Lecture Notes in Mathematics 2067

Editors:
J.-M. Morel, Cachan
B. Teissier, Paris

For further volumes:
http://www.springer.com/series/304

Frederik S. Herzberg

Stochastic Calculus with Infinitesimals

Springer

Frederik S. Herzberg
Bielefeld University
Institute of Mathematical Economics
Bielefeld, Germany

and

Ludwig Maximilian University of Munich
Munich Center for Mathematical Philosophy
Munich, Germany

ISBN 978-3-642-33148-0 ISBN 978-3-642-33149-7 (eBook)
DOI 10.1007/978-3-642-33149-7
Springer Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2012951626

Mathematics Subject Classification (2010): 03H05; 60G05; 91B25; 81Q30; 60G51; 60H05; 60H10

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Dedicated to Professor Edward Nelson on the occasion of his eightieth birthday in 2012

Abstract

This short monograph develops basic stochastic analysis—including Itô's formula, Girsanov's theorem, the Feynman–Kac formula, and results about Lévy processes with finite-variation jump part—and select applications in the framework of Edward Nelson's *Radically elementary probability theory* [Annals of Mathematics Studies, 117, Princeton, NJ: Princeton University Press, 1987]. This approach requires neither measure-theoretic probability theory nor functional analysis, but is based on a rigorous, yet elementary use of unlimited natural numbers and infinitesimals.

The underlying axiomatic framework, a modest subsystem of Nelson's Internal Set Theory (IST) [Bulletin of the American Mathematical Society, 83(6):1165–1198, 1977] and hence called Minimal Internal Set Theory, is truly elementary and can be easily motivated through the incompleteness of the Peano axioms or an ultrapower construction. (As a subsystem of IST, it is also conservative over—and hence consistent relative to—conventional mathematics, i.e. ZFC; moreover, a substantial fragment of it also admits an accessible relative consistency proof.)

In an excursion, the "radically elementary" approach to stochastic analysis will be employed to provide a "radically elementary" proof of the fundamental theorems of asset pricing. As an example for applications of Minimal Internal Set Theory in mathematical physics, a fully rigorous "radically elementary" definition of the Feynman path integral is proposed.

All these features recommend Minimal Internal Set Theory as a suitable framework for teaching stochastic analysis to finance or physics students without previous training in pure mathematics. The book is self-contained and written in expository style; in particular, it assumes no prior knowledge of nonstandard analysis.

Keywords Internal Set Theory; Infinitesimals; Nonstandard analysis; Itô's formula; Girsanov's theorem; Dynkin's formula; Feynman–Kac formula; Lévy processes; Fundamental theorems of asset pricing; Feynman path integral

Preface

This work continues Edward Nelson's programme of devising "radically elementary" approaches to analysis broadly conceived. This research agenda was initiated by Nelson in the mid-seventies through the invention of Internal Set Theory (**IST**) [59] and reached a first climax with the publication of *Radically Elementary Probability Theory*, which appeared in 1987 in the *Annals of Mathematics Studies* monograph series [60].

The objective of Nelson's 1987 monograph was to make the theory of stochastic processes (including continuous-time processes!) "readily available to anyone who can add, multiply, and reason" (from the preface [60, p. vii]) through an elementary, yet fully rigorous use of infinitesimals and unlimited numbers by invoking a very modest and easily accessible fragment of nonstandard analysis. The core concepts which make this possible are (a) the notion of a finite set with an unlimited number of elements and (b) the notion of a positive infinitesimal number; the point is that the employment of these concepts enables one to treat stochastic continuous-time phenomena as stochastic processes on finite probability spaces with discrete time lines of infinitesimal spacing.

This work extends Nelson's elementarization even to stochastic analysis, covering topics such as stochastic integration and differentiation (Itô's formula), change of measure (Girsanov's theorem), the link between diffusions and semi-elliptic partial differential equations (Dynkin's formula, Feynman–Kac formula), jump-diffusion processes (Lévy processes) as well as applications of stochastic analysis in financial economics (fundamental theorems of asset pricing), financial engineering (volatility invariance in the Black–Scholes model), and mathematical physics (rigorous definition of the Feynman path integral).

Viewed from an axiomatic perspective, we shall follow Nelson's example in assuming not just the axioms of conventional mathematics (say, Zermelo–Fraenkel set theory with Choice, **ZFC**) but also some elementary axioms that allow for basic nonstandard analysis; the resulting extension of **ZFC** will be called Minimal Internal Set Theory and is a subsystem of **IST**. Nelson [59] showed through an elaborate set-theoretic argument that **IST** is a conservative extension of **ZFC**; in Appendix A, we shall give a simple proof for the fact that at least a powerful

subsystem of Minimal Internal Set Theory is a conservative extension of **ZFC** and hence consistent relative to **ZFC**. In Appendix B, the relation of Minimal Internal Set Theory to Robinsonian nonstandard analysis is briefly discussed. The remainder of the text, however, requires no acquaintance with model theory or any other part of mathematical logic whatsoever.

Munich, May 2012 Frederik S. Herzberg

Acknowledgments

Most of this short book was written while I was a Visiting Fellow, at Professor Edward Nelson's invitation, at the Mathematics Department of Princeton University, during the six months leading up to Easter 2010. My work was financially supported by a scholarship of the German Research Foundation (DFG), which I hereby acknowledge with thanks. The preparation of a revised version took place while I was a Visiting Fellow at the Munich Center for Mathematical Philosophy of Ludwig Maximilian University Munich, on a scholarship funded by the Alexander von Humboldt Foundation, which I also hereby gratefully acknowledge.

Words can hardly express how thankful I am to Edward and Sarah Jones Nelson for the warm and kind way in which they welcomed my wife and me to Princeton, for their manifold support throughout our sojourn and beyond, and not least for numerous extremely helpful discussions, both mathematically and otherwise. Furthermore, I also thank the Mathematics Department of Princeton University for their hospitality during my stay at Princeton. Moreover, I would like to thank five anonymous referees and the Editorial Board of Lecture Notes in Mathematics for their thorough perusal of this book and their many valuable suggestions. Thanks are also due to Ute McCrory and the staff at Springer for steering this manuscript through the reviewing and production process.

I take this opportunity to express my sincere gratitude to Professors Sergio Albeverio, Robert M. Anderson, Peter Koepke, Hannes Leitgeb, and Frank Riedel—both for discussions on the subject of this book and related topics as well as for their generous academic support in general over several years past.

Finally, I would like to include two more personal words of thanksgiving: to my family, above all my wonderful wife Angélique and my son Christian, for their love and faithfulness. And not least,

It is right and good
That with full hearts and minds and voices
We should praise You, Father Almighty, the unseen God,
Through Your only Son, Jesus Christ our Lord,

Who has saved us by His death, paid the price of Adam's sin,
And reconciled us once again to You.
Glory be to You forever.
Amen.

From the *Exsultet* of the Anglican Easter liturgy

Contents

Introduction

In a visionary monograph, Edward Nelson [60] has constructed the fundamental building blocks of a "radically elementary" theory of continuous-time stochastic processes, based on a simplified axiomatic version of nonstandard analysis, viz. a subsystem of Internal Set Theory (**IST**, also introduced by Nelson one decade earlier [59]). Nelson [60] extensively studied the Wiener process, including Donsker's invariance principle and Lévy's martingale characterization of the Wiener process (*nota bene:* in a single theorem [60, Theorem 18.1]), in such a "radically elementary" setting. However, he left the—significantly simpler—task of developing a radically elementary stochastic analysis from these building blocks to others.

The first and thus far only paper on radically elementary stochastic calculus was written by Benoît [10], who proved basic versions of both Itô's formula and Girsanov's theorem in a radically elementary setting. Benoît's [10] main concern, however, was the characterization of the measure induced by the Wiener walk. van den Berg [13] has authored a finance course based on radically elementary probability theory, but does not develop a fully fledged stochastic calculus therein. Moreover, after the first draft of this work had been written, the author came across the research by van den Berg and Amaro [16] who build upon Benoît's [10] work and link stochastic differential equations with partial differential equations—however, within the full framework of Internal Set Theory rather than within the framework of radically elementary probability theory, and without providing a systematic treatment of Itô diffusions.[1]

[1]There is, of course, also a significant body of research on stochastic integration and stochastic differential equations within the Robinsonian framework of nonstandard analysis (based on saturated enlargements of superstructures, cf. Robinson and Zakon [68]), starting from the seminal work of Loeb [51] and Anderson [4]. Major contributions to this area of research include those by Lindstrøm [45–48], Keisler [41], Hoover and Perkins [37, 38], Stroyan and Bayod [74], Capiński and Cutland [21–23], and Osswald [64]. A survey of some of the earlier results as well as a nonstandard approach to potential theory and the theory of Dirichlet forms can be found in the volume by Albeverio et al. [3]. The very first application of nonstandard analysis to (the foundations of) probability theory was given by Robinson's student Allen R. Bernstein and Frank

In this book, we develop basic stochastic analysis in the framework of radically elementary probability theory. First, we shall define (and briefly discuss) the axiomatic system of radically elementary probability theory. This axiom system will be a small subsystem of Nelson's Internal Set Theory [59] and thus a moderate extension of the conventional Zermelo–Fraenkel set theory including the Axiom of Choice (**ZFC**). This new axiomatic system, henceforth referred to as Minimal Internal Set Theory, comes in three variants of slightly different strength, viz. **minIST**$^+$, **minIST** and **minIST**$^-$, where **minIST**$^+$ contains **minIST** and **minIST** contains **minIST**$^-$. The results of this work will only depend on **minIST**, and much of radically elementary stochastic analysis can even be developed in **minIST**$^-$, the weakest of these axiom systems.

A short review of radically elementary probability theory—which is nothing more than finite probability theory with the additional axioms of Minimal Internal Set Theory at hand—will follow. After defining Wiener walks, Wiener processes and recalling some important results such as the radically elementary equivalent of Lévy's characterization of Wiener processes (Nelson's "de Moivre–Laplace–Lindeberg–Feller–Wiener–Lévy–Doob–Erdős–Kac–Donsker–Prokhorov theorem" [60, Theorem 18.1]), we will present the original contributions of this work.

These new results include radically elementary versions of the martingale representation theorem, Itô's formula, Girsanov's theorem, the diffusion invariance principle, the Markov property of Itô diffusions, Dynkin's formula, and the Feynman–Kac formula. Finally, we shall propose a radically elementary theory of Lévy processes. In addition, the book includes various excursions: a radically elementary discussion of certain "geometric" Itô processes (Sect. 3.4 of Chap. 3), a radically elementary approach to the fundamental theorems of asset pricing (Chap. 5), a rigorous radically elementary definition of the Feynman path integral (Chap. 8) as well as a proof of the conservativity of **minIST**$^-$ as an extension of **ZFC** (Appendix A). One of the excursions in this book (Chap. 8) suggests another area of application of Minimal Internal Set Theory within mathematical physics. We shall provide a rigorous, yet radically elementary definition of the Feynman path integral.

Most challenging to prove among these results is the radically elementary version of Girsanov's theorem. Just as Lévy's [44] classical martingale characterization of the Wiener process is a pivotal ingredient in the classical proof of Girsanov's theorem [27], we shall use the aforementioned radically elementary analogue of Lévy's martingale characterization of the Wiener process established by Nelson [60, Theorem 18.1]) in order to prove our radically elementary version of Girsanov's theorem.

The logical interdependence of the various parts of the book is as follows. Chapter 1 (axiomatic framework), Sect. 2.1 of Chap. 2 (Random variables and stochastic processes), Sect. 2.3 of Chap. 2 (Wiener walks), and the definitions

---

Wattenberg [17], not long after the appearance of Robinson's groundbreaking monograph *Nonstandard analysis* [67].

from Sect. 3.1 of Chap. 3 are basic and will be needed throughout this work. The discussion of Lévy processes (Chap. 9) assumes Sect. 3.5 of Chap. 3 (Lévy's characterization of Wiener processes). The proof of Girsanov's theorem (Chap. 4) assumes all of Chaps. 1–3, with the sole exception of Sect. 2.4 of Chap. 2 (which is optional). In particular, none of the results in the excursions will be used elsewhere in the text. The only exception to this rule is Sect. 3.4 of Chap. 3 (the excursion on certain "geometric" Itô processes), which will be used towards the end of the excursion on the fundamental theorems of asset pricing (Chap. 5). The brief informal introduction to Lévy finance in Sect. 9.5 of Chap. 9 assumes, of course, some familiarity with mathematical finance or financial economics, such as can be found in Chap. 5.

Thus the logical interdependencies within this book, excluding the contents of Chap. 4, may be visualized as follows:

$$\begin{tikzcd}
 & & \text{Chap. 1} \arrow[d] \arrow[rrr] & & & \text{Chap. 8} \\
 & & \text{Sects. 2.1 and 2.3 of Chap. 2} \arrow[d] & & & \\
 & & \text{def.s from Sect. 3.1 of Chap. 3} \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] \arrow[drrr] \arrow[rrr] & & & \text{Sect. 3.4 of Chap. 3} \arrow[d] \\
\begin{array}{c}\text{Chap. 5}\\ \text{except}\\ \text{Lemma 5.9}\end{array} & \text{Chap. 6} & & \text{Chap. 7} & \text{Chap. 9} & \begin{array}{c}\text{end of Chap. 5}\\ \text{(Lemma 5.9)}\end{array}
\end{tikzcd}$$

For Chap. 4, we have the following, very simple, chart (which we only include for the sake of completeness):

$$\begin{tikzcd}
\text{Chap. 1} \arrow[d] \\
\text{Chap. 2 w/o Sect. 2.4} \arrow[d] \\
\text{Chap. 3} \arrow[d] \\
\text{Chap. 4}
\end{tikzcd}$$

This work is self-contained, except for occasional references to some important results from Nelson's monograph [60], the content of which is fully described in this book. These are:

- The underspill/overspill principle [60, Theorem 5.4] (see Remark 1.1)
- The radically elementary characterization of a.s. convergence [60, Theorem 7.1] (see Remark 2.4)
- The radically elementary Radon–Nikodym theorem [60, Theorem 8.1] (see Remark 2.2)

- The radically elementary Lebesgue theorem [60, Theorem 8.2] (see Remark 2.3)
- The (near) equivalence of a.s. infinitely close processes [60, Theorem 17.2] (see Remark 2.1)
- A radically elementary martingale inequality [60, paragraph following Theorems 11.1 and 11.2] (see Remark 2.12)
- The a.s. continuity of normalized martingales with infinitesimal increments [60, paragraph following Theorem 13.1] (see Remark 3.5)
- The unified "de Moivre–Laplace–Lindeberg–Feller–Wiener–Lévy–Doob–Erdős–Kac–Donsker–Prokhorov theorem" [60, Theorem 18.1] (see Remark 3.13)

For those readers who intend to study some or all of the above results in greater detail by consulting Nelson's original text [60], we briefly summarize the logical interdependencies:

(1) The radically elementary characterization of a.s. convergence [60, Theorem 13.1] follows from the underspill/overspill principle [60, Theorem 5.4].
(2) The radically elementary Lebesgue theorem [60, Theorem 8.2] is a consequence of the radically elementary versions of the Radon–Nikodym theorem [60, Theorem 8.1] and the characterization of a.s. convergence [60, Theorem 7.1].
(3) The (near) equivalence of a.s. infinitely close processes [60, Theorem 17.2] follows from the radically elementary Lebesgue theorem [60, Theorem 8.2].
(4) The proof of the unified "de Moivre–Laplace–Lindeberg–Feller–Wiener–Lévy–Doob–Erdős–Kac–Donsker–Prokhorov theorem" uses the following results:

    (a) The underspill/overspill principle [60, Theorem 5.4]
    (b) The radically elementary Lebesgue theorem [60, Theorem 8.2]
    (c) A radically elementary supermartingale inequality [60, Theorem 11.1]
    (d) A continuity result for martingales [60, Theorem 13.1], which in turn depends on the limited-fluctuation criterion [60, Theorem 12.3] and by that means on [60, Theorem 11.1] and some upcrossing inequalities [60, Theorem 12.1–12.2]
    (e) The fact that the Lindeberg condition makes (a.s.) increments infinitesimal [60, Theorem 14.1], which depends on the radically elementary characterization of a.s. convergence [60, Theorem 7.1]
    (f) A truncation lemma for martingales satisfying the Lindeberg condition [60, Theorem 14.3], which depends again on [60, Theorem 7.1], on [60, Theorem 14.1] and on the underspill/overspill principle [60, Theorem 5.4]
    (g) The fact that a small change of the probability measure transforms a process into a (nearly) equivalent one [60, Theorem 17.1]
    (h) The (near) equivalence of a.s. infinitely close processes [60, Theorem 17.2]
    (i) The fact that near equivalence respects continuity [60, Corollary 2 to Theorem 17.3], which depends on the underspill/overspill principle [60, Theorem 5.4]