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Discretization Methods and Iterative Solvers Based on Domain Decomposition

With 82 Figures and 25 Tables

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Preface

DOMAINE: [dɔmɛn]
Ce domaine est encore fermé aux savants

DÉCOMPOSER: [dekɔ̃poze]
Décomposer un problème pour mieux le résoudre

*Micro Robert: Dictionnaire du français primordial

The numerical approximation of partial differential equations, very often, is a challenging task. Many such problems of practical interest can only be solved by means of modern supercomputers. However, the efficiency of the simulation depends strongly on the use of special numerical algorithms. Domain decomposition methods provide powerful tools for the numerical approximation of partial differential equations arising in the modeling of many interesting applications in science and engineering. Although the first domain decomposition techniques were used successfully more than hundred years ago, these methods are relatively new for the numerical approximation of partial differential equations. The possibilities of high performance computations and the interest in large-scale problems have led to an increased research activity in the field of domain decomposition.

However, the meaning of the term "domain decomposition" depends strongly on the context. It can refer to optimal discretization techniques for the underlying problems, or to efficient iterative solvers for the arising large systems of equations, or to parallelization techniques. In many modern simulation codes, different aspects of domain decomposition techniques come into play, and the overall efficiency depends on a smooth interaction between these different components. The coupling of different discretization schemes, the coupling of different physical models, and many efficient preconditioners for the algebraic systems can be analyzed within an abstract framework. At first glance these aspects seem to be rather independent. However, all have one central idea in common: The decomposition of the underlying global problem into suitable subproblems of smaller complexity. In general, a complete decoupling of the global problem into many independent subproblems, which are easy to solve, is not possible. Since, the subproblems are very often coupled, there has to be communication between the different subproblems. Although the term optimal depends on the context, the proper handling of the information transfer across the interfaces between the subproblems is of major importance for the design of optimal methods. In the case of discretization techniques, a priori estimates for the discretization errors have to be considered. They very much depend on the appropriate couplings across the interfaces which are often realized by matching conditions. The jump across the interfaces which measures the nonconformity of the method has to be bounded in a suitable way. In the case of iterative solvers, the convergence rate and the computational effort for one iteration step measure the quality of a method. To obtain scalable iteration schemes, very often, one has to include a suitable global problem of small complexity.

In this work, both discretization techniques and iterative solvers are addressed. A brief overview of different approaches is given and new techniques and ideas are proposed. An abstract framework for domain decomposition methods is presented and an analysis is carried out for new techniques of special interest. Optimal estimates for the methods considered are established and numerical results confirm the theoretical predictions.

Chapter 1 concerns special discretization methods based on domain decomposition techniques. In particular, the decomposition of geometrical complex structures into subdomains of simple shape is of special interest. Another example is the decomposition into substructures on which different physical models are relevant. Then, for each of these subproblems, an optimal approximation scheme involving the choice of the triangulation as well as the discretization can be chosen. However to obtain optimal discretizations for the global problem, the discrete subproblems have been glued together appropriately. Here, we focus on mortar finite element methods.

To start, we review the standard mortar setting for the coupling of Lagrangian conforming finite elements in Sect. 1.1. Both standard mortar formulations – the nonconforming positive definite problem and the saddle point problem based on the unconstrained product space – are given.

In Sect. 1.2, we introduce and analyze alternative Lagrange multiplier spaces. We derive abstract conditions on the Lagrange multiplier spaces such that the nonconforming discretization schemes obtained yield optimal a priori results. Lagrange multiplier spaces based on a dual basis are of special interest. In such a case, a biorthogonality relation between the nodal basis functions of these spaces and the finite element trace spaces holds. A main advantage of these new Lagrange multiplier spaces is that the locality of the support of the nodal basis functions of the constrained space can be preserved.

With this observation in mind, we introduce a new equivalent mortar formulation defined on the unconstrained product space in Sect. 1.3. We show that the non-symmetric formulation can be analyzed as a Dirichlet–Neumann coupling. Based on the elimination of the Lagrange multiplier, we derive a symmetric positive definite formulation on the unconstrained product space, and the equivalence to the positive definite problem on the constrained space is shown. Two formulations, a variational as well as an algebraic one, are presented and discussed. A standard nodal basis for the unconstrained product space can be used in the implementation. The stiffness matrix associated with our new variational form can be obtained from the standard one on the unconstrained space by local operations.

Section 1.4 concerns two examples of non-standard mortar situations. Each of them reflects an interesting feature of the abstract general framework, and illustrates the flexibility of the method. We start with the coupling of two different discretization schemes. The matching at the interface is based on the dual role of Dirichlet and Neumann boundary conditions. Two different equivalent formulations are given for the coupling of mixed and standard conforming finite elements. In our second example, we rewrite the nonconforming Crouzeix–Raviart finite elements as mortar finite elements. We consider the extreme case that the decomposition of the domain is given by the fine triangulation and that therefore the number of subdomains tends to infinity as the discretization parameter of the triangulation tends to zero.

Finally in Sect. 1.5, we present several series of numerical results. In particular, we study the influence of the choice of the Lagrange multiplier space on the discretization errors. Examples with several crosspoints, a corner singularity, discontinuous coefficients, a rotating geometry, and a linear elasticity problem are considered. A second test series concerns the influence of the choice of the non-mortar side. Adaptive and uniform refinement techniques are applied. In our last test series, we consider the influence of jumps in the coefficient on an adaptive refinement process at the interface.

Chapter 2 concerns iterative solution techniques based on domain decomposition. A brief overview of general Schwarz methods, including multigrid techniques, is given in Sect. 2.1. Examples for the standard H^1-case illustrate overlapping, non-overlapping, and hierarchical decomposition techniques. The following sections contain new results on non-standard situations; we discuss vector field discretizations as well as mortar methods.

Section 2.2 focuses on an iterative substructuring and a hierarchical basis method for Raviart–Thomas finite elements in 3D. We start with the definition of the local spaces and the relevant bilinear forms and subspaces. The central result of this section is established in Subsect. 2.2.2; it is a polylogarithmical bound independent of the jumps of the coefficients across the subdomain boundaries of our iterative substructuring method. The technical tools are discussed in detail with particular emphasis on the role of trace theorems, harmonic extensions, and dual norms applied to finite element spaces. As in the 2D case for standard Lagrangian finite elements, we introduce three different types of subspaces called V_H, V_F, and V_T. We cannot avoid the use of a global space to obtain quasi-optimal bounds. But in contrast to the standard Lagrangian finite elements in 3D, the low dimensional Raviart–Thomas space associated with the macro-triangulation formed by the subregions can be used to obtain quasi-optimal results where the constant does not depend on the jumps of the coefficients across the subdomain boundaries.

Sections 2.3–2.5 concern different iterative solvers for mortar finite element formulations. In Sect. 2.3, we combine the idea of dual basis functions for the Lagrange multiplier space with standard multigrid techniques for symmetric positive definite systems. The new mortar formulation, analyzed in Sect. 1.3, is the point of departure for the introduction of our iterative solver. We define and analyze our multigrid method in terms of level dependent bilinear forms, modified transfer operators, and a special class of smoothers which includes a standard Gauß–Seidel smoother. Convergence rates independent of the number of refinement steps are established for the $\mathcal{W}$-cycle provided that the number of smoothing steps is large enough. The numerical results confirm the theory. Moreover asymptotically constant convergence rates are obtained for the $\mathcal{V}$-cycle with one pre- and one postsmoothing step.

Section 2.4 concerns a Dirichlet–Neumann type algorithm for the mortar method. It turns out to be a block Gauß–Seidel solver for the unsymmetric mortar formulation on the product space. Numerical results illustrate the influence of the choice of the damping parameter. The transfer of the boundary values at the interface is realized in terms of a scaled mass matrix. This matrix is sparse if and only if dual Lagrange multiplier spaces are used.

In Sect. 2.5, we study a multigrid method for the saddle point formulation. Two different types of smoothers are discussed; a block diagonal and one reflecting the saddle point structure. In the second case, the exact solution of the modified Schur complement system is replaced by an iteration, resulting in an inner and an outer iteration. This multigrid method is given for the standard mortar formulation as presented in Sect. 1.1. In contrast to the two previous sections, the use of dual Lagrange multiplier spaces does, in general, not reduce the computational costs for one iteration step.

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Contents