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Yves Achdou • Guy Barles • Hitoshi Ishii
Grigory L. Litvinov

Hamilton-Jacobi Equations: Approximations, Numerical Analysis and Applications

Cetraro, Italy 2011

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Foreword

These lecture notes contain the material relative to the courses given at the CIME Summer School held in Cetraro, Italy, from August 29 to September 3, 2011. The topic was *Hamilton–Jacobi Equations: Approximations, Numerical Analysis and Applications*

The courses dealt mostly with the following subjects: first-order and second-order Hamilton–Jacobi–Bellman equations, properties of viscosity solutions, homogenization and asymptotic behaviours, mean field games, approximation and numerical methods, and idempotent analysis. The content of the courses went from an introduction to viscosity solutions to quite advanced topics, at the cutting edge of the research in the field. We believe that they opened perspectives on new and delicate issues.

This volume contains four courses

- Finite Difference Methods for Mean Field Games
Yves Achdou
- An Introduction to the Theory of Viscosity Solutions for First-Order Hamilton–Jacobi Equations and Applications
Guy Barles
- A Short Introduction to Viscosity Solutions and the Large Time Behavior of Solutions of Hamilton–Jacobi Equations
Hitoshi Ishii
- Idempotent/Tropical Analysis, the Hamilton–Jacobi and Bellman Equations
Grigory L. Litvinov

A fifth course held at the workshop by Panagiotis E. Souganidis of the University of Chicago (Homogenization and Approximation for Hamilton–Jacobi Equations) is not included in this volume.

The participants came from several countries (ordered decreasingly with the number of participants): Italy, France, the USA, Argentina, Austria, Chile, China, Germany, Japan, Greece, Iran, Rumania, Russia, Sweden and Vietnam.

On September 1st, Paola Loreti, Elvira Mascolo and Nicoletta Tchou organized a session open to the younger researchers. This “CIME-young” session allowed the doctoral students and posdoctoral researchers to present their new results.

Young Speakers

- Moreno Concezzi
Università Degli Studi Roma Tre, Italy
Numerical methods and applications-dynamic programming for HCS and fractional laplacian approximation
- Jean-Paul Daniel
Laboratoire Jacques-Louis Lions—Université Paris 6, France
A game interpretation for fully non linear equations with Neumann condition
- Tiziano De Angelis
Sapienza Università di Roma, Italy
Optimal stopping of a Hilbert space valued diffusion process
- Joscha Diehl
University of Berlin, Germany
Pathwise approach to rough Burger’s PDEs
- Benjamin Fehrman
University of Chicago, USA
Homogenization of systems of viscous Hamilton–Jacobi equations
- Giulio Galise
Università degli Studi di Salerno, Italy
Viscosity solutions of uniformly elliptic equations without boundary and growth conditions at infinity
- Anna Chiara Lai
Dipartimento di Scienze di Base e Applicate per l’Ingegneria, Sapienza Università di Roma, Italy
A multi-phalanx self-similar robot finger
- Roberto Mecca
Dipartimento di Matematica “G. Castelnuovo”, Sapienza Università di Roma, Italy
Shape from shading via photometric stereo technique a new differential approach
- Cristina Pucci
Dipartimento di Scienze di Base e Applicate per l’Ingegneria, Sapienza Università di Roma, Italy
Propagation of fronts in nonlinear diffusion equations
- F.J. Silva,
Dipartimento di Matematica “G. Castelnuovo”, Sapienza Università di Roma, Italy
A semi-Lagrangian scheme for a 1st order-infinite horizon mean field game model

Authors' Short Vitae

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- Hitoshi Ishii was a professor at Chuo University (Tokyo) (April 1989–March 1996) and at Tokyo Metropolitan University (April 1996–March 2001). He has been a professor at Waseda University, Tokyo, since April 2001 and also adjunct professor at King Abdulaziz University, Jeddah, since August 2011. His research interest has been in nonlinear partial differential equations and their applications.
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Contents

Finite Difference Methods for Mean Field Games 1

Yves Achdou

1 Introduction 1

2 Finite Difference Schemes 4

2.1 Description of the Schemes..... 4

2.2 Existence and A priori Bounds 9

2.3 A Fundamental Identity..... 14

2.4 Uniqueness 16

2.5 A priori Estimates for (21)–(22) with Local Operators Φ 16

3 Examples of Convergence Results 19

4 Algorithms for Solving the Discrete Linear Systems 24

4.1 Newton Methods for Solving (21)–(22) 24

4.2 Iterative Strategies for Solving (54) Based
on Eliminating \mathcal{U} 26

5 Some Simulations 31

6 The Planning Problem 35

6.1 Description of the Planning Problem 35

6.2 The Finite Difference Scheme and an Optimal
Control Formulation 36

6.3 Uniqueness 44

6.4 A Penalty Method 45

References 45

**An Introduction to the Theory of Viscosity Solutions
for First-Order Hamilton–Jacobi Equations and Applications** 49

Guy Barles

1 Introduction 49

2 Preliminaries: A Running Example..... 51

3	The Notion of Continuous Viscosity Solutions: Definition(s) and First Properties	53
3.1	Why a “Good” Notion of Weak Solution is Needed?	53
3.2	Continuous Viscosity Solutions	54
3.3	Back to the Running Example (I): The Value Function U is a Viscosity Solution of (7).....	56
3.4	An Equivalent Definition and Its Consequences	58
4	The First Stability Result for Viscosity Solutions.....	60
5	Uniqueness: The Basic Arguments and Additional Recipes	64
5.1	A First Basic Result.....	64
5.2	Several Variations	70
5.3	Finite Speed of Propagation	72
6	Discontinuous Viscosity Solutions, Discontinuous Nonlinearities and the “Half-Relaxed Limits” Method	74
6.1	Discontinuous Viscosity Solutions	74
6.2	Back to the Running Example (II): The Dirichlet Boundary Condition for the Value-Function.....	76
6.3	The Half-Relaxed Limit Method	77
6.4	Strong Comparison Results.....	81
7	Existence of Viscosity Solutions: Perron’s Method.....	82
8	Regularity Results	86
9	Convex Hamiltonians, Barron–Jensen Solutions.....	89
10	Large Time Behavior of Solutions of Hamilton–Jacobi Equations	92
10.1	Introduction	92
10.2	Existence and Regularity of the Solution	93
10.3	Ergodic Behavior.....	94
10.4	Asymptotic Behavior of $u(x, t) - ct$	97
10.5	The Namah–Roquejoffre Framework	98
10.6	The “Strictly Convex” Framework	100
10.7	Concluding Remarks.....	106
	References	107

A Short Introduction to Viscosity Solutions and the Large Time Behavior of Solutions of Hamilton–Jacobi Equations

Hitoshi Ishii

1	Introduction to Viscosity Solutions	114
1.1	Hamilton–Jacobi Equations	114
1.2	An Optimal Control Problem.....	115
1.3	Characterization of the Value Function	121
1.4	Semicontinuous Viscosity Solutions and the Perron Method	128
1.5	An Example	139
1.6	Sup-convolutions.....	141
2	Neumann Boundary Value Problems	143
3	Initial-Boundary Value Problem for Hamilton–Jacobi Equations	148
3.1	Initial-Boundary Value Problems	148

3.2	Additive Eigenvalue Problems	152
3.3	Proof of Comparison Theorem	155
4	Stationary Problem: Weak KAM Aspects	165
4.1	Aubry Sets and Representation of Solutions.....	166
4.2	Proof of Theorem 4.2	174
5	Optimal Control Problem Associated with (ENP)–(ID)	185
5.1	Skorokhod Problem	185
5.2	Value Function I.....	191
5.3	Basic Lemmas	194
5.4	Value Function II.....	202
5.5	Distance-Like Function d	208
6	Large-Time Asymptotic Solutions	211
6.1	Preliminaries to Asymptotic Solutions.....	214
6.2	Proof of Convergence	219
6.3	Representation of the Asymptotic Solution u_∞	222
6.4	Localization of Conditions (A9) $_{\pm}$	226
A.1	Local maxima to global maxima.....	229
A.2	A Quick Review of Convex Analysis.....	230
A.3	Global Lipschitz Regularity	235
A.4	Localized Versions of Lemma 4.2	238
A.5	A Proof of Lemma 5.4.....	242
A.6	Rademacher's Theorem	245
	References	247

Idempotent/Tropical Analysis, the Hamilton–Jacobi and Bellman Equations

Grigory L. Litvinov

1	Introduction	251
2	The Maslov Dequantization	253
3	Semirings and Semifields: The Idempotent Correspondence Principle.....	254
4	Idempotent Analysis	255
5	The Superposition Principle and Linear Equations	256
5.1	Heuristics	256
5.2	The Cauchy Problem for the Hamilton–Jacobi Equations	259
6	Convolution and the Fourier–Legendre Transform	260
7	Idempotent Functional Analysis	261
7.1	Idempotent Semimodules and Idempotent Linear Spaces	262
7.2	Basic Results	265
7.3	Idempotent b -semialgebras	266
7.4	Linear Operator, b -semimodules and Subsemimodules	267
7.5	Functional Semimodules	268
7.6	Integral Representations of Linear Operators in Functional Semimodules.....	270

7.7	Nuclear Operators and Their Integral Representations	272
7.8	The b -approximation Property and b -nuclear Semimodules and Spaces	272
7.9	Kernel Theorems for Functional b -Semimodules	273
7.10	Integral Representations of Operators in Abstract Idempotent Semimodules	273
8	The Dequantization Transform, Convex Geometry and the Newton Polytopes	275
8.1	Dequantization Transform: Algebraic Properties	276
8.2	Generalized Polynomials and Simple Functions	277
8.3	Subdifferentials of Sublinear Functions	278
8.4	Newton Sets for Simple Functions	279
9	Dequantization of Set Functions and Measures on Metric Spaces	280
10	Dequantization of Geometry	281
11	Some Semiring Constructions and the Matrix Bellman Equation	282
11.1	Complete Idempotent Semirings and Examples	282
11.2	Closure Operations	282
11.3	Matrices Over Semirings	283
11.4	Discrete Stationary Bellman Equations	284
11.5	Weighted Directed Graphs and Matrices Over Semirings	284
12	Universal Algorithms	287
13	Universal Algorithms of Linear Algebra Over Semirings	288
14	The Correspondence Principle for Computations	293
15	The Correspondence Principle for Hardware Design	293
16	The Correspondence Principle for Software Design	295
17	Interval Analysis in Idempotent Mathematics	296
	References	297

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