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Wavelets in Numerical Simulation

Problem Adapted Construction and Applications

With 61 Figures

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To Almut, Tobias, Niklas and Annika.

Preface

Sapere aude!

Immanuel Kant (1724-1804)

Numerical simulations play a key role in many areas of modern science and technology. They are necessary in particular when experiments for the underlying problem are too dangerous, too expensive or not even possible. The latter situation appears for example when relevant length scales are below the observation level. Moreover, numerical simulations are needed to control complex processes and systems. In all these cases the relevant problems may become highly complex. Hence the following issues are of vital importance for a numerical simulation:

- Efficiency of the numerical solvers:
  Efficient and fast numerical schemes are the basis for a simulation of 'real world' problems. This becomes even more important for realtime problems where the runtime of the numerical simulation has to be of the order of the time span required by the simulated process. Without efficient solution methods the simulation of many problems is not feasible. 'Efficient' means here that the overall cost of the numerical scheme remains proportional to the degrees of freedom, i.e., the numerical approximation is determined in *linear time* when the problem size grows e.g. to upgrade accuracy. Of course, as soon as the solution of large systems of equations is involved this requirement is very demanding.
- Accuracy and reliability of the numerical approximation:
  Even though in many cases it is possible to validate numerical simulations by comparisons with experimental data, these methods will be applied to problems where no knowledge of the behavior of the solution is available. In particular when design processes are based upon numerical simulations, the method has to produce a reliable and accurate approximation.
- Data compression:
  Given efficient numerical schemes, the complexity of 'real world' problems

from science and engineering is often still so large that 'ad hoc' discretization methods would overflow memory capacities even of computers of the next generations. Without efficient strategies for keeping the discrete problems as small as possible without sacrificing too much accuracy, these large scale and sometimes high-dimensional problems can not be treated appropriately. Of course, a sound mathematical analysis is required to determine the best accuracy/work balance that can be realized for a given problem.

All these issues are partly correlated and closely intertwined with the mathematical tools a numerical method is based upon. It is desirable that a tool allows the combination of these issues. In particular, the last issue in the above wish lists calls for the design of schemes that automatically adapt to a given problem at hand, so that the desired overall accuracy of the approximate solution is obtained at the expense of possibly few degrees of freedom in the discretization. For instance, in the context of finite elements one would want to employ a small mesh size where ever the approximated object exhibits small scale features or singularities. Such concepts have been studied and developed in various discretization settings such as finite volume, finite difference or finite element methods. Although by no means common yet in industrial applications these methods have achieved already a certain level of practical maturity that strongly encourages further efforts in this direction. On the other hand, on the level of a rigorous analysis many problems appear to remain widely open. Given such adaptive concepts, again it remains to efficiently solve the resulting hopefully reduced systems of equations which however may now exhibit somewhat different features than their counterparts based on uniform refinements. The multigrid method is a prominent example of a general methodology to deal with this task by exploiting the interaction of different scales. In many cases this works very well although matters are no longer covered by the available analysis.

Wavelets offer an alternative that perhaps to a somewhat larger extent offers a synthesis of the discretization and solution process in an adaptive solution framework. This is mainly due to what we will sometimes refer to as 'strong analytical properties' of wavelet bases mainly based on their good *localization* in frequency and physical space, their *cancellation properties* that entail quasi sparse representations of many relevant operators and of functions with isolated singularities, and last but not least on the fact that wavelet expansions induce isomorphisms between many function spaces and sequence spaces which offers a natural coupling between the continuous and discrete realm. Based on these properties a rigorous convergence and cost analysis for adaptive wavelet methods has been recently developed, [28, 29, 39, 42, 43]. In particular, it was proven in [28, 29] that for a large class of operator equations an appropriate adaptive wavelet method converges at an optimal rate with almost optimal complexity.

The prize to be paid for these features is the higher sophistication of the tool itself. This accounts to some extent for the fact that the application of these concepts to real life problems is still much less advanced than for the above mentioned more conventional schemes. In fact, the applicability of a wavelet basis for the numerical simulation of a given problem demands, of course, the availability of an appropriate basis. This requirement is twofold: First the bases should be adapted to the considered *problem* and second to the *domain* of interest. Adapting a basis to a problem is not only done by choosing functions with the appropriate smoothness. Many problems pose extra conditions that have to be met by a discretization. In this monograph we focus on both aspects, namely on the construction of appropriate wavelet bases even on complex domains and on the adaptation of such wavelet bases to differential operators involving the div- and **curl**-operator.

We are now going to describe the organization of this monograph in somewhat more detail:

In Chap. 1 we describe a construction of wavelet bases also on fairly general domains. This construction is based upon a domain decomposition and mapping strategy. The domain of interest is subdivided into non-overlapping subdomains which are mapped to a single reference cube. On the cube, tensor products of appropriate wavelet systems on the interval $(0,1)$ are used. This is described in Sect. 1.3. However, the understanding of the subsequent developments does not require working through all details of the construction. The important point is that these systems of functions are built in such a manner that certain relevant properties hold. We collect in Sect. 1.1 those properties that will later be needed for our analysis. Those who are not interested in the technical details of the construction are therefore invited to consult just this section before moving on to the subsequent topics. It should be mentioned that most of the described constructions are already realized in public domain software packages.

Chap. 2 is devoted to the adaptation of wavelet bases to differential problems in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$. In fact, another potential advantage is the flexibility of biorthogonal wavelet bases. As opposed to *orthonormal* wavelet bases the concept of *biorthogonal* systems leaves some freedom in the construction while still ensuring all relevant properties. This freedom can be used to fulfill additional requirements that are needed for the discretization of problems in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$. These spaces arise naturally in the variational formulation of a whole variety of partial differential equations. We construct biorthogonal wavelet bases for these spaces. In particular, we construct divergence- and curl-free wavelet bases which give rise to an orthogonal Hodge decomposition of spaces of vector fields. It should be noted that the properties of this Hodge decomposition are inherited by any finite dimensional subspaces spanned by any subsets of the full wavelet bases. This implies that for each (adaptive) selection of wavelet functions certain operators can be decoupled. Moreover, divergence- and curl-free wavelets can directly be used to discretize problems involving this as an extra condition. For instance, with regard to the incompressible Navier-Stokes equations the use of divergence-free wavelets allows one (at least on 'simple' domains) to replace the somewhat more complex saddle point problem by an elliptic one. By a fictitious domain approach these bases can also be used on complex domains – at the expense of having to deal again with a saddle point problem induced by the Lagrange multipliers for the boundary conditions. However, the dimension of the Lagrange multiplier spaces is in this case much smaller than that for the trial spaces on the domain.

In Chap. 3 we describe various applications of our constructions. We consider examples such as the Lamé equations for nearly incompressible material in linear elasticity, the incompressible Navier-Stokes equations in fluid dynamics and Maxwell's equations in electromagnetism. In all these equations, certain parameters appear that may come from the problem itself or from a discretization in time. First, we use the analytical properties of the wavelet bases used for the discretization in order to prove *robustness* and *optimality* of certain wavelet preconditioners. By *robust* we mean, that the condition of the preconditioned system should be independent of the involved parameters. By *optimal* we mean that the condition number of the arising linear systems is independent of the number of unknowns. This follows from the fact that (a properly scaled) wavelet representation of the differential operator defines a boundedly invertible mapping on ℓ_2 combined with the stability of Galerkin discretizations (which is for granted in connection with definite problems). Realizing these properties for the above function spaces is an important prerequisite for the application of the results in [28, 29] in this context.
While these applications are of more theoretical nature, the next two are of more experimental character. In Sect. 3.2 we use divergence-free wavelets for the analysis and simulation of turbulent flows. We show that the analytical properties of these wavelets offer a powerful tool both for the simulation of incompressible flows as well as for the analysis of given flow data. Using these bases allows one to resolve local effects and to separate errors in the data due to inexact measuring or approximate modeling. The multiscale structure of such flow fields is shown. Finally, we investigate the smoothness of the given flow fields in a certain *Besov* scale which is an indicator for the potential complexity reduction offered by an adaptive Wavelet-Galerkin method.

Finally, in Sect. 3.3 we study the hardening of an elastoplastic rod. The occurance of plasticity is due to certain local conditions on the stress of the material. These conditions involve point values of the stress. On the other hand, the displacement of the elastic part is determined by an elliptic partial differential equation. In order to combine the advantages of wavelet discretizations of elliptic problems with the pointwise correction, we use an elastic predictor-plastic corrector method with interpolatory wavelets for the stress correction. In this framework, we use the analytical properties of wavelet bases in order to detect plastic waves, i.e., regions of plasticity, directly from the wavelet coefficients. It turns out that a certain discrete sequence of weightened wavelet coefficients allows us to clearly identify plastic regions.

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Work is important, but it is not all. My family has always been the center of my life. My parents, my wife Almut and my children Tobias, Niklas and Annika are giving me the energy also for my scientific work.

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