Lecture Notes in Mathematics

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Proper Forcing

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INTRODUCTION

These notes can be viewed and used in several different ways, each has some justification, a collection of papers, a research monograph or a text book.

The author has lectured variants of several of the chapters several times: in University of California, Berkeley, 1978, Ch. III , IV, V in Ohio State University in Columbus, Ohio 1979, Ch. I,II and in the Hebrew University 1979/80 Ch. I, II, III, V, and parts of VI.

Moreover Azriel Levi, who has a much better name than the author in such matters, made notes from the lectures in the Hebrew University, rewrote them, and they are Chapters I, II and part of III , and were somewhat corrected and expanded by D. Drai, R. Grossberg and the author. Also most of XI §1-5 were lectured on and written up by Shai Ben David.

Also our presentation is quite self-contained. We adopted an approach I heard from Baumgartner and may have been used by others: not proving that forcing work, rather take axiomatically that it does and go ahead to applying it. As a result we assume only knowledge of naive set theory (except some isolated points later on in the book). The idea of this approach is that otherwise when the student learns what is axiomatic set theory and how you can show by forcing that CH may fail (and that CH holds by learning something on L) the course is finished. But he has only a vague idea of the rich possibilities in forcing, and no idea how to use them. I think the direct approach is more appealing.

Also some other parts were written and rewritten Ch. IV, was written from notes of Rens from the lecture in Berkeley suffered a heavy criticism of a referee, and thus was rewritten, then Ron Holtzman corrected it and expanded it. In Chapter IX, Uri Avraham found various errors, it appeared in [Sh 81d] and was revised later. Chapter X appeared in [Sh 81] and was revised later.

On the other hand most material was accumulated and not rewritten after the author's knowledge expanded (this is the true reason why some central theorems are not immediately proved in the general form, each Chapter is in fact a paper, though sometimes with references to previous ones. This may serve a didactical purpose. In fact the chapter on the strong covering lemma, XIII and on uniformization properties XIV does not depend on the previous chapters (though in XIII the reader has to take on faith theorems quoted there).

Also many chapters were written and circulated as hand written notes in various times, and were not changed much. The letter to Wimmers from July 78 is essentially Chapter IV, the letters from August and September 78 are Ch. VII (§5 and §1-§4 resp.) the letter from October 78 is Ch. VIII, November 78 is Ch. IX. Chapter X was written in the summer 79, and Ch. XI in November 79, and Ch. XII were "Notes on forcing, April, 80".

Chapter XIV has a long history: §1 grows from a remark on extending [DS] in [Sh 198?] §6 was written up by Rami Grossberg, rewritten by the author and then again by Grossberg, who also revised the other sections.

Still the book is really a research monograph.

There are few themes dominant to the book: it concentrates on independence results on small uncountable cardinals, by iterated forcing, and try to give some general tools in this area. The hope is that someone trying to prove such independence results could use them, but it is not intended to someone who wants to get independence results using some consistent axioms without learning forcing. This research start in [Sh 80] which introduce proper forcing and oracle c.c. and whose motivation strongly enough was to show some restrictions in model theoretic results in the author's book in model theory were necessary.

The last two chapters , XIII , XIV, can stand alone. But in addition to their separate interests, they are complementary to the previous ones. The strong covering lemma from Ch. XIII, show that it is quite hard to produce non-proper (more exactly, non E-proper) forcing notions, so (hard means need large cardinals) our preoccupation with them. The work on the uniformization properties serve as a complement to the work on forcing notions not adding reals showing the limitations on theorems saying "in a CS iteration $\bar{Q}$ of each Q_i satisfies X, then so does Lim $\bar{Q}$".

I have mentioned most people's help and I thank them all, in addition I thank Baumgartner, Harrington and Magidor for helpful discussions. Wimmers for pointing out the problem in a first try of VI §5 (= Ramsey ultrafilter), Harrington for hearing patiently the proofs while growing. Grossberg for reading carefully and correcting many Chapters, and last but not least, Danit Sharon for typing the manuscript, Grossberg and Drai for proof reading those notes and Gitik for doing the same for several chapters.

Last remark, I use the privilege of writing lecture notes and not a book, to be lazy with references, when not harming the readability: some results are attributed to people without the references and some others are called "well-known" (and I sincerely think they are). As Ch. I,II contains no new material we almost have no references and usually do not name the originator.

Also, while working on these notes the author has got more relevant results, but had already sworn in July 1981 not to add anything, and they will appear in [Sh 198? d] , so a reader who wants to work on extensions and solutions is advised to ask for it, and it should exist when these notes will appear.

Notation: Natural numbers are denoted by k, ℓ, m, n and sometimes ϑ.

Ordinals are denoted by $i, j, \alpha, \beta, \gamma, \delta, \xi, \zeta$, where δ is reserved for limit ordinals.

Cardinals (usually infinite) are denoted by $\lambda, \mu, \kappa, \chi$ let $\aleph_\alpha$ be the α-th infinite cardinal, $\omega_\alpha = \aleph_\alpha$ $\omega = \omega_0$. Let ${}^\beta\alpha = \{f : \beta \to \alpha, f :$ function $\}$, ${}^{\beta >}\alpha = \bigcup_{\gamma < \beta} {}^\gamma\alpha$ For sequences of ordinals (i.e., members of some ${}^\beta\alpha$), $\ell(\eta) = \mathrm{Dom}\,\eta$, $\eta(i)$ the ith ordinals so $\eta = \langle \eta(i) \colon i < \ell(\eta) \rangle$. We write also $\langle \eta(0), \ldots, \eta(n) \rangle$ or $\eta(0), \ldots, \eta(n)$ as seems fit. We denote sequences, usually of ordinals, by η, ν, ρ and also τ. Let c.l.u.b. or club mean closed unbounded.

Let $|A|$ denote the cardinality of the set A, $\mathcal{P}(A)$ denote the power set of A, and cf (α) the cofinality of α.

A real means here a subset of ω, or its characteristic functions.

Let φ, ψ denote first order formulas, $\varphi(x_0, \ldots, x_{n-1})$, means every free variable of φ appear in $\{x_0, \ldots, x_{n-1}\}$.

Let P (and also Q and R) denote poset, i.e., a partially ordered set or even a quasi ordered set (i.e., $p \le q \le p$ does not necessarily imply $p = q$). We call such P a forcing notion, and assume ϕ is a minimal member of P. We use G for a generic subset of P, (usually G_α for a generic subset of P_α), (for definition of generic see I §1). Let p, q, r denote members of forcing notions, we say p, q are incompatible in P if they have no common upper bound in P.

We do not distinguish strictly between a model M or a poset P and their universe. $\times_{i \in I} A_i$ is the cartesian product sometimes also denoted by $\prod_{i \in I} A_i$, distinguish it from $\prod_{i \in I} \mu(i)$ (multiplication of cardinals). We shall not distinguish

between multiplication of cardinals or ordinals. For an uncountable cardinal λ whose cofinality $> \aleph_0$ D_λ stand for the filter generated by the closed unbounded subsets of λ; When $S \subseteq \lambda$ is stationary (also denoted by $S \not\equiv 0 \bmod D_\lambda$) $D_\lambda + S$ is the filter generated by $D_\lambda \cup \{S\}$.

Let α, β be ordinals such that $\beta < \alpha$ then $S^\alpha_\beta = \{\gamma < \aleph_\alpha : \mathrm{cf}\ \gamma = \aleph_\beta\}$ but when μ, λ are cardinals such that $\mu < \lambda$ then $S^\lambda_\mu = \{\gamma < \lambda : \mathrm{cf}\ \lambda = \mu\}$.

We made a special effort to uniformize the notation we use but still there maybe some exceptions in the chapters, for example in chapter III $\mathcal{S}_{\aleph_0}(A)$ is used to denote the family of countable subsets of A (see Definition III 1.2) but the same family in chapter V is denoted by $\mathcal{S}_{<\aleph_1}(A)$ (see Definition V 3.3). So since some of the notions are redefined the reader is advised to check the nearest definition and not the first.

Models are denoted by the letters M, N perhaps with an index we shall not always distinguish between the model and its universe, but always $|M|$ will denote the universe of M and $\|M\|$ its cardinality.

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CONTENT BY SUBJECT

1. Results outside Set Theory

In Chapter XIV, section 2 there are results on the power of $Ext(G,\mathbb{Z})$ assuming various weak diamonds.

2. Results in naive Set theory

In XIII §5 we prove that $(\aleph_\delta)^{\mathrm{cf}\,\delta} < \aleph_{(|\delta|^{\mathrm{cf}\,\delta})^+}$ for every limit ordinal δ (including cf $\delta = \aleph_0$). In XIII §6 more information and problems are discussed, and XIII. §0 tell the history of the singular cardinal problem.

In XIV §1 we investigate weak variants of the diamond (continuing Devlin and Shelah [DS].)

In XIV §3 we prove that CH implies some kind of weak diamond to S_1^2.

In IX §3 we discuss various specializations of Aronszajn trees.

3. Other non forcing results

In XIII §1, §2, §4 we deal with the strong covering lemma, and its applications to the value of $2^{\aleph_0}$ in $V[r]$.

4. Basics of Forcing

In Ch. I we explain how to use forcing and discuss some basic forcing. In Ch. II we explain iteration with $< \kappa$-support, (in particular finite support) deal for example with Martin Axiom, and more examples.

5. Specific Independence results on trees

In III 5.4 we present a proof that every $\aleph_1$-Aronszajn tree can be specialized by a c.c.c. forcing notions (and generally in III §5 deal with κ-trees).

In III §6 we present a proof of the consistency of "$ZFC + 2^{\aleph_0} = \aleph_2 +$ there is no $\aleph_2$-Aronszajn trees."

In V §6, §7, we prove CON (ZFC + G.C.H. + every Aronszajn tree is special.)

In VII §3 F we prove consistency results on Aronszajn trees strengthening the previous ones, motivated by general topology.

In VII §3 G we present a proof of the consistency of a strengthening of CON (ZFC + G.C.H. + there is no $\aleph_1$-Kurepa tree.)

6. Theorem on $\aleph_1$- complete forcing

In VIII §1 we prove that we can iterate $\aleph_2$-complete forcing and $\aleph_1$-complete forcing satisfying a strong $\aleph_2$-chain condition, with collapsing $\aleph_1$ and $\aleph_2$.

In VIII 27A we remark on another strong $\aleph_2$-chain condition preserved by CS iterations.

7. Chain conditions

The c.c.c and κ-c.c.c. are introduced in II and the preservation of the c.c.c. by FS is proved there.

In III 4.1 we prove that a CS iteration of proper forcing notions of power $< \kappa$, κ regular $(\forall \mu < \kappa)$ $\mu^{\aleph_0} < \kappa$, of length κ satisfies the κ-c.c. (by proving that if the length is $< \kappa$, it has a dense subset $< \kappa$).

Of course IV is dedicated to oracle c.c.

Lemma V 1.5 proves the $\aleph_2$-c.c. of a CS iteation of E-complete forcing notions each of power $\aleph_1$. (assuming CH).

In VII §1 deals with a strong κ-chain condition (e.c.c.) similar to the one from [Sh 78a], such that if we have a CS iteration of length κ with the conditions we use for not adding reals (in V §7, VIII §4) then the forcing satisfies

the κ-c.c. So this helps to get consistency results with ZFC + CH.

In VII §2 we deal with κ*-pic* (= κ-properness isomorphism conditions). If P satisfies it, then P satisfies by κ-c.c., and add $<\kappa$ reals, and an iteration of length κ still satisfies the κ-c.c. It helps to get consistency results with $ZFC + 2^{\aleph_0} = \aleph_2 + 2^{\aleph_1} = \lambda$ (so we start with $V \models$ "$CH + 2^{\aleph_1} = \kappa$" and this holds for the intermediate stages). application is starting with $V \models$ "$CH + 2^{\aleph_1} = \kappa$" and use a CS iteration $\bar{Q} = \langle P_i, Q_i : i < \omega_2 \rangle$ Q_i specialize all Aronszajn trees without adding reals.

On the κ-c.c. for RCS iteration of length κ see X §5 (5.3, 5.4) and XI 6.3 (2).

8. Preservation of properness and variants

As the book was written in the generic way, there are several such proofs. In III §3 the preservation of properness by CS iteration is proved. In IX §2 the preservation of properness under $\aleph_1$-free limit is proved. In X §2 the preservation of properness and semi-properness under RCS is proved. (remember that semi-proper forcing may change the cofinality of some regular $\lambda > \aleph_1$ to ω). In XII §2 the preservation of properness under CS iteration is reproved, using the definition of properness by games.

We deal with preservation of α-properness and $(\omega,1)$-properness in V §2, §3, X §7.

In VIII §3 we prove the consistency of $ZFC + CH + SH + 2^{\aleph_1} > \aleph_2$.

In IX §4 we prove of that SH $\not\Rightarrow$ "every Aronszajn tree is special".

9. Consistency results on the uniformization properties are variant

In II §4 we prove the consistency of "some family of $\aleph_1$ subsets of ω have the uniformization properties"

In V §1, we prove the consistency with GCH "of for some stationary $S \subset \omega_1$, $\langle A_\delta : \delta < \omega_1 \rangle$ have the uniformization property if each $A_\delta$is a set of order type ω, $\mathrm{Sup}\,(A_\delta) = \delta$."

In VIII §4 we prove the consistency with G.C.H. of $\neg\ \Phi^{S}_{\aleph_2}$.

10. Other consistency results

In III §7 we deal with "for a family of $\aleph_1$ countable subsets A_i of ω_1 $(i < \omega_1)$, order type $(A_i) \subset \mathrm{Sup}\, A_i$, there is a club $C \subseteq \omega_1$ $\bigwedge_i$ Sup $(C \cap A_i) \subset \mathrm{Sup}\, A_i$.

In VI §4 we prove the consistency of "there is no P-point".

In VI §5 we prove the consistency of "there is no Ramsey ultrafilter".

In X, XI we prove various independence results on ω_2, read XI §1, X 8.4, and XI §7's theorem.

In VII §3, §4, many applications are listed.

11. Other preservation of properness

In V §3 we prove the preservation of ω-properness + the ω^ω-bounding property.

In VI §1 and X §7 we deal with preservation of covering model, hence of various properties which can be formulated this way (Sacks property, Laver property, PP-property, D generates a Ramsey ultrafilter, D generates a P-point (ultrafilter)).

In IX §4 we deal with the preservation of "(T^*,S)-preserving" which means T^* look like Souslin trees at levels $\delta \notin S$, and in the end we comment on possible generalizations.

Not adding reals is dealt with in V §1, §2, and mainly V §7, VIII §4 and X §7.

In V §1 and X §3 we deal with preservation of generalizations of $\aleph_1$-completeness.

In XI §5, §6 we deal with the preservation of the S-condition (always satisfied by Nm' which change the cofinality of $\aleph_2$ to $\aleph_0$ but guarantees reals

are not added) and in XII §3 with a generalization including proper forcings.

12. Examples

In VII §5 we build of iteration $\langle P_n, \underset{\sim}{Q}_n : n < \omega \rangle$ such that each $\underset{\sim}{Q}_n$ does not collapse any stationary subset of ω_1; but any limit we take collapse $\aleph_1$.

In III §4 we build (in ZFC) a forcing notion of power $\aleph_1$, not collapsing $\aleph_1$ but also not preserving the stationarity of some $S \subseteq \omega_1$.

Examples of iteration $\langle P_n, \underset{\sim}{Q}_n : n < \omega \rangle$ each $\underset{\sim}{Q}_n$ is α-proper for each α not adding reals, but any limit we take collapse $\aleph_1$ is presented in V 5.1 using XIV §1 (really the previous result of Devlin and Shelah [DS]).

In XII §1 we show that many forcing notions satisfy the conditions from XI but are not semi-proper. In fact if there is an $\{\aleph_1\}$-semi proper forcing notion changing the cofinality of $\aleph_1$ to $\aleph_0$ then Chang conjecture holds.

ANNOTATED CONTENT

I FORCING, BASIC FACTS

§1. Introducing forcing 1

[We define generic sets, names for forcing a notion, and formulate Cohen's theorems]

§2. The consistency of CH 9

[Our aim is to construct by forcing a model of ZFC were CH holds, first we explain the problem of not collapsing cardinals, and second prove that $\aleph_1$-complete forcing notion does not add reals]

§3. On the consistency 14

[We construct a model of ZFC in which the Continuum Hypothesis fails; define the c.c.c., prove that forcing with c.c.c. forcing preserves cardinalities and cofinalities, and prove also the Δ-system lemma for finite sets]

§4. More on the cardinality $2^{\aleph_0}$ and Cohen reals 21

[We construct for every cardinal λ in V which satisfies $\lambda^{\aleph_0} = \lambda$ a model $V[G]$ such that $V[G] \models 2^{\aleph_0} = \lambda$. Also define Cohen reals]

§5. Equivalence of forcing notions, and canonical names 26

[Define when two forcing notions are equivalent. Introduce canonical names and prove that for every P-name τ there is a canonical P-name σ

such that $\Vdash_P$ "$\tau = \sigma$ "]

§6. **Random reals collapsing cardinals and diamonds** 31

[Introduce random reals; Levy collapse. Prove that for regular λ $\mathcal{A}(\aleph_0, < \lambda)$ satisfies the λ-c.c. For every uncountable regular λ and a stationary $S \subseteq \lambda$ define a forcing notion P which does not collapse S such that $V^P \models \diamondsuit_S$]

II ITERATION OF FORCING

§1. **The composition of two forcing notions** 39

[Define composition of two notions and state the associativity lemma]

§2. **Iterated forcing** 44

[Define it, and prove that the c.c.c is preserved by FS iteration]

§3. **Martin Axiom and few applications** 48

[Prove that ZFC $+ 2^{\aleph_0} > \aleph_1 + MA$ is consistent. Use MA to prove many simple uniformization properties]

§4. **The uniformization properties** 57

[Here we deal with more general uniformization properties, we weaken the demand of almost disjointness to a kind of tree]

§5. **Maximal almost disjoint families of subsets of ω** 68

[A maximal almost disjoint (mad) subset of $\mathcal{P}(\omega)$ is a family of infinite

subsets of ω such that the intersection of any two members is finite and maximal with this property. We prove using MA every mad set has cardinality $2^{\aleph_0}$. The other direction: For every $\aleph_1 \leq \lambda < 2^{\aleph_0}$ there exists a generic extension of V by c.c.c. forcing such that in it there exist mad set of power λ]

III PROPER FORCING

§1. **Introducing properness** 73

[We define "P is a proper forcing notion " prove some definitions are equivalent (and deal with the closed unbounded filter $\mathcal{D}_{\aleph_0}(\lambda)$)]

§2. **More on properness** 82

[We define "p is (N,P)-generic " and deal more with equivalent definitions of properness]

§3. **Preservation of properness under CS iteration** 90

[We prove the theorem mentioned in the title]

§4. **Martin Axiom revisited** 95

[We discuss the popularity of the c.c.c., whether we can replace it by a more natural and weaker condition. We give a sufficient condition for a CS iteration of length κ to satisfy the κ-c.c. We prove the consistency (assuming existence of an inaccessible cardinal) of "ZFC + $2^{\aleph_0} = \aleph_1$ + MA for forcing notions not destroying stationary subsets of ω_1". We show that the last demand cannot be replaced by "not collapsing

cardinalities or cofinalities"]

§5. **Aronszajn trees** 100

[We define κ-Aronszajn, κ-Souslin, present existence theorems (for λ^+ when $\lambda = \lambda^{<\lambda}$) and prove that under MA every Aronszajn tree is special]

§6. **Maybe there is no $\aleph_2$–Aronszajn tree** 104

[We prove the consistency of ZFC $+ 2^{\aleph_0} = \aleph_2$ + there is no $\aleph_2$-Aronszajn tree , the method collapsing successively all $\lambda, \aleph_1 < \lambda < \kappa$ (κ a weakly compact cardinal) treating every potential initial segment of an $\aleph_2$-Aronszajn tree so that it cannot actually be so.]

§7. **Closed unbounded sets of ω_1 can run away from many sets** 109

[We prove the consistency of ZFC $+ 2^{\aleph_0} = \aleph_2$ with if for $i < \omega_1$, $A_i \subset \omega_1$ has order type C Sup A_i, then for some closed unbounded $C \subseteq \omega_1$, $(\forall i)$[Sup $(C \cap A_i) <$ Sup A_i]

IV ON ORACLE-C.C. AND "$\mathcal{P}(\omega)$ /FINITE HAS NO NON-TRIVIAL AUTOMORPHISM"

§0. **Introduction** 114

[The oracle-c.c. method enables us to start with $V \models \diamondsuit_{\aleph_1}$ extend the set of reals ω_2-times (by iterated forcing), in the intermediate stages $\diamondsuit_{\aleph_1}$ holds, and we omit types of power $\aleph_1$ along the way.]

§1. **On oracle chain condition** 117

[One way to build forcing notions satisfying the $\aleph_1$-c.c., is by successive countable approximations including promises to maintain the predensity of countably many subset, many times using the diamond. we formalize a corresponding property ($\bar{M}$-c.c., $\bar{M}$ an oracle) and prove the equivalence of some variants of the definition]

§2. **The omitting type theorem** 122

[We prove that if the intersection of $\aleph_1$ Borel sets is empty and even if we add a Cohen real it remains empty , (and $\diamondsuit_{\aleph_1}$) *then* for some oracle $\bar{M}$, for every P satisfying the $\bar{M}$-c.c., in V^P the intersection of the Borel sets (reinterpreted) is still empty]

§3. **Iterations of $\bar{M}$-c.c.** 124

[We show that for Finite Support iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \leq \omega_2 \rangle$, if $\bar{M}_i \in V^{P_i}$ is an $\aleph_1$-oracle "large enough for $\langle \langle \bar{M}_j P_j, \bar{Q}_j \rangle : j < i \rangle$," and Q_i satisfies the $\bar{M}_i$-c.c. *then* $P_\alpha = \operatorname{Lim} \bar{Q}$ satisfies the $\bar{M}_0$-c.c. The first three sections give the exact formulation of the aim stated in the introduction and prove that it works]

§4. **Reduction of the Main Theorem to the Main Lemma** 129

[We show how to apply the method described in §1 - §3 in order to get a model in which the Boolean algebra $\mathcal{P}(\omega)$/finite has no non-trivial automorphism, i.e., one induces by permutations of ω]

§5. **Proof of the Main Lemma 4.6** 134

[The point missing in §4 is: if F is an automorphism of $\mathcal{P}(\omega)$/finite, $\bar{M}$ an

$\aleph_1$-oracle, then there is forcing notion P satisfying the $\widetilde{M}$-c.c., and a P name $\underset{\sim}{Y}$ of a real such that in V^P, for no $Y \subseteq x$, $(\forall A, B \in \mathcal{P}(\omega)^V)[x \cap A = B = Y \cap F(A) = F(B)]$ (even a Cohen forcing does not introduce such a Y). We try to build such $P, \underset{\sim}{Y}$ and prove that if we always fail F is trivial]

V α- PROPERNESS AND NOT ADDING REALS

§1. **Iterations of forcing notions which does not add real** 153

[We define what it means to be E-complete e.g., if $P \subseteq ({}^{\omega_1 >}2, <)$, $E \subseteq \omega_1$ stationary and $f_n \subseteq f_{n+1} \in P$, Sup (Dom f_n) $\in E$. We show that properness + E-completeness are preserved by CS iteration and get corresponding Axiom. Also introduce a form of MA which is consistent with CH and prove using it a uniformization property which imply existence of a non free Whitehead group]

§2. **(E,α,ℓ)- properness** 162

[We introduce various variants of properness]

§3. **α- properness and (E,α)- properness revisited** 164

[We repeat the previous section in more details]

§4. **Preservation of "ω- properness + the ω^ω- bounding property** 169

[P satisfies the ω^ω-bounding property if $[\forall f \in ({}^\omega\omega)^{V^P}][\exists g \in ({}^\omega)^V]$ $(\bigwedge_n f(n) < g(n))$. **We prove in great detail the theorem stated in the**

title]

§5. **What forcing can we iterate without adding reals** 177

[We explain why "not adding reals" is not preserved by any kind of iteration, and suggest a remedy - $\triangleleft$-completeness]

§6. **Specializing Aronszajn trees without adding reals** 181

[We prove that every Aronszajn tree can be specialized by a nice forcing α-proper for every $\alpha < \omega_1$ and $\triangleleft$-complete for some $\aleph_1$-completeness system $\triangleleft$ together with the next section this gives a proof $Con(ZFC + \exists\kappa[\kappa$ inaccessble$]) \Longrightarrow Con(ZFC + G.C.H. + SH)$ and with Chapter VIII a new proof of Jensen's $Con(ZFC + G.C.H. + SH)$.]

§7. **Iteration of $(E,\triangleleft)$- complete forcing notions** 189

[We prove the limit of a CS iteration of Q_i each is α-proper for every $\alpha < \omega_1$, and $\triangleleft$-complete for some simple $\aleph_1$-completeness]

VI P-POINTS AND PRESERVATION THEOREM

§1. **A general preservation theorem** 195

[We present a way to prove preservation of "$(\omega,1)$-properness $+ x$" for properties x restricting our set of reals. Our hope is that this framework is easy to be applied to many properties]

§2. **Three known properties** 203

[We prove that the ω^ω-bounding property, the Sacks property and the

Laver property comes under the framework of §1; the "Sacks" and "Laver" property appear first and most characteristically in the forcing notions bearing the respective names]

§3. **PP (P-point) property** 209

[We introduce a new property (*PP*) which comes under the framework of §1, and some variants of it]

§4. **There may be no P-point** 213

[We present another proof of this theorem, using the preservation of the *PP*-property. This may serve as a preliminary test, whether our general machinery simplifies and clarifies proofs]

§5. **There may be a unique Ramsey ultrafilter** 221

[The main result is the consistency of $ZFC + 2^{\aleph_0} = \aleph_2 +$ "there is a unique Ramsey ultrafilter on ω up to permutations of ω". For this we have to prove that "D generates a Ramsey ultrafilter " is preserved - by another application of §1, and of course to work on each iterand]

VII THE SEPTEMBER NOTES ON PROPER FORCING

§1 **On the $\aleph_2$- c.c.** 233

[When we iterate $\aleph_2$ times forcings not adding real, (but not necessarily $\aleph_1$-complete) we suggest a condition called $\aleph_2$-e.c.c. so that if each Q_i satisfies the $\aleph_2$-e.c.c., then P_{ω_2} satisfies the $\aleph_2$-c.c.]

§2. **The axioms** 236

[We suggest some axioms whose consistency follows from the theorem on preservation under iteration of various properties.]

§3. **Applications of Ax II** 241

[We prove several applications of an axiom consistent with G.C.H.]

§4. **Applications of Ax I** 253

[We prove some applications and mention others of an axiom consistent with $2^{\aleph_0} = \aleph_2$.]

§5. **An Example.** 255

[An example is given of a countable support iteration of length ω of forcing not collapsing stationary subsets of ω_1, but the limit collapse $\aleph_1$]

VIII THE OCTOBER NOTES ON PROPER FORCING

§1. **Mixed Iteration** 258

[We prove that we can iterate $\aleph_2$-complete forcings and $\aleph_1$-complete forcings satisfying a strong $\aleph_2$-chain condition, without collapsing $\aleph_1$ and $\aleph_2$]

§2. **Chains conditions revisited** 262

[We suggest another condition, κ - pic, to ensure the limit of the iteration P_κ satisfies the κ-c.c. The aim is e.g., we start with $V \models$ "$2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} > \aleph_2$", and use CS iteration $\bar{Q}$ of length ω_2 , each

time dealing with "all problems" (there are $2^{\aleph_1}$) at once]

§3. **The Axioms Revisited** 266

[We discuss what axioms we can get according to the four possibilities of the truth of $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ but assuming always $2^{\aleph_0} \leq \aleph_2$]

§4. **More on forcings not adding** ω-sequences and on the diagonal arguments 269

[We prove e.g., that CH does not imply $\Psi^{3}_{\aleph_1}$ by dealing with 2-complete systems]

IX SOUSLIN HYPOTHESIS DOES NOT IMPLY "EVERY ARONSZAJN TREE IS SPECIAL"

§1. **Free limit** 278

[We look at Boolean algebras generated by a set of sentences in infinitary propositional calculus (mainly $L_{\omega_1,\omega}$). This enables us to define free limit]

§2. **Preservation by free limit** 281

[We prove that an iteration in which we use $L_{\omega_1,\omega}$-free limit at limit stages, preserve properness]

§3. **Aronszajn trees: various ways to specialize** 285

[We introduce some new ways to specialize Aronszajn trees, and present the old ones, as well as the connection between those properties]

§4 Independence results 291

[Here are the main results. We use an iterated forcing $S\text{-}st$-specializing any Aronszajn tree. The problem is to make sure that some fixed tree T^* will remain not special. They introduce such a property of forcing "(T^*,S)-preserving and show that is is preserved in iteration. There is a discussion of the problem and our strategy in the beginning of the section and discussion of open problems and how can the preservation theorem be generalized]

X SEMI-PROPER FORCINGS

§0. Introduction 304

§1. Iterated forcing with RCS (revised countable support) 304

[The standard countable support iteration cannot be spoiled when cofinalities are changed to ω, we introduce the revised version suitable for this case.]

§2. Proper forcing revisited 313

[We define semi-properness, and prove that it is strongly preserved by RCS iteration.]

§3. Pseudo-completeness 320

[We prove that a weakening of $\aleph_1$-completeness is strongly preserved by RCS iteration.]

§4. **Specific forcings** 326

[We deal with Prikry forcing, Namba forcing and generalizations which are semi-proper when we use Galvin filter.]

§5. **Chain conditions and Avraham's problem** 335

[We prove that under reasonable conditions the κ-c.c. holds and get its first application: a universe V in which for every $A \subseteq \omega_1$ there is a countable subset of ω_2^V which does not belong to $L(A)$.]

§6. **Reflection properties of S_0^2. Refining Avraham's problem and precipitous ideals** 338

[For some large cardinal κ, by iteration we find a forcing notion P, such that $V^P \models$ "$\kappa = \aleph_2$ and $A = \{\delta < \kappa : \mathrm{cf}\,\delta = \aleph_0\,, \delta$ regular in $V\}$ is stationary ". So we may make A large in some sense, as mentioned in the title.]

§7. **Strong preservation and properness** 346

[We present some properties strongly preserved by RCS iteration; the most important is a strengthening of not adding reals. This continues VI §1.]

§8. **Friedman's problem** 347

[We collapse some large κ, by iterated forcing, which sometimes collapses $(2^{\aleph_2})^+$ to $\aleph_1$, sometimes change the cofinality of $\aleph_2$ to $\aleph_0$, and sometimes add a closed unbounded $C \subseteq S$ of order type ω_1, where $S \subseteq S_0^2$ is stationary. We get a model V in which every stationary $S \subseteq S_0^2 = \{\delta < \aleph_2 : \mathrm{cf}\,\delta = \aleph_0\}$ contains a closed copy of ω_1. By stronger hypothesis we get it for every stationary $S \subseteq S_0^\alpha$, $\mathrm{cf}\,\aleph_\alpha > \aleph_0$.]

XI CHANGING COFINALITIES: EQUI-CONSISTENCY RESULTS

§1. **The theorems** 354

[Here we describe what kind of a condition on forcing notions we want. Then we proceed to get consistency results. The proof uses RCS-iteration of length κ, κ a strongly inaccessible cardinal. In each step, we allow Namba forcing. The consistency results are mostly from X but here we use the minimal large cardinals required.]

§2. **The condition** 359

[We describe here the conditions, and some helping definitions and conventions]

§3. **The preservation properties guaranteed by the S-condition** 362

[We prove that such a condition implies $\aleph_1$ is not collapsed, and (assuming CH) no real is added; and for it partitions theorems on trees]

§4. **Forcing notions satisfying the S-condition** 366

[We show that Namba forcing, Nm' satisfies the $\{\aleph_2\}$-condition that Nm and Nm' are really different forcing notions that Nm, Nm' may satisfy the $\aleph_4$-c.c. (while $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1}$ is large) We also prove $\aleph_1$-complete forcing and a forcing notion shooting a closed unbounded subset of order type ω_1 through a stationary $S \subseteq S_0^2$ satisfies our condition]

§5. **Finite composition** 372

[We prove that under suitable hypothesis, a composition of forcing satisfying an S-condition satisfies it. For this we prove a combinatorial

theorem on trees]

§6. **Preservation of the I- condition by iteration** 375

[Here we prove that if we iterate forcing notions satisfying our conditions, but enough times collapse the present $2^{|P|}$ to $\aleph_1$, the composite forcing satisfies the condition. So usually we have large segments of cardinals which we have to collapse by $\aleph_1$-complete forcings, but for strongly inaccessible we can use Nm' straight away (by 6.5)]

§7. **Further independence results** 388

[We prove the equiconsistency of ZFC + "κ is Mahlo + $\aleph_2$ has the Friedman property", and a further result using weakly compact cardinal. We also prove the equiconsistency of ZFC + "κ is 2-Mahlo " and ZFC+ there is the club of $\aleph_2$ consisting of regular cardinals of L]

XII IMPROPER FORCING

§0. **Introduction** 394

§1. **When Namba forcing is semi-proper, Chang's Conjecture and games** 395

[We prove e.g., that if some $\{\aleph_1\}$-semi proper forcing changes the cofinality of $\aleph_2$ to ω then Namba forcing is semi-proper, and Chang's Conjecture holds hence $0^\#\in V$.]

§2. **Games and properness** 400

[Equivalent definitions of variants of properness by games are given, and

it is exemplified how the proofs of the preservation theorems in this context look like]

§3. **Amalgamating properness with the S-condition** 406

[We show how we can extend the results of the previous Chapter to more forcing notions]

XIII THE STRONG COVERING LEMMA ND THE G.C.H.

§0. **Introduction** 410

[Explanation of the history of the singular cardinals problem, and the significance of the strong covering lemma and its relation to proper forcing]

§1. **The strong covering lemma: Definitions and implications** 416

[Here we introduce the notions connected with the strong covering lemma and notice some trivial connections. Note that if (W,V) satisfies the hypothesis of the strong covering lemma, then so does $(W^{\dagger},V)$ whenever $W \subseteq W^{\dagger} \subseteq V$.]

§2. **Proof of the strong covering lemma** 420

[Here we prove the statement on games phrased above (and obviously implying the strong covering lemma) by induction on α. We prove e.g., that if $0^{\#} \notin L$, then (L,V) satisfies the $\aleph_3$-covering lemma. This part is somewhat harder than the rest of the paper.]

§3. **A counterexample** 435

[We prove that we may extend L to V (by forcing) collapsing $\aleph_2$ only, cf $^V(\aleph_2^L) = \aleph_1$, so that the $\aleph_1$-covering lemma holds but the strong $\aleph_1$-covering lemma fail.]

§4. **When adding a real cannot destroy CH** 437

[We deal with variants of "can adding a real violate CH while preserving cardinals", and prove that each implies an inner model with suitable large cardinals. For getting sharper result we have to improve the results of §2 getting e.g., that if $A \subseteq \omega_2$, $A \in V$, $\aleph_2^{L[A]} = \aleph_2^V$ and $0^\# \notin V$ then $(L[A], V)$ satisfies the strong λ-covering lemma for every λ.]

§5. **Bound on $2^{\aleph_\alpha}$ for $\aleph_\alpha$ singular** 444

[We give bound to $(\aleph_\delta)^{\mathrm{cf}\,\delta}$ in the way we deal with scales and cofinalities of ultraproducts. This section can be read alone].

§6. **Concluding remarks and questions** 453

[This section continues §0. We make some remarks giving some claims we can prove but their value is not clear, and discussing the open questions, and explain how to get simpler proofs for weaker theorems.]

XIV ON WEAK DIAMONDS AND THE POWER OF EXT

§0. **Introduction** 461

§1. **Unif-strong negation of the weak diamonds fR** 463

[Introduce a generalization of the negation of the weak diamond (i.e., $\Phi^2_{\aleph_1}$) and prove it from an appropriate replacement of $2^{\aleph_0} < 2^{\aleph_1}$].

§2. **On the power of Ext and Whitehead problem** 474

§3 **Weak diamond for $\aleph_2$ assuming CH** 485

[We prove that every ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S^2_1 \rangle$ when η_δ is *continuous* cannot be uniformized assuming $2^{\aleph_0} = \aleph_1$].

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