II. Preschemes

The most satisfactory type of object yet derived in which to carry out all the operations natural to algebraic geometry is the prescheme. I fully agree that it is painful to go back again to the foundations and redefine our basic objects after we have built them up so carefully in the last Chapter. The motivation for doing this comes from many directions. For one thing, we have so far neglected a very essential possibility inherent in our subject, which is, after all, a marriage of algebra and geometry: that is to examine and manipulate algebraically with the coefficients of the polynomials which define our varieties. It does not make much sense to say that a differentiable manifold is defined by integral equations; it makes good sense to say that an affine variety is the locus of zeroes of a set of integral polynomials. Another motivation for preschemes comes from the possibility of constructing via schemes an explicit and meaningful theory of infinitesimal objects. This is based on the idea of introducing nilpotent functions into the structure sheaf, whose values are everywhere zero, but which are still non-zero sections. Schemes with nilpotents are not only useful for many applications, but they come up inevitably when you examine the fibres of morphisms between quite nice varieties. Thirdly, it is only when you use schemes that the full analogy between arithmetic and geometric questions becomes explicit. For example, there is the connection given by the general theory of Dedekind domains which unites the theory of a) rings of integers in a number field and b) rings of algebraic functions of one complex variable. A much deeper connection is given by class field theory, between the tower of number fields and the tower of coverings of an algebraic curve defined over a finite field. The analogies suggested by this approach can be carried so far that they even give a definition of the higher homotopy groups of the integers, (i.e., of Spec(Z)): The vision of combined arithmetic-geometric objects goes back to Kronecker. It is interesting to read Felix Klein describing what to all intents is nothing but the theory of schemes:

"Ich beschränke mich darauf, noch einmal das allgemeinste Problem, welches hier vorliegt, im Anschluß an Kroneckers Festschrift von 1881 zu charakterisieren. Es handelt sich nicht nur um die reinen Zahlkörper oder Körper, die von einem Parameter Z abhängen, oder um die Analogisierung dieser Körper, sondern es handelt sich schließlich darum, für Gebilde, die gleichzeitig arithmetisch und funktiontheoretisch sind, also von gegebenen algebraischen Zahlen und gegebenen algebraischen Funktionen irgendwelcher Parameter algebraisch abhängen, das selbe zu leisten, was mehr oder weniger vollständig in den einfachsten Fällen gelungen ist.