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Topological Vector Spaces I

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Preface to the First Edition

It is the author's aim to give a systematic account of the most important ideas, methods and results of the theory of topological vector spaces. After a rapid development during the last 15 years, this theory has now achieved a form which makes such an account seem both possible and desirable.

This present first volume begins with the fundamental ideas of general topology. These are of crucial importance for the theory that follows, and so it seems necessary to give a concise account, giving complete proofs. This also has the advantage that the only preliminary knowledge required for reading this book is of classical analysis and set theory. In the second chapter, infinite dimensional linear algebra is considered in comparative detail. As a result, the concept of dual pair and linear topologies on vector spaces over arbitrary fields are introduced in a natural way. It appears to the author to be of interest to follow the theory of these linearly topologised spaces quite far, since this theory can be developed in a way which closely resembles the theory of locally convex spaces. It should however be stressed that this part of chapter two is not needed for the comprehension of the later chapters.

Chapter three is concerned with real and complex topological vector spaces. The classical results of Banach's theory are given here, as are fundamental results about convex sets in infinite dimensional spaces. The subsequent chapters contain a full account of the properties of locally convex spaces. This account is concerned above all with the general theory, but some important classes of spaces, such as for example (F)-spaces, barrelled spaces and bornological spaces, are considered in greater detail. A large number of examples and counterexamples are intended to enable both the scope and the limits of the theory to be seen.

The second volume will contain the theory of linear mappings and the special spaces and classes of spaces which are important in analysis. The theory of Hilbert space will not be dealt with, since there are plenty of excellent textbooks on this topic.

Information about the contents of the book is given in the detailed table of contents at the beginning of the book, and in the short summaries at the beginning of each chapter. No claim for completeness is made for the bibliography at the end of the book, but it should nevertheless be detailed enough to enable further independent work to be done.

My teacher O. TOEPLITZ provided the first impulse for work on the theme of this book. In § 30, I have endeavoured to give an account of the theory of perfect spaces, which was developed by us together, I have to thank repeated personal contact with my French colleagues J. DIEUDONNÉ, A. GROTHENDIECK and L. SCHWARTZ since the war, for detailed knowledge of the theory developed by them; this forms the main subject-matter of this book. The present account is frequently based on the two volumes of BOURBAKI (BOURBAKI [6] in the bibliography) and on the lectures of GROTHENDIECK [11].

I am particularly indebted to W. NEUMER and H. G. TILLMANN who have respectively read through the first half, and the whole of the manuscript, carefully and critically. M. LANDSBERG, H. SCHAEFER and J. WLOKA have made important suggestions and observations.

Finally I thank the publishers for their speedy and excellent printing.

Heidelberg, August 1960 G. KÖTHE

Preface to the Second Edition

The second edition contains a number of corrections, the need for which was kindly pointed out to me by various readers, together with reference to recent articles in which some of the open problems mentioned in the first edition are solved. Apart from this, the text remains unaltered.

Frankfurt, October 1965 G. KÖTHE

Preface to the English Edition

This English edition is a translation of the second German edition. It differs from the German edition only in several corrections, mainly due to Dr. D. J. H. Garling.

I wish to express my sincere gratitude to Dr. Garling for the excellent and careful translation. I am also indebted to Dr. D. Findley for preparing the index.

Frankfurt, July 1969 G. KÖTHE

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