(\varepsilon_{j}\right)= & \omega\left\{(1) \varepsilon_{1}+(0) \varepsilon_{2}+(0) \varepsilon_{3}+(2) \varepsilon_{4}\right\} \\
& +(1-\omega)\left\{(1) \varepsilon_{9}+(0) \varepsilon_{10}+(0) \varepsilon_{11}+(2) \varepsilon_{12}\right\} \\
\Pi_{\mathrm{B}_{2}}\left(\varepsilon_{j}\right)= & \omega\left\{(0) \varepsilon_{5}+(2) \varepsilon_{6}+(1) \varepsilon_{7}+(0) \varepsilon_{8}\right\} \\
& +(1-\omega)\left\{(0) \varepsilon_{13}+(2) \varepsilon_{14}+(1) \varepsilon_{15}+(0) \varepsilon_{16}\right\}, \tag{21}
\end{align*}
$$

where $1 \leq j \leq 16$ and $\Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right), \Pi_{\mathrm{A}_{2}}\left(\varepsilon_{j}\right), \Pi_{\mathrm{B}_{1}}\left(\varepsilon_{j}\right)$, and $\Pi_{\mathrm{B}_{2}}\left(\varepsilon_{j}\right)$ are the payoffs to Alice of type 1, Alice of type 2, Bob of type 1 , and Bob of type 2, respectively, expressed in terms of the probabilities $\varepsilon_{j}$ that are defined in (17).

Note that the payoffs (21) are reduced to the payoffs in the mixed-strategy game (14) when the probability set $\varepsilon_{j}(1 \leq j \leq 16)$ is factorizable in terms of the probabilities $p, q, p^{\prime}, q^{\prime} \in[0,1]$ as given by Eq. (22).

## 5 Nash equilibrium inequalities in Bayesian game of Battle of Sexes with EPR probabilities

So as to find the Nash equilibria when the probability set $\varepsilon_{j}(1 \leq j \leq 16)$ are nonfactorizable, we notice that when $\varepsilon_{j}$ are factorizable in terms of the probabilities $p$, $q, p^{\prime}$, and $q^{\prime}$, we can write

$$
\begin{align*}
\varepsilon_{1} & =p p^{\prime}, \varepsilon_{2}=p\left(1-p^{\prime}\right), \varepsilon_{3}=(1-p) p^{\prime}, \varepsilon_{4}=(1-p)\left(1-p^{\prime}\right), \\
\varepsilon_{5} & =p q^{\prime}, \varepsilon_{6}=p\left(1-q^{\prime}\right), \varepsilon_{7}=(1-p) q^{\prime}, \varepsilon_{8}=(1-p)\left(1-q^{\prime}\right), \\
\varepsilon_{9} & =q p^{\prime}, \varepsilon_{10}=q\left(1-p^{\prime}\right), \varepsilon_{11}=(1-q) p^{\prime}, \varepsilon_{12}=(1-q)\left(1-p^{\prime}\right),  \tag{22}\\
\varepsilon_{13} & =q q^{\prime}, \varepsilon_{14}=q\left(1-q^{\prime}\right), \varepsilon_{15}=(1-q) q^{\prime}, \varepsilon_{16}=(1-q)\left(1-q^{\prime}\right)
\end{align*}
$$

With this, the payoff relations (21) are reduced to the payoffs given by (14). Also, when $\varepsilon_{j}$ are factorizable, and can be expressed in terms of $p, q, p^{\prime}$, and $q^{\prime}$, they satisfy the constraints given by Eqs. $(18,19)$ and $p, q, p^{\prime}$, and $q^{\prime}$ can then be expressed in terms of the probabilities $\varepsilon_{j}$ as follows

$$
\begin{align*}
p & =\frac{1}{2}\left(\varepsilon_{1}+\varepsilon_{2}+\varepsilon_{5}+\varepsilon_{6}\right), q=\frac{1}{2}\left(\varepsilon_{9}+\varepsilon_{10}+\varepsilon_{13}+\varepsilon_{14}\right), \\
p^{\prime} & =\frac{1}{2}\left(\varepsilon_{1}+\varepsilon_{3}+\varepsilon_{9}+\varepsilon_{11}\right), q^{\prime}=\frac{1}{2}\left(\varepsilon_{5}+\varepsilon_{7}+\varepsilon_{13}+\varepsilon_{15}\right) . \tag{23}
\end{align*}
$$

From the set of inequalities (15), the expressions describing the Nash equilibria in the factorizable game are

$$
\begin{align*}
& \left.\frac{\partial \Pi_{\mathrm{A}_{1}}}{\partial p}\right|_{*}\left(p^{*}-p\right) \geq 0,\left.\frac{\partial \Pi_{\mathrm{A}_{2}}}{\partial q}\right|_{*}\left(q^{*}-q\right) \geq 0, \\
& \left.\frac{\partial \Pi_{\mathrm{B}_{1}}}{\partial p^{\prime}}\right|_{*}\left(p^{*}-p^{\prime}\right) \geq 0,\left.\frac{\partial \Pi_{\mathrm{B}_{2}}}{\partial q^{\prime}}\right|_{*}\left(q^{* *}-q^{\prime}\right) \geq 0 \tag{24}
\end{align*}
$$

To evaluate these inequalities, we use the relations (20) to express the payoff relations (21) in terms of the elements from the set $\mu$ as follows

$$
\begin{align*}
& \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)=\frac{1}{2}\left(2 \varepsilon_{1}+\varepsilon_{4}+2 \varepsilon_{5}+\varepsilon_{8}\right) \\
& \Pi_{\mathrm{A}_{2}}\left(\varepsilon_{j}\right)=\frac{1}{4}\left(3-\varepsilon_{1}+\varepsilon_{4}+\varepsilon_{5}-\varepsilon_{8}-3 \varepsilon_{9}-3 \varepsilon_{12}+5 \varepsilon_{14}+\varepsilon_{15}\right)  \tag{25}\\
& \Pi_{\mathrm{B}_{1}}\left(\varepsilon_{j}\right)=\omega\left(\varepsilon_{1}+2 \varepsilon_{4}-\varepsilon_{9}-2 \varepsilon_{12}\right)+\left(\varepsilon_{9}+2 \varepsilon_{12}\right) \\
& \Pi_{\mathrm{B}_{2}}\left(\varepsilon_{j}\right)=\frac{\omega}{2}\left(3+\varepsilon_{1}-\varepsilon_{4}-3 \varepsilon_{5}-3 \varepsilon_{8}-\varepsilon_{9}+\varepsilon_{12}-3 \varepsilon_{14}-3 \varepsilon_{15}\right)+\left(2 \varepsilon_{14}+\varepsilon_{15}\right)
\end{align*}
$$

Similarly, using the relations (20), we express $p, q, p^{\prime}, q^{\prime}$ in (23) in terms of the elements from the set $\mu$ as

$$
\begin{align*}
p & =\frac{1}{2}\left(1+\varepsilon_{1}-\varepsilon_{4}+\varepsilon_{5}-\varepsilon_{8}-\varepsilon_{9}+\varepsilon_{12}+\varepsilon_{14}-\varepsilon_{15}\right), \\
q & =\frac{1}{2}\left(1-\varepsilon_{1}+\varepsilon_{4}+\varepsilon_{5}-\varepsilon_{8}+\varepsilon_{9}-\varepsilon_{12}+\varepsilon_{14}-\varepsilon_{15}\right),  \tag{26}\\
p^{\prime} & =\frac{1}{2}\left(1+\varepsilon_{1}-\varepsilon_{4}-\varepsilon_{5}+\varepsilon_{8}+\varepsilon_{9}-\varepsilon_{12}-\varepsilon_{14}+\varepsilon_{15}\right), \\
q^{\prime} & =\frac{1}{2}\left(1-\varepsilon_{1}+\varepsilon_{4}+\varepsilon_{5}-\varepsilon_{8}+\varepsilon_{9}-\varepsilon_{12}-\varepsilon_{14}+\varepsilon_{15}\right) .
\end{align*}
$$

This allows us to use the chain rule to evaluate

$$
\begin{align*}
\frac{\partial \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)}{\partial p}= & \frac{\partial \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)}{\partial \varepsilon_{1}} \frac{\partial \varepsilon_{1}}{\partial p}+\frac{\partial \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)}{\partial \varepsilon_{4}} \frac{\partial \varepsilon_{4}}{\partial p}+\frac{\partial \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)}{\partial \varepsilon_{5}} \frac{\partial \varepsilon_{5}}{\partial p}+\frac{\partial \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)}{\partial \varepsilon_{8}} \frac{\partial \varepsilon_{8}}{\partial p} \\
& +\frac{\partial \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)}{\partial \varepsilon_{9}} \frac{\partial \varepsilon_{9}}{\partial p}+\frac{\partial \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)}{\partial \varepsilon_{12}} \frac{\partial \varepsilon_{12}}{\partial p}+\frac{\partial \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)}{\partial \varepsilon_{14}} \frac{\partial \varepsilon_{14}}{\partial p} \\
& +\frac{\partial \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)}{\partial \varepsilon_{15}} \frac{\partial \varepsilon_{15}}{\partial p} \tag{27}
\end{align*}
$$

that gives $\frac{\partial \Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}\right)}{\partial p}=2$. Similarly, we obtain

$$
\begin{equation*}
\frac{\partial \Pi_{\mathrm{A}_{2}}\left(\varepsilon_{j}\right)}{\partial q}=4, \frac{\partial \Pi_{\mathrm{B}_{1}}\left(\varepsilon_{j}\right)}{\partial p^{\prime}}=-2, \frac{\partial \Pi_{\mathrm{B}_{2}}\left(\varepsilon_{j}\right)}{\partial q^{\prime}}=-2(2 \omega+1) \tag{28}
\end{equation*}
$$

The inequalities (24) are now written as

$$
\begin{align*}
& 2\left(p^{*}-p\right) \geq 0,4\left(q^{*}-q\right) \geq 0, \\
& -2\left(p^{*}-p^{\prime}\right) \geq 0,-2(2 \omega+1)\left(q^{*}-q^{\prime}\right) \geq 0, \tag{29}
\end{align*}
$$

giving

$$
\begin{align*}
p^{*} & =\frac{1}{2}\left(\varepsilon_{1}^{*}+\varepsilon_{2}^{*}+\varepsilon_{5}^{*}+\varepsilon_{6}^{*}\right)=1, q^{*}=\frac{1}{2}\left(\varepsilon_{9}^{*}+\varepsilon_{10}^{*}+\varepsilon_{13}^{*}+\varepsilon_{14}^{*}\right)=1, \\
p^{*} & =\frac{1}{2}\left(\varepsilon_{1}^{*}+\varepsilon_{3}^{*}+\varepsilon_{9}^{*}+\varepsilon_{11}^{*}\right)=0, q^{*}=\frac{1}{2}\left(\varepsilon_{5}^{*}+\varepsilon_{7}^{*}+\varepsilon_{13}^{*}+\varepsilon_{15}^{*}\right)=0, \tag{30}
\end{align*}
$$

as a unique Bayesian Nash equilibrium of the game. Using Eqs. (23, 26), we then obtain $\varepsilon_{2}^{*}=\varepsilon_{6}^{*}=\varepsilon_{10}^{*}=\varepsilon_{14}^{*}=1$ with all the rest of $\varepsilon_{j}^{*}$ being zeros. The players' payoffs at this equilibrium are obtained as

$$
\begin{equation*}
\Pi_{\mathrm{A}_{1}}\left(\varepsilon_{j}^{*}\right)=0, \Pi_{\mathrm{A}_{2}}\left(\varepsilon_{j}^{*}\right)=2, \Pi_{\mathrm{B}_{1}}\left(\varepsilon_{j}^{*}\right)=0, \Pi_{\mathrm{B}_{2}}\left(\varepsilon_{j}^{*}\right)=2, \tag{31}
\end{equation*}
$$

presenting a dramatic contrast to what is the case when the underlying probabilities are factorizable.

## 6 Discussion

As is well known, Bell's inequality presents stark difference between the classical and quantum world. The CHSH form of Bell's inequality can be expressed as a constraint on probabilities without referring to the formalism and mathematical machinery of quantum mechanics. Game theory is based on the theory of probability, and this suggests that the quantum contents of quantum games can be given an expression that only employ probabilities without referring to the formalism of quantum mechanics. A probabilistic approach to quantum games is only a matter of perspective as the quantum Bayesian game discussed in this paper can be physically implemented using EPR-type experiments that result in non-factorizable probabilities.

Essentially, this paper discusses a physical implementation of a quantum Bayesian game using EPR experiments. We find that these experiments provide a natural setting for analyzing a quantum Bayesian game. Our analysis uses only probabilities as they facilitate wider access to the area of quantum games. The physical realization of our game is provided by actual EPR experiments where quantum mechanics makes the difference and which can also be expressed in terms of probabilities only.

We study the game of Battle of Sexes with incomplete information that has both pure and mixed Bayesian Nash equilibria. We investigate the situation when the underlying probabilities of this game can become non-factorizable. As is known, the probabilities in generalized EPR experiments can become non-factorizable and in certain cases can maximally violate the corresponding CHSH version of Bell's inequality. When the quantum mechanical probabilities are factorizable, the game attains a classical interpretation. However, when the probabilities are allowed to become non-factorizable, we find that the solution of the game turns out to be entirely different and, in contrast to the classical game that has both pure and mixed Bayesian Nash equilibria, the quantum game has a unique Bayesian Nash equilibrium.

A natural question to ask here is whether any set of probabilities that satisfies the constraints $(18,19)$ is physically realizable? Quantum mechanics is known to impose further constraints, and one such constraint is given by the CHSH version of Bell's inequality $[58,70,71]$. This constraint states that the quantity $\Delta$ defined by

$$
\begin{equation*}
\Delta=2\left(\varepsilon_{1}+\varepsilon_{4}+\varepsilon_{5}+\varepsilon_{8}+\varepsilon_{9}+\varepsilon_{12}+\varepsilon_{14}+\varepsilon_{15}-2\right) \tag{32}
\end{equation*}
$$

is restricted in the range $\left|\Delta_{Q M}\right| \leq 2 \sqrt{2}$ by the laws of quantum mechanics, and the CHSH version of Bell's inequality is violated in quantum mechanical experiments when $2<|\Delta|$. These constraints are imposed by physical realizations, and a maximum value of $\Delta=4$ emerges when only nonnegative probabilities are considered. It is known that this value is not physically realizable.

Our analysis for a Bayesian game of Battle of Sexes shows that when the underlying probabilities are obtained from generalized EPR experiments, and thus can be non-factorizable, a unique Bayesian Nash equilibrium of the game emerges. This equilibrium corresponds to $\varepsilon_{2}^{*}=\varepsilon_{6}^{*}=\varepsilon_{10}^{*}=\varepsilon_{14}^{*}=1$ with the remaining $\varepsilon_{j}^{*}$ being zeros. Substituting these values in Eq. (32) gives $\Delta=-2$, i.e., the unique Bayesian Nash equilibrium is obtained without violating the CHSH version of Bell's inequality.

A particularly interesting situation would be when the violation of CHSH version of Bell's inequality leads to a new outcome of the game. Quantum mechanics is known to put its own constraints on the allowed ranges of the variables $\left(\varepsilon_{1}+\varepsilon_{2}+\varepsilon_{5}+\varepsilon_{6}\right)$, $\left(\varepsilon_{9}+\varepsilon_{10}+\varepsilon_{13}+\varepsilon_{14}\right),\left(\varepsilon_{1}+\varepsilon_{3}+\varepsilon_{9}+\varepsilon_{11}\right)$, and $\left(\varepsilon_{5}+\varepsilon_{7}+\varepsilon_{13}+\varepsilon_{15}\right)$, as described in Eq. (23), and they are different from those imposed by just not permitting probabilities to have negative values. An important question would then be to ask whether this would change or affect the outcome of the game. A consideration of Eqs. $(29,30)$ shows that the conditions yielding the Bayesian Nash equilibrium are simply too strong to be affected by extra constraints that quantum mechanics can impose on the allowed ranges of the variables $\left(\varepsilon_{1}+\varepsilon_{2}+\varepsilon_{5}+\varepsilon_{6}\right),\left(\varepsilon_{9}+\varepsilon_{10}+\varepsilon_{13}+\varepsilon_{14}\right),\left(\varepsilon_{1}+\varepsilon_{3}+\varepsilon_{9}+\varepsilon_{11}\right)$, and $\left(\varepsilon_{5}+\varepsilon_{7}+\varepsilon_{13}+\varepsilon_{15}\right)$. However, it is possible that for other Bayesian games, the conditions giving the outcome are not so strong and it then would be worthwhile to investigate this question further.

Acknowledgments We acknowledge helpful discussions with Andrew Allison.

## 7 Appendix

Using the standard notation, the set of 16 probabilities in (17) is obtained from a pure state of two qubits

$$
\begin{equation*}
\left|\psi_{0}\right\rangle=\alpha|00\rangle+\beta|01\rangle+\gamma|10\rangle+\delta|11\rangle \tag{33}
\end{equation*}
$$

where $\alpha, \beta, \gamma, \delta \in \boldsymbol{C}$ and $|\alpha|^{2}+|\beta|^{2}+|\gamma|^{2}+|\delta|^{2}=1$. We assume that observer 1's directions $D_{1}$ and $D_{2}$ are along the unit vectors $\hat{a}$ and $\hat{c}$, respectively. Similarly, observer 2's directions $D_{1}^{\prime}$ and $D_{2}^{\prime}$ are along the unit vectors $\hat{b}$ and $\hat{d}$, respectively. Without the loss of generality, we also assume that the unit vectors $\hat{a}=\left[a_{x}, a_{y}\right]$, $\hat{b}=\left[b_{x}, b_{y}\right], \hat{c}=\left[c_{x}, c_{y}\right]$, and $\hat{d}=\left[d_{x}, d_{y}\right]$ are all located in the $\mathrm{x}-\mathrm{y}$ plane. Observer 1's measurement operators are then $\hat{\sigma} \cdot \hat{a}$ and $\hat{\sigma} \cdot \hat{c}$, respectively. Similarly, observer 2's measurement operators are $\hat{\sigma} \cdot \hat{b}$ and $\hat{\sigma} \cdot \hat{d}$, respectively. Here, $\hat{\sigma}=\left[\sigma_{x}, \sigma_{y}, \sigma_{z}\right]$ and $\sigma_{x}, \sigma_{y}, \sigma_{z}$ are Pauli spin matrices.

Consider the probability $\varepsilon_{1}$ in the (17) that corresponds to observers 1 and 2 measuring along the directions $\hat{a}$ and $\hat{b}$, respectively, and both obtaining the outcome +1 . For this situation, we require the eigenstates of the operators $(\hat{\sigma} \cdot \hat{a})$ and $(\hat{\sigma} \cdot \hat{b})$, with the eigenvalue of +1 for both. These are found to be $\frac{|0\rangle+\left(a_{x}+i a_{y}\right)|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle+\left(b_{x}+i b_{y}\right)|1\rangle}{\sqrt{2}}$, respectively. From these, the eigenstate of the measurement operator $(\hat{\sigma} \cdot \hat{a}) \otimes(\hat{\sigma} \cdot \hat{b})$, with the eigenvalue +1 , is obtained as

$$
\begin{equation*}
\left|\psi_{1}\right\rangle=\frac{1}{2}\left(|00\rangle+\left(b_{x}+i b_{y}\right)|01\rangle+\left(a_{x}+i a_{y}\right)|10\rangle+\left(a_{x}+i a_{y}\right)\left(b_{x}+i b_{y}\right)|11\rangle\right), \tag{34}
\end{equation*}
$$

and the probability $\varepsilon_{1}$ is then obtained from $\left|\left\langle\psi_{1} \mid \psi_{0}\right\rangle\right|^{2}$. For the pure state (33), this becomes

$$
\begin{equation*}
\varepsilon_{1}=\frac{1}{4}\left|\alpha+\beta\left(b_{x}-i b_{y}\right)+\gamma\left(a_{x}-i a_{y}\right)+\delta\left(a_{x}-i a_{y}\right)\left(b_{x}-i b_{y}\right)\right|^{2} . \tag{35}
\end{equation*}
$$

Similarly, for the probability $\varepsilon_{2}$, along with the eigenstate of $(\hat{\sigma} \cdot \hat{a})$ with eigenvalue +1 obtained above, we require the eigenstate of the operator $(\hat{\sigma} \cdot \hat{b})$ with the eigenvalues -1 , which is $\frac{|0\rangle-\left(b_{x}+i b_{y}\right)|1\rangle}{\sqrt{2}}$. From these, the eigenstate of the measurement operator $(\hat{\sigma} \cdot \hat{a}) \otimes(\hat{\sigma} \cdot \hat{b})$, with the eigenvalue -1 , is obtained as

$$
\begin{equation*}
\left|\psi_{2}\right\rangle=\frac{1}{2}\left(|00\rangle-\left(b_{x}+i b_{y}\right)|01\rangle+\left(a_{x}+i a_{y}\right)|10\rangle-\left(a_{x}+i a_{y}\right)\left(b_{x}+i b_{y}\right)|11\rangle\right), \tag{36}
\end{equation*}
$$

and, as before, the probability $\varepsilon_{2}$ is then obtained from $\left|\left\langle\psi_{2} \mid \psi_{0}\right\rangle\right|^{2}$. For the pure state (33), this becomes

$$
\begin{equation*}
\varepsilon_{2}=\frac{1}{4}\left|\alpha-\beta\left(b_{x}-i b_{y}\right)+\gamma\left(a_{x}-i a_{y}\right)-\delta\left(a_{x}-i a_{y}\right)\left(b_{x}-i b_{y}\right)\right|^{2}, \tag{37}
\end{equation*}
$$

and the remaining probabilities $\varepsilon_{3}, \varepsilon_{4} \cdots \varepsilon_{16}$ can be obtained similarly.

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