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Stochastic Geometry, Spatial Statistics and Random Fields

Asymptotic Methods



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*... Geometry is the knowledge of the
eternally existent.*

Plato, "Republic", 527.

Foreword

Geometric intuition is central to many areas of statistics and probabilistic arguments. This is particularly true in the areas covered by this book, namely stochastic geometry, random fields and random geometric graphs. Nevertheless, intuition must be followed with rigorous arguments if it is to become part of the general literature of probability and statistics. Many such arguments in this area deviate from traditional statistics, requiring special (and often beautiful) tools outside a working statistician's usual toolbox.

Professor Spodarev assembled a very impressive cast of instructors for a workshop in Söllerhaus in September 2009 in order to further the literature in this area and to introduce the topics to participating graduate students. I previously had the pleasure of hosting several of the contributors of this volume at workshop at BIRS in February 2009 on Random Fields and Stochastic Geometry and am certain the workshop in Söllerhaus was a tremendous success.

The success of the workshop is further evidenced from this volume of lecture notes. Professor Spodarev has managed to produce a volume that combines both introductory material and current research in these notes. This book will be a useful reference for myself in this area, as well as to all researchers with an interest in stochastic geometry.

Stanford, CA
March 2012

Jonathan Taylor

Preface

This volume is an attempt to write a textbook providing a modern introduction to stochastic geometry, spatial statistics, the theory of random fields and related topics. It has been made out of selected contributions to the Summer Academy on Stochastic Geometry, Spatial Statistics and Random Fields

<http://www.uni-ulm.de/summeracademy09>

which took place during 13–26 Sep 2009 at Söllerhaus, an Alpine conference centre of the University of Stuttgart and RWTH Aachen, in the village Hirschegg (Austria). It was organized by the Institute of Stochastics of Ulm University in cooperation with the Chair of Probability Theory of Lomonosov Moscow State University. In contrast with previous schools on this subject (Sandbjerg 2000, Martina Franca 2004, Sandbjerg 2007), this summer academy concentrated on the asymptotic theory of random sets, fields and geometric graphs. At the same time, it provided an introduction to more classical subjects of stochastic geometry and spatial statistics, giving (post)graduate students an opportunity to start their own research within a couple of weeks. The summer academy hosted 38 young participants from 13 countries (Australia, Austria, Denmark, Germany, France, Mongolia, Russia, Romania, Sweden, Switzerland, UK, USA and Vietnam). Twelve experts gave lectures on various domains of geometry, probability theory and mathematical statistics. Moreover, students and young researchers had the possibility to give their own short talks.

As it was pointed out above, this volume is focused on the asymptotic methods in the theory of random geometric objects (point patterns, sets, graphs, trees, tessellations and functions). It reflects advances in this domain made within the last two decades. This especially concerns the limit theorems for random tessellations, random polytopes, finite point processes and random fields.

The book is organized as follows. The first chapter provides an introduction to the theory of random closed sets (RACS). It starts with the foundations of geometric probability (Buffon needle problem, Bertrand's paradox) and continues with the classical theory of random sets by Matheron. Then it gives laws of large numbers and limit theorems for Minkowski sums and unions of independent

identically distributed (i.i.d.) RACS. Chapter 2 provides basics of the classical integral geometry and its applications to stereology, a part of spatial stochastics which deals with the reconstruction of the higher-dimensional properties of geometric objects from lower-dimensional sections. In Chap. 3, principal classes of spatial point processes (Poisson-driven point processes, finite point processes) are introduced. Their simulation and statistical inference techniques (partially using the Markov Chain Monte Carlo (MCMC) methods) are discussed as well. Chapter 4 provides an account of the theory of marked point processes and the asymptotic statistics for them in growing domains. Ergodicity, mixing and m -dependence properties of marked point processes are studied in detail. Random tessellation models are the matter of Chap. 5. There, Poisson-driven tessellations as well as Cox processes on them and hierarchical networks constructed on their basis are considered. Scaling limits for some characteristics of these networks are found. Applications to telecommunication networks are also discussed. Distribution tail asymptotics and limit theorems for the characteristics of the (large) typical cell of Poisson hyperplane and Poisson–Voronoi tessellations are given in Chap. 6. The shape of large cells of hyperplane tessellations as well as limit theorems for some geometric functionals of convex hulls of a large number of i.i.d. points within a convex body and of random polyhedra are dealt with in Chap. 7. Weak laws of large numbers and central limit theorems for functionals of finite point patterns are discussed in Chap. 8. Additionally, their applications to various topics ranging from optimization to sequential packings of convex bodies are touched upon. Chapter 9 surveys the elementary theory of random functions with the focus on random fields. Basic classes of random field models as well as an account of the correlation theory, statistical inference and simulation techniques are provided. Special attention is paid to infinitely divisible random functions. Dependence concepts for random fields (such as mixing, association, (BL, θ) -dependence) as well as central limit theorems for weakly dependent random fields are the subject of Chap. 10. They are applied to establish the limiting behaviour of the volume of excursion sets of weakly dependent stationary random fields. Chapter 11 focuses on almost sure limit theorems for partial sums (or increments) of random fields on \mathbb{N}^d such as laws of large numbers, laws of single or iterated logarithm and others. In the final chapter, the geometry of large rooted plane random trees with nearest neighbour interaction is studied. A law of large numbers, a large deviation principle for the branching type statistics and scaling limits of the tree are considered. Connections of these results with the solutions of some partial differential equations are discussed as well. Some of the chapters are written in a more formal and rigorous way than others which reflects the personal taste and style of the authors.

The topics of this volume are (almost) self-contained. Thus, we recommend Chaps. 1, 2 and 7 for the first acquaintance with the theory of random sets. A reader interested in the (asymptotic theory of) point processes might start reading with Chap. 3 and continue with Chaps. 4, 5 and 8 following the references to Chap. 1 if needed. Readers with an interest in tessellations and random polytopes might additionally read Chaps. 2, 6 and 7. To get a concise introduction to random fields and limit theorems for them, one could read Chaps. 9–11 occasionally following the

references to earlier chapters. For random graphs and trees, Chaps. 8 and 12 are a good starting point.

All in all, the authors hope that the present volume will be helpful to graduates and PhD students in mathematics to get a first glance of the geometry of random objects and its asymptotical methods. Written in the spirit of a textbook (with a significant number of proofs and exercises for active reading), it might be also instrumental to lecturers in preparing their own lecture courses on this subject.

Söllerhaus at Hirschegg
September 2011

Evgeny Spodarev

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Contents

1	Foundations of Stochastic Geometry and Theory of Random Sets	1
	Ilya Molchanov	
1.1	Geometric Probability and Origins of Stochastic Geometry	1
1.1.1	Random Points and the Buffon Problem	1
1.1.2	Random Lines	3
1.1.3	Sets Constructed from Random Points	6
1.2	Distributions of Random Sets	7
1.2.1	Definition of a Random Closed Set	7
1.2.2	Capacity Functional and the Choquet Theorem	10
1.2.3	Selections and Measurability Issues	12
1.3	Limit Theorems for Random Sets	14
1.3.1	Expectation of a Random Set	14
1.3.2	Law of Large Numbers and the Limit Theorem for Minkowski Sums	16
1.3.3	Unions of Random Sets	18
2	Introduction into Integral Geometry and Stereology	21
	Markus Kiderlen	
2.1	Integral Geometric Foundations of Stereology	21
2.1.1	Intrinsic Volumes and Kinematic Integral Formula	22
2.1.2	Blaschke–Petkantschin Formulae	29
2.2	Stereology	35
2.2.1	Motivation	35
2.2.2	Model-Based Stereology	36
2.2.3	Design-Based Stereology	42
3	Spatial Point Patterns: Models and Statistics	49
	Adrian Baddeley	
3.1	Models	50
3.1.1	Point Processes	50
3.1.2	Poisson Processes	53

3.1.3	Intensity	57
3.1.4	Poisson-Driven Processes	58
3.1.5	Finite Point Processes	64
3.2	Simulation	76
3.2.1	Basic Simulation Principles	76
3.2.2	Simulating the Poisson Process	87
3.2.3	Simulating Poisson-Driven Processes	90
3.2.4	Markov Chain Monte Carlo	92
3.3	Inference	100
3.3.1	Maximum Likelihood	100
3.3.2	MCMC Maximum Likelihood	104
3.3.3	Fitting Models Using Summary Statistics	106
3.3.4	Estimating Equations and Maximum Pseudolikelihood	109
4	Asymptotic Methods in Statistics of Random Point Processes	115
	Lothar Heinrich	
4.1	Marked Point Processes: An Introduction.....	115
4.1.1	Marked Point Processes: Definitions and Basic Facts	116
4.1.2	Higher-Order Moment Measures and Palm Distributions	123
4.1.3	Different Types of Marking and Some Examples	130
4.2	Point Process Statistics in Large Domains	132
4.2.1	Empirical K -Functions and Other Summary Statistics of Stationary PP's	133
4.2.2	The Role of Ergodicity and Mixing in Point Process Statistics	139
4.2.3	Kernel-Type Estimators for Product Densities and the Asymptotic Variance of Stationary Point Processes	140
4.3	Mixing and m -Dependence in Random Point Processes	142
4.3.1	Poisson-Based Spatial Processes and m -Dependence	142
4.3.2	Strong Mixing and Absolute Regularity for Spatial Processes.....	145
4.3.3	Testing CSR Based on Empirical K -Functions.....	148
5	Random Tessellations and Cox Processes	151
	Florian Voss, Catherine Gloaguen, and Volker Schmidt	
5.1	Random Tessellations	151
5.1.1	Deterministic Tessellations.....	152
5.1.2	Random Tessellations	154
5.1.3	Tessellation Models of Poisson Type	156
5.2	Cox Processes	159
5.2.1	Cox Processes and Random Measures.....	159
5.2.2	Cox Processes on the Edges of Random Tessellations....	161

5.3	Cox–Voronoi Tessellations.....	163
5.3.1	Local Simulation of the Typical Poisson– Voronoi Cell	164
5.3.2	Cox Processes on the Edges of PLT	165
5.4	Typical Connection Lengths in Hierarchical Network Models	168
5.4.1	Neveu’s Exchange Formula	168
5.4.2	Hierarchical Network Models.....	169
5.4.3	Distributional Properties of D^o and C^o	172
5.5	Scaling Limits	176
5.5.1	Asymptotic Behaviour of D^o	176
5.5.2	Asymptotic Behaviour of C^o	179
5.6	Monte Carlo Methods and Parametric Approximations	180
5.6.1	Simulation-Based Estimators	180
5.6.2	Parametric Approximation Formulae	181
6	Asymptotic Methods for Random Tessellations	183
	Pierre Calka	
6.1	Random Tessellations: Distribution Estimates	183
6.1.1	Definitions	184
6.1.2	Empirical Means and Typical Cell	186
6.1.3	Examples of Distribution Tail Estimates	191
6.2	Asymptotic Results for Zero-Cells with Large Inradius.....	196
6.2.1	Circumscribed Radius	196
6.2.2	Limit Theorems for the Number of Hyperfaces.....	201
7	Random Polytopes	205
	Daniel Hug	
7.1	Random Polytopes	205
7.1.1	Introduction	206
7.1.2	Asymptotic Mean Values	207
7.1.3	Variance Estimates and Limit Results	215
7.1.4	Random Polyhedral Sets	218
7.2	From Random Polytopes to Random Mosaics	223
7.2.1	Hyperplane Tessellations	225
7.2.2	Poisson–Voronoi Mosaics	230
7.2.3	The Shape of Typical Faces	232
8	Limit Theorems in Discrete Stochastic Geometry	239
	Joseph Yukich	
8.1	Introduction.....	239
8.1.1	Functionals of Interest.....	240
8.1.2	Examples	242
8.2	Subadditivity	244
8.2.1	Subadditive Functionals	244
8.2.2	Superadditive Functionals	245

8.2.3	Subadditive and Superadditive Euclidean Functionals ...	245
8.2.4	Examples of Functionals Satisfying Smoothness (8.20)	248
8.2.5	Laws of Large Numbers for Superadditive Euclidean Functionals	249
8.2.6	Rates of Convergence of Euclidean Functionals	251
8.2.7	General Umbrella Theorem for Euclidean Functionals ...	251
8.3	Stabilization	253
8.3.1	Homogeneous Stabilization	253
8.3.2	Stabilization with Respect to the Probability Density κ	255
8.3.3	A Weak Law of Large Numbers for Stabilizing Functionals	255
8.3.4	Variance Asymptotics and Central Limit Theorems for Stabilizing Functionals	258
8.3.5	Proof of Asymptotic Normality in Theorem 8.5; Method of Cumulants	261
8.3.6	Central Limit Theorem for Functionals of Binomial Input	266
8.4	Applications	269
8.4.1	Random Packing	269
8.4.2	Convex Hulls	271
8.4.3	Intrinsic Dimension of High Dimensional Data Sets	272
8.4.4	Clique Counts, Vietoris–Rips Complex	274
9	Introduction to Random Fields	277
	Alexander Bulinski and Evgeny Spodarev	
9.1	Random Functions	277
9.2	Some Basic Examples	283
9.2.1	White Noise	283
9.2.2	Gaussian Random Functions	283
9.2.3	Lognormal and χ^2 Random Functions	284
9.2.4	Cosine Fields	285
9.2.5	Shot-Noise Random Fields and Moving Averages	285
9.2.6	Random Cluster Measures	288
9.2.7	Stable Random Functions	288
9.2.8	Random Fields Related to Random Closed Sets	290
9.3	Markov and Gibbs Random Fields	290
9.3.1	Preliminaries	290
9.3.2	Energy and Potential	293
9.3.3	Averintsev–Clifford–Hammersley Theorem	298
9.3.4	Gaussian Markov Random Fields	301
9.3.5	Final Remarks	304
9.4	Moments and Covariance	304

9.5	Stationarity, Isotropy and Scaling Invariance	306
9.6	Positive Semi-definite Functions	309
9.6.1	General Results	309
9.6.2	Isotropic Case	311
9.6.3	Construction of Positive Semi-definite Functions	312
9.7	Infinitely Divisible Random Functions	314
9.7.1	Infinitely Divisible Distributions	314
9.7.2	Infinitely Divisible Random Measures	315
9.7.3	Infinitely Divisible Stochastic Integrals	318
9.7.4	Spectral Representation of Infinitely Divisible Random Functions	319
9.8	Elementary Statistical Inference for Random Fields	321
9.8.1	Estimation of the Mean	321
9.8.2	Estimation of the Covariance Function and Related Characteristics	323
9.8.3	Estimation of the Asymptotic Covariance Matrix	325
9.9	Simulation of Random Fields	328
9.9.1	Gaussian Random Fields	328
9.9.2	Shot-Noise Random Fields	331
9.9.3	Infinitely Divisible Random Fields	331
10	Central Limit Theorems for Weakly Dependent Random Fields	337
	Alexander Bulinski and Evgeny Spodarev	
10.1	Dependence Concepts for Random Fields	337
10.1.1	Families of Independent and Dependent Random Variables	338
10.1.2	Mixing Coefficients and m -Dependence	338
10.1.3	Association, Positive Association and Negative Association	340
10.1.4	Association on Partially Ordered Spaces	344
10.1.5	FKG-Inequalities and Their Generalizations	346
10.1.6	Quasi-association and Further Extensions	348
10.2	Moment and Maximal Inequalities for Partial Sums	350
10.2.1	Regularly Growing Sets	350
10.2.2	Variances of Partial Sums	352
10.2.3	Moment Inequalities for Partial Sums	355
10.2.4	Bounds Based on Supermodular Results	360
10.2.5	The Móricz Theorem	363
10.2.6	Rosenthal Type Inequalities	366
10.2.7	Estimates Obtained by Randomization Techniques	367
10.2.8	Estimates for the Distribution Functions of Partial Sums	368
10.3	Limit Theorems for Partial Sums of Dependent Random Variables	369
10.3.1	Generalization of the Newman Theorem	369
10.3.2	Corollaries of the CLT	374

10.3.3	Application to the Excursion Sets	375
10.3.4	The Newman Conjecture and Some Extensions of CLT	377
10.3.5	Concluding Remarks	382
11	Strong Limit Theorems for Increments of Random Fields	385
	Ulrich Stadtmüller	
11.1	Introduction.....	385
11.2	Classical Laws for Random Fields	387
11.3	Chow-Type Laws for Random Fields	389
11.4	Proofs	390
11.4.1	Sufficiency: The Upper Bound.....	393
11.4.2	Necessity	396
11.5	Boundary Cases	396
11.6	Exercises	397
12	Geometry of Large Random Trees: SPDE Approximation	399
	Yuri Bakhtin	
12.1	Infinite Volume Limit for Random Plane Trees.....	399
12.1.1	Biological Motivation	400
12.1.2	Gibbs Distribution on Trees. Law of Large Numbers and Large Deviation Principle for Branching Type	402
12.1.3	Infinite Volume Limit for Random Trees	405
12.2	From Discrete to Continuum Random Trees.....	408
12.2.1	Diffusion Processes	409
12.2.2	Convergence of the Process of Generation Sizes	410
12.2.3	Trees as Monotone Flows	412
12.2.4	Structure of the Limiting Monotone Flow.....	418
	References.....	421
	Index	441

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Acronyms

Sets

\mathbb{C}	Complex numbers
\mathbb{N}	Positive integer numbers
\mathbb{Q}	Rational numbers
\mathbb{R}	Real numbers
\mathbb{Z}	All integer numbers
\mathcal{K}^d	Class of all compact sets
$\mathcal{K}_{\text{conv}}^d$	Class of all convex bodies in \mathbb{R}^d
\mathcal{E}_k^d	Family of affine k -planes in \mathbb{R}^d
$\mathcal{B}_0(\mathbb{R}^d)$	Family of bounded Borel sets in \mathbb{R}^d
\mathbb{S}^{d-1}	Unit sphere in \mathbb{R}^d

Probability Theory

ξ, η	Random variables
X, Y	Random sets
\mathbf{P}_ξ	Probability measure of ξ
$\mathbf{E} \xi$	Expectation of ξ
$\text{corr}(\xi, \eta)$	Correlation of ξ and η
$\text{cov}(\xi, \eta)$	covariance of ξ and η
$\text{var } \xi$	Variance of ξ
\mathcal{N}	Set of all locally finite simple point patterns
\mathfrak{N}	Set of all locally finite counting measures
$\mathbf{1}$	Indicator function
$\text{Ber}(p)$	Bernoulli distribution with parameter p
$\text{Binom}(n, p)$	Binomial distribution with parameters n and p
$\text{Exp}(\lambda)$	Exponential distribution with parameter λ
$N(\mu, \sigma^2)$	Normal distribution with parameters μ and σ^2

$\text{Pois}(\lambda)$	Poisson distribution with parameter λ
$\text{Unif}(a, b)$	Uniform distribution on $[a, b]$

Graph Theory

\mathbb{G}	Graph
\mathbb{V}	Set of vertices
\mathbb{E}	Set of edges

Other Notations

E	Energy
V	Potential
T	Tessellation
diag	Diagonal matrix
dist	Distance function
conv	Convex hull
$ A $	Cardinality of a set A
$ a $	Absolute value of a number a
$ x , \ x\ _2$	Euclidean norm of a vector x
diam	Diameter
SO	Rotation group
$\text{Im } z$	Imaginary part of a complex number
$\text{Re } z$	Real part of a complex number
sgn	Signum function
Lip	Lipschitz operator
span	Linear hull
ν_d	Lebesgue measure
V_0, \dots, V_d	Intrinsic volumes
χ	Euler characteristic
K^r	r -neighbourhood of K
$\text{Int } K$	Interior of K
$K L$	Orthogonal projection of K onto L
o	The origin of \mathbb{R}^d
$B_r(o)$	Ball of radius r centred around the origin
T_x	Shift operator