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David Mumford

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P R E F A C E

These notes originated in several classes that I taught in the mid 60's to introduce graduate students to algebraic geometry. I had intended to write a book, entitled "Introduction to Algebraic Geometry", based on these courses and, as a first step, began writing class notes. The class notes first grew into the present three chapters. As there was a demand for them, the Harvard mathematics department typed them up and distributed them for a while (this being in the dark ages before Springer Lecture Notes came to fill this need). They were called *"Introduction to Algebraic Geometry: preliminary version of the first 3 chapters"* and were bound in red. The intent was to write a much more inclusive book, but as the years progressed, my ideas of what to include in this book changed. The book became two volumes, and eventually, with almost no overlap with these notes, the first volume appeared in 1976, entitled *"Algebraic Geometry I: complex projective varieties"*. The present plan is to publish shortly the second volume, entitled *"Algebraic Geometry II: schemes and cohomology"*, in collaboration with David Eisenbud and Joe Harris.

David Gieseker and several others have, however, convinced me to let Springer Lecture Notes reprint the original notes, long out of print, on the grounds that they serve a quite distinct purpose. Whereas the longer book *"Algebraic Geometry"* is a systematic and fairly comprehensive exposition of the basic results in the field, these old notes had been intended only to explain in a quick and informal way what varieties and schemes are, and give a few key examples illustrating their simplest properties. The hope was to make the basic objects of algebraic geometry as familiar to the reader as the basic objects of differential geometry and topology: to make a variety as familiar as a manifold or a simplicial complex. This volume is a reprint of the old notes without change, except that the title has been changed to clarify their aim.

The weakness of these notes is what had originally driven me to undertake the bigger project: there is no real *theorem* in them! I felt it was hard to convince people that algebraic geometry was a great and glorious field unless you offered them a theorem for their money, and that takes a much longer book. But for a puzzled non-algebraic geometer who wishes to find the facts needed to make sense of some algebro-geometric statement that they want to apply, these notes may be a convenient way to learn quickly the basic definitions. In twenty years of giving colloquium talks about algebraic geometry to audiences of mostly non-algebraic geometers, I have learned only too well that algebraic geometry is not so easily accessible, nor are its basic definitions universally known.

It may be of some interest to recall how hard it was for algebraic geometers, even knowing the phenomena of the field very well, to find a satisfactory language in which to communicate to each other. At the time these notes were written, the field was just emerging from a twenty-year period in which every researcher used his own definitions and terminology, in which the "foundations" of the subject had been described in at least

half a dozen different mathematical "languages". Classical style researchers wrote in the informal geometric style of the Italian school, Weil had introduced the concept of *specialization* and made this the cornerstone of his language and Zariski developed a hybrid of algebra and geometry with valuations, universal domains and generic points relative to various fields k playing important roles. But there was a general realization that not all the key phenomena could be clearly expressed and a frustration at sacrificing the suggestive geometric terminology of the previous generation.

Then Grothendieck came along and turned a confused world of researchers upside down, overwhelming them with the new terminology of schemes as well as with a huge production of new and very exciting results. These notes attempted to show something that was still very controversial at that time: that schemes really were the most natural language for algebraic geometry and that you did not need to sacrifice geometric intuition when you spoke "scheme". I think this thesis is now widely accepted within the community of algebraic geometry, and I hope that eventually schemes will take their place alongside concepts like Banach spaces and cohomology, i.e. as concepts which were once esoteric and abstruse, but became later an accepted part of the kit of the working mathematician. Grothendieck being sixty this year, it is a great pleasure to dedicate these notes to him and to send him the message that his ideas remain the framework on which subsequent generations will build.

Cambridge, Mass.
Feb. 21, 1988

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