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Luigi Ambrosio • Alberto Bressan  
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# Modelling and Optimisation of Flows on Networks

Cetraro, Italy 2009

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# Preface

The present volume collects notes from lectures delivered for the CIME course on Modelling and optimisation of flows on networks, held in Cetraro in the summer of 2009.

In recent years modelling of flows on networks has been the subject of many investigations leading to an increasing number of research papers. Moreover, a wide set of possible applications, such as vehicular traffic, blood flow, supply chains and others, has directed the attention of mathematicians towards research domains usually populated by engineers, physicists or researchers with other expertise.

The aim of the CIME school was to gather summer courses which could give a wide view of modelling, analysis, numerics and control for dynamic flows on networks. Encompassing all application domains (including irrigation channels, data networks, air traffic management and others) was impossible, thus we focused on mathematical approaches, which are feasible for a number of applications, and a restricted set of specific applications, in particular vehicular traffic and supply chains. The attempt of finding a common ground, for different mathematical techniques to treat flows on networks, was already successful in a number of cases both at the level of research projects (such as the Italian national INDAM project 2005) and editorial initiatives (the foundation in 2006 of a new applied math journal entitled *Networks and Heterogeneous Media*).

The school took place in Cetraro, Italy, on June 15–19 2009. The course subjects were the following:

1. Introduction to conservation laws: Alberto Bressan (PennState)
2. Optimal transportation: Luigi Ambrosio (SNS, Pisa)
3. Pedestrian motions and vehicular traffic: Dirk Helbing (ETH)
4. Control and stabilization of waves on 1-D networks: Enrique Zuazua (BCAM)
5. Modelling and optimization of scalar flows on networks: Axel Klar (Kaiserlautern)
6. Fluid dynamic and kinetic models for supply chains: Christian Ringhofer (Arizona State)

## Rationale Behind the Choice of Courses for CIME School

Taking into account the above-mentioned scientific background, courses for the CIME school were chosen in order to give a wide view over main mathematical techniques and their applications in specific contexts.

1. Analysis and control of linear PDEs on networks
2. Analysis of nonlinear PDEs on networks
3. Optimization techniques for complex networks
4. Numerical methods for PDEs on networks

To cover the first topic and last one for the linear PDE aspect, we decided to focus on wave equations on networks of one-dimensional structures and, in particular, on the use of spectral methods. Therefore, the choice was made to contact Enrique Zuazua, Director of the Basque Center for Applied Mathematics and a world leader on the subject. Prof. Zuazua also authored a volume on the subject (SMAI series, Springer-Verlag, 2006).

In many applications it is natural to use conservation laws to model flows on networks, thus for the second course we contacted Alberto Bressan of PennState University, who was one of the major contributors of the theory of systems of conservation laws in last 20 years and author of a well-known monograph (Cambridge University Press, 2000).

The fourth topic for the nonlinear aspect was covered in courses dealing also with applications, and illustrated below, of Klar and Ringhofer. Finally, for the third topic, we individuated optimal transportation as one of the most suited mathematical framework, and thus decided to contact Luigi Ambrosio of Scuola Normale Superiore of Pisa, who authored various recent fundamental papers in the subject and a monograph on the related topics of gradient flows in metric spaces (Birkhauser, 2008).

For what concerns applications related to our main theme, Dirk Helbing of ETH of Zurich accepted to deliver a course covering both pedestrian dynamics and vehicular traffic. Helbing was one of the pioneers in providing advanced mathematical modelling for pedestrians with celebrated papers in *Nature*.

Then we focused on supply chain dynamics and thus contacted Christian Ringhofer of Arizona State University, who coauthored a pioneering paper in 2006 providing the first model of supply chains using PDEs. The course of Ringhofer dealt also with kinetic approaches.

Finally, Axel Klar of Kaiserlautern Technical University provided a course not only dealing with general modelling and numerics of conservation laws on networks but also treating coupled systems of ODEs and PDEs with examples from vehicular traffic, supply chains and sewer systems.

The present volume contains lecture notes from the first five courses of the CIME school. We wish readers a pleasant and fruitful reading.

Camden, NJ  
Nice, France

Benedetto Piccoli  
Michel Rascle



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