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On the Estimation of Multiple Random Integrals and U-Statistics

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Preface

This lecture note has a fairly long history. Its starting point was an attempt to solve some limit problems about the behaviour of non-linear functionals of a sequence of independent random variables. These problems could not be solved by means of classical probabilistic methods. I tried to solve them with the help of some sort of Taylor expansion. The idea was to represent the functional we are investigating as a sum with a leading term whose asymptotic behaviour can be well described by means of classical results of probability theory and with some error terms whose effect is negligible. This approach worked well, but to bound the error terms I needed some non-trivial estimates. The proof of these estimates was interesting in itself, it was a problem worth of a closer study on its own right. So I tried to work out the details and to present the most important and most interesting results I met during this research. This lecture note is the result of these efforts.

To solve the problems I met I had to give a good estimate on the tail distribution of the integral of a function of several variables with respect to the appropriate power of a normalized empirical distribution. Beside this I also had to consider a generalized version of this problem when the tail distribution of the supremum of such integrals has to be bounded. The difficulties in these problems concentrate around two points.

(a) We consider non-linear functionals of independent random variables, and we have to work out some techniques to deal with such problems.
(b) The idea behind several arguments is the observation that independent random variables behave in many respects almost as if they were Gaussian. But we have to understand how strong this similarity is, and when we can apply the techniques worked out for Gaussian random variables. Beside this we have to find methods to deal with our problems also in such cases when the techniques related to Gaussian and almost Gaussian random variables do not work.

To deal with problem (a) I have discussed the theory of multiple random integrals and their most important properties together with the properties of the so-called (degenerate) U-statistics. I considered the Wiener–Itô integrals which are multiple Gaussian type integrals and provide a useful tool to handle non-linear functionals

of Gaussian sequences. I also proved some results about a good representation of the product of Wiener–Itô integrals or degenerate U-statistics as a sum of Wiener–Itô integrals or degenerate U-statistics. A comparison of these results indicates some similarity between the behaviour of Wiener–Itô integrals and degenerate U-statistics. I tried to present a fairly detailed discussion of Wiener–Itô integrals and degenerate U-statistics which contains their most important properties.

Problem (b) appeared in particular in the study of the supremum of a class of random integrals. It may be worth mentioning that there is a deep theory worked out mainly by Michel Talagrand which gives good estimates in such problems, at least in the case if only onefold integrals are considered. It turned out however that the results and methods of this theory are not appropriate to prove such estimates that I needed in this work. Roughly speaking, the problems I met have a different character than those investigated in Talagrand's theory. This point is discussed in more detail in the main text of this work, in particular in Chap. 18, which gives an overview of the problems investigated in this work together with their history. The problems get even harder if the supremum not only of onefold but also of multiple random integrals has to be estimated. Here some new methods are needed which we can find by refining some symmetrization arguments appearing in the theory of the so-called Vapnik–Červonenkis classes.

I have also considered an example in Chap. 2 which shows how to apply the estimates proved in this work in the study of some limit theorem problems in mathematical statistics. Actually this was the starting point of the research described in this work. I discussed only one example, but I consider it more than just an example. My goal was to explain a method that can help in solving some non-trivial limit problems and to show why the results of this lecture notes are useful in their investigation. I think that this approach works in a very general setting, but this is the task of future research. Let me also remark that to understand how this method works and how to apply it one does not have to learn the whole material of this lecture note. It is enough to understand the content of the results in Chap. 8 together with some results of Chap. 9 about the properties of U-statistics.

I had two kinds of readers in mind when writing this lecture note. The first kind of them would like to learn more about such problems in which relatively few independence is available, and as a consequence the methods of classical probability theory do not work in their study. They would like to acquire some results and methods useful in such cases, too. The second kind of readers would not like to go into the details of complicated, unpleasant arguments. They would restrict their attention to some useful methods which may help them in proving the limit theorem problems of probability theory they meet also in such cases when the standard methods do not work. This lecture note can be considered as an attempt to satisfy the wishes of both kinds of readers.

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Contents

Acronyms

$\Phi(u)$ Standard normal distribution function.

$\mathscr{F}$ It denotes generally a class of functions with some nice property. See e.g. page 22.

$S_n(f)$ The normalized sum $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(\xi_k)$ of independent identically distributed random variables with some test function f.

$\mu_n(A)$ The value of the empirical distribution on the set A.

$J_n(f)$ Onefold random integral with respect to a normalized empirical distribution.

$J_{n,k}(f)$ k-fold random integral with respect to a normalized empirical distribution.

$\int'$ The prime in the integral means that the diagonals are omitted from the domain of integration of a multiple integral.

$|S|$ The cardinality of a (finite) set S.

$I_{n,k}(f)$ U-statistic of order k with n sample points and kernel function f.

$I_{n,0}(c)$ U-statistic of order zero, where c is a constant.

$\mathrm{Sym}\, f$ Symmetrization of the function f.

μ_W White noise with reference measure μ.

$Z_{\mu,k}(f)$ k-fold Wiener–Itô integral with respect of a white noise with reference measure μ.

$P_j f$ The projection of the function f defined in the Euclidean space R^k to the subspace consisting of the functions not depending on the j-th coordinate.

$Q_j f$ The projection orthogonal to the projection P_j in the space of functions on R^k.

$f_V(x_{j_1}, \dots, x_{j_{|V|}})$ The canonical function depending on the arguments indexed by the set V which appears in the Hoeffding decomposition of the U-statistic $I_{n,k}(f)$.

$\mathscr{H}_{\mu,k}$	The class of functions which can be chosen as the kernel function of a k-fold Wiener–Itô integral with respect to a white noise with reference measure μ.
$\Gamma(k,l)$	The class of diagrams in the diagram formula for the product of a k-fold and an l-fold Wiener–Itô integral.
$F_\gamma(f,g)$	The kernel function of the Wiener–Itô integral corresponding to the diagram γ in the diagram formula for the product of two Wiener–Itô integrals. The kernel function $F_\gamma(f_1,f_2)$ corresponding to the coloured diagram γ in the diagram formula for the product of two degenerate U-statistics appears at page 125.
$\Gamma(k_1,\dots,k_m)$	The class of diagrams in the diagram formula for the product of Wiener–Itô integrals of order $k_1,k_2,\dots k_m$. The same notation is applied for the class of coloured diagrams in the diagram formula for the product of degenerate U-statistics.
$F_\gamma(f_1,\dots,f_m)$	The kernel function of the Wiener–Itô integral in the general form of the diagram formula corresponding to the diagram γ. The same notation is applied for the kernel function corresponding to a coloured diagram γ in the diagram formula for the product of degenerate U-statistics.
$\bar\Gamma(k_1,\dots,k_m)$	The class of closed diagrams in the diagram formula. The same notation for the class of closed coloured diagrams.
$H_k(u)$	The k-th Hermite polynomial with leading coefficient 1.
$\operatorname{Exp}(\mathscr{H}_\mu)$	The Fock space.
$O(\gamma)$ and $C(\gamma)$	The open and closed chains of a coloured diagram γ.
$O_2(\gamma)$	The set of open chains of length 2 in a coloured diagram with two rows.
$W(\gamma)$	An appropriate function of a coloured diagram γ appearing in the diagram formula for the product of degenerate U-statistics. It is defined in the case of the product of two degenerate U-statictics at page 126, in the general case at page 132.
$\bar I_{n,k}(f)$	Decoupled U-statistic of order k with n sample points.
$\bar I^{\varepsilon}_{n,k}(f)$	Randomized decoupled U-statistic of order k with n sample points.
$H_{n,k}(f)$	A random variable appearing in the definition of good tail behaviour for a class of integrals of decoupled U-statistics in Chap. 15.
$H_{n,k}(f\|G,V_1,V_2)$	A random variable playing central role in the proofs of Chaps. 16 and 17. It depends of a function of k variables, a diagram G and two subsets V_1 and V_2 of the set $\{1,\dots,k\}$.

$I_{n,k}(f(\ell))$	Generalized U-statistics.
$\bar{I}_{n,k}(f(\ell))$	Generalized decoupled U-statistics.
$\tilde{I}_{n,k}(f)$ and $\tilde{I}^{\varepsilon}_{n,k}(f)$	Some linear combinations of decoupled U-statistics and randomized decoupled U-statistics applied in the symmetrization argument of Chapter 15.
$\mathscr{G}$	A class of diagram defined in Chapter 16. applied in the proof of the main result.