

**Editors-in-Chief:**

J.-M. Morel, Cachan

B. Teissier, Paris

**Advisory Board:**

Camillo De Lellis, Zurich

Mario di Bernardo, Bristol

Alessio Figalli, Austin

Davar Khoshnevisan, Salt Lake City

Ioannis Kontoyiannis, Athens

Gábor Lugosi, Barcelona

Mark Podolskij, Aarhus

Sylvia Serfaty, Paris and New York

Catharina Stroppel, Bonn

Anna Wienhard, Heidelberg

More information about this series at <http://www.springer.com/series/304>

Toshiyuki Kobayashi • Toshihisa Kubo  
Michael Pevzner

# Conformal Symmetry Breaking Operators for Differential forms on Spheres

 Springer

Toshiyuki Kobayashi  
Kavli IPMU and  
Graduate School of Mathematical Sciences  
The University of Tokyo  
3-8-1 Komaba, Meguro, Tokyo, Japan

Toshihisa Kubo  
Ryukoku University  
Kyoto, Japan

Michael Pevzner  
Mathematics Laboratory, FR 3399 CNRS  
University of Reims-Champagne-Ardenne  
Reims, France

ISSN 0075-8434                      ISSN 1617-9692 (electronic)  
Lecture Notes in Mathematics  
ISBN 978-981-10-2656-0              ISBN 978-981-10-2657-7 (eBook)  
DOI 10.1007/978-981-10-2657-7

Library of Congress Control Number: 2016955062

Mathematics Subject Classification (2010): 22E47, 22E46, 53A30, 53C10, 58J70

© Springer Nature Singapore Pte Ltd. 2016

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made.

Printed on acid-free paper

This Springer imprint is published by Springer Nature  
The registered company is Springer Nature Singapore Pte Ltd.  
The registered company address is: 152 Beach Road, #22-06/08 Gateway East, Singapore 189721, Singapore

# Contents

<b>Summary</b> .....	ix
<b>1 Introduction</b> .....	1
<b>2 Symmetry Breaking Operators and Principal Series</b>	
<b>Representations of <math>G = O(n+1, 1)</math></b> .....	13
2.1 Principal Series Representations of $G = O(n+1, 1)$ .....	13
2.2 Conformal View on Principal Series Representations of $O(n+1, 1)$ ..	17
2.3 Representation Theoretic Properties of $(\varpi_{u,\delta}^{(i)}, \mathcal{E}^i(S^n))$ .....	18
2.4 Differential Symmetry Breaking Operators for Principal Series ....	21
2.5 Symmetry Breaking Operators for Connected Group $SO_0(n, 1)$ ....	25
2.6 Branching Problems for Verma Modules .....	28
<b>3 F-method for Matrix-Valued Differential Operators</b> .....	31
3.1 Algebraic Fourier Transform .....	31
3.2 Differential Operators Between Two Manifolds .....	32
3.3 F-method for Principal Series Representations .....	32
3.4 Matrix-Valued Differential Operators in the F-method .....	37
<b>4 Matrix-Valued F-method for <math>O(n+1, 1)</math></b> .....	41
4.1 Strategy of Matrix-Valued F-method for $(G, G') = (O(n+1, 1), O(n, 1))$ .....	41
4.2 Harmonic Polynomials .....	42
4.3 Description of $\text{Hom}_L(V, W \otimes \text{Pol}(\mathfrak{n}_+))$ .....	43
4.4 Decomposition of the Equation $(d\pi_{(\sigma,\lambda)^*}(N_1^+) \otimes \text{id}_W)\psi = 0$ .....	45
4.5 Matrix Coefficients in the F-method .....	47
<b>5 Application of Finite-Dimensional Representation Theory</b> .....	51
5.1 Signatures in Index Sets .....	51
5.2 Action of $O(N)$ on the Exterior Algebra $\wedge^*(\mathbb{C}^N)$ .....	52
5.3 Construction of Intertwining Operators .....	53

5.4	Application of Finite-Dimensional Representation Theory	55
5.5	Classification of $\text{Hom}_{O(n-1)}(\wedge^i(\mathbb{C}^n), \wedge^j(\mathbb{C}^{n-1}) \otimes \mathcal{H}^k(\mathbb{C}^{n-1}))$	60
5.6	Descriptions of $\text{Hom}_{O(n-1)}(\wedge^i(\mathbb{C}^n), \wedge^j(\mathbb{C}^{n-1}) \otimes \text{Pol}[\xi_1, \dots, \xi_n])$	63
5.7	Proof of the Implication (i) $\Rightarrow$ (iii) in Theorem 2.8	63
<b>6</b>	<b>F-system for Symmetry Breaking Operators (<math>j = i - 1, i</math> case)</b>	67
6.1	Proof of Theorem 2.8 for $j = i - 1, i$	68
6.2	Reduction Theorem	69
6.3	Step 2: Matrix Coefficients $M_{IJ}$ for $d\widehat{\pi_{(i,\lambda)^*}}(N_1^+)\psi$	72
6.4	Step 3: Case-Reduction for $M_{IJ}^{\text{vect}}$	74
6.5	Step 4 - Part I: Formulæ for Saturated Differential Equations	79
6.6	Step 4 - Part II: Explicit Formulæ for $M_{IJ}$	80
6.7	Step 5: Deduction from $M_{IJ} = 0$ to $L_r(g_0, g_1, g_2) = 0$	84
<b>7</b>	<b>F-system for Symmetry Breaking Operators (<math>j = i - 2, i + 1</math> case)</b>	87
7.1	Proof of Theorem 7.1	88
<b>8</b>	<b>Basic Operators in Differential Geometry and Conformal Covariance</b>	93
8.1	Twisted Pull-Back of Differential Forms by Conformal Transformations	93
8.2	Hodge Star Operator Under Conformal Transformations	94
8.3	Normal Derivatives Under Conformal Transformations	98
8.4	Basic Operators on $\mathcal{E}^i(\mathbb{R}^n)$	101
8.5	Transformation Rules Involving the Hodge Star Operator and $\text{Rest}_{x_n=0}$	103
8.6	Symbol Maps for Differential Operators Acting on Forms	106
<b>9</b>	<b>Identities of Scalar-Valued Differential Operators <math>\mathcal{D}_\ell^\mu</math></b>	111
9.1	Homogeneous Polynomial Inflation $I_a$	111
9.2	Identities Among Juhl's Conformally Covariant Differential Operators	112
9.3	Proof of Proposition 1.4	114
9.4	Two Expressions of $\mathcal{D}_{u,a}^{i \rightarrow i-1}$	117
<b>10</b>	<b>Construction of Differential Symmetry Breaking Operators</b>	121
10.1	Proof of Theorem 2.9 in the Case $j = i - 1$	121
10.2	Proof of Theorem 2.9 in the Case $j = i + 1$	125
10.3	Application of the Duality Theorem for Symmetry Breaking Operators	125
10.4	Proof of Theorem 2.9 in the Case $j = i$	127
10.5	Proof of Theorem 2.9 in the Case $j = i - 2$	128
<b>11</b>	<b>Solutions to Problems A and B for <math>(S^n, S^{n-1})</math></b>	131
11.1	Problems A and B for Conformal Transformation Group $\text{Conf}(X; Y)$	131
11.2	Model Space $(X, Y) = (S^n, S^{n-1})$	132
11.3	Proof of Theorem 1.1	134

11.4	Proof of Theorems 1.5–1.8	136
11.5	Change of Coordinates in Symmetry Breaking Operators	136
<b>12</b>	<b>Intertwining Operators</b>	<b>141</b>
12.1	Classification of Differential Intertwining Operators Between Forms on $S^n$	142
12.2	Differential Symmetry Breaking Operators Between Principal Series Representations	143
12.3	Description of $\text{Hom}_L(V, W \otimes \text{Pol}(\mathfrak{n}_+))$	145
12.4	Solving the F-system when $j = i + 1$	146
12.5	Solving the F-system when $j = i$	148
12.6	Solving the F-system when $j = i - 1$	151
12.7	Proof of Theorem 12.1	151
12.8	Hodge Star Operator and Branson's Operator $\mathcal{S}_{2\ell}^{(i)}$	152
<b>13</b>	<b>Matrix-Valued Factorization Identities</b>	<b>155</b>
13.1	Matrix-Valued Factorization Identities	156
13.2	Proof of Theorem 13.1 (1)	159
13.3	Proof of Theorem 13.1 (2)	161
13.4	Proof of Theorem 13.2 (1)	162
13.5	Proof of Theorem 13.2 (2)	164
13.6	Proof of Theorem 13.3	164
13.7	Proof of Theorem 13.4	166
13.8	Renormalized Factorization Identities	167
<b>14</b>	<b>Appendix: Gegenbauer Polynomials</b>	<b>173</b>
14.1	Normalized Gegenbauer Polynomials	173
14.2	Derivatives of Gegenbauer Polynomials	175
14.3	Three-Term Relations Among Renormalized Gegenbauer Polynomials	176
14.4	Duality of Gegenbauer Polynomials for Special Values	178
14.5	Proof of Theorem 6.7	179
	<b>References</b>	<b>185</b>
	<b>List of Symbols</b>	<b>187</b>
	<b>Index</b>	<b>191</b>

# Summary

We make a systematic study of all possible conformally covariant differential operators that transform differential forms on a Riemannian manifold  $X$  into those on a submanifold  $Y$  with focus on the model space  $(X, Y) = (S^n, S^{n-1})$ .

We accomplish the complete classification of all such conformally covariant differential operators, and find explicit formulæ for these new matrix-valued operators in the flat coordinates in terms of basic operators in differential geometry and classical hypergeometric polynomials. Resulting families of operators are natural generalizations of the Rankin–Cohen brackets for modular forms and Juhl’s operators from conformal holography.

The matrix-valued factorization identities are also established for all possible combinations of these new conformally covariant differential operators with known operators for the  $X = Y$  case such as the Yamabe, Paneitz–Fradkin–Tseytlin, GJMS, and Branson operators.

The main machinery of the proof is the “F-method” which has been recently introduced in [Contemp. Math., 2013] and [Differential Geom. Appl., 2014] by the first author and developed in Kobayashi–Ørsted–Somberg–Souček [Adv. Math., 2015] and Kobayashi–Pevzner [Selecta Math., 2016] in various settings. It is a general method to construct intertwining operators between  $C^\infty$ -induced representations or to find singular vectors of generalized Verma modules in the context of branching laws, as solutions to differential equations given by the “algebraic Fourier transform of Verma modules”. We extend the F-method to the matrix-valued case in the general setting, which could be applied to other problems as well.

A short summary of the main results was announced in [C. R. Acad. Sci. Paris, 2016].